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The Faltings-Moriwaki modular height and isogenies of elliptic curves

By

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Abstract

We estimate the variation of the Faltings-Moriwaki modular height under isogenies of elliptic curves. In particular, we reprove the Faltings finiteness theorem for elliptic curves over fields of finite type over $\mathbb{Q}$. Our approach is "effective" in the sence that we generalize Raynaud's upper bound to higher dimensional cases.

1. Introduction

Let K be a field of finite type over $\mathbb{Q}$ with $\mathrm{tr.deg}_{\mathbb{Q}} K = d$, S a normal projective arithmetic variety with function field isomorphic to K, and $\bar{H}_1, \dots, \bar{H}_d$ C^∞-hermitian $\mathbb{Q}$-line bundles on S. Then the collection $\hat{S} = (S; \bar{H}_1, \dots, \bar{H}_d)$ of the data $S, \bar{H}_1, \dots, \bar{H}_d$ is called a polarization of K if $\bar{H}_1, \dots, \bar{H}_d$ are all nef (cf. Conventions and terminology below). We say that a polarization $\hat{S}$ is ample if $\bar{H}_1, \dots, \bar{H}_d$ are all ample (cf. *loc. cit.*). For an abelian variety A over K, there exists a Néron model $\mathcal{N}_S^1(A)$ of A in codimension one over S and we can define the L^1_{loc}-hermitian Hodge reflexive sheaf $\bar{\lambda}^{\mathrm{Fal}}(A/K; S)$ on S by extending the Hodge line bundle of $\mathcal{N}_S^1(A)$ endowed with the Faltings metric. Following [18], we define the Faltings-Moriwaki modular height by

$$h^{\hat{S}}(A) = \widehat{\deg}(\hat{c}_1(\bar{H}_1) \cdots \hat{c}_1(\bar{H}_d) \cdot \hat{c}_1(\bar{\lambda}^{\mathrm{Fal}}(A/K; S))),$$

which is a natural generalization of the Faltings modular height over algebraic number fields. In [18], Moriwaki proved the following finiteness property; if $\hat{S}$ satisfies a certain positivity condition, then, for any positive real number c, the number of isomorphism classes of abelian varieties over K with $h^{\hat{S}}(A) \le c$ is finite.

Let F be the algebraic closure of $\mathbb{Q}$ in K and U a non-empty open subscheme of S over which A extends to an abelian scheme. Then any abelian variety B over K which is K-isogenous to A extends to an abelian scheme over $U_{\mathbb{Q}} = U \times_{\mathbb{Z}} \mathrm{Spec}(\mathbb{Q})$. The main theorem of this paper is the following; (The meanings of "birationally semi-abelian reduction" and "ordinary reduction" are found in §2.1 and §3.3 respectively.)

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Theorem. *Suppose that the following conditions are satisfied.*

(1) $\hat{S}$ *is ample.*

(2) A *has birationally semi-abelian reduction over* S.

(3) A *is an elliptic curve over* K (*resp. there is a prime number* p *at which* A *has ordinary reduction*).

Then there exist mutually distinct horizontal arithmetic curves $C_1, \dots, C_n$ *on* S *with generic points* $\xi_1, \dots, \xi_n \in U$ *respectively such that, for any abelian variety* B *over* K *which is* K*-isogenous* (*resp.* p*-isogenous*) *to* A, *we have*

$$h^{\hat{S}}(A) - h^{\hat{S}}(B) = \frac{1}{m} \sum_{i=1}^{n} (h(\mathcal{N}_S^1(A)_{\xi_i}) - h(\mathcal{N}_S^1(B)_{\xi_i})),$$

where $m \cdot \deg(\hat{S}) = [F : \mathbb{Q}] \cdot \sum_{i=1}^{n} [k(\xi_i) : F]$ *and* $h(\mathcal{N}_S^1(A)_{\xi_i})$ *and* $h(\mathcal{N}_S^1(B)_{\xi_i})$ *are the Faltings modular heights of* $\mathcal{N}_S^1(A)_{\xi_i}$ *and* $\mathcal{N}_S^1(B)_{\xi_i}$ *respectively.*

In particular, by using Moriwaki's finiteness theorem and Raynaud's upper bounds, we have the following result.

Corollary. *If* A *is an elliptic curve over* K (*resp. there is a normal projective arithmetic variety* S *with function field isomorphic to* K *such that* A *has ordinary reduction at* p), *then the number of* K*-isomorphism classes in the* K*-isogeny class of* A (*resp. the* p*-isogeny class of* A) *is finite.*

As a corollary of these results, we obtain the Faltings-Zarhin theorem and the Šafarevič conjecture for elliptic curves over K.

It is well-known that these corollaries have been already proved by Faltings in [6] for all abelian varieties A over K and all prime numbers p. By using the complex Hodge theory (P. Deligne, Théorie de Hodge, II, §4, Publ. Math. I.H.É.S. **40** (1971)), the higher dimensional cases can be reduced to the case of algebraic number fields. We do not need to treat higher dimensional arithmetic varieties in Faltings's proof. It must be convenient for readers to recall his proof briefly. Let X be a smooth, geometrically integral, affine scheme over F, with function field isomorphic to K. Suppose that A extends to an abelian scheme $\mathcal{A}$ over X and X has an F-rational point $x \in X(F)$. We fix an embedding $F \hookrightarrow \mathbb{C}$ and denote by X^{an} the complex manifold associated to $X \times_F \mathrm{Spec}(\mathbb{C})$. Let $\bar{X} = X \times_F \mathrm{Spec}(\bar{F})$, $\mathcal{A}_x$ the fiber of $\mathcal{A}$ at x, and $\mathcal{A}_x^{\mathrm{an}}$ the complex abelian variety $\mathcal{A}_x \times_F \mathrm{Spec}(\mathbb{C})$. Then $\mathrm{Gal}(\bar{K}/K)$ acts on $T_p(A) = T_p(\mathcal{A}_x)$ via the étale fundamental group, $\mathrm{Gal}(\bar{K}/K) \to \pi_1^{\text{ét}}(X, x)$, which fits into the split exact sequence

$$0 \to \pi_1^{\text{ét}}(\bar{X}, \bar{x}) \to \pi_1^{\text{ét}}(X, x) \to \mathrm{Gal}(\bar{F}/F) \to 0.$$

Here $\pi_1^{\text{ét}}(\bar{X}, \bar{x})$ is isomorphic to the profinite completion of the topological fundamental group $\pi_1(X^{\mathrm{an}}, \bar{x})$. Since $T_p(\mathcal{A}_x) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = \mathrm{H}_1(\mathcal{A}_x^{\mathrm{an}}, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}_p$ is a semi-simple $\pi_1(X^{\mathrm{an}}, \bar{x})$-module (cf. *loc. cit.*Theorem (4.2.6)) and the Tate conjecture holds for $\mathcal{A}_x/F$, $T_p(A) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is a semi-simple $\mathrm{Gal}(\bar{K}/K)$-module.

Moreover, by using [8, p.60, Théorème], we have

$$\begin{aligned}\mathrm{End}_K(A) \otimes_{\mathbb{Z}} \mathbb{Z}_p &= \mathrm{End}_X(\mathcal{A}) \otimes_{\mathbb{Z}} \mathbb{Z}_p \\ &= \mathrm{End}_{\pi_1^{\text{ét}}(X)}(T_p(\mathcal{A})) \cap \mathrm{End}_F(\mathcal{A}_x) \otimes_{\mathbb{Z}} \mathbb{Z}_p \\ &= \mathrm{End}_{\pi_1^{\text{ét}}(X)}(T_p(\mathcal{A})) \cap \mathrm{End}_{\mathrm{Gal}(\bar{F}/F)}(T_p(\mathcal{A}_x)) \\ &= \mathrm{End}_{\mathrm{Gal}(\bar{K}/K)}(T_p(A)).\end{aligned}$$

The key step to prove our main theorem is to extend semi-abelian schemes over two dimensional local schemes; i.e.

Lemma. *Let S be a regular local scheme of dimension two, s the closed point of S, and K the function field of S. Let A, B be two abelian varieties over K and $\phi : A \to B$ a K-isogeny. Suppose that A extends to a semi-abelian scheme $\mathcal{A}$ over S and $\mathcal{A}_s$ satisfies one of the following three conditions;*
(i) *$\mathcal{A}_s$ is a torus over $k(s)$.*
(ii) *$\mathcal{A}_s$ is an ordinary abelian variety over $k(s)$.*
(iii) *$\mathcal{A}_s$ is an elliptic curve over $k(s)$.*
Then B extends to a semi-abelian scheme over S.

The case (iii) is a consequence of the purity theorem of de Jong-Oort (cf. [13, Theorem 5.1]). We can verify that, by using the Faltings-Moriwaki modular height, the Tate conjecture for any abelian variety over K follows from the purity theorem for semi-abelian schemes [5, Chapter V, Theorem 6.7]. However, it unfortunately does not hold in general by the example due to Raynaud, Ogus, Gabber, de Jong, and Oort (cf. [13, §6]). Similarly, the Tate conjecture for elliptic curves over K follows from the purity theorem for semi-stable curves, which is proved by de Jong and Oort in [13]. The above lemma shall be proved in §2.3 in a sufficiently general form, which is essentially due to Faltings and Chai, and allows us to prove a slightly stronger result than using the purity theorem of de Jong-Oort. The author hopes that our result would be used to give an effective version of the Šafarevič conjecture for elliptic curves over K.

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[*1] Moriwaki informed the author that G. Freixas i Montplet gave a geometric proof to a special case of the Tate conjecture (the case where A has abelian reduction over a regular projective arithmetic surface S) by using his original height morphism on suitable cycles of A_g^*.

1.1. Conventions and terminology

We fix several conventions and terminology of this paper.

(1.1.1) In this paper, all schemes are assumed to be separated. A scheme S is said to be of characteristic zero if the structure morphism $S \to \mathrm{Spec}(\mathbb{Z})$ factors through $\mathrm{Spec}(\mathbb{Q})$. Let G be a group scheme, n a positive integer, and p a prime number. Then $[n] : G \to G$ denotes the multiplication by n, $G[n]$ the kernel of $[n]$, and $G[p^\infty] = \varinjlim G[p^n]$.

(1.1.2) An *arithmetic variety* is an integral scheme S which is flat, quasi-projective, and generically smooth over $\mathrm{Spec}(\mathbb{Z})$. We denote the associated complex manifold $(S \times_{\mathbb{Z}} \mathrm{Spec}(\mathbb{C}))^{\mathrm{an}}$ by S_∞. Throughout this paper, the Krull dimension of $S \times_{\mathbb{Z}} \mathrm{Spec}(\mathbb{Q})$ is denoted by d.

(1.1.3) The set of isomorphism classes of C^∞-hermitian line bundles on S is denoted by $\widehat{\mathrm{Pic}}(S)$. Let $\bar{L}_1, \dots, \bar{L}_r$ be C^∞-hermitian line bundles on S and $m_1, \dots, m_r$ integers. The tensor product $\bar{L}_1^{\otimes m_1} \otimes \cdots \otimes \bar{L}_r^{\otimes m_r}$ is denoted by

$$m_1 \bar{L}_1 + \cdots + m_r \bar{L}_r$$

in the additive way like divisors.

An element $\bar{L}$ of $\widehat{\mathrm{Pic}}(S) \otimes_{\mathbb{Z}} \mathbb{Q}$ is called a C^∞-hermitian $\mathbb{Q}$-line bundle on S.

- We say that $\bar{L}$ is *ample* if there is a positive integer m such that mL is an ample line bundle on S, $c_1(m\bar{L})$ is positive on S_∞, and mL is generated by $\{\sigma \in \mathrm{H}^0(S, mL) \mid \|\sigma\|_{\sup} < 1\}$ at each point.
- Suppose that S is projective. $\bar{L}$ is said to be *nef* if $c_1(\bar{L})$ is semipositive on S_∞ and, for any one-dimensional closed integral subscheme $\Gamma \subseteq S$, $\widehat{\deg}(\bar{L}|_\Gamma) \geq 0$.
- $\bar{L}$ is said to be *big* if there is a positive integer m such that $mL_{\mathbb{Q}}$ is a big line bundle on $S_{\mathbb{Q}}$ and mL has a non-zero global section σ with $\|\sigma\|_{\sup} < 1$.
- We say $\bar{L}$ is $\mathbb{Q}$-*effective* if there is a positive integer m such that mL has a non-zero global section σ with $\|\sigma\|_{\sup} \leq 1$. We write $\bar{L} \geq_{\mathbb{Q}} 0$ if $\bar{L}$ is $\mathbb{Q}$-effective.

(1.1.4) Let $\widehat{\mathrm{CH}}^q_{L^1}(S)$ be the group of arithmetic L^1_{loc}-cycles of codimension q on S. Then we have the intersection product

$$\widehat{\mathrm{Pic}}(S) \otimes_{\mathbb{Z}} \widehat{\mathrm{CH}}^q_{L^1}(S) \to \widehat{\mathrm{CH}}^{q+1}_{L^1}(S)$$

and, if S is projective, the degree map

$$\widehat{\deg} : \widehat{\mathrm{CH}}^{\dim S}_{L^1}(S) \to \widehat{\mathrm{CH}}^1(\mathrm{Spec}(\mathbb{Z})) \simeq \mathbb{R}.$$

For detailed expositions on the arithmetic intersection theory, we refer to [7] and [15].

2. Extension of semi-abelian schemes

2.1. Semi-abelian reduction

Let K be a field. A *model* S of K is a noetherian integral scheme whose function field (that is, the local ring at the generic point of S) is isomorphic to K. Let A be an abelian variety over K and S a normal model of K. A *big* open subscheme V of S is a non-empty open subscheme V of S with $\mathrm{codim}(S\backslash V, S) \geq 2$ or, equivalently, with $\mathrm{depth}(\mathcal{O}_{S,x}) \geq 2$ for any $x \in S \setminus V$. A *Néron model* of A over S is a quasi-compact smooth group scheme $\mathcal{N}_S(A)$ over S such that

(i) the generic fiber of $\mathcal{N}_S(A)$ is K-isomorphic to A and

(ii) $\mathcal{N}_S(A)$ has the Néron mapping property [2, 1.2/1] over S; i.e. for each smooth scheme $\mathcal{X}$ over S,

$$\mathrm{Hom}_S(\mathcal{X}, \mathcal{N}_S(A)) \simeq \mathrm{Hom}_K(\mathcal{X}_K, A)$$

holds.

Proposition 2.1.1. *The naturally orderd set*

$$\left\{ V \subseteq S \;\middle|\; \begin{array}{c} V \textit{ is an open subscheme over which} \\ \textit{a Néron model } \mathcal{N}_V(A) \textit{ of } A \textit{ exists} \end{array} \right\}$$

has the maximal element $V_A = V_{A,S}$ and V_A is big in S. We denote a Néron model $\mathcal{N}_{V_A}(A)$ of A over V_A by $\mathcal{N}_S^1(A)$, which is unique up to canonical S-isomorphism, and call it a Néron model of A in codimension one over S ([18, Proposition 1.1.1]).

A split torus over S is a group scheme over S which is S-isomorphic to a product of finitely many copies of the multiplicative group and a torus over S is a group scheme over S which is étale locally isomorphic to a split torus. A semi-abelian scheme over S is a quasi-compact smooth commutative group scheme over S whose geometric fiber at each point of S is an extension of an abelian variety by a split torus. In particular, every geometric fiber of a semi-abelian scheme is connected.

• We say that A has abelian (resp. semi-abelian) reduction over S if there is an abelian scheme (resp. a semi-abelian scheme) over S whose generic fiber is K-isomorphic to A.

• We say that A has abelian (resp. semi-abelian) reduction in codimension one over S if there is a big open subscheme $V \subseteq S$ over which A has abelian (resp. semi-abelian) reduction.

• Moreover, A is said to have *birationally* abelian (resp. semi-abelian) reduction over S if there is a birational proper morphism $T \to S$ of normal models of K and A has abelian (resp. semi-abelian) reduction over T. We note that $T \to S$ is isomorphic over a big open subscheme of S.

Let S_1 be an integral closed subscheme of S of codimension one with generic point ξ and S_1' the normalization of S_1. Suppose that $\mathcal{N}_S^1(A)_\xi$ is an abelian variety over $k(\xi)$. Then it is clear that, if A has birationally semi-abelian reduction over S, then $\mathcal{N}_S^1(A)_\xi$ has also birationally semi-abelian reduction over S'_1.

Our results in this paper are owed to the following basic fact, which is one of the key steps in the proof of Gabber's lemma (cf. [3, Lemme 1.15] or [24, Exposé V]). For reader's convenience, we would like to include a proof of it here.

Lemma 2.1.1. *Let S be a noetherian scheme, U a dense open subscheme of S, G a group scheme of finite type over S, and H a closed subscheme of G. If $H|_U$ is a subgroup scheme of $G|_U$ and H is flat over S, then H is a subgroup scheme of G.*

Proof. Let $\tau : H \times_S H \to G \times_S H$ be the translation corresponding to the inclusion $H \to G$ and $\tilde{H}$ the closed subscheme of $G \times_S H$ defined by the kernel of $\mathcal{O}_{G\times_S H} \to \tau_* \mathcal{O}_{H\times_S H}$. Then $\tilde{H}$ is flat over S. Since τ coincides with the translation $(H|_U) \times_U (H|_U) \to (H|_U) \times_U (H|_U)$ over U, we have $\tilde{H}|_U = (H|_U) \times_S (H|_U)$. Consider the cartesian square

$$\begin{array}{ccc} \tilde{H}|_U & \longrightarrow & U \\ \downarrow & & \downarrow \\ \tilde{H} & \longrightarrow & S. \end{array}$$

By the "flat base chage" [9, IV$_2$, Proposition (2.3.2)], we have

$$\begin{aligned} \tilde{H} &= \text{the scheme-theoretic closure of } (H|_U) \times_S (H|_U) \text{ in } \tilde{H} \\ &= \text{the scheme theoretic closure of } (H|_U) \times_S (H|_U) \text{ in } G \times_S H \\ &\subseteq H \times_S H \end{aligned}$$

because the scheme-theoretic closure of U in S is S. Considering the composition

$$H \times_S H \xrightarrow{\ \tau\ } G \times_S H \xrightarrow{\ \mathrm{pr}_1\ } G$$

and using the transitivity [9, I, Proposition (9.5.5)], we can conclude that the multiplication $H \times_S H \to G$ factors through H. By the same arguments, we can easily verify that both the identity section $S \to G$ and the inverse $H \to G$ also factor through H and these three morphisms satisfy the axioms of group schemes. □

Lemma 2.1.2. *Let K be a field, S a normal model of K, and A, B two abelian varieties over K. We suppose that there is a K-isogeny $\phi : A \to B$ and A extends to a semi-abelian scheme $\mathcal{A}$ over S. Then there is a blow-up $\pi : \tilde{S} \to S$ with center in $S \setminus V_B$ such that B extends to a semi-abelian scheme over $\tilde{S}$.*

Proof. ϕ extends to an isogeny $\mathcal{A}|_{V_B} \to \mathcal{N}_S^1(B)^\circ$ over V_B and we denote its kernel by $\mathcal{G}$. Due to [22, Théorème (5.2.2)], there is a blow-up $\pi : \tilde{S} \to S$

with center in $S \setminus V_B$ such that the scheme-theoretic closure $\tilde{\mathcal{G}}$ of $\mathcal{G}$ in $\pi^*\mathcal{A}$ is flat over $\tilde{S}$. By the above lemma, $\tilde{\mathcal{G}}$ is a closed subgroup scheme of $\pi^*\mathcal{A}$, flat over $\tilde{S}$. The quotient $\pi^*\mathcal{A}/\tilde{\mathcal{G}}$ exists as an algebraic space (cf. [2, 8.4/9]) and it is actually a scheme by the same argument as in [24, Exposé V, Proposition 4.10]. □

2.2. Raynaud's criterion

Let K be a field, S a normal model of K, and A an abelian variety over K of dimension g. We assume further that S is excellent. (Actually, it suffices to assume that S is universally Japanese; cf. [9, 0_{IV}, Définition (23.1.1)].) Let m_1, m_2 be integers not divisible by $\mathrm{char}(K)$ with $m_1, m_2 \geq 3$, $(m_1, m_2) = 1$. Let $m_3 = m_1 m_2$. Raynaud's criterion (cf. [18, Proposition 1.2.1]) implies that, if A has a level m_3 structure over K, then A has semi-abelian reduction in codimension one over S. In this case, due to the existence of the troidal compactification of A_g, we can prove the following weak extension theorem, which is slightly stronger than Gabber's lemma, in which π is assumed to be generically finite.

Lemma 2.2.1. *Let $U \subseteq S$ be a non-empty open subscheme over which A has abelian reduction. If A has a level m_3 structure defined over K, then there is a birational projective morphism $\pi : T \to S$ of normal models of K such that π is isomorphic over U and A has semi-abelian reduction over T.*

Proof. Let $\mathcal{A}_U$ be an abelian scheme over U with generic fiber K-isomorphic to A. First, we assume that $\mathcal{A}_U$ has a principal polarization. The level m_1 (resp. m_2, m_3) structure of A over K uniquely extends to that of $\mathcal{A}_U$ over $U \times_{\mathbb{Z}} \mathrm{Spec}(\mathbb{Z}[1/m_1])$ (resp. $U \times_{\mathbb{Z}} \mathrm{Spec}(\mathbb{Z}[1/m_2])$, $U \times_{\mathbb{Z}} \mathrm{Spec}(\mathbb{Z}[1/m_3])$). For an integer m with $m \geq 3$, $\mathsf{A}_{g,m}$ denotes the fine moduli scheme, over $\mathbb{Z}$, of principally polarized abelian schemes of dimension g with a level m structure. By [5, Chapter IV, Theorem 6.7 and Remark 6.12], there exists a projective troidal compactification $\bar{\mathsf{A}}_{g,m}$ of $\mathsf{A}_{g,m}$ over $\mathbb{Z}[1/m]$ (i.e. in the notation of [5], there exists a projective $GL(X)$-admissible polyhedral cone decomposition $\{\sigma_\alpha\}$ of $C(X)$ which is smooth with respect to $B(X)$ and has an additional natural property; cf. [5, Chapter IV, §2] and [1, Chapter II]). We fix such a $\{\sigma_\alpha\}$. By [5, Chapter V, Theorem 5.8], $\bar{\mathsf{A}}_{g,m}$ is represented by a scheme. Let T_1 (resp. T_2, T_3) be the scheme-theoretic closure of the graph of $U \times_{\mathbb{Z}} \mathrm{Spec}(\mathbb{Z}[1/m_1]) \to \mathsf{A}_{g,m_1}$ (resp. $U \times_{\mathbb{Z}} \mathrm{Spec}(\mathbb{Z}[1/m_2]) \to \mathsf{A}_{g,m_2}$, $U \times_{\mathbb{Z}} \mathrm{Spec}(\mathbb{Z}[1/m_3]) \to \mathsf{A}_{g,m_3}$) in $(S \times_{\mathbb{Z}} \mathrm{Spec}(\mathbb{Z}[1/m_1])) \times_{\mathbb{Z}[1/m_1]} \bar{\mathsf{A}}_{g,m_1}$ (resp. $(S \times_{\mathbb{Z}} \mathrm{Spec}(\mathbb{Z}[1/m_2])) \times_{\mathbb{Z}[1/m_2]} \bar{\mathsf{A}}_{g,m_2}$, $(S \times_{\mathbb{Z}} \mathrm{Spec}(\mathbb{Z}[1/m_3])) \times_{\mathbb{Z}[1/m_3]} \bar{\mathsf{A}}_{g,m_3}$). We may assume that T_1, T_2, and T_3 are reduced and normal. Since the natural morphisms $\bar{\mathsf{A}}_{g,m_3} \to \bar{\mathsf{A}}_{g,m_1}$ and $\bar{\mathsf{A}}_{g,m_3} \to \bar{\mathsf{A}}_{g,m_2}$ are respectively obtained by the normalization of $\mathsf{A}_{g,m_3} \to \bar{\mathsf{A}}_{g,m_1}$ and $\mathsf{A}_{g,m_3} \to \bar{\mathsf{A}}_{g,m_2}$, they are finite morphisms. Hence, by Zariski's Main Theorem, we have the natural isomorphisms $T_3 \to T_1 \times_{\mathbb{Z}[1/m_1]} \mathrm{Spec}(\mathbb{Z}[1/m_3])$ and $T_3 \to T_2 \times_{\mathbb{Z}[1/m_2]} \mathrm{Spec}(\mathbb{Z}[1/m_3])$. Thus we can find a normal integral scheme T and a projective and surjective morphism $\pi : T \to S$ such that π is isomorphic over U and A has semi-abelian reduction over T.

In general, take a polarization of $\mathcal{A}_U$ over U and apply Zarhin's trick [16, Chapitre IX, Lemme 1.1]. Then $\mathcal{B}_U = (\mathcal{A}_U \times_U \mathcal{A}_U^t)^4$ has a principal polarization, where $\mathcal{A}_U^t$ is the dual abelian scheme of $\mathcal{A}_U$ over U. Assume that $\mathcal{B}_U$ extends to a semi-abelian scheme $\mathcal{B}_T$ over T. We may replace T with a blow-up with center in $T \setminus \pi^{-1}(U)$ and A extends to a semi-abelian scheme over T by Lemma 2.1.1. □

2.3. Isogeny and extension

We prove the Key Lemma in Introduction in a general situation (Theorem 2.3.1). The proofs are essentially in [5, Chapter V, §6].

Proposition 2.3.1. *Let $S = \mathrm{Spec}(R)$ be a regular local scheme of dimension ≥ 2, $s = \mathrm{Spec}(R/\mathfrak{m})$ the closed point of S, $V = S \setminus \{s\}$, and $\eta = \mathrm{Spec}(K)$ the generic point of S. Let $\mathcal{A}_V$ be a semi-abelian scheme over V and assume that $(\mathcal{A}_V)_\eta$ is an abelian variety over K. Let $\tilde{S} \to S$ be the simple blow-up with center $\{s\}$ and E the exceptional fiber. If $\mathcal{A}_V$ extends to a semi-abelian scheme $\mathcal{A}_{\tilde{S}}$ over $\tilde{S}$ and the restriction $\mathcal{A}_E$ of $\mathcal{A}_{\tilde{S}}$ to E is trivial, then $\mathcal{A}_V$ extends to a semi-abelian scheme $\mathcal{A}_S$ over S.*

Proof. We borrow the proof from [5, Chapter V, Proposition 6.10]. First, we may assume that S is complete and strictly local. In fact, let $\mu : S^* \to S$ be the completion of the strict henselization of S and consider the strict transform

$$\begin{array}{ccc} (S^*)\tilde{} & \longrightarrow & \tilde{S} \\ \downarrow & & \downarrow \\ S^* & \xrightarrow{\mu} & S. \end{array}$$

Since V is normal, there is a cubical symmetric V-ample invertible sheaf $\mathcal{L}_V$ on $\mathcal{A}_V$ [21, Théorème XI 1.13]. Set $W = \mu^{-1}(V)$ and suppose that the pull-back $\mu^*\mathcal{A}_V$ of $\mathcal{A}_V$ extends to a semi-abelian scheme $\mathcal{A}_{S^*}$ over S^*.

$$\begin{array}{ccc} \mu^*\mathcal{A}_V & \xrightarrow{\mu} & \mathcal{A}_V \\ \downarrow & & \downarrow \\ W & \xrightarrow{\mu} & V. \end{array}$$

By [16, Chapitre II, Théorème 3.3], $\mu^*\mathcal{L}_V$ extends uniquely to a cubical symmetric S^*-ample invertible sheaf $\mathcal{L}_{S^*}$ on $\mathcal{A}_{S^*}$. We consider the two polarized semi-abelian schemes $(\mathcal{A}_1, \mathcal{L}_1) = \mathrm{pr}_1^*(\mathcal{A}_{S^*}, \mathcal{L}_{S^*})$ and $(\mathcal{A}_2, \mathcal{L}_2) = \mathrm{pr}_2^*(\mathcal{A}_{S^*}, \mathcal{L}_{S^*})$ over $S^* \times_S S^*$. There is an isomorphism $\varphi : \mathcal{A}_1 \to \mathcal{A}_2$ over $\mathrm{pr}_1^{-1}(W) \cap \mathrm{pr}_2^{-1}(W)$ and it extends uniquely to an isomorphism φ over $S^* \times_S S^*$ since $\mathrm{pr}_1^{-1}(W) \cap \mathrm{pr}_2^{-1}(W)$ contains all the points $x \in S^* \times_S S^*$ with $\mathrm{depth}(\mathcal{O}_{S^* \times_S S^*, x}) \leq 1$ [21, Corollaire IX 1.4]. Similarly, the isomorphism $\psi : \mathcal{L}_1 \to \varphi^*\mathcal{L}_2$ over $\mathrm{pr}_1^{-1}(W) \cap \mathrm{pr}_2^{-1}(W)$ extends to an isomorphism ψ over $S^* \times_S S^*$ [9, IV_4, Proposition (19.9.8)]. It is clear that φ and ψ satisfy the cocycle condition over $S^* \times_S S^* \times_S S^*$. Thus the pair $(\mathcal{A}_{S^*}, \mathcal{L}_{S^*})$ descends to a pair $(\mathcal{A}_S, \mathcal{L}_S)$ over S and $\mathcal{A}_S$ is a semi-abelian scheme over S extending $\mathcal{A}_V$.

Let $\mathcal{I}$ be the ideal sheaf on $\tilde{S}$ defining E, $\tilde{S}_n$ the closed subscheme of $\tilde{S}$ defined by $\mathcal{I}^{n+1}$, $(\tilde{S})^\wedge$ the $\mathcal{I}$-adic completion of $\tilde{S}$, and $S_n = \operatorname{Spec}(R/\mathfrak{m}^{n+1})$. We denote by $\mathcal{A}_{\tilde{S}_n}$ the restriction of $\mathcal{A}_{\tilde{S}}$ to $\tilde{S}_n$. Then $\mathcal{A}_{\tilde{S}_0} = \mathcal{A}_E$ comes from a semi-abelian scheme $\mathcal{A}_{S_0}$ over $S_0 = \operatorname{Spec}(k(s))$ which fits into the extension

$$0 \to \mathcal{H}_{S_0} \to \mathcal{A}_{S_0} \to \mathcal{G}_{S_0} \to 0,$$

where $\mathcal{H}_{S_0}$ is a split torus and $\mathcal{G}_{S_0}$ is an abelian variety over S_0. By the infinitesimal lifting of tori [5, Chapter I, Theorem 2.2], there is a unique lifting $\mathcal{H}_{\tilde{S}_n} \subseteq \mathcal{A}_{\tilde{S}_n}$ of $\mathcal{H}_{S_0} \times_{S_0} \tilde{S}_0 \subseteq \mathcal{A}_{\tilde{S}_0}$ such that the quotient $\mathcal{G}_{\tilde{S}_n}$ of $\mathcal{A}_{\tilde{S}_n}$ by $\mathcal{H}_{\tilde{S}_n}$ is an abelian scheme over $\tilde{S}_n$. Let $\mathcal{L}_{\tilde{S}}$ be a cubical $\tilde{S}$-ample invertible sheaf on $\mathcal{A}_{\tilde{S}}$ and $\mathcal{L}_{\tilde{S}_n}$ the restriction of $\mathcal{L}_{\tilde{S}}$ to $\mathcal{A}_{\tilde{S}_n}$. Then we can descend $\mathcal{L}_{\tilde{S}_n}$ to a cubical $\tilde{S}_n$-ample invertible sheaf $\mathcal{M}_{\tilde{S}_n}$ on $\mathcal{G}_{\tilde{S}_n}$. The adic formal polarized abelian scheme $(\mathcal{G}^\wedge_{\tilde{S}}, \mathcal{M}^\wedge_{\tilde{S}})$ is algebraizable [9, III_1, Théorème (5.4.5)] and thereby we have the Raynaud extension associated to $\mathcal{A}_{\tilde{S}}$;

$$0 \to \mathcal{H}_{\tilde{S}} \to \tilde{\mathcal{A}}_{\tilde{S}} \to \mathcal{G}_{\tilde{S}} \to 0.$$

We prove, by induction, that the adic formal polarized semi-abelian scheme $(\mathcal{A}^\wedge_{\tilde{S}}, \mathcal{L}^\wedge_{\tilde{S}})$ over $(\tilde{S})^\wedge$ comes from an adic formal polarized semi-abelian scheme $(\mathcal{A}^\wedge_{S}, \mathcal{L}^\wedge_{S})$ over $\operatorname{Spf}(R)$. Suppose that we have a semi-abelian scheme $\mathcal{A}_{S_{n-1}}$ over S_{n-1} attaining $\mathcal{A}_{\tilde{S}_{n-1}}$. There is no obstruction to lift $\mathcal{A}_{S_{n-1}}$ to a semi-abelian scheme over S_n and the set of isomorphism classes of semi-abelian schemes over S_n lifting $\mathcal{A}_{S_{n-1}}$ is an affine space under

$$\mathrm{H}^1(\mathcal{A}_{S_{n-1}}, T_{\mathcal{A}_{S_{n-1}}/S_{n-1}}) \otimes \mathfrak{m}^n/\mathfrak{m}^{n+1} = \mathbf{t}_{\mathcal{A}_{S_{n-1}}} \otimes \mathbf{t}_{\mathcal{G}^t_{S_{n-1}}} \otimes \mathfrak{m}^n/\mathfrak{m}^{n+1},$$

where $T_{\mathcal{A}_{S_\bullet}/S_\bullet}$ is the tangent bundle and $\mathbf{t}_{\mathcal{A}_{S_\bullet}}$ is the pull-back of $T_{\mathcal{A}_{S_\bullet}/S_\bullet}$ to $S_\bullet$ (cf. [24, Exposé VI, Appendice 1, Théorème A 1.1]). Consider an affine covering of $\tilde{S}$ and we know that the lifting $\mathcal{A}_{\tilde{S}_n}$ of $\mathcal{A}_{\tilde{S}_{n-1}}$ corresponds to an element in an affine space under

$$\mathrm{H}^0(\tilde{S}_{n-1}, \mathbf{t}_{\mathcal{A}_{\tilde{S}_{n-1}}} \otimes \mathbf{t}_{\mathcal{G}^t_{\tilde{S}_{n-1}}} \otimes \mathcal{I}^n/\mathcal{I}^{n+1}) = \mathbf{t}_{\mathcal{A}_{S_{n-1}}} \otimes \mathbf{t}_{\mathcal{G}^t_{S_{n-1}}} \otimes \mathfrak{m}^n/\mathfrak{m}^{n+1}.$$

Thus there is a lifting $\mathcal{A}_{S_n}$ over S_n attaining $\mathcal{A}_{\tilde{S}_n}$. The invertible sheaf $\mathcal{L}_{\tilde{S}_n}$ comes from an invertible sheaf $\mathcal{L}_{S_n}$ on $\mathcal{A}_{S_n}$ and $\mathcal{L}_{S_n}$ can be descended to an invertible sheaf $\mathcal{M}_{S_n}$ on $\mathcal{G}_{S_n}$. Since $\mathcal{M}_{S_0}$ is ample, the adic formal polarized abelian scheme $(\mathcal{G}^\wedge_{S}, \mathcal{M}^\wedge_{S})$ is algebraizable and we have the Raynaud extension

$$0 \to \mathcal{H}_{S} \to \tilde{\mathcal{A}}_{S} \to \mathcal{G}_{S} \to 0.$$

The rest of the proof is due to the Mumford-Faltings-Chai construction. (We use the equivalence of categories due to Raynaud, Mumford, Faltings, and Chai. See [5, Chapter II, III].) Take a regular parameter a of R and let $\tilde{R} = (\bigoplus_{k\geq 0} \mathfrak{m}^k)[1/a]_0$ $(\deg(a) = 1)$, $U = U_a = \operatorname{Spec}(\tilde{R})$, and $I = \mathfrak{m}\tilde{R}$ the ideal of $\tilde{R}$ defining $E \cap U$. Since $\bigoplus_{k\geq 0} \mathfrak{m}^k$ is $(\bigoplus_{k\geq 1} \mathfrak{m}^k)$-adic comlete,

$\tilde{R}$ is I-adic complete. Let $\tilde{\mathcal{L}}_{\tilde{S}}$ (resp. $\tilde{\mathcal{L}}_S$) be the cubical $\tilde{S}$-ample (resp. S-ample) invertible sheaf on $\tilde{\mathcal{A}}_{\tilde{S}}$ (resp. $\tilde{\mathcal{A}}_S$) obtained by pulling-back $\mathcal{M}_{\tilde{S}}$ on $\mathcal{G}_{\tilde{S}}$ (resp. $\mathcal{M}_S$ on $\mathcal{G}_S$) and $\underline{X}$ (resp. $\underline{Y}$) the character group scheme of $\mathcal{H}_U$ (resp. $\mathcal{H}_U^t$). Then we can reconstruct $\mathcal{A}_U$ from an ample degeneration data; $\tilde{\mathcal{A}}_U$, periods $\iota : \underline{Y}_\eta \to (\tilde{\mathcal{A}}_U)_\eta = (\tilde{\mathcal{A}}_S)_\eta$, polarization $\underline{Y}_\eta \to \underline{X}_\eta$, and an action of $\underline{Y}_\eta$ on $(\tilde{\mathcal{L}}_U)_\eta = (\tilde{\mathcal{L}}_S)_\eta$ which lifts ι. Let $\mathcal{A}_S$ be the semi-abelian scheme over S obtained by applying the Mumford-Faltings-Chai construction to the above ample degeneration data. Then $(\pi^*\mathcal{A}_S)|_U$ is isomorphic to $\mathcal{A}_U$ and we have the assertion. □

For an abelian variety A over a field k of characteristic $p > 0$, we write $f(A)$ for the p-rank of A, i.e.

$$A[p](\bar{k}) \simeq (\mathbb{Z}/p\mathbb{Z})^{f(A)}.$$

We say that A is ordinary if $f(A) = \dim A$.

Theorem 2.3.1. *Let S be a regular local scheme, s the closed point of S, $V = S\backslash\{s\}$, and $\eta = \operatorname{Spec}(K)$ the generic point of S. Let A, B be two abelian varieties over K and $\phi : A \to B$ a K-isogeny. We suppose that A extends to a semi-abelian scheme $\mathcal{A}$ over S and one of the following two conditions is satisfied.*

(i) $\operatorname{char}(k(s)) = 0$.

(ii) *For any $x \in S$, $\mathcal{A}_x$ sits in an extension*

$$0 \to H_x \to \mathcal{A}_x \to G_x \to 0,$$

where H_x is a torus over $k(x)$ and G_x is an ordinary abelian variety or an elliptic curve over $k(x)$.

Then B extends to a semi-abelian scheme over S.

Proof. By the same arguments as in the proof of Proposition 2.3.1, we may assume that S is complete and strictly local. We prove the assertion by induction on $\dim S$. In particular, we may assume that B extends to a semi-abelian scheme $\mathcal{B}_V$ over V. First, we suppose $\dim S = 2$. By Lemma 2.1.2, B extends to a semi-abelian scheme $\mathcal{B}_{\tilde{S}}$ after blowing-up, $\pi : \tilde{S} \to S$, with center $\{s\}$ (which may not be reduced). We can decompose π into a sequence of simple blow-ups and, by induction on the length of the sequence, we may assume that π is the simple blow-up. Let $E = \mathbb{P}^1_{k(s)}$ be the exceptional fiber. By Proposition 2.3.1, what we have to show is that the restriction $\mathcal{B}_E$ of $\mathcal{B}_{\tilde{S}}$ to E is trivial. The K-isogeny $\phi : A \to B$ induces isogenies $\pi^*\mathcal{A} \to \mathcal{B}_{\tilde{S}}$, $\phi_E : \mathcal{A}_s \times_{k(s)} E \to \mathcal{B}_E$, and

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_s \times_{k(s)} E & \longrightarrow & \mathcal{A}_s \times_{k(s)} E & \longrightarrow & G_s \times_{k(s)} E & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \phi_E} & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{H}_E & \longrightarrow & \mathcal{B}_E & \longrightarrow & \mathcal{G}_E & \longrightarrow & 0, \end{array}$$

where $\mathcal{H}_E$ is a torus over E, and $\mathcal{G}_E$ is an abelian scheme over E (cf. [2, 7.3/5 and 7.3/6]).

(i): If $\mathcal{A}$ is an abelian scheme over S, then the assertion is clear by the purity theorem of Zariski-Nagata. In general, the kernel of ϕ_E is finite étale over $E = \mathbb{P}^1_{k(s)}$ and, thus, is trivial.

(ii): Let m be an integer ≥ 3, not divisible by $\mathrm{char}(k(s))$ and assume that $(m, \deg(\phi_E)) = 1$. Since $k(s)$ is separably closed, G_s has a level m structure defined over $k(s)$ and, via the isomorphism $G_s[m] \times_{k(s)} E \to \mathcal{G}_E[m]$, a level m structure α of $\mathcal{G}_E$ over E is induced. Let λ be a polarization on $\mathcal{G}_E$. Since G_s is an ordinary abelian variety or an elliptic curve over $k(s)$, $\mathcal{G}_E$ is isotrivial (cf. [20, (2.4)] or [16, Chapitre XI, Théorème 5.1]). Hence, by [16, Chapitre IX, Théorème 3.1], $(\mathcal{G}_E, \lambda, \alpha)$ is trivial. The extension

$$0 \to \mathcal{H}_E \to \mathcal{B}_E \to \mathcal{G}_E \to 0$$

is coded by a homomorphism $\underline{X}(\mathcal{H}_E) \to \mathcal{G}_E^t$, which comes from a homomorphism over $k(s)$. Namely, $\mathcal{B}_E$ is trivial.

Next, we consider the case of $\dim S \geq 3$. Let $\pi : \tilde{S} \to S$ be the simple blow-up with reduced center $\{s\}$ and E the exceptional fiber.

Claim 1. B extends to a semi-abelian scheme over $\tilde{S}$.

Proof. By the induction hypothesis, B extends to a semi-abelian scheme $\mathcal{B}_{\tilde{S}\setminus Q}$ over $\tilde{S} \setminus Q$, where $Q \subseteq E$ is a finite set of closed points. Let U be an affine open subscheme of $\tilde{S}$ containing points in Q, $U' = U \setminus Q$, I (resp. I') the ideal sheaf defining the closed subscheme $E \cap U \subseteq U$ (resp. $E \cap U' \subseteq U'$), and U_n (resp. U'_n) the closed subscheme of U (resp. U') defined by I^{n+1} (resp. $(I')^{n+1}$). Let $\mathcal{B}_{U'_n}$ be the restriction of $\mathcal{B}_{\tilde{S}\setminus Q}$ to U'_n. Then $\mathcal{B}_{U'_0}$ extends uniquely to a semi-abelian scheme $\mathcal{B}_{U_0}$ over U_0. We can lift $\mathcal{B}_{U_0}$ to a semi-abelian scheme $\mathcal{B}_{U_n}$ over U_n which attains $\mathcal{B}_{U'_n}$ by the same arguments as in the proof of Proposition 2.3.1 since we have

$$\mathbf{t}_{\mathcal{B}_{U_{n-1}}} \otimes \mathbf{t}_{\mathcal{G}^t_{U_{n-1}}} \otimes I^n/I^{n+1} \simeq \mathrm{H}^0(U'_{n-1}, \mathbf{t}_{\mathcal{B}_{U'_{n-1}}} \otimes \mathbf{t}_{\mathcal{G}^t_{U'_{n-1}}} \otimes (I')^n/(I')^{n+1}).$$

The adic formal semi-abelian scheme $\mathcal{B}_U^\wedge$ is algebraizable and we have the Raynaud extension

$$0 \to \mathcal{H}_U \to \tilde{\mathcal{B}}_U \to \mathcal{G}_U \to 0.$$

By shrinking U', we may assume that U' is affine and reconstruct $\mathcal{B}_{U'}$ from an ample degeneration data; $\tilde{\mathcal{B}}_{U'}$, periods $\iota : \underline{Y}_\eta \to (\tilde{\mathcal{B}}_{U'})_\eta = (\tilde{\mathcal{B}}_U)_\eta$, polarization $\underline{Y}_\eta \to \underline{X}_\eta$, and an action of $\underline{Y}_\eta$ on $(\tilde{\mathcal{L}}_{U'})_\eta = (\tilde{\mathcal{L}}_U)_\eta$ which lifts ι (see [5, Chapter III, §2] and the proof of Proposition 2.3.1). Let $\mathcal{B}_U$ be the semi-abelian scheme over U obtained by applying the Mumford-Faltings-Chai construction to the above ample degeneration data. We can glue $\mathcal{B}_U$'s up to a semi-abelian scheme $\mathcal{B}_{\tilde{S}}$ over $\tilde{S}$, which extends B. □

By the same arguments as above, the restriction $\mathcal{B}_E$ of $\mathcal{B}_{\tilde{S}}$ to E is trivial and $\mathcal{B}_{\tilde{S}}$ comes from a semi-abelian scheme over S which extends $\mathcal{B}_V$. □

Remark 1. Raynaud, Ogus, Gabber, de Jong, and Oort constructed an abelian scheme $\mathcal{A}$ of relative dimension two over an excellent regular local scheme S of dimension two of mixed characteristic such that an isogeny image of $\mathcal{A}_K$ does not extend to an abelian scheme over S [13, §6]. We note that $\mathcal{A}_s$ is super-singular in their example.

Corollary 2.3.1. *Let K be a field, S a regular model of K, A, B two abelian varieties over K, and $\phi : A \to B$ a K-isogeny. Suppose that one of the following two conditions is satisfied.*
(i) *S is of characteristic zero.*
(ii) *A is an elliptic curve over K.*
If $\mathcal{N}_S^1(A)$ is a semi-abelian scheme over V_A, then we have $V_A \subseteq V_B$.

This corollary follows from the following lemma.

Lemma 2.3.1. *Let K be a field, S a normal model of K, V a big open subscheme of S, and A an abelian variety over K. If a Néron model $\mathcal{N}_V(A)$ of A exists and the neutral component $\mathcal{N}_V(A)^\circ$ extends to a quasi-compact smooth group scheme $\mathcal{A}$ over S, then a Néron model $\mathcal{N}_S(A)$ of A exists.*

Proof. First, we note that the extension $\mathcal{A}$ is unique by [21, Corollaire IX 1.4]. Since $\mathcal{A}_V$ is isomorphic to $\mathcal{N}_V(A)^\circ$, we have a quasi-compact smooth scheme $\mathcal{N}$ over S by gluing up. We can also prolong the group laws to a group law $\mathcal{N} \times_S \mathcal{N} \to \mathcal{N}$ on $\mathcal{N}$ and $\mathcal{N}$ is a Néron model of A over S. □

3. Main Theorem

3.1. Review of the Faltings-Moriwaki modular height

Let K be a field of finite type over $\mathbb{Q}$ with $\mathrm{tr.deg}_{\mathbb{Q}} K = d$, S a projective arithmetic variety with function field isomorphic to K, and $\bar{H}_1, \dots, \bar{H}_d$ C^∞-hermitian $\mathbb{Q}$-line bundles on S. We call a collection $\hat{S} = (S; \bar{H}_1, \dots, \bar{H}_d)$ a *polarization* of K if $\bar{H}_1, \dots, \bar{H}_d$ are all nef. The degree of the polarization $\hat{S}$ is given by

$$\deg(\hat{S}) = \int_{S_\infty} c_1(\bar{H}_1) \wedge \cdots \wedge c_1(\bar{H}_d).$$

We say that $\hat{S}$ is *ample* if $\bar{H}_1, \dots, \bar{H}_d$ are all ample. $\hat{S}$ is said to be *strictly fine* if there are two generically finite morphisms $\pi : \tilde{S} \to S$ and $\mu : \tilde{S} \to (\mathbb{P}^1_{\mathbb{Z}})^d$ and ample C^∞-hermitian $\mathbb{Q}$-line bundles $\bar{L}_1, \dots, \bar{L}_d$ on $\mathbb{P}^1_{\mathbb{Z}}$ such that $\pi^*(\bar{H}_i) \geq_{\mathbb{Q}} (\mathrm{pr}_i \circ \mu)^*(\bar{L}_i)$ for every i. We note that there always exists an ample polarization of K and ample polarizations (and, more generally, big polarizations) are strictly fine.

Let $G \to S$ be a smooth group scheme over S. The *Hodge line bundle* of $G \to S$ is given by

$$\lambda_{G/S} = \det \epsilon^*(\Omega^1_{G/S}),$$

where $\epsilon : S \to G$ is the identity-section. Suppose that S is normal. Let A be an abelian variety over K and $\mathcal{N}_S^1(A)$ a Néron model of A in codimension one over S. The $L^1_{\rm loc}$-hermitian Hodge reflexive sheaf $\bar{\lambda}^{\rm Fal}(A/K;S)$ of A is defined by

$$\bar{\lambda}^{\rm Fal}(A/K;S) = (\lambda(A/K;S), \|\cdot\|_{{\rm Fal},L^1_{\rm loc}}) = \iota_*(\lambda_{\mathcal{N}^1_S(A)/V_A}, \|\cdot\|_{\rm Fal}),$$

where $\iota : V_A \to S$ is the inclusion. By [16, Chapitre IX, Théorème 2.1], $c_1(\lambda(A/K;S))$ is effective if A has semi-abelian reduction in codimension one over S. Now, we define the Faltings-Moriwaki modular height of A with respect to $\hat{S}$ by

$$h^{\hat{S}}(A) = \widehat{\deg}(\hat{c}_1(\bar{H}_1)\cdots\hat{c}_1(\bar{H}_d)\cdot\hat{c}_1(\bar{\lambda}^{\rm Fal}(A/K;S))).$$

Let B be another abelian variety over K and $\phi : A \to B$ a K-isogeny. Then ϕ induces an injection $\phi^* : \lambda(B/K;S) \to \lambda(A/K;S)$, so that there is an effective Weil divisor $D_{\phi,S}$ satisfying

$$c_1(\lambda(A/K;S)) = c_1(\lambda(B/K;S)) + D_{\phi,S}.$$

Since ϕ^* is isomorphic over $S \times_{\mathbb{Z}} {\rm Spec}(\mathbb{Z}[1/\deg(\phi)])$, $D_{\phi,S}$ is vertical.

Proposition 3.1.1 ([18, Proposition 3.2]). *With the same notation as above, we have the isogeny formula*:

$$h^{\hat{S}}(A) - h^{\hat{S}}(B) = \widehat{\deg}(\hat{c}_1(\bar{H}_1)\cdots\hat{c}_1(\bar{H}_d)|_{D_{\phi,S}}) - \frac{1}{2}\deg(\hat{S})\log(\deg(\phi)).$$

We abbreviate $\widehat{\deg}(\hat{c}_1(\bar{H}_1)\cdots\hat{c}_1(\bar{H}_d)|_{D_{\phi,S}})$ to $\widehat{\deg}(D_{\phi,S};\hat{S})$. The main result of [18] is the following.

Theorem 3.1.1 ([18, Theorem 6.1]). *Suppose that $\hat{S}$ is strictly fine. Then, for any positive real number c, the number of isomorphism classes of abelian varieties defined over K with $h^{\hat{S}}(A) \le c$ is finite.*

3.2. Induction step

Let K be a field of finite type over $\mathbb{Q}$ with ${\rm tr.deg}_{\mathbb{Q}}(K) = d$, S a normal projective arithmetic variety with function field isomorphic to K, and $\hat{S} = (S;\bar{H}_1,\ldots,\bar{H}_d)$ a polarization of K. For a positive integer l, $P_l \subseteq {\rm Spec}(\mathbb{Z})$ denotes the zero locus of l. Let A be an abelian variety over K and P a finite set of prime numbers. We define ${\rm NR}_{S,P}(A)$ to be

$${\rm NR}_{S,P}(A) = \{x \in S \mid {\rm codim}(S,\overline{\{x\}}) = 2, \overline{\{x\}} \subseteq S \setminus V_A, \text{ and } {\rm char}(k(x)) \in P\}.$$

Let $u : S \to {\rm Spec}(\mathcal{O}_F)$ be a Stein factorization of the structure morphism $S \to {\rm Spec}(\mathbb{Z})$, where $\mathcal{O}_F$ is the maximal order of an algebraic number field F, and W_S a non-empty open subscheme of ${\rm Spec}(\mathcal{O}_F)$ such that $u_{W_S} : S|_{W_S} \to W_S$ is smooth. We note that F is uniquely determined as the algebraic closure of $\mathbb{Q}$ in K.

Proposition 3.2.1. *Let H be an ample $\mathbb{Q}$-line bundle on S, U a non-empty open subscheme of S, and Δ a finite subset of $\{x \in S \mid \dim \mathcal{O}_{S,x} \geq 2, \mathrm{char}(k(x)) > 0\}$. Then, for sufficiently large m, there is a non-zero global section σ of mH with the following properties*;

(I) *each generic point of the irreducible components of* $\mathrm{div}(\sigma)_{\mathrm{hor}}$ *is contained in U,*

(II) *each irreducible component of* $\mathrm{div}(\sigma)_{\mathrm{hor}}$ *does not contain any point in Δ,*

(III) *each irreducible component of* $\mathrm{div}(\sigma)_{\mathrm{ver}}$ *is not contained in* $u^{-1}(\mathrm{Spec}(\mathcal{O}_F) \setminus W_S)$*, and*

(IV) $\mathrm{div}(\sigma)_\infty$ *is smooth,*

where $\mathrm{div}(\sigma)_{\mathrm{hor}}$ *is the horizontal part and* $\mathrm{div}(\sigma)_{\mathrm{ver}}$ *is the vertical part of* $\mathrm{div}(\sigma)$*. Moreover, if $d \geq 2$ and mH is very ample, then $S_1 := \mathrm{div}(\sigma)_{\mathrm{hor}}$ is a prime divisor on S. If $d = 1$ and mH is very ample, we can write* $\mathrm{div}(\sigma)_{\mathrm{hor}} = \sum_{i=1}^n C_i$*, where $C_1, \dots, C_n$ are mutually distinct horizontal arithmetic curves on S with $\sum_{i=1}^n \sharp C_{i,\infty} = m \deg(H_\infty)$.*

Proof. Let $(S \setminus U) \cup u^{-1}(\mathrm{Spec}(\mathcal{O}_F) \setminus W_S) = \bigcup_{i=1}^{t_1} D'_i \cup \bigcup_{j=1}^{t_2} D_j$ be the irreducible decomposition such that $D'_1, \dots, D'_{t_1}$ are horizontal and $D_1, \dots, D_{t_2}$ are vertical, x_j the generic point of D_j for $j = 1, \dots, t_2$, and $y_1 \dots, y_{t_1}$ mutually distinct closed points of S_∞ with $y_i \in D'_{i,\infty}$ for $i = 1, \dots, t_1$. Let m be a positive integer and Z_m the hypersurface of the projective space $|mH_\infty|$ which consists of all singular divisors and divisors passing at least one of $\{y_1, \dots, y_{t_1}\}$. Then we can apply Moriwaki's arithmetic Bertini theorem [17, Theorem 5.3] to $\Delta, x_1, \dots, x_{t_2}$ and Z_m (see also [17, Corollary 4.3]). The latter half of the proposition follows from [11, Chapter III, Corollary 7.9]. □

Theorem 3.2.1. *Let A, B be two abelian varieties over K and $\phi : A \to B$ a K-isogeny. Let U be a non-empty open subscheme of S over which A has abelian reduction. We assume that the following two conditions are satisfied*;

(1) *A has birationally semi-abelian reduction over S, and*

(2) *H_d is ample and there is a non-zero global section σ of H_d with the properties* (I), (II)*, and* (III) *of Proposition* 3.2.1 *for U and*

$$\Delta := \mathrm{NR}_{S,P_{\deg(\phi)}}(A) \cup \mathrm{NR}_{S,P_{\deg(\phi)}}(B).$$

We write $\mathrm{div}(\sigma)_{\mathrm{hor}} = \sum_i a_i S_i$*. Let K_i be the function field of S_i, S'_i the normalization of a generic desingularization of S_i, and $\hat{S}'_i = (S'_i; \bar{H}_1|_{S'_i}, \dots, \bar{H}_{d-1}|_{S'_i})$ the polarization of K_i for all i. Then, we have*

$$h^{\hat{S}}(A) - h^{\hat{S}}(B) = \sum_i a_i (h^{\hat{S}'_i}(\mathcal{N}^1_S(A)_{K_i}) - h^{\hat{S}'_i}(\mathcal{N}^1_S(B)_{K_i})).$$

Proof. There are a big open subscheme V of S and a projective morphism $\mu : S^* \to S$ of normal models of K such that μ is isomorphic over V and A extends to a semi-abelian scheme over S^*. Moreover, by using Lemma 2.1.2, there is a normalized blow-up $T \to S^*$ with center in $S^* \setminus V_{B,S^*}$ such that A

and B extend to semi-abelian schemes $\mathcal{A}_T$ and $\mathcal{B}_T$ over T respectively. Due to Hironaka's desingularization theorem, we may assume that T is generically smooth over $\mathrm{Spec}(\mathbb{Z})$. We denote the composition of the morphisms $T \to S^* \to S$ by π and we may assume that π is isomorphic over V after shrinking V. Let $\hat{T}$ be the polarization of K defined by $(T; \pi^*(\bar{H}_1), \dots, \pi^*(\bar{H}_d))$ and T_i the strict transform of S_i in T, i.e. the scheme-theoretic closure of $S_i \cap V$ in T. Let T'_i be the normalization of a generic desingularization of T_i and $\hat{T}'_i$ the polarization of K_i defined by $(T'_i; \pi^*(\bar{H}_1)|_{T'_i}, \dots, \pi^*(\bar{H}_{d-1})|_{T'_i})$. Let $\phi_i : \mathcal{N}^1_S(A)_{K_i} \to \mathcal{N}^1_S(B)_{K_i}$ be the K_i-isogeny induced by ϕ. Since $\mathcal{N}^1_S(A)_{K_i}$ has semi-abelian reduction in codimension one over S'_i, $\mathcal{N}^1_S(B)_{K_i}$ has semi-abelian reduction in codimension one over S'_i for all i. Hence, due to [18, Proposition 3.1], we have

$$h^{\hat{S}}(A) - h^{\hat{S}}(B) = h^{\hat{T}}(A) - h^{\hat{T}}(B)$$

and

$$h^{\hat{S}'_i}(\mathcal{N}^1_S(A)_{K_i}) - h^{\hat{S}'_i}(\mathcal{N}^1_S(B)_{K_i}) = h^{\hat{T}'_i}(\mathcal{N}^1_S(A)_{K_i}) - h^{\hat{T}'_i}(\mathcal{N}^1_S(B)_{K_i})$$

for each i. In order to compute $\widehat{\deg}(D_{\phi,T}; \hat{T})$, we need the following lemma.

Lemma 3.2.1. *Let $\mathfrak{q}_0, \dots, \mathfrak{q}_n$ be mutually distinct prime ideals of F. Then there is a rational function $\alpha \in F^\times$ having the following properties;*

(i) $\|\alpha\|_{\sup} < 1$, *where $\|\cdot\|$ is the canonical metric on $\mathcal{O}_F$.*

(ii) *α has no poles except at $\mathfrak{q}_0$.*

(iii) *α has no zeros at $\mathfrak{q}_1, \dots, \mathfrak{q}_n$.*

Proof. We prove the lemma by induction on n. If $n = 0$, the assertion is due to Minkowski (cf. [19, Kapitel I, §4 und Kapitel III, §3]). By the induction hypothesis, there are $\beta, \gamma \in F^\times$ such that $\|\beta\|_{\sup}, \|\gamma\|_{\sup} < 1$ and β (resp. γ) has no poles except at $\mathfrak{q}_0$ (resp. $\mathfrak{q}_n$) and no zeros at $\mathfrak{q}_1, \dots, \mathfrak{q}_{n-1}$. If $\mathrm{ord}_{\mathfrak{q}_n}(\beta) = 0$, there is nothing left to be shown. If not, we put $e_1 = \mathrm{ord}_{\mathfrak{q}_n}(\beta)$, $e_2 = -\mathrm{ord}_{\mathfrak{q}_n}(\gamma)$, and $\alpha = \beta^{e_2}\gamma^{e_1}$ and, then, α has the desired properties. □

By the above lemma, there are a rational function $\alpha \in F^\times$, prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ of F, and ratinal numbers $r, r_1 \dots, r_n$ such that $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ do not divide $\deg(\phi)$ and

$$\mathrm{div}(\sigma) + ru^*(\mathrm{div}(\alpha)) = \sum_i a_i S_i + \sum_{j=1}^n r_j u^*(\mathfrak{p}_j).$$

We may treat $\alpha^r \sigma$ as a global section of the $\mathbb{Q}$-line bundle $H_d + r\mathcal{O}_S$ on S. Let $\pi^*(\alpha^r\sigma)$ be the section of $\pi^*(H_d + r\mathcal{O}_S)$ obtained by pulling-back. Then we can write

$$\mathrm{div}(\pi^*(\alpha^r\sigma)) = \sum_i a_i T_i + E + \sum_{j=1}^n r_j (u \circ \pi)^*(\mathfrak{p}_j),$$

where E is a π-exceptional effective Weil divisor on T. Since $D_{\phi,T}$ is a vertical Cartier divisor, we have

$$\begin{aligned}\hat{c}_1(\pi^*(\bar{H}_d))|_{D_{\phi,T}} &= \hat{c}_1(\mathcal{O}_T(D_{\phi,T}), \|\cdot\|_{\mathrm{can}}) \cdot \hat{c}_1(\pi^*(\bar{H}_d + r\bar{\mathcal{O}}_S^{\mathrm{can}})) \\ &= \sum_i a_i(D_{\phi,T}\cdot T_i, 0) + (D_{\phi,T}\cdot E, 0) \\ &\qquad + \sum_{j=1}^n r_j(D_{\phi,T}\cdot (u\circ\pi)^*(\mathfrak{p}_j), 0) \\ &= \sum_i a_i(D_{\phi_i,T_i}, 0) + (D_{\phi,T}\cdot E, 0)\end{aligned}$$

in $\widehat{\mathrm{CH}}^2(T)\otimes_{\mathbb{Z}}\mathbb{Q}$. Thus we have

$$\begin{aligned}&\widehat{\deg}(D_{\phi,T};\hat{T}) \\ &= \sum_i a_i\,\widehat{\deg}(D_{\phi_i,T_i};\hat{T}_i) + \widehat{\deg}(\hat{c}_1(\pi^*(\bar{H}_1))\cdots\hat{c}_1(\pi^*(\bar{H}_{d-1}))\cdot(D_{\phi,T}\cdot E, 0)) \\ &= \sum_i a_i\,\widehat{\deg}(D_{\phi_i,T_i'};\hat{T}_i') + \widehat{\deg}(\hat{c}_1(\bar{H}_1)\cdots\hat{c}_1(\bar{H}_{d-1})\cdot\pi_*(D_{\phi,T}\cdot E, 0)).\end{aligned}$$

We show $\widehat{\deg}(\hat{c}_1(\bar{H}_1)\cdots\hat{c}_1(\bar{H}_{d-1})\cdot\pi_*(D_{\phi,T}\cdot E,0)) = 0$. Let E_0 be an irreducible component of E and $\pi(E_0)$ the integral closed subscheme of S. First, we may assume that $\pi(E_0)$ have codimension two in S and is vertical. Since $\pi(E_0)\subseteq (S\setminus V)\cap(\bigcup_i S_i)$, $\pi(E_0)$ does not contain any point in $\mathrm{NR}_{S,P_{\deg(\phi)}}(A)$ and $\mathrm{NR}_{S,P_{\deg(\phi)}}(B)$. Thus there is a non-empty open subscheme U_0 of $\pi(E_0)$ such that the restrictions $\mathcal{A}_{(\pi|_{E_0})^{-1}(U_0)}$ and $\mathcal{B}_{(\pi|_{E_0})^{-1}(U_0)}$ of $\mathcal{A}_T$ and $\mathcal{B}_T$ come from semi-abelian schemes over U_0. Hence we have

$$\begin{aligned}&\widehat{\deg}(\hat{c}_1(\bar{H}_1)\cdots\hat{c}_1(\bar{H}_{d-1})\cdot\pi_*(D_{\phi,T}\cdot E, 0)) \\ =&\widehat{\deg}(\hat{c}_1(\bar{H}_1)\cdots\hat{c}_1(\bar{H}_{d-1})\cdot\pi_*\hat{c}_1(\bar{\lambda}^{\mathrm{can}}_{\mathcal{A}_E/E})) \\ &\qquad - \widehat{\deg}(\hat{c}_1(\bar{H}_1)\cdots\hat{c}_1(\bar{H}_{d-1})\cdot\pi_*\hat{c}_1(\bar{\lambda}^{\mathrm{can}}_{\mathcal{B}_E/E})) \\ =&0,\end{aligned}$$

where $\lambda_{\mathcal{A}_{E_0}/E_0}$ and $\lambda_{\mathcal{B}_{E_0}/E_0}$ are the Hodge line bundles of $\mathcal{A}_{E_0}$ and $\mathcal{B}_{E_0}$ respectively. Since $\deg(\hat{T}) = \sum_i a_i\deg(\hat{T}_i')$ and $\deg(\phi) = \deg(\phi_i)$ hold, we have

$$h^{\hat{S}}(A) - h^{\hat{S}}(B) = \sum_i a_i(h^{\hat{S}_i'}(\mathcal{N}_S^1(A)_{K_i}) - h^{\hat{S}_i'}(\mathcal{N}_S^1(B)_{K_i})).$$

□

3.3. Main Theorem

Let K be a field of finite type over $\mathbb{Q}$ with $\mathrm{tr.deg}_{\mathbb{Q}} K = d$, S a normal projective arithmetic variety whose function field is isomorphic to K, and $\hat{S} =$

$(S;\bar{H}_1,\dots,\bar{H}_d)$ a polarization of K. Let A, B be two abelian varieties over K and p a prime number. A p-isogeny $\phi : A \to B$ is a K-isogeny such that $P_{\deg(\phi)} = \{p\}$. We say that A has ordinary reduction at p if $\mathcal{N}_S^1(A)_\eta$ is an ordinary abelian variety over $k(\eta)$ for all generic points η of $S(p) = S \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}(k(p))$.

Theorem 3.3.1. *Let U be a non-empty open subscheme of S over which A has abelian reduction. Suppose that the following conditions are satisfied.*

(1) *$\hat{S}$ is ample.*

(2) *A has birationally semi-abelian reduction over S.*

(3) *A is an elliptic curve over K (resp. A has ordinary reduction at p).*

Then there exist mutually distinct horizontal arithmetic curves $C_1,\dots,C_n$ on S with generic points $\xi_1,\dots,\xi_n \in U$ respectively such that, for any abelian variety B over K which is K-isogenous (resp. p-isogenous) to A, we have

$$h^{\hat{S}}(A) - h^{\hat{S}}(B) = \frac{1}{m}\sum_{i=1}^{n}(h(\mathcal{N}_S^1(A)_{\xi_i}) - h(\mathcal{N}_S^1(B)_{\xi_i})),$$

where $m \cdot \deg(\hat{S}) = [F:\mathbb{Q}] \cdot \sum_{i=1}^{n}[k(\xi_i):F]$.

Proof. First, we consider the elliptic curve case. Let P_A be the set-theoretic image of the vertical part $(S \setminus V_A)_{\mathrm{ver}}$ by the structure morphism $S \to \mathrm{Spec}(\mathbb{Z})$,

$$\Delta = \mathrm{NR}_{S,P_A}(A) \cup \{x \in S \mid \dim \mathcal{O}_{S,x} = 2 \text{ and } \mathcal{O}_{S,x} \text{ is not regular}\},$$

and $\sigma \in \mathrm{H}^0(S, m_d H_d)$ a non-zero section with the properties (I), (II), (III), and (IV) of Proposition 3.2.1 for U and Δ. We suppose that $m_d H_d$ is very ample and replace $m_d H_d$ with H_d. Since $U_{\mathbb{Q}}$ is smooth, B has abelian reduction over each irreducible component of $U_{\mathbb{Q}}$. From now on, we use the same notation as in Proposition 3.2.1. We assume $d = 1$. Let ξ_i be the generic point of C_i for $i = 1,\dots,n$. Then we have

$$h^{\hat{S}}(A) - h^{\hat{S}}(B) = \sum_{i=1}^{n}(h(\mathcal{N}_S^1(A)_{\xi_i}) - h(\mathcal{N}_S^1(B)_{\xi_i}))$$

by Theorem 3.2.1 and Theorem 2.3.1. Thus the assertion holds in this case. Next we assume $d \geq 2$. Let S_1' be the normalization of S_1, K_1 the function field of S_1, and $\hat{S}_1' = (S_1'; \bar{H}_1|_{S_1'},\dots,\bar{H}_{d-1}|_{S_1'})$ the polarization of K_1. By Theorem 3.2.1 and Theorem 2.3.1, we have

$$h^{\hat{S}}(A) - h^{\hat{S}}(B) = h^{\hat{S}_1'}(\mathcal{N}_S^1(A)_{K_1}) - h^{\hat{S}_1'}(\mathcal{N}_S^1(B)_{K_1})$$

and $\deg(\hat{S}_1') = \deg(\hat{S})$. We regard the normal open subscheme $U_1 = \mathrm{Nor}(S_1) \cap U$ as an open subscheme of S_1'. Note that U_1 is dense in S_1' since $S_{1,\infty}$ is smooth over $\mathbb{C}$, the generic point of S_1 is contained in U, and the normalization $S_1' \to S_1$ is isomorphic over U_1. By the induction hypothesis, there are positive integers

m, n and mutually distinct horizontal arithmetic curves $C_1, \dots, C_n$ on S'_1 with generic points $\xi_1, \dots, \xi_n \in U_1$ respectively such that

$$h^{\hat{S}'_1}(\mathcal{N}^1_S(A)_{K_1}) - h^{\hat{S}'_1}(\mathcal{N}^1_S(B)_{K_1}) = \frac{1}{m}\sum_{i=1}^{n}(h(\mathcal{N}^1_S(A)_{\xi_i}) - h(\mathcal{N}^1_S(B)_{\xi_i}))$$

and $m \cdot \deg(\hat{S}'_1) = [F : \mathbb{Q}] \cdot \sum_{i=1}^{n}[k(\xi_i) : F]$ hold. Thus the assertion is proved.

We can prove the ordinary reduction case in the same manner. We put

$$\Delta = \mathrm{NR}_{S,P_A}(A) \cup \{x \in S \mid \dim \mathcal{O}_{S,x} = 2 \text{ and } \mathcal{O}_{S,x} \text{ is not regular}\} \cup \left\{ x \in S(p) \,\middle|\, \begin{array}{c} \dim \mathcal{O}_{S(p),x} = 1 \text{ and } \mathcal{N}^1_S(A)_x \text{ exists} \\ \text{but is not an ordinary abelian variety} \end{array} \right\}.$$

Then we can take a non-zero global section $\sigma \in \mathrm{H}^0(S, m_d H_d)$ with the properties (I), (II), (III), and (IV) of Proposition 3.2.1 for U and Δ since Δ is a finite set. We suppose that $m_d H_d$ is very ample and replace $m_d H_d$ with H_d. In the same notation as before, we have

$$h^{\hat{S}}(A) - h^{\hat{S}}(B) = \sum_{i=1}^{n}(h(\mathcal{N}^1_S(A)_{\xi_i}) - h(\mathcal{N}^1_S(B)_{\xi_i}))$$

if $d = 1$, and

$$h^{\hat{S}}(A) - h^{\hat{S}}(B) = h^{\hat{S}'_1}(\mathcal{N}^1_S(A)_{K_1}) - h^{\hat{S}'_1}(\mathcal{N}^1_S(B)_{K_1})$$

if $d \geq 2$ by Theorem 3.2.1 and Theorem 2.3.1. Since $S'_1(p) \to S_1(p)$ is finite and $\mathcal{N}^1_S(A)_\eta$ is ordinary for every generic point η of $S_1(p)$, $\mathcal{N}^1_S(A)_{K_1}$ has ordinary reduction at p over S'_1. By the induction hypothesis, there are positive integers m, n and mutually distinct horizontal arithmetic curves $C_1, \dots, C_n$ on S'_1 with generic points $\xi_1, \dots, \xi_n \in U_1$ respectively, such that

$$h^{\hat{S}'_1}(\mathcal{N}^1_S(A)_{K_1}) - h^{\hat{S}'_1}(\mathcal{N}^1_S(B)_{K_1}) = \frac{1}{m}\sum_{i=1}^{n}(h(\mathcal{N}^1_S(A)_{\xi_i}) - h(\mathcal{N}^1_S(B)_{\xi_i}))$$

and $m \cdot \deg(\hat{S}'_1) = [F : \mathbb{Q}] \cdot \sum_{i=1}^{n}[k(\xi_i) : F]$ hold. □

Remark 2. Whether A has ordinary reduction at a prime number p is a difficult problem. If K is a number field, a result due to Ogus is that an elliptic curve or an abelian surface over $\mathrm{Spec}(\mathcal{O}_K)$ has ordinary reduction at almost all prime of degree one (after enlarging K); cf. Lecture Notes in Math. **900**, VI.

Corollary 3.3.1. *Let S be normal, $\hat{S}$ a polarization of K, and A, B two elliptic curves over K. If B is K-isogenous to A and A has semi-abelian reduction in codimension one over S, then $|h^{\hat{S}}(A) - h^{\hat{S}}(B)|$ can be bounded from above by a constant depending only on $\hat{S}$ and A.*

Proof. We reduce the assertion to Theorem 3.3.1 as follows.

Claim 2. We may assume that A has birationally semi-abelian reduction over S and $\hat{S}$ is ample.

Proof. Suppose that $\bar{H}_1, \ldots, \bar{H}_i$ is ample for $0 \leq i \leq d-1$. We prove the claim by induction on i. Let $\bar{H}$ be an ample C^∞-hermitian line bundle on S. For any nef C^∞-hermitian $\mathbb{Q}$-line bundle $\bar{L}$ on S, we abbreviate $(S; \bar{H}_1, \ldots, \bar{H}_i, \bar{L}, \bar{H}_{i+2}, \ldots, \bar{H}_d)$ to $\hat{S}\langle \bar{L} \rangle_{i+1}$. Then we have

$$\begin{aligned} &|h^{\hat{S}}(A) - h^{\hat{S}}(B)| \\ \leq &|h^{\hat{S}\langle \bar{H}_{i+1}+\bar{H} \rangle_{i+1}}(A) - h^{\hat{S}\langle \bar{H}_{i+1}+\bar{H} \rangle_{i+1}}(B)| + |h^{\hat{S}\langle \bar{H} \rangle_{i+1}}(A) - h^{\hat{S}\langle \bar{H} \rangle_{i+1}}(B)|. \end{aligned}$$

Namely, we may assume that $\bar{H}_{i+1}$ is an ample C^∞-hermitian $\mathbb{Q}$-line bundle due to the Nakai-Moishezon-Kleiman criterion on arithmetic variety [26, Corollary (4.8)]. □

Hence, by Theorem 3.3.1, there are positive integers m, n and mutually distinct horizontal arithmetic curves $C_1, \ldots, C_n$ on S with generic points $\xi_1, \ldots, \xi_n$ respectively such that

$$|h^{\hat{S}}(A) - h^{\hat{S}}(B)| \leq \frac{1}{m} \sum_{i=1}^{n} |h(\mathcal{N}_S^1(A)_{\xi_i}) - h(\mathcal{N}_S^1(B)_{\xi_i})|$$

holds and the right-hand-side of the inequality can be bounded effectively by the Raynaud constants [24, Exposé VII, Théorème 4.4.9], which depend only on $\hat{S}$ and A. □

Remark 3. As we noted in the introduction, we can prove Corollary 3.3.1 by using de Jong's alteration theorem and the purity theorem of de Jong-Oort ([13, Theorem 5.1]).

The following three results were proved by Faltings in [6, Chapter VI] for all abelian varieties A and all prime numbers p.

Corollary 3.3.2 (cf. [6, Chapter VI §3, Corollary]). *If A is an elliptic curve over K (resp. there is a normal projective arithmetic variety S with function field isomorphic to K such that A has ordinary reduction at p), then the number of K-isomorphism classes in the K-isogeny class of A (resp. the p-isogeny class of A) is finite.*

Proof. By the same arguments as in [24, Exposé VIII, Théorème 2], using Theorem 3.3.1. See also [18, Lemma 5.1]. □

Corollary 3.3.3 (Tate conjecture). *Let $T_p(A)$ the Tate module of A at p. Suppose that one of the following conditions is satisfied.*

(i) *A is an elliptic curve over K.*

(ii) *There is a normal projective arithmetic variety S with function field isomorphic to K such that A has ordinary reduction at p.*
Then

(1) $T_p(A) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ *is a semi-simple* $\mathrm{Gal}(\bar{K}/K)$*-module and*

(2) *the natural morphism* $\mathrm{End}_K(A) \otimes_{\mathbb{Z}} \mathbb{Z}_p \to \mathrm{End}_{\mathrm{Gal}(\bar{K}/K)}(T_p(A))$ *is an isomorphism.*

Proof. It is derived from Corollary 3.3.2 by the same arguments as in [24, Exposé VIII, Théorème 3 et Proposition 3]. □

Corollary 3.3.4 (Šafarevič conjecture)**.** *Let S be a normal arithmetic variety with function field isomorphic to K. Then, there are finitely many elliptic curves over K, up to isomorphism, having abelian reduction in codimension one over S.*

Proof. It is also derived from Corollary 3.3.2 by the same arguments as in [6, Chapter VI, §4 Theorem 2]. □

3.4. A remark on the general case to Corollary 3.3.3

We can prove the following result without using [8, p.60, Théorème] (cf. Introduction of this paper or [6, Chapter VI]). The key point is the extension theorem on homomorphisms of p-divisible groups due to Tate [25, (4.2), Theorem 4].

Theorem 3.4.1. *For any abelian variety A over K, the injection*

$$\mathrm{End}_K(A) \otimes_{\mathbb{Z}} \mathbb{Z}_p \to \mathrm{End}_{\mathrm{Gal}(\bar{K}/K)}(T_p(A))$$

is an isomorphism for all but finitely many prime numbers p (the prime numbers at which A has "abelian reduction").

Let S be a normal projective arithmetic variety with function field isomorphic to K and $\hat{S} = (S; \bar{H}_1, \ldots, \bar{H}_d)$ an ample polarization of K. We assume that A extends to a semi-abelian scheme $\mathcal{A}$ over S and $\mathcal{A}_\eta$ is an abelian variety over $k(\eta)$ for all generic points η of $S(p)$. Let $\psi_K : A[p^\infty] \to A[p^\infty]$ be a homomorphism of p-divisible groups, $\Gamma_K \subseteq A[p^\infty] \oplus A[p^\infty]$ the graph of ψ_K, and $B_n = (A \times_K A)/\Gamma_K[p^n]$. Then the above theorem follows from the following fact (cf. [6, Chapter IV, (2.4)]).

Theorem 3.4.2. *With the same notation and assumptions as above, we have*

$$h^{\hat{S}}(A \times_K A) = h^{\hat{S}}(B_n)$$

for every n.

Proof. Let $U \subseteq S$ be the maximal open subscheme over which A has abelian reduction and

$$\Delta = \{x \in S(p) \mid \dim \mathcal{O}_{S(p),x} = 1 \text{ and } \mathcal{A}_x \text{ is not an abelian variety}\}.$$

We take a non-zero global section σ of $m_d H_d$ with the properties (I), (II), (III) and (IV) of Proposition 3.2.1 for U and Δ. Due to [25, (4.2), Theorem 4], ψ_K extends to a homomorphism $\psi_U : (\mathcal{A}|_U)[p^\infty] \to (\mathcal{A}|_U)[p^\infty]$ of p-divisible groups and the graph of ψ_U, $\Gamma_U \subseteq (\mathcal{A}|_U)[p^\infty] \oplus (\mathcal{A}|_U)[p^\infty]$, is a p-divisible group over U. In particular, B_n extends to an abelian scheme $\mathcal{B}_{n,U} = ((\mathcal{A}|_U) \times_U (\mathcal{A}|_U))/\Gamma_U[p^n]$ over U for all n. Using the same arguments as in the proof of Theorem 3.3.1, there are positive integers m, n and mutually distinct horizontal arithmetic curves $C_1, \ldots, C_n$ on S with generic points $\xi_1, \ldots, \xi_n \in U$ respectively such that

$$h^{\hat{S}}(A \times_K A) - h^{\hat{S}}(B_n) = \frac{1}{m} \sum_{i=1}^{n} (h((\mathcal{A} \times_S \mathcal{A})_{\xi_i}) - h(\mathcal{N}_S^1(B_n)_{\xi_i}))$$

holds. Both $(\mathcal{A} \times_S \mathcal{A})_{\xi_i}$ and $\mathcal{N}_S^1(B_n)_{\xi_i}$ have abelian reduction at every prime ideal over p and the $k(\xi_i)$-isogeny $(\mathcal{A} \times_S \mathcal{A})_{\xi_i} \to \mathcal{N}_S^1(B_n)_{\xi_i}$ preserves the Faltings modular height by using the arguments due to Faltings. □

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