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On the damped nonlinear Schrödinger equation with delta functions as initial data

By

Kazuyuki Doi

Abstract

In this paper, we consider the Cauchy problem for the damped nonlinear Schrödinger equation with superposed delta functions as initial data. The aim of this paper is to study the influence of nonlinearity and damping (absorption) on the solution. The case where the nonlinearity makes the solution unstable is of special interest, because the dissipative nature from the damping comes into play in this case.

1. Introduction

We consider the Cauchy problem for the damped nonlinear Schrödinger equation

$$(1.1)\qquad \begin{cases} i\partial_t u + \Delta u + idu = \lambda \mathcal{N}(u) & \text{in } [0,\infty)\times \mathbf{R}^n, \\ u(0,x) = \sum_{j\in\mathbf{Z}} \mu_j \delta_{ja}(x) & \text{for } x \in \mathbf{R}^n, \end{cases}$$

where $i = \sqrt{-1}$, $n \geq 1$, $\partial_t = \partial/\partial t$, $\Delta = \sum_{k=1}^n \partial^2/\partial x_k^2$, and the unknown function $u = u(t,x)$ is complex-valued. The nonlinearity $\mathcal{N}(u)$ is given by

$$\mathcal{N}(u) = |u|^{p-1}u, \quad p > 1.$$

For $b \in \mathbf{R}^n$, let $\delta_b(x)$ denote the Dirac mass at b. In this paper we assume the condition

$$(1.2)\qquad p < 1 + \frac{2}{n}, \quad d \in \mathbf{R}, \quad a \in \mathbf{R}^n \setminus \{0\}, \quad \lambda \in \mathbf{C}, \quad \{\mu_j\} \in \ell_1^2,$$

where ℓ_α^2 is defined by

$$\ell_\alpha^2 := \{\{A_j\}_{j\in\mathbf{Z}}\,;\, \|\{A_j\}_{j\in\mathbf{Z}}\|_{\ell_\alpha^2} < \infty\},$$
$$\|\{A_j\}_{j\in\mathbf{Z}}\|_{\ell_\alpha^2} := \Big(\sum_{j\in\mathbf{Z}} (1+|j|^2)^\alpha |A_j|^2\Big)^{1/2}.$$

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We will abbreviate $\{A_j\}_{j\in\mathbf{Z}}$ to $\{A_j\}$ for simplicity of description.

This work is inspired by the pioneering work of Kita [3], [4], where the evolution of the superposition of delta functions under the nonlinear Schrödinger flow was studied. More precisely, when $d = 0$, Kita [3], [4] shows, among other things, the following results under the assumption (1.2): (i) there exists a local solution of (1.1) which has the special form like (1.3) below; (ii) if $\mathrm{Im}\lambda > 0$, then the local solution blows up in finite time; (iii) if $\mathrm{Im}\lambda \le 0$, then the solution exists globally in time. In particular, we see that if $\mathrm{Im}\lambda > 0$, then the nonlinearity makes the solution unstable. On the other hand, if $d > 0$, then the linear term idu has the stabilizing property, so that it is often called the damping term or absorbing term (for the literature on the damped nonlinear Schrödinger equation, we refer to [1], [2], [5], [6], [7] for instance). Therefore, the following question naturally arises: When $d > 0$ and $\mathrm{Im}\lambda > 0$, does the dissipative nature coming from the damping term violate the nonlinear effect or not? In fact, an analogous question was posed by Tsutsumi [7], where the Cauchy problem for the damped Schrödinger equation with supercritical nonlinearity was studied when the initial data is more regular and decays at spatial infinity, say $u(0) \in H^s(\mathbf{R}^n)$ $(s > n/2)$ and $|x|u(0,x) \in L^2(\mathbf{R}^n)$. In this situation, the answer is completely negative. However, Theorems 1.1 and 1.2 below tell us that the answer is different from the above result from [7], if the initial data is a superposition of delta functions.

First of all, we state the result on the local solution. For $d \in \mathbf{R}$ and $b \in \mathbf{R}^n$, let $U_d(t)\delta_b$ denote the solution of the damped linear Schrödinger equation

$$\begin{cases} i\partial_t u + \Delta u + idu = 0 & \text{in } [0,\infty)\times\mathbf{R}^n, \\ u(0,x) = \delta_b(x) & \text{for } x \in \mathbf{R}^n, \end{cases}$$

that is, $U_d(t)\delta_b = \exp((i\Delta - d)t)\delta_b = \exp(-dt)(4\pi it)^{-n/2}\exp(i|x-b|^2/4t)$.

Proposition 1.1 (local result)**.** *Assume* (1.2)*. Then there exist a time* $T > 0$ *and a unique solution of the Cauchy problem* (1.1) *described as*

$$u(t,x) = \sum_{j\in\mathbf{Z}} A_j(t)U_d(t)\delta_{ja} \in L^\infty_{\mathrm{loc}}((0,T];L^\infty(\mathbf{R}^n)), \tag{1.3}$$

such that $\{A_j(t)\} \in C([0,T];\ell^2_1) \cap C^1((0,T];\ell^2_1)$ *with* $A_j(0) = \mu_j$ $(j \in \mathbf{Z})$*.*

The idea of the proof of Proposition 1.1 is based on [4]. Specifically, we shall reduce the PDE (1.1) to the ODE system for $\{A_j\}$ and use the contraction mapping principle (see Section 3).

Next we collect some notation to state our main results. Let $\mathbf{T} = \mathbf{R}/2\pi\mathbf{Z}$ where $\mathbf{Z}$ stands for the set of integers. The Lebesgue space $L^q = L^q(\mathbf{T})$ is the Banach space of complex-valued q-summable functions on $\mathbf{T}$. The norm $\|\cdot\|_{L^q}$ denotes $\|g\|_{L^q} := \left(\int_0^{2\pi}|g(\theta)|^q\,d\theta\right)^{1/q}$ if $1 < q < \infty$, and $\|g\|_{L^\infty} := \mathrm{ess.sup}_{\theta\in\mathbf{T}}|g(\theta)|$. The Sobolev space $H^s = H^s(\mathbf{T})$ is defined by $H^s = \{g(\theta) \in L^2;\ \|g\|_{H^s} < \infty\}$, where $\|g\|_{H^s} = \|\{C_j\}\|_{\ell^2_s}$ with $C_j = (2\pi)^{-1/2}\int_0^{2\pi} g(\theta)e^{-ij\theta}\,d\theta$. Let us write $\langle g,h\rangle_\theta = \int_0^{2\pi} g(\theta)\overline{h(\theta)}\,d\theta$ for g and $h \in L^2$, where $\overline{h(\theta)}$ denotes the complex conjugate of $h(\theta)$. In addition, we define positive constants under the assumptions $d > 0$ and $\mathrm{Im}\lambda > 0$ (recall the assumption (1.2)):

$$\begin{aligned}
\varepsilon_0 &:= \Big(C_* \frac{d^{1-n(p-1)/2}}{\mathrm{Im}\lambda}\Big)^{1/(p-1)}, \\
\varepsilon_1 &:= \frac{1}{\sqrt{2\pi}}\Big(C_* \frac{d^{1-n(p-1)/2}}{C_0}\Big)^{1/(p-1)}, \\
C_* &:= \Big(\frac{4\pi}{p-1}\Big)^{n(p-1)/2} \Gamma\Big(1 - \frac{n(p-1)}{2}\Big)^{-1}, \\
C_0 &:= C_1 + C_2, \quad C_1 := \gamma_{p+1}^{p+1}\mathrm{Im}\lambda, \quad C_2 := p\gamma_\infty^{p-1}|\lambda|,
\end{aligned} \tag{1.4}$$

where Γ is the gamma function, and γ_{p+1}, γ_∞ are the best constants in Gagliardo-Nirenberg inequalities, namely, they are the smallest constants C satisfying

$$\|v\|_{L^{p+1}} \le C\|v\|_{L^2}^{(p+3)/(2(p+1))}\|v\|_{H^1}^{(p-1)/(2(p+1))}, \tag{1.5}$$

$$\|v\|_{L^\infty} \le C\|v\|_{L^2}^{1/2}\|v\|_{H^1}^{1/2} \tag{1.6}$$

for all $v \in H^1$, respectively.

Now we state the main results of this paper.

Theorem 1.1 (blowup result). *Assume* (1.2)*,* $d > 0$ *and* $\mathrm{Im}\lambda > 0$*. If* $\|\{\mu_j\}\|_{\ell_0^2} > \varepsilon_0$*, then the solution* (1.3) *blows up in finite time, that is, there exists a finite time* $T^* > 0$ *such that* $\lim_{t\to T^*-0} \|\{A_j(t)\}\|_{\ell_1^2} = \infty$.

Theorem 1.2 (global result). *Assume* (1.2)*,* $d > 0$ *and* $\mathrm{Im}\lambda > 0$*. If* $\|\{\mu_j\}\|_{\ell_1^2} \le \varepsilon_1$*, then there exists a unique global solution of* (1.1) *described as* (1.3)*, such that* $\{A_j(t)\} \in C([0,\infty);\ell_1^2) \cap C^1((0,\infty);\ell_1^2)$*. Moreover, if* $\|\{\mu_j\}\|_{\ell_1^2} < \varepsilon_1$*, then there exists a unique* $\{\nu_j\} \in \ell_1^2$ *such that* $\lim_{t\to\infty} \|\{A_j(t)\} - \{\nu_j\}\|_{\ell_1^2} = 0$.

Remark 1. Both ε_0 and ε_1 are independent of a, which is related to the support of the initial data $u(0,x) = \sum_{j\in\mathbf{Z}} \mu_j\delta_{ja}(x)$ (see (4.2) and (4.3)).

Remark 2. Theorems 1.1 and 1.2 can be rephrased in a slightly different form by treating the conditions on the initial value $\{\mu_j\} \in \ell_1^2$ as those on the damping coefficient. We define

$$\begin{aligned}
d_0 &:= \big(C_*^{-1}(\mathrm{Im}\lambda)\|\{\mu_j\}\|_{\ell_0^2}^{p-1}\big)^{1/(1-n(p-1)/2)}, \\
d_1 &:= \big(C_*^{-1}(2\pi)^{(p-1)/2}C_0\|\{\mu_j\}\|_{\ell_1^2}^{p-1}\big)^{1/(1-n(p-1)/2)}.
\end{aligned}$$

Then we have the following.

Corollary 1.1 (blowup result; see Remarks 4 and 7). *Assume* (1.2) *and* $\mathrm{Im}\lambda > 0$*. If* $d < d_0$*, then the solution* (1.3) *blows up in finite time, that is, there exists a finite time* $T^* > 0$ *such that* $\lim_{t\to T^*-0} \|\{A_j(t)\}\|_{\ell_1^2} = \infty$.

Corollary 1.2 (global result). *Assume* (1.2) *and* $\mathrm{Im}\lambda > 0$. *If* $d \geq d_1$, *then there exists a unique global solution of* (1.1) *described as* (1.3), *such that* $\{A_j(t)\} \in C([0,\infty);\ell_1^2) \cap C^1((0,\infty);\ell_1^2)$. *Moreover, if* $d > d_1$, *then there exists a unique* $\{\nu_j\} \in \ell_1^2$ *such that* $\lim_{t\to\infty} \|\{A_j(t)\} - \{\nu_j\}\|_{\ell_1^2} = 0$.

Remark 3. When $\|\{\mu_j\}\|_{\ell_0^2} \leq \varepsilon_0$ and $\varepsilon_1 < \|\{\mu_j\}\|_{\ell_1^2}$, we have not known in general whether the family $\{A_j(t)\}$ is global or not. However, in Section 2 we shall give an example, for which we can find the critical value of the size of the initial data.

Remark 4. In Corollary 1.1, we observe that the conditions $\mathrm{Im}\lambda > 0$ and $d \leq 0$ imply that $\{A_j(t)\}$ blows up in finite time for all nontrivial initial data $\{\mu_j\} \in \ell_1^2$ (see Remark 7). This remark and Theorem 1.2 signify that the signature of d determines the nature of the stability of the zero solution when $\mathrm{Im}\lambda > 0$.

2. An example: The special case $u(0,x) = \mu_0\delta_0(x)$

When the initial data of (1.1) is of a special form $u(0,x) = \mu_0\delta_0(x)$, we have the following stronger version of Proposition 1.1, Theorems 1.1 and 1.2.

Proposition 2.1. *Assume* (1.2) *and* $u(0,x) = \mu_0\delta_0(x)$.

1. *There exist a time* $T > 0$ *and a unique solution of the Cauchy problem* (1.1) *described as*

$$u(t,x) = A(t)U_d(t)\delta_0 \in L^\infty_{\mathrm{loc}}((0,T];L^\infty(\mathbf{R}^n)), \tag{2.1}$$

such that

$$A(t) = \begin{cases} \mu_0\Big(1-(p-1)(\mathrm{Im}\lambda)|\mu_0|^{p-1}\displaystyle\int_0^t f(\tau)\,d\tau\Big)^{i\lambda/((p-1)\mathrm{Im}\lambda)} & \\ & \text{if } \mathrm{Im}\lambda \neq 0, \\ \mu_0 \exp\Big(-i\lambda|\mu_0|^{p-1}\displaystyle\int_0^t f(\tau)\,d\tau\Big) & \\ & \text{if } \mathrm{Im}\lambda = 0, \end{cases}$$

where

$$f(t) := \big(\exp(-dt)(4\pi t)^{-n/2}\big)^{p-1}. \tag{2.2}$$

2. *Suppose* $d > 0$ *and* $\mathrm{Im}\lambda > 0$. *If* $|\mu_0| > \varepsilon_0$, *then the solution* (2.1) *blows up in finite time. More precisely, there exists a finite time* $T^* > 0$ *such that* $\lim_{t\to T^*-0} |A(t)| = \infty$.

3. *Suppose* $d > 0$ *and* $\mathrm{Im}\lambda > 0$. *If* $|\mu_0| \leq \varepsilon_0$, *then there exists a unique global solution of* (1.1) *described as* (2.1). *Moreover, if* $|\mu_0| < \varepsilon_0$, *then there exists a unique* $\nu_0 \in \mathbf{C}$ *such that* $\lim_{t\to\infty} A(t) = \nu_0$.

In this section we prove the above proposition.

Proof of the part 1. Substitution of (2.1) into (1.1) yields the ODE of $A(t)$:

$$\begin{cases} i\dfrac{dA}{dt} = \lambda f(t)\mathcal{N}(A), \quad t \in [0,T], \\ A(0) = \mu_0. \end{cases} \tag{2.3}$$

To solve (2.3), we multiply $\overline{A(t)}$ to both sides of the equation. Taking the imaginary part, we have

$$\frac{d}{dt}|A(t)|^2 = 2(\mathrm{Im}\lambda) f(t)|A(t)|^{p+1},$$

and hence we obtain

$$|A(t)|^{p-1} = \frac{|\mu_0|^{p-1}}{1-(p-1)(\mathrm{Im}\lambda)|\mu_0|^{p-1}\int_0^t f(\tau)\,d\tau}. \tag{2.4}$$

Note that the integral in (2.4) makes a sense when $p < 1 + 2/n$, and we find $\sup_{0\le t\le T} |A(t)| < C$ for sufficiently small $T > 0$. We have the solution $A(t)$ by substituting (2.4) into (2.3). This completes the proof of the part 1.

Proof of the part 2. By the assumptions $\mathrm{Im}\lambda > 0$ and $d > 0$, (2.4) and an elementary computation with a change of variables as $d(p-1)\tau = \sigma$, one obtains that $A(t)$ blows up in finite time if

$$1-(p-1)(\mathrm{Im}\lambda)|\mu_0|^{p-1}\int_0^\infty f(\tau)\,d\tau < 0, \quad \text{i.e. } |\mu_0| > \varepsilon_0.$$

Thus we have proven the part 2.

Proof of the part 3. From the argument in the proof of the part 2, we find immediately that if $|\mu_0| \le \varepsilon_0$, then $A(t)$ is global in time. Moreover if $|\mu_0| < \varepsilon_0$, then $A(t)$ converges to a $\nu_0 \in \mathbf{C}$:

$$\begin{aligned} \lim_{t\to\infty} A(t) &= \mu_0\Big(1-(p-1)(\mathrm{Im}\lambda)|\mu_0|^{p-1}\int_0^\infty f(\tau)\,d\tau\Big)^{i\lambda/((p-1)\mathrm{Im}\lambda)} \\ &= \mu_0\Big(1-\Big(\frac{|\mu_0|}{\varepsilon_0}\Big)^{p-1}\Big)^{i\lambda/((p-1)\mathrm{Im}\lambda)} =: \nu_0. \end{aligned}$$

This completes the proof. □

Remark 5. When $|\mu_0| = \varepsilon_0$, using l'Hôpital's rule yields

$$\begin{aligned} \lim_{t\to\infty}\Big(1-(p-1)(\mathrm{Im}\lambda)|\mu_0|^{p-1}\int_0^t f(\tau)\,d\tau\Big)^{-i\mathrm{Re}\lambda/\mathrm{Im}\lambda}\Big(\frac{A(t)}{e^{dt}t^{n/2}}\Big)^{p-1} \\ = (4\pi)^{n(p-1)/2}\frac{\mu_0^{p-1}d}{|\mu_0|^{p-1}\mathrm{Im}\lambda}. \end{aligned}$$

By taking the absolute value of both sides of the identity, in view of (2.4), we have the following relation:

$$\lim_{t\to\infty}\Big(\frac{|A(t)|}{e^{dt}t^{n/2}}\Big)^{p-1}=(4\pi)^{n(p-1)/2}\frac{d}{\mathrm{Im}\lambda},$$

that is, $|A(t)|=O(e^{dt}t^{n/2})$ as $t\to\infty$. It is the same order as $(U_d(t)\delta_0)^{-1}$. Therefore we conclude that $|u(t,x)|\to(d/\mathrm{Im}\lambda)^{1/(p-1)}$ as $t\to\infty$ for all $x\in\mathbf{R}^n$, that is, the absolute value of the solution of (1.1) described as (2.1) converges to the constant.

Remark 6. Analogously to the above argument, we can show that there exists a unique solution of the final value problem

$$\begin{cases} i\dfrac{dA}{dt}=\lambda f(t)\mathcal{N}(A), & t\in[0,\infty),\\ \lim\limits_{t\to\infty}A(t)=\nu_0, \end{cases}$$

such that

$$A(0)=\nu_0\Big(1+\Big(\frac{|\nu_0|}{\varepsilon_0}\Big)^{p-1}\Big)^{i\lambda/((p-1)\mathrm{Im}\lambda)}.$$

3. Proof of Proposition 1.1

In order to prove Proposition 1.1, we use the following lemma:

Lemma 3.1. *Let* $\{A_j(t)\}\in C([0,T];\ell^2_1)$. *Then we have*

$$\mathcal{N}\Big(\sum_{j\in\mathbf{Z}}A_j(t)U_d(t)\delta_{ja}\Big)=f(t)\sum_{j\in\mathbf{Z}}\widetilde{A}_j(t)U_d(t)\delta_{ja}, \tag{3.1}$$

where $\widetilde{A}_j(t)=(2\pi)^{-1}e^{-i|ja|^2/4t}\langle\mathcal{N}(v),e^{-ij\theta}\rangle_\theta$ *with*

$$v=v(t,\theta)=\sum_{j\in\mathbf{Z}}A_j(t)e^{-ij\theta}e^{i|ja|^2/4t},$$

and $f(t)$ *is defined by* (2.2).

The proof of this lemma can be reduced to the case $d=0$, and such an assertion is proved by Kita [4].

Now we shall show Proposition 1.1 following the argument in [4].

Proof of Proposition 1.1. Substituting (1.3) into (1.1), and using Lemma 3.1 and the fact that $i\partial_tU_d(t)\delta_{ja}+\Delta U_d(t)\delta_{ja}+idU_d(t)\delta_{ja}=0$, we see that

$$\sum_{j\in\mathbf{Z}}i\frac{dA_j}{dt}U_d(t)\delta_{ja}=\sum_{j\in\mathbf{Z}}\lambda f(t)\widetilde{A}_jU_d(t)\delta_{ja}.$$

Equating each term on both hand sides, we arrive at the following ODE system:

$$\begin{cases} i\dfrac{dA_j}{dt}=\lambda f(t)\widetilde{A}_j, & t\in[0,T], \quad j\in\mathbf{Z},\\ A_j(0)=\mu_j. \end{cases} \tag{3.2}$$

It suffices to prove the existence and uniqueness of (3.2) for proving Proposition 1.1. To solve (3.2), we deal with the following integral equation:

$$\begin{aligned}\{A_j(t)\} &= \{\mu_j\} - i\lambda \int_0^t f(\tau)\{\widetilde{A}_j(\tau)\}\,d\tau \\ &=: \Phi(\{A_k(t)\}), \quad t \in [0,T].\end{aligned} \tag{3.3}$$

By noting an inequality $e^{-d(p-1)t} \le \max\{1, e^{-d(p-1)T}\}$ for all $0 \le t \le T$, the same argument as [4] shows that for $T > 0$ sufficiently small, the mapping Φ is a contraction on the complete metric space

$$\big\{\{A_j\} \in L^\infty([0,T];\ell_1^2);\ \|\{A_j\}\|_{L^\infty([0,T];\ell_1^2)} \le 2\|\{\mu_j\}\|_{\ell_1^2}\big\}$$

furnished with the norm of $L^\infty([0,T];\ell_0^2)$. Hence a solution of (3.3) exists in $L^\infty([0,T];\ell_1^2)$. Furthermore, we have $\{A_j(t)\} \in C([0,T];\ell_1^2) \cap C^1((0,T];\ell_1^2)$ and the uniqueness of $\{A_j(t)\}$ in $C([0,T];\ell_0^2)$ in the same way as [4]. This completes the proof. □

4. Proofs of Theorems 1.1 and 1.2

From (3.2), $v = v(t,\theta) = \sum_{j\in\mathbf{Z}} A_j(t)e^{-ij\theta}e^{i|ja|^2/4t}$ defined in Lemma 3.1 satisfies

$$i\partial_t v = -\frac{|a|^2}{4t^2}\partial_\theta^2 v + \lambda f(t)\mathcal{N}(v), \tag{4.1}$$

where $f(t)$ is defined by (2.2). One can justify $\partial_t v$, $\partial_\theta^2 v$ and so on, by using the mollifier $P_r(\theta) = (2\pi)^{-1}\sum_{j\in\mathbf{Z}} r^{|j|}e^{ij\theta}$ $(0 < r < 1)$. However we avoid this kind of issue, for the sake of simplicity. Note that $\|\{A_j(t)\}\|_{\ell_0^2} = (2\pi)^{-1/2}\|v(t)\|_{L^2}$ and $\|\{jA_j(t)\}\|_{\ell_0^2} = (2\pi)^{-1/2}\|\partial_\theta v(t)\|_{L^2}$ by Parseval's identity. Thus it is sufficient to consider the norm $\|v(t)\|_{H^1}^2 (= \|v(t)\|_{L^2}^2 + \|\partial_\theta v(t)\|_{L^2}^2 = 2\pi\|\{A_j(t)\}\|_{\ell_1^2}^2)$. Now multiplying (4.1) by $\overline{v}$, integrating on $\mathbf{T}$ and taking the imaginary part, we obtain

$$\frac{d}{dt}\|v(t)\|_{L^2}^2 = 2(\mathrm{Im}\lambda)f(t)\|v(t)\|_{L^{p+1}}^{p+1}. \tag{4.2}$$

Applying ∂_θ to (4.1), multiplying $\overline{\partial_\theta v}$, integrating on $\mathbf{T}$ and taking the imaginary part, we obtain

$$\frac{d}{dt}\|\partial_\theta v(t)\|_{L^2}^2 = 2f(t)\mathrm{Im}\big(\lambda\langle\partial_\theta\mathcal{N}(v(t)), \partial_\theta v(t)\rangle_\theta\big). \tag{4.3}$$

We see that

$$\partial_\theta\mathcal{N}(v) = \frac{p+1}{2}|v|^{p-1}\partial_\theta v + \frac{p-1}{2}|v|^{p-3}v^2\overline{\partial_\theta v}$$

by simple calculations. Therefore we have

$$\|\partial_\theta\mathcal{N}(v)\|_{L^2} \le p\|v\|_{L^\infty}^{p-1}\|\partial_\theta v\|_{L^2}. \tag{4.4}$$

End of the proof of Theorem 1.1. Assume that the conclusion is false, that is, $\|\{A_j(t)\}\|_{\ell_1^2} < \infty$ for all $t \geq 0$. We shall show that this leads to contradiction. Notice that Hölder's inequality gives $\|v\|_{L^{p+1}}^{p+1} \geq (2\pi)^{-(p-1)/2}\|v\|_{L^2}^{p+1}$. Therefore (4.2) and the assumption $\mathrm{Im}\lambda > 0$ imply

$$\frac{d}{dt}\|v(t)\|_{L^2}^2 \geq 2(2\pi)^{-(p-1)/2}(\mathrm{Im}\lambda)f(t)\|v(t)\|_{L^2}^{p+1}.$$

Hence we have

$$\|v(t)\|_{L^2}^{p-1} \geq \frac{\|v(0)\|_{L^2}^{p-1}}{1-(p-1)(\mathrm{Im}\lambda)\|\{\mu_j\}\|_{\ell_0^2}^{p-1}\int_0^t f(\tau)\,d\tau}.$$

Thus we obtain the following: if the condition

$$1-(p-1)(\mathrm{Im}\lambda)\|\{\mu_j\}\|_{\ell_0^2}^{p-1}\int_0^\infty f(\tau)\,d\tau < 0, \tag{4.5}$$

i.e. $\|\{\mu_j\}\|_{\ell_0^2} > \varepsilon_0$ holds, then there exists a $T^* > 0$ such that $\lim_{t\to T^*-0}\|v(t)\|_{L^2} = \infty$. Therefore we find $\lim_{t\to T^*-0}\|\{A_j(t)\}\|_{\ell_0^2} = \infty$, which implies a contradiction. This completes the proof. □

Remark 7. The conditions $\mathrm{Im}\lambda > 0$ and $d \leq 0$ imply that $\{A_j(t)\}$ blows up in finite time for all nontrivial initial data $\{\mu_j\} \in \ell_1^2$. Indeed, (4.5) holds under the conditions, because $\int_0^\infty f(\tau)\,d\tau$ does not converge.

End of the proof of Theorem 1.2. We shall derive an *a priori* bound on $\|\{A_j(t)\}\|_{\ell_1^2}^2 (= (2\pi)^{-1}(\|v(t)\|_{L^2}^2 + \|\partial_\theta v(t)\|_{L^2}^2))$. Applying the Gagliardo-Nirenberg inequalities (1.5)-(1.6), Schwarz's inequality, and (4.4) to (4.2) and (4.3), we have

$$\begin{aligned}
\frac{d}{dt}\|v\|_{L^2}^2 &\leq 2(\mathrm{Im}\lambda)f(t)\gamma_{p+1}^{p+1}\|v\|_{L^2}^{(p+3)/2}\|v\|_{H^1}^{(p-1)/2} \\
&\leq 2C_1 f(t)\|v\|_{H^1}^{p+1}, \\
\frac{d}{dt}\|\partial_\theta v\|_{L^2}^2 &\leq 2f(t)|\lambda|\|\partial_\theta\mathcal{N}(v)\|_{L^2}\|\partial_\theta v\|_{L^2} \\
&\leq 2f(t)|\lambda|p\|v\|_{L^\infty}^{p-1}\|\partial_\theta v\|_{L^2}\|\partial_\theta v\|_{L^2} \\
&\leq 2f(t)|\lambda|p\gamma_\infty^{p-1}\|v\|_{L^2}^{(p-1)/2}\|v\|_{H^1}^{(p-1)/2}\|\partial_\theta v\|_{L^2}^2 \\
&\leq 2C_2 f(t)\|v\|_{H^1}^{p+1},
\end{aligned}$$

where the positive constants C_1 and C_2 are defined in (1.4). Thus we obtain

$$\frac{d}{dt}\|v\|_{H^1}^2 \leq 2C_0 f(t)\|v\|_{H^1}^{p+1}$$

with $C_0 = C_1 + C_2$. Hence we find

$$\|v(t)\|_{H^1}^{p-1} \leq \frac{\|v(0)\|_{H^1}^{p-1}}{1-(2\pi)^{(p-1)/2}(p-1)C_0\|\{\mu_j\}\|_{\ell_1^2}^{p-1}\int_0^t f(\tau)\,d\tau}. \tag{4.6}$$

Therefore we conclude the following: if the condition

$$1 - (2\pi)^{(p-1)/2}(p-1)C_0\|\{\mu_j\}\|_{\ell_1^2}^{p-1} \int_0^\infty f(\tau)\,d\tau \geq 0,$$

i.e. $\|\{\mu_j\}\|_{\ell_1^2} \leq \varepsilon_1$ holds, then $\{A_j(t)\} \in C([0,\infty);\ell_1^2) \cap C^1((0,\infty);\ell_1^2)$.

Furthermore we suppose $\|\{\mu_j\}\|_{\ell_1^2} < \varepsilon_1$. By (4.6) we have $\|\{A_j(t)\}\|_{\ell_1^2} = (2\pi)^{-1/2}\|v(t)\|_{H^1} < C$ for all $t \geq 0$. Define

$$\{\nu_j\} := \{\mu_j\} - i\lambda \int_0^\infty f(\tau)\{\widetilde{A}_j(\tau)\}\,d\tau.$$

Note that the integral above converges in ℓ_1^2 due to the following lemma:

Lemma 4.1 (Kita [4])**.** *Let $I = [0,T]$ or $I = [0,\infty)$. Then,*

$$\|\{\widetilde{A}_j\}\|_{L^\infty(I;\ell_1^2)} \leq C\|\{A_j\}\|_{L^\infty(I;\ell_1^2)}^p \tag{4.7}$$

holds.

Hence we have $\{\nu_j\} \in \ell_1^2$. Moreover by (4.7), we obtain

$$\begin{aligned}\|\{A_j(t)\} - \{\nu_j\}\|_{\ell_1^2} &\leq C\|\{A_j(t)\}\|_{L^\infty(0,\infty;\ell_1^2)}^p \int_t^\infty f(\tau)\,d\tau \\ &\leq C\int_t^\infty f(\tau)\,d\tau \longrightarrow 0 \quad (t \to \infty).\end{aligned}$$

The proof is complete. □

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Department of Mathematics
Graduate School of Science
Osaka University
Toyonaka, Osaka 560-0043
Japan
e-mail: k-doi@cr.math.sci.osaka-u.ac.jp

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