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On a generalization of Massey products

By

Takashi Mizuno

1. Introduction

We first recall the definition of Massey products. Let $\mathcal{A}$ be a DGA whose differential is of degree -1. Given homogeneous elements $a, b, c \in H_*(\mathcal{A})$, we suppose that

$$ab = bc = 0.$$

We first choose representatives $\alpha, \beta, \gamma \in \mathcal{A}$ of a, b, c respectively. We next choose $X, Y \in \mathcal{A}$ such that $(-1)^{|\alpha|+1}\alpha\beta = dX$ and $(-1)^{|\beta|+1}\beta\gamma = dY$ which realize the relation above. Then we obtain a cycle

$$(-1)^{|a|+1}\alpha Y + (-1)^{|a|+|b|+2}X\gamma$$

of $\mathcal{A}$ and its representing homology class is the Massey product of a, b, c with respect to the choice $\alpha, \beta, \gamma, X, Y$, here we follow the sign convension of May [3]. Massey products are one of the most classical higher operations and generalized by several authors. Kraines [2] generalized Massey products for ordered n-tuple of elements of $H_*(\mathcal{A})$ satisfying an analogous condition, which is called Massey higher products. Furthermore, May [3] generalized Kraines' construction of Massey higher products by replacing homology classes with matrices whose entries are homology classes. These generalizations are linear in the following two sense. May's generalization replaces scalar values by matrices and, in this sense, May's generalization is linear. On the other hand, Kraines' generalization deals with ordered linear sequences of homology classes satisfying conditions on sub-linear sequences and then, in this sense, Kraines' generalization is also linear. However, there are many higher operations which are not linear.

The aim of this article is to generalize Massey products by using graphs and to establish basic formulae in those higher operations such as linearity, associativity and Jacobi identity. Actually, Kraines' Massey higher products of order n are described by a linear directed graph with n-arcs and May's matrix Massey products are described by gluing linearly special directed graphs.

In the sequel paper, we will generalize Toda brackets analogously using graphs and give applications to the stable homotopy groups of spheres.

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2. Hammock

In this section, we introduce special graphs called hammocks by their shapes, which form a basis of our study, and consider their decompositions.

Recall first that a directed graph D consists of the set of vertices $V(D)$, the set of arcs $A(D)$ and the coincidence function $\psi = (\psi_1, \psi_2) : A(D) \to V(D) \times V(D)$ (See [1]). An arc $a \in A(D)$ is said to join u to v if $\psi_1(a) = u$ and $\psi_2(a) = v$. In the usual manner, we draw a directed graph by a dot for each vertex and a directed curve from a dot of a vertex u to a dot of a vertex v for each arc joining u to v. A directed graph is called finite if both of the sets of vertices and arcs are finite. We can consider a directed subgraph of a directed graph in the obvious sense.

We introduce some notations and terminologies for directed graphs which we will make use of. Let D be a directed graph. For vertices $u, v \in V(D)$, a directed path joining v to u in D of length n is a subgraph ℓ of D such that $V(\ell) = \{u = v_1, v_2, \dots, v_n, v_{n+1} = v\}$, $A(\ell) = \{a_1, \dots, a_{n-1}\}$ such that $\psi(a_i) = (v_{i+1}, v_i)$ for $i = 1, \dots, n$. We denote the union of directed paths from v to u in D by D_{uv}, which is a subgraph of D. We ambiguously write by D_{uv} the set of all directed paths from v to u in D. We write $u < v$ for $u, v \in D$ if $D_{uv} \neq \emptyset$.

Now we define special graphs called hammocks.

Definition 2.1.

1. A quasi-hammock is a finite directed graph without loops whose set of vertices form a poset by the above $<$.
2. A hammock is a quasi-hammock without isolated vertices.
3. A hammock is said to be short if each arc joins a maximal vertex to a minimal vertex.
4. A hammock is said to be simple if the numbers of minimal vertices and of maximal vertices are one.

Let H be a quasi-hammock. We can consider sub-quasi-hammocks or sub-hammocks of H in the obvious sense. We denote the set of minimal vertices and the set of maximal vertices of $V(H)$ by $H^{\min}$ and $H^{\max}$ respectively. We also denote the unions of all directed paths of length 1 starting from $H^{\max}$ and of those terminated at $H^{\min}$ by H^+ and H^- respectively. Then H^+ and H^- are short sub-hammocks of H. We write by $\mu(H)$ the set of nonextremal vertices, that is, vertices neither minimal nor maximal. We make a convention of drawing quasi-hammocks with arcs from right to left. Following this convention, when we write the pair of the initial vertex and the terminal vertex of a directed path, we put the initial vertex on right.

Let us consider decompositions of quasi-hammocks by cutting transversely to the direction from minimal vertices to maximal vertices. To do so, we introduce cutting functions on quasi-hammocks. Let H be a quasi-hammock. A function $c : A(H) \to \mathbf{Z}$ is called a cutting function if $h(a) \leq h(b)$ for $a, b \in A(H)$ such that $\psi_2(a) = \psi_1(b)$. Then we can regard that a cutting function provides a grading of arcs respecting the direction of H. For a cutting function $c : A(H) \to \mathbf{Z}$ and a sequence of integers $n_1 < \ldots < n_k$, we define subgraphs $c^{-1}(-\infty, n_1]$ and $c^{-1}(n_k, +\infty)$ by the minimum sugraphs containing $\left(\bigcup_{i \leq n_1} c^{-1}(i)\right) \cup H^{\min}$ and $\left(\bigcup_{i > n_k} c^{-1}(i)\right) \cup H^{\max}$ respectively. We also define a subgraph $c^{-1}(n_i, n_{i+1}]$ for $i = 1, \ldots, k-1$ by the minimum subgraph containing $\left(\bigcup_{n_i < i \leq n_{i+1}} c^{-1}(i)\right) \cup (c^{-1}(-\infty, n_i] \cap c^{-1}(n_{i+1}, +\infty))$. Note that each of $c^{-1}(-\infty, n_1], c^{-1}(n_1, n_2], \ldots, c^{-1}(n_{k-1}, n_k], c^{-1}(n_k, +\infty)$ is a quasi-hammock.

Definition 2.2. If, for sub-hammocks $H_1, \ldots, H_{k+1}$ of a quasi-hammock H, there are a cutting function $c : A(H) \to \mathbf{Z}$ and a sequence of integers $n_1 < \ldots < n_k$ such that $c^{-1}(-\infty, n_1] = H_1, c^{-1}(n_1, n_2] = H_2, \ldots, c^{-1}(n_{k-1}, n_k] = H_k, c^{-1}(n_k, +\infty) = H_{k+1}$, then we write

$$H = (H_1, \ldots, H_{k+1}).$$

Let H be a quasi-hammock. If $H = (H_1, \ldots, H_n)$ for quasi-hammocks $H_1, \ldots, H_n$, then all isolated vertices of H are contained in each H_i for $i = 1, \ldots, n$. In particular, if one of $H_1, \ldots, H_n$ is a hammock, then so is H. However, even if H is a hammock, H_i needs not be a hammock. Note that, for H^- and H^+, there are quasi-hammocks F_1 and F_2 such that $H = (H^-, F_1) = (F_2, H^+)$.

The easiest case of the above decomposition is:

Proposition 2.1. *Let H and G be quasi-hammocks. For each bijection from $G^{\min}$ to $H^{\max}$, there is a quasi-hammock F such that $F = (H, G)$.*

Modifying the conditions in the above proposition, we define:

Definition 2.3. Let H, H_1, H_2, H_3 be quasi-hammocks such that $H = (H_1, H_2, H_3)$.

1. Suppose that H is simple. Then H_2 is called full in H is the following conditions are satisfied.
   - (a) H_2 is a hammock.
   - (b) For each directed path ℓ joining $v \in H^{\max}$ to $u \in H^{\min}$, $\ell \cap H_2^{\max} \neq \emptyset$ and $\ell \cap H_2^{\min} \neq \emptyset$.
2. H_2 is called locally full in H if, for each $(u, v) \in H^{\min} \times H^{\max}$, $H_2 \cap H_{uv}$ is full in H_{uv}.

Let H, H_1, H_2, H_3, H_4 be quasi-hammocks such that $H = (H_1, H_2, H_3, H_4)$. Note that if H_2, H_3 are locally full in H, then so is (H_2, H_3).

We next consider decompositions of quasi-hammocks by cutting in the parallel direction from minimal vertices to maximal vertices.

Definition 2.4.

1. Let H and G be quasi-hammocks. Suppose that there are bijections $H^{\min} \to G^{\min}$ and $H^{\max} \to G^{\max}$. Then we can glue together H and G by these bijections and obtain a new quasi-hammocks which we denote by $H \oplus G$.

Let $H_1, \ldots, H_4$ be quasi-hammocks which form a quasi-hammock $H = (H_1, \ldots, H_4)$. Note that if H_2, H_3 are full in H, then so is (H_2, H_3).

For the last of this section, we introduce a way to construct a new quasi-hammock out of given one.

Definition 2.5. Let H be a quasi-hammock. A quasi-hammock $\widehat{H}$ is given by deleting arcs in H which joins v to u such that there is a vertex w with $u < w < v$.

3. Hammock Massey product

In this section, we define hammock Massey product using hammocks which generalize the classical Massey products and May's matrix Massey products [3].

Hereafter, we fix a DGA $\mathcal{A}$ over a commutative ring R whose differential d is of degree -1. For a homogeneous element $a \in \mathcal{A}$, we put $\bar{a} = (-1)^{|a|+1}a$. We will often use the following formulae:

$$\bar{\bar{a}} = a, \ \overline{ab} = -\bar{a}\bar{b}, \ d\bar{a} = -\overline{da}$$

for $a, b \in \mathcal{A}$.

Definition 3.1.

1. An $\mathcal{A}$-colored quasi-hammock $\mathcal{H}$ consists of a quasi-hammock H and a function $\alpha : A(H) \to H_*(\mathcal{A})$ such that:
   (a) $\alpha(a)$ is homogeneous for each $a \in A(H)$.
   (b) For any $u, v \in V(H)$, $\sum_{a \in A(\ell)}(|\alpha(a)|+1)$ is independent from choice of a directed path $\ell \in H_{uv}$.
2. An $\mathcal{A}$-colored hammock is an $\mathcal{A}$-colored quasi-hammock whose underlying quasi-hammock is a hammock.

We can consider an $\mathcal{A}$-colored sub-quasi-hammock in the obvious sense. For an $\mathcal{A}$-colored hammock $\mathcal{H}$, if the underlying quasi-hammock of $\mathcal{H}$ satisfies some conditions such as hammock and locally full, then we say that $\mathcal{H}$ satisfies those conditions.

Let $\mathcal{H} = (\alpha, H)$ be an $\mathcal{A}$-colored quasi-hammock. We denote H by $h(\mathcal{H})$. If $\alpha = 0$, then we denote $\mathcal{H}$ by 0. We also denote by $\deg(u, v)$ the integer $\sum_{a \in \ell}(|\alpha(a)| + 1) - 2$ for $\ell \in H_{uv}$ with $u < v$, which is independent from choice of $\ell \in H_{uv}$. We write by $\mathcal{H}_{uv}$, $\mathcal{H}^-$ and $\mathcal{H}^+$ the $\mathcal{A}$-colored sub-hammocks of $\mathcal{H}$ whose underlying hammocks are H_{uv}, H^- and H^+ respectively. If $\mathcal{A}$-colored quasi-hammocks $\mathcal{H}_1 = (\alpha_1, H_1), \ldots, \mathcal{H}_n = (\alpha_n, H_n)$ satisfy $H = (H_1, \ldots, H_n)$ and $\alpha_i = \alpha|_{A(H_i)}$ for $i = 1, \ldots, n$, then we write

$$\mathcal{H} = (\mathcal{H}_1, \ldots, \mathcal{H}_n).$$

Analogously, if $\mathcal{A}$-colored quasi-hammocks $\mathcal{G}_1 = (\beta_1, G_1), \mathcal{G}_2 = (\beta_2, G_2)$ satisfy $H = G_1 \oplus G_2$ and $\beta_i = \alpha|_{A(G_i)}$ for $i = 1, 2$, then we write

$$\mathcal{H} = \mathcal{G}_1 \oplus \mathcal{G}_2.$$

Definition 3.2. Let $\mathcal{H} = (\alpha, H)$ be an $\mathcal{A}$-colored hammock and let D be a subset of $P\mathcal{H} = \{(u, v) \in (V(H) \times V(H)) \setminus (H^{\min} \times H^{\max}) | u < v\}$.

1. A partial defining system on $\mathcal{H}$ over D is a function $\Omega : D \to \mathcal{A}$ such that:
   (a) $\Omega(u, v) \in \mathcal{A}_{\deg(u,v)+1}$.
   (b) If H_{uv} is a short hammock, $\Omega(u, v)$ is a cycle representing $\sum_{a \in A(H_{uv})} \alpha(a)$.
   (c) If H_{uv} is not a short hammock, $\Omega(u, v)$ satisfies
$$d\Omega(u, v) = \sum_{u<w<v} \overline{\Omega(u, w)}\Omega(w, v).$$
2. A defining system on $\mathcal{H}$ is a partial defining system over $P\mathcal{H}$.

Remark 3.1. If there is a directed path from u to v in a hammock H which is not a single arc, then there is no contribution of arcs from u to v to any partial defining systems on an $\mathcal{A}$-colored hammock (α, H). Then a partial defining system on (α, H) is actually defined on $(\alpha|_{A(\widehat{H})}, \widehat{H})$.

We define a higher operation in $\mathcal{A}$ using defining systems on $\mathcal{A}$-colored hammocks. To do so, we need the following lemma. Let Ω be a defining system on an $\mathcal{A}$-colored hammock (α, H). We define $\Omega_{uv} \in \mathcal{A}$ for $u, v \in V(H)$ with $u < v$, possibly $(u, v) \in H^{\min} \times H^{\max}$, by:
$$\Omega_{uv} = \begin{cases} 0 & H_{uv} \text{ is a short hammock} \\ \sum_{u<w<v} \overline{\Omega(u, w)}\Omega(w, v) & \text{otherwise} \end{cases}$$

Lemma 3.1. *Let $\mathcal{H} = (\alpha, H)$ be an $\mathcal{A}$-colored hammock and let Ω be a defining system on $\mathcal{H}$. For $(u, v) \in V(H) \times V(H)$ with $u < v$, Ω_{uv} is a cycle.*

Proof. We may assume that H_{uv} is not a short hammock. For vertices x, z of H_{uv}, if there is no y such that $x < y < z$, then $d\overline{\Omega(x, y)} = 0$ by definition. Thus we have:
$$\begin{aligned} d\Omega_{uv} &= \sum_{u<w<v} (d(\overline{\Omega(u, w)})\Omega(w, v) + (-1)^{|\overline{\Omega(u,w)}|}\overline{\Omega(u, w)}d\Omega(w, v)) \\ &= \sum_{u<w<v} \left(\sum_{u<x<w} (-1)^{|\Omega(u,w)|+1}\overline{\Omega(u, x)}\Omega(x, w) \right) \Omega(w, v) \\ &\qquad + \sum_{u<w<v} (-1)^{2|\Omega(u,w)|+1}\Omega(u, w) \sum_{w<y<v} \overline{\Omega(w, y)}\Omega(y, v) \\ &= \sum_{u<x<y<v} \epsilon_{uxy}\Omega(u, x)\Omega(x, y)\Omega(y, v), \end{aligned}$$
here $\epsilon_{uxy} = ((-1)^{|\Omega(u,y)|+|\Omega(u,x)|+2} + (-1)^{2|\Omega(u,x)|+|\Omega(x,y)|+2})$. By definition of an $\mathcal{A}$-colored hammock, we have $|\Omega(u, x)| + |\Omega(x, y)| + 1 = |\Omega(u, y)|$ and thus $d\Omega_{uv} = 0$. ∎

Now we define a hammock Massey product as follows.

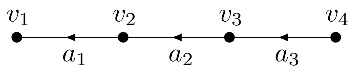
Definition 3.3. Let $\mathcal{H} = (\alpha, H)$ be an $\mathcal{A}$-colored hammock and let Ω be a defining system on $\mathcal{H}$. A hammock Massey product $\langle \mathcal{H} | \Omega \rangle$ is an $\mathcal{A}$-colored short hammock $(\alpha^{\Omega}, H^{\Omega})$ defined by:

1. $V(H^{\Omega}) = H^{\min} \cup H^{\max}$, $A(H^{\Omega}) = \{(u, v) \in H^{\min} \times H^{\max} | u < v\}$ and $\psi(u, v) = (u, v)$.
2. $\alpha^{\Omega}(u, v) = [\Omega_{uv}]$.

We denote the set of all hammock Massey products of an $\mathcal{A}$-colored hammock $\mathcal{H}$ by $\langle \mathcal{H} \rangle$. In this sense, if there is a defining system on an $\mathcal{A}$-colored hammock $\mathcal{H}$, then we say that $\langle \mathcal{H} \rangle$ is defined. In particular, $\langle \mathcal{S}_1, \mathcal{S}_2 \rangle$ is always defined when $\mathcal{S}_1$ and $\mathcal{S}_2$ are $\mathcal{A}$-colored short hammocks. Note that, by definition, $\langle \mathcal{S}_1, \mathcal{S}_2 \rangle$ consists of a single element.

We give examples describing Massey products and May's matrix Massey products by hammock Massey products.

Example 3.1. Let H be the following hammock.

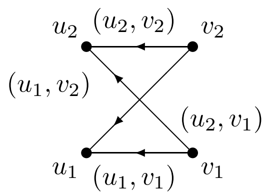


Define $\alpha : A(H) \to H_*(A)$ by $\alpha(a_i) = x_i \in H_{n_i}(\mathcal{A})$ for $i = 1, 2, 3$. Then $\mathcal{H} = (\alpha, H)$ is an $\mathcal{A}$-colored hammock. Suppose that $x_1 x_2 = x_2 x_3 = 0$. A function $\Omega : P\mathcal{H} \to \mathcal{A}$ is a defining system on $\mathcal{H}$ if and only if $X_i = \Omega(a_i, a_{i+1})$ is a cycle representing x_i for $i = 1, 2, 3$ and $Y_i = \Omega(v_i, v_{i+2})$ satisfies $dY_i = (-1)^{|x_i|+1} X_i X_{i+1}$ for $i = 1, 2$. Then, by definition, a defining system Ω on $\mathcal{H}$ defines

$$\Omega_{v_1 v_4} = (-1)^{|x_1|+1} X_1 Y_2 + (-1)^{|x_1|+|x_2|+2} Y_1 X_3.$$

Thus $\langle \mathcal{H} | \Omega \rangle$ is an $\mathcal{A}$-colored hammock such that the underlying hammock is a simple hammock with a single arc and the Massey product $[\Omega_{v_1 v_4}]$ is put on this arc.

Example 3.2. Let $H(m, n)$ be a short hammock such that $V(H(m, n)) = \{u_1, \ldots, u_m, v_1, \ldots, v_n\}$ and $A(H(m, n)) = \{(u_k, v_l) \in V(H(m, n)) \times V(H(m, n)) | 1 \le k \le m, 1 \le l \le n\}$. For example, $H(2, 2)$ is:



In particular, we have a hammock $(H(n_1, n_2), H(n_2, n_3), \ldots, H(n_k, n_{k+1}))$.

Note that an $\mathcal{A}$-colored hammock $\mathcal{H}(m, n) = (\alpha, H(m, n))$ can be identified with the $m \times n$ matrix $(\alpha(u_k, v_l))_{1 \le k \le m, 1 \le l \le n}$. It is easy to see that $\mathcal{A}$-colored hammocks $\mathcal{H}(l, m)$ and $\mathcal{H}(m, n)$ form an $\mathcal{A}$-colored hammock $(\mathcal{H}(l, m), \mathcal{H}(m, n))$ if and only if, by the above identification, $\mathcal{H}(m, n)$ and $\mathcal{H}(m, l)$ are multipliable in the sense of May [3]. A defining system on

$(\mathcal{H}(n_1, n_2)), \dots, \mathcal{H}(n_k, n_{k+1}))$ in our sense also coincides with that for matrices $(\mathcal{H}(n_1, n_2), \dots, \mathcal{H}(n_k, n_{k+1}))$ in the sense of May [3]. Moreover, by definition, the hammock Massey product $\langle \mathcal{H}(n_1, n_2), \dots, \mathcal{H}(n_k, n_{k+1}) \rangle$ is identified with May's matrix Massey product $\langle \mathcal{H}(n_1, n_2), \dots, \mathcal{H}(n_k, n_{k+1}) \rangle$.

Let $\mathcal{L}, \mathcal{U}$ be $\mathcal{A}$-colored quasi-hammocks and let $\mathcal{H}$ be an $\mathcal{A}$-colored hammock. Suppose that $L, \mathcal{U}, \mathcal{H}$ form an $\mathcal{A}$-colored hammock $(\mathcal{L}, \mathcal{H}, \mathcal{U})$. One of the advantage of hammock Massey products is that we can construct a new $\mathcal{A}$-colored hammock $(\mathcal{L}, \langle \mathcal{H} \rangle, \mathcal{U})$ in some cases, which produces basic formulae in hammock Massey products.

Lemma 3.2. *Let $\mathcal{L}, \mathcal{H}, \mathcal{U}$ be $\mathcal{A}$-colored quasi-hammocks such that there is an $\mathcal{A}$-colored hammock $(\mathcal{L}, \mathcal{H}, \mathcal{U})$ in which $\mathcal{H}$ is locally full. For each defining system Ω on $\mathcal{H}$, $\mathcal{A}$-colored quasi-hammocks $\mathcal{L}, \langle \mathcal{H} | \Omega \rangle, \mathcal{U}$ form an $\mathcal{A}$-colored hammock $(\mathcal{L}, \langle \mathcal{H} | \Omega \rangle, \mathcal{U})$.*

Proof. Note that $V(h(\mathcal{H})) \cap V(h(\mathcal{L})) = h(\mathcal{H})^{\min}$ and $V(h(\mathcal{H})) \cap V(h(\mathcal{U})) = h(\mathcal{H})^{\max}$ since $\mathcal{H}$ is locally full in $(\mathcal{L}, \mathcal{H}, \mathcal{U})$. We construct a new hammock out of a quasi-hammock $h(\mathcal{L}) \cup h(\mathcal{U})$ by joining $v \in V(h(\mathcal{U}))$ to $u \in V(h(\mathcal{L}))$ with a single arc if $u < v$. Then we obtain a new hammock $H' = h(\mathcal{L}) \cup h(\langle \mathcal{H} | \Omega \rangle) \cup h(\mathcal{U})$ hence $h(\langle \mathcal{H} | \Omega \rangle)$ is equal to the short hammock consists of the joining arcs. Let $c : A(h((\mathcal{L}, \mathcal{H}, \mathcal{U}))) \to \mathbf{Z}$ be a cutting function giving the decomposition $(\mathcal{L}, \mathcal{H}, \mathcal{U})$ and let $\mathcal{H} = c^{-1}(n, m]$. Define a function $c' : A(H') \to \mathbf{Z}$ by $c'|_{A(h(\mathcal{L})) \cup A(h(\mathcal{U}))} = c|_{A(h(\mathcal{L})) \cup A(h(\mathcal{U}))}$ and $c'(a) = m$ for $a \in h(\langle \mathcal{H} | \Omega \rangle)$. Then c' gives a decomposition $H' = (h(\mathcal{L}), h(\langle \mathcal{H} \rangle), h(\mathcal{U}))$.

Define a function $\alpha' : A(H') \to H_*(\mathcal{A})$ by $\alpha'|_{A(h(\mathcal{L})) \cup A(h(\mathcal{U}))} = \alpha|_{A(h(\mathcal{L})) \cup A(h(\mathcal{U}))}$ and $\alpha'(a)$ to be the homology class represented by Ω_{uv} for $a \in A(\langle \mathcal{H} | \Omega \rangle)$ joining v to u. Then it follows from the definition of a hammock being locally full that α' satisfies the degree condition and thus $\mathcal{L}, \langle \mathcal{H} | \Omega \rangle, \mathcal{U}$ form an $\mathcal{A}$-colored hammock $(\mathcal{L}, \langle \mathcal{H} | \Omega \rangle, \mathcal{U})$. □

The following lemma allows us to start with any choice in constructing defining systems.

Lemma 3.3. *Let $\mathcal{H} = (\alpha, H)$ be an $\mathcal{A}$-colored hammock such that $\langle \mathcal{H} \rangle$ is defined and let $\tilde{\alpha} : A(H) \to \mathcal{A}$ be such that $\tilde{\alpha}(a)$ represents $\alpha(a)$ for $a \in A(H)$. Then, for each $x \in \langle \mathcal{H} \rangle$, there is a defining system Ω on $\mathcal{H}$ such that $x = \langle \mathcal{H} | \Omega \rangle$ and $\Omega(u, v) = \sum_{a \in H_{uv}} \tilde{\alpha}(a)$ when H_{uv} is a short hammock.*

Proof. Let Ω' be a defining system on $\mathcal{H}$ such that $x = \langle \mathcal{H} | \Omega' \rangle$. We construct the desired defining system Ω on $\mathcal{H}$ out of Ω'. Let $S\mathcal{H} = \{(u, v) \in P\mathcal{H} | H_{uv}$ is a short hammock$\}$. First, we put $\Omega(u, v) = \sum_{a \in A(H_{uv})} \tilde{\alpha}(a)$ and $\Delta(u, v) = \Omega'(u, v) - \Omega(u, v)$ for $(u, v) \in S\mathcal{H}$. Next, we inductively put, for $(u, v) \in P\mathcal{H}$ but $(u, v) \notin S\mathcal{H}$,

$$\Omega(u, v) = \Omega'(u, v) - \sum_{(u,w) \in S\mathcal{H}} \overline{\Delta(u, w)} \Omega(w, v) - \sum_{(w,v) \in S\mathcal{H}} \Omega'(u, w) \Delta(w, v).$$

Then it is straightforward to check that Ω is a defining system on $\mathcal{H}$. If $(u,v) \in H^{\min} \times H^{\max}$ belongs to $S\mathcal{H}$, we have $\Omega'_{uv} - \Omega_{uv}$ is a boundary by definition. If $(u,v) \in H^{\min} \times H^{\max}$ does not belong to $S\mathcal{H}$, we have

$$\Omega'_{uv} - \Omega_{uv} = d(\sum_{(u,w)\in S\mathcal{H}} \overline{\Delta(u,w)}\Omega(w,v) + \sum_{(w,v)\in S\mathcal{H}} \Omega'(u,w)\Delta(w,v))$$

and then the proof is completed. □

In order to consider extending partial defining systems to defining systems, we introduce the following (See [3]).

Definition 3.4. An $\mathcal{A}$-colored hammock $\mathcal{H} = (\alpha, H)$ is called strictly defined if, for each $(u,v) \notin H^{\min} \times H^{\max}$, $\langle \mathcal{H}_{uv} \rangle$ is defined such that $\langle \mathcal{H}_{uv} \rangle = \{0\}$, where $\mathcal{H}_{uv}$ is an $\mathcal{A}$-colored hammock $(\alpha|_{A(H_{uv})}, H_{uv})$.

Let $\mathcal{H}$ be an $\mathcal{A}$-colored hammock. Note that if $\langle \mathcal{H} \rangle$ is defined, we always have $0 \in \langle \mathcal{H}_{uv} \rangle$ for each $(u,v) \notin H^{\min} \times H^{\max}$ but $\langle \mathcal{H}_{uv} \rangle = \{0\}$ is not always true.

Lemma 3.4. *Let $\mathcal{H}$ be an $\mathcal{A}$-colored hammock such that $\langle \mathcal{H} \rangle$ is strictly defined. Then every partial defining system on $\mathcal{H}$ extends to a defining system on $\mathcal{H}$.*

Proof. Let $\mathcal{H} = (\alpha, H)$ and let Ω be a partial defining system on $\mathcal{H}$ over $D \subset P\mathcal{H}$. For $(u,v) \notin D$ such that H_{uv} is a short hammock, we put $\Omega(u,v)$ to be an arbitrary representative of $\sum_{a \in A(H_{uv})} \alpha(a)$. Suppose that we can extend Ω to all $(u,v) \in P\mathcal{H}$ such that each directed path joining v to u is of length less than n. For $(u,v) \in P\mathcal{H}$ such that each directed path joining u to v is of length less than or equal to n, we can define Ω_{uv} and, by Lemma 3.1, $d\Omega_{uv} = 0$. Then it follows from $\langle H_{uv} \rangle = \{0\}$ that there exists $x \in \mathcal{A}$ such that $\Omega_{uv} = dx$. We put $\Omega(u,v) = x$. Thus the proof is completed by induction on n. □

Corollary 3.1. *Let $\mathcal{H}$ be an $\mathcal{A}$-colored hammock such that $\langle \mathcal{H} \rangle$ is strictly defined. Then $\langle \mathcal{H} \rangle$ is defined.*

Proof. Let $\mathcal{H} = (\alpha, H)$ and $S\mathcal{H}$ be as in the proof of Lemma 3.3. By putting $\Omega(u,v)$ to be an arbitrary representative of $\sum_{a \in A(H_{uv})} \alpha(a)$ for $(u,v) \in S\mathcal{H}$, we have a partial defining system on $\mathcal{H}$ over $S\mathcal{H}$. Then the proof is completed by Lemma 3.4. □

It is easy to check that:

Corollary 3.2. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be $\mathcal{A}$-colored hammocks such that $\mathcal{H} = (\mathcal{H}^-, \mathcal{H}_1) = (\mathcal{H}_2, \mathcal{H}^+)$. Suppose that $\langle \mathcal{H}_1 \rangle$ (resp. $\langle \mathcal{H}_2 \rangle$) is strictly defined and $\langle \mathcal{H}_1 \rangle = \{0\}$ (resp. $\langle \mathcal{H}_2 \rangle = \{0\}$). Then every defining system Ω^2 on $\mathcal{H}_2$ such that $\langle \mathcal{H}_2 | \Omega^2 \rangle = 0$ (resp. Ω^1 on $\mathcal{H}_1$ such that $\langle \mathcal{H}_1 | \Omega^1 \rangle = 0$) extends to a defining system on $\mathcal{H}$.*

4. Indeterminacy

As ordinary higher operations, hammock Massey products are defined with some ambiguity which is called the indeterminacy. In this section, we consider the indeterminacy of hammock Massey products. Let us first introduce an operation in $\mathcal{A}$-colored short hammocks.

Definition 4.1. Let $\mathcal{H} = (\alpha, H)$ and $\mathcal{G} = (\beta, H)$ be $\mathcal{A}$-colored short hammocks such that $|\alpha(a)| = |\beta(b)|$ for each $\alpha \in A(H)$. For $r, s \in R$, we define

$$r\mathcal{H} + s\mathcal{G} = (r\alpha + s\beta, H).$$

Now we define the indeterminacy of hammock Massey products.

Definition 4.2. Let $\mathcal{H}$ be an $\mathcal{A}$-colored hammock such that $\langle \mathcal{H} \rangle$ is defined. The indeterminacy of $\langle \mathcal{H} \rangle$ is

$$\mathrm{In}\langle \mathcal{H} \rangle = \{x - y | x, y \in \langle \mathcal{H} \rangle\}.$$

We give a lower bound for the indeterminacy of hammock Massey products when the underlying hammocks are simple.

Proposition 4.1. *Let $\mathcal{H} = (\alpha, H)$ be an $\mathcal{A}$-colored simple hammock such that $H^{\min} = \{u\}$ and $H^{\max} = \{v\}$. Suppose that $H^{\min}$ and $H^{\max}$ are separated, that is, $H^- \cap H^{\max} = H^+ \cap H^{\min} = \emptyset$. Then we have*

$$\mathrm{In}\langle \mathcal{H} \rangle \supset \sum_{w \in V(H^-) \setminus H^{\min}} \left(\sum_{a \in A(H_{uw})} \alpha(a) \right) H_{\deg(w,v)+1}(\mathcal{A}) + \sum_{w \in V(H^+) \setminus H^{\max}} H_{\deg(u,w)+1} \left(\sum_{a \in A(H_{wv})} \alpha(a) \right).$$

Proof. Let Ω be a defining system on $\mathcal{H}$ and let A_x and B_y be arbitrary cycles in $\mathcal{A}_{\deg(x,v)+1}$and $\mathcal{A}_{\deg(u,y)+1}$ respectively for $x \in V(H^-) \setminus H^{\min}$ and $y \in V(H^+) \setminus H^{\max}$. Define a function $\Omega' : P\mathcal{H} \to \mathcal{A}$ by:

$$\Omega'(x,y) = \begin{cases} \Omega(x,y) + A_x & x \in V(H^-) \setminus H^{\min},\ y = v \\ \Omega(x,y) + B_y & y \in V(H^+) \setminus H^{\max},\ x = u \\ \Omega(x,y) & \text{otherwise} \end{cases}$$

Then it is obvious that Ω' is a defining system on $\mathcal{H}$ such that

$$\langle \mathcal{H} | \Omega' \rangle - \langle \mathcal{H} | \Omega \rangle = \pm \sum_{w \in V(H^-) \setminus H^{\min}} \left(\sum_{a \in A(H_{uw})} \alpha(a) \right) A_w \pm \sum_{w \in V(H^+) \setminus H^{\max}} B_w \left(\sum_{a \in A(H_{wv})} \alpha(a) \right).$$

Thus the proof is completed. □

Let $\mathcal{H} = (\alpha, H)$ be an $\mathcal{A}$-colored hammock. In order to give an upper bound for hammock Massey products, we construct a new $\mathcal{A}$-colored hammock out of $\mathcal{H}$. This is a generalization of Proposition 2.3 in [3]. Define a hammock $H^{\oplus}$ by deleting H^- and all arcs in $\widehat{H}$ terminated at $V(H^-)\setminus H^{\min}$ from $\widehat{H}$, here we do not delete the initial vertices of arcs terminated at $V(H^-)\setminus H^{\min}$. We also define $H^{\ominus}$ by deleting H^+ and all arcs in $\widehat{H}$ terminated at $V(H^+)\setminus H^{\max}$ from $\widehat{H}$, here we do not delete the initial vertices of arcs terminated at $V(H^+)\setminus H^{\max}$ as well. We construct a new hammock $\widetilde{H}$ by attaching to the disjoint union $H^{\oplus} \sqcup H^{\ominus}$ the arcs joining $u \in H^{\oplus}$ to $v \in H^{\ominus}$ if there is a directed path joining u to v of length 2. Define an $\mathcal{A}$-colored hammock $\widetilde{\mathcal{H}}_{\tilde{\alpha}} = (\tilde{\alpha}, \widetilde{H})$ by $\tilde{\alpha}|_{A(H^{\oplus})} = \alpha|_{A(H^{\oplus})}$, $\tilde{\alpha}|_{A(H^{\ominus})} = \alpha|_{A(H^{\ominus})}$ and $\tilde{\alpha}(a)$ being an arbitrary element in $H_{\deg(u,v)+1}(\mathcal{A})$ for $a \notin A(H^{\oplus} \sqcup H^{\ominus})$.

Proposition 4.2. *Let $\mathcal{H} = (\alpha, H)$ be an $\mathcal{A}$-colored hammock such that $\langle\mathcal{H}\rangle$ is defined. Then we have*

$$\mathrm{In}\langle\mathcal{H}\rangle \subset \bigcup_{\widetilde{\mathcal{H}}_{\tilde{\alpha}} \textit{ is defined.}} \langle\widetilde{\mathcal{H}}_{\tilde{\alpha}}\rangle.$$

Proof. Let Ω be a defining system on $\mathcal{H}$. Since $\langle\mathcal{H}|\Omega\rangle = \langle(\alpha|_{A(\widehat{H})}, \widehat{H})|\Omega\rangle$, we assume that $H = \widehat{H}$. Let Ω' be another defining system on $\mathcal{H}$. It follows from Lemma 3.3 that we can put $\Omega(u,v) = \Omega'(u,v)$ when H_{uv} is short. Then $\Omega'(u,v) - \Omega(u,v)$ is a cycle if v is joined to u only by length 2 directed paths in $\hat{H}$. Consider the set of vertices $W_{uv} = \{w \in V(H) | u < w < v, H(u,w)$ and $H(w,v)$ are short$\}$. Then we can choose $\tilde{\alpha}$ such that

$$\tilde{\alpha}(a) = \left[\sum_{w \in W_{uv}} \{\Omega'(u,w)\Omega'(w,v) - \Omega(u,w)\Omega(w,v)\}\right]$$

for $a \notin A(H^{\oplus} \sqcup H^{\ominus})$. Define a function $\widetilde{\Omega} : P\widetilde{\mathcal{H}} \to \mathcal{A}$ by:

$$\widetilde{\Omega}(u,v) = \begin{cases} \Omega(u,v) & u \in V(H^{\oplus}) \\ \Omega'(u,v) & v \in V(H^{\ominus}) \\ \Omega'(u,v) - \Omega(u,v) & (u,v) \in V(H^{\ominus}) \times V(H^{\oplus}) \end{cases}$$

Let us check that $\widetilde{\Omega}$ is a defining system on $\widetilde{H}$. We only have to check $\widetilde{\Omega}(u,v)$ for $(u,v) \in V(H^{\ominus}) \times V(H^{\oplus})$.

$$\begin{aligned} d\widetilde{\Omega}(u,v) &= d(\Omega'(u,v) - \Omega(u,v)) \\ &= \sum_{w \in V(H_{uv})} (\overline{\Omega'(u,w)}\Omega'(w,v) - \overline{\Omega(u,w)}\Omega(w,v)) \\ &= \sum_{w \in V(H_{uv})} ((\overline{\Omega'(u,w)} - \overline{\Omega(u,w)})\Omega'(w,v) + \overline{\Omega(u,w)}(\Omega'(w,v) - \Omega(w,v))) \end{aligned}$$

When H_{uw} is short, we have $\Omega'(u,w) = \Omega(u,w)$ and then the above sum $\sum_{w\in V(H_{uv})}(\overline{\Omega'(u,w)}-\overline{\Omega(u,w)})\Omega'(w,v)$ is assumed to be taken over $w \in V(H_{uv}) \cap V(H^{\oplus})$. Analogously, the above sum $\sum_{w\in V(H_{uv})} \overline{\Omega(u,w)}(\Omega'(w,v)-\Omega(w,v))$ is also assume to be taken over $w \in V(H_{uv}) \cap V(H^{\ominus})$. Thus we have:

$$\begin{aligned}
d\widetilde{\Omega}(u,v) = &\sum_{w\in V(H_{uv})\cap V(H^{\oplus})} (\overline{\Omega'(u,w)} - \overline{\Omega(u,w)})\Omega'(w,v) \\
&\quad + \sum_{w\in V(H_{uv})\cap V(H^{\ominus})} \overline{\Omega(u,w)}(\Omega'(w,v) - \Omega(w,v)) \\
= &\sum_{w\in V(H_{uv})\cap V(H^{\oplus})} \overline{\widetilde{\Omega}(u,w)}\widetilde{\Omega}(w,v) + \sum_{w\in V(H_{uv})\cap V(H^{\ominus})} \overline{\widetilde{\Omega}(u,w)}\widetilde{\Omega}(w,v) \\
= &\sum_{u<w<v} \overline{\widetilde{\Omega}(u,w)}\widetilde{\Omega}(w,v)
\end{aligned}$$

Then $\widetilde{\Omega}$ is a defining system on $\widetilde{\mathcal{H}}$. Now we have:

$$\begin{aligned}
\langle\widetilde{\mathcal{H}}|\widetilde{\Omega}\rangle = &\sum_{u<w<v} \overline{\widetilde{\Omega}(u,w)}\widetilde{\Omega}(w,v) \\
= &\sum_{w\in V(H^{\ominus})} \overline{\Omega(u,w)}(\Omega'(w,v) - \Omega(w,v)) \\
&\quad + \sum_{w\in V(H^{\oplus})} (\overline{\Omega'(u,w)} - \overline{\Omega(u,w)})\Omega'(w,v) \\
= &- \sum_{w\in V(H^{\ominus})} \overline{\Omega(u,w)}\Omega(w,v) + \sum_{w\in V(H^{\oplus})} \overline{\Omega'(u,w)}\Omega'(w,v) \\
&\quad - \sum_{w\in V(H^{\oplus})\setminus V(H^{\ominus})} \overline{\Omega(u,w)}\Omega'(w,v) \\
&\qquad + \sum_{w\in V(H^{\ominus})\setminus V(H^{\oplus})} \overline{\Omega(u,w)}\Omega'(w,v)
\end{aligned}$$

Note that $\Omega(w,v) = \Omega'(w,v)$ for $w \in V(H^{\oplus}) \setminus V(H^{\ominus})$ and $\Omega(u,w) = \Omega'(u,w)$ for $w \in V(H^{\ominus}) \setminus V(H^{\oplus})$. Therefore we obtain

$$\langle\widetilde{\mathcal{H}}|\widetilde{\Omega}\rangle = \sum_{u<w<v} \overline{\Omega'(u,w)}\Omega'(w,v) - \sum_{u<w<v} \overline{\Omega(u,w)}\Omega(w,v) = \langle\mathcal{H}|\Omega'\rangle - \langle\mathcal{H}|\Omega\rangle$$

and this completes the proof. □

We can give a stronger lower bound for hammock Massey products by imposing strictly definedness.

Proposition 4.3. *Let $\mathcal{H}, \mathcal{L}, \mathcal{G}, \mathcal{U}$ be $\mathcal{A}$-colored hammocks such that $\mathcal{H} = (\mathcal{L}, \mathcal{G}, \mathcal{U})$ in which $\mathcal{G}$ is locally full. Suppose that*:

1. $\langle\mathcal{H}\rangle$ *is strictly defined.*

2. *For each* $(u,v) \in h(\mathcal{G})^{\min} \times h(\mathcal{G})^{\max}$, $\mathcal{G}_{uv}$ *is not short.*
Given a defining system Ω *on* $\mathcal{G}$, *we define an* $\mathcal{A}$*-colored hammock* $(\alpha', \mathcal{H}')$ *by defining* $\mathcal{H}' = (h(\mathcal{L}), h(\langle \mathcal{G}|\Omega\rangle), h(\mathcal{U}))$, $\alpha' : P(\mathcal{L}, \langle \mathcal{G}|\Omega\rangle, \mathcal{U}) \to H_*(\mathcal{A})$ *such that* $\alpha'(a) = \alpha(a)$ *for* $a \notin A(h(\langle \mathcal{G}|\Omega\rangle))$ *and* $\alpha'(a)$ *is arbitrary element in* $H_{\deg(\psi_1(a),\psi_2(a))+1}(A)$ *for* $a \in A(h(\langle \mathcal{G}|\Omega\rangle))$. *Then* $\langle \mathcal{H}'\rangle$ *is strictly defined and*

$$\langle \mathcal{H}'\rangle \subset \mathrm{In}\langle \mathcal{H}\rangle.$$

Proof. As in the proof of Lemma 3.2, $\mathcal{H}'$ is an $\mathcal{A}$-colored hammock. We proceed the proof by induction on the number of vertices of $\mathcal{H}$. If the number of vertices of $\mathcal{H}$ is three, then $\mathcal{H}' = \langle \mathcal{G}\rangle$ is short so that $\langle \mathcal{H}'\rangle = \{0\} \subset \mathrm{In}\langle \mathcal{H}\rangle$. Suppose that, for $\mathcal{H}$ with vertices less than n which satisfies all conditions, $\langle \mathcal{H}'\rangle \subset \mathrm{In}\langle \mathcal{H}\rangle$. Let the number of the vertices of $\mathcal{H}$ be n and let $u, v \in V(h(\mathcal{H}))$ such that $(u,v) \in h(\mathcal{H})^{\min} \times h(\mathcal{H})^{\max}$. Note yhat $\mathcal{H}_{uv} = (\mathcal{H}_{uv} \cap \mathcal{L}, \mathcal{H}_{uv} \cap \mathcal{G}, \mathcal{H}_{uv} \cap \mathcal{U})$ satisfies all the above conditions. Then, by hypothesis of induction, we obtain $\langle \mathcal{H}'_{uv}\rangle \subset \mathrm{In}\langle \mathcal{H}_{uv}\rangle = \{0\}$. Thus $\langle \mathcal{H}'\rangle$ is strictly defined. Let Ω' be a defining system on $\mathcal{H}'$. Then $\Omega'|_{P\mathcal{L} \cup P\mathcal{U}}$ extends to a defining system Λ on $\mathcal{H}$. Define a function $\Lambda' : P\mathcal{H} \to \mathcal{A}$ by:

$$\Lambda'(u,v) = \begin{cases} \Lambda(u,v) - \Omega'(u,v) & u \in V(h(\mathcal{L})), v \in V(h(\mathcal{U})) \\ \Lambda & \text{otherwise} \end{cases}$$

then we can easily verify that Λ' is a defining system on $\mathcal{H}$ and $\langle \mathcal{H}|\Lambda\rangle - \langle \mathcal{H}|\Lambda'\rangle = \langle \mathcal{H}'|\Omega'\rangle$. □

5. Linearity and associativity

In this section, we consider linearity and associativity in hammock Massey products. Before doing so, let us first give the easiest relation in hammock Massey products. For an $\mathcal{A}$-colored quasi-hammock $\mathcal{H} = (\alpha, H)$, we define an $\mathcal{A}$-colored quasi-hammock $\overline{\mathcal{H}}$ by $\overline{\mathcal{H}} = (\bar{\alpha}, H)$, here $\bar{\alpha}(a) = \overline{\alpha(a)}$ for $a \in A(H)$. It is easily checked that:

Proposition 5.1. *Let* $\mathcal{H}$ *be an* $\mathcal{A}$*-colored hammock such that* $\langle \mathcal{H}\rangle$ *is defined. Then* $\langle \overline{\mathcal{H}}\rangle$ *is defined and*

$$\langle \overline{\mathcal{H}}\rangle = -\overline{\langle \mathcal{H}\rangle}.$$

Now we consider linearity in hammock Massey products.

Theorem 5.1.
1. *Let* $\mathcal{L}, \mathcal{S}, \mathcal{U}$ *be* $\mathcal{A}$*-colored quasi-hammocks such that there is an* $\mathcal{A}$*-colored hammock* $(\mathcal{L}, \mathcal{S}, \mathcal{U})$ *and* $\mathcal{S}$ *is short. Suppose that* $\langle \mathcal{L}, \mathcal{S}, \mathcal{U}\rangle$ *is defined. Then* $\langle \mathcal{L}, r\mathcal{S}, \mathcal{U}\rangle$ *is defined for* $r \in R$ *and*

$$r\langle \mathcal{L}, \mathcal{S}, \mathcal{U}\rangle \subset \langle \mathcal{L}, r\mathcal{S}, \mathcal{U}\rangle.$$

2. *Let $\mathcal{L}, \mathcal{U}$ be $\mathcal{A}$-colored quasi-hammocks and $\mathcal{S}_1, \mathcal{S}_2$ be $\mathcal{A}$-colored short hammocks such that there is an $\mathcal{A}$-colored hammock $(\mathcal{L}, \mathcal{S}_1 \oplus \mathcal{S}_2, \mathcal{U})$. Suppose that $\langle \mathcal{L}, \mathcal{S}_1 \oplus \mathcal{S}_2, \mathcal{U}\rangle$ is defined and $\langle \mathcal{L}, \mathcal{S}_1, \mathcal{U}\rangle$ is strictly defined. Then $\langle \mathcal{L}, \mathcal{S}_2, \mathcal{U}\rangle$ is defined and*

$$\langle \mathcal{L}, \mathcal{S}_1 \oplus \mathcal{S}_2, \mathcal{U}\rangle \subset \langle \mathcal{L}, \mathcal{S}_1, \mathcal{U}\rangle + \langle \mathcal{L}, \mathcal{S}_2, \mathcal{U}\rangle.$$

3. *Let $\mathcal{L}, \mathcal{U}$ be $\mathcal{A}$-colored quasi-hammocks and let $\mathcal{S}_1, \mathcal{S}_2$ be $\mathcal{A}$-colored short hammocks such that there are $\mathcal{A}$-colored hammocks $(\mathcal{L}, \mathcal{S}_1\mathcal{U}), (\mathcal{L}, \mathcal{S}_2\mathcal{U})$. Suppose that $\mathcal{H}_1$ and $\mathcal{H}_2$ satisfy the following conditions*:

(a) *$\mathcal{H}_1$ and $\mathcal{H}_2$ form an $\mathcal{A}$-colored hammock $\mathcal{H}$ by gluing together along $\mathcal{L}$ and $\mathcal{U}$.*

(b) *$h(\mathcal{H}_1)_{uv} = h(\mathcal{H})_{uv}$ for $u, v \in V(h(\mathcal{H}_1))$ such that $h(\mathcal{H}_1)_{uv} \neq \emptyset$ with $u \notin h(\mathcal{L})$ or $v \notin h(\mathcal{U})$.*

(c) *$\langle \mathcal{H}\rangle$ is defined and $\langle \mathcal{H}_1\rangle$ is strictly defined.*

Then $\langle \mathcal{H}_2\rangle$ is defined and

$$\langle \mathcal{H}\rangle \subset \langle \mathcal{H}_1\rangle + \langle \mathcal{H}_2\rangle.$$

Proof. 1. Let Ω be a defining system on $(\mathcal{L}, \mathcal{S}, \mathcal{H})$. Define a function $\Omega^r : P(\mathcal{L}, r\mathcal{S}, \mathcal{U}) \to \mathcal{A}$ by:

$$\Omega^r(u,v) = \begin{cases} r\Omega(u,v) & v \in h(\mathcal{U}), u \in h(\mathcal{L}) \\ \Omega(u,v) & \text{otherwise} \end{cases}$$

Then it is easily seen that Ω^r is a defining system on $(\mathcal{L}, r\mathcal{S}, \mathcal{U})$ such that $\langle \mathcal{L}, r\mathcal{S}, \mathcal{U} | \Omega^r\rangle = r\langle \mathcal{L}, \mathcal{S}, \mathcal{U}|\Omega\rangle$. This completes the proof.

3. Let Ω be a defining system on $\mathcal{H}$. The condition (b) allows us to restrict Ω to a partial defining system on $\mathcal{H}_1$ over $\{(u,v) \in P\mathcal{H}_1 | v \notin h(\mathcal{U}) \text{ or } u \notin h(\mathcal{L})\}$. Since $\langle \mathcal{H}_1\rangle$ is strictly defined, the above partial defining system extends to a defining system Ω^1 on $\mathcal{H}_1$. Define a function $\Omega^2 : P\mathcal{H}_2 \to \mathcal{A}$ by

$$\Omega^2(u,v) = \begin{cases} \Omega(u,v) - \Omega^1(u,v) & v \in h(\mathcal{U}), u \in h(\mathcal{L}) \\ \Omega(u,v) & \text{otherwise.} \end{cases}$$

It is easy to verify that Ω^2 is a defining system on $\mathcal{H}_2$ and $\langle \mathcal{H}|\Omega\rangle = \langle \mathcal{H}_1|\Omega^1\rangle + \langle \mathcal{H}_2|\Omega^2\rangle$. This completes the proof of 3. The case 2 follows from 3. □

We next show associativity of hammock Massey products.

Theorem 5.2. *Let $\mathcal{S}_1, \mathcal{S}_2$ be $\mathcal{A}$-colored short hammocks, let $\mathcal{H}_1, \mathcal{H}_3$ be $\mathcal{A}$-colored quasi-hammocks and let $\mathcal{H}_2$ be an $\mathcal{A}$-colored hammock such that there is an $\mathcal{A}$-colored hammock $\mathcal{H} = (\mathcal{S}_1, \mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \mathcal{S}_2)$ in which $(\mathcal{S}_1, \mathcal{H}_1, \mathcal{H}_2)$ and $(\mathcal{H}_2, \mathcal{H}_3, \mathcal{S}_2)$ are locally full. Suppose that there is a partial defining system on $\mathcal{H}$ over $P(\mathcal{S}_1, \underline{\mathcal{H}_2, \mathcal{H}_3}) \cup P(\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3) \cup P(\mathcal{H}_2, \mathcal{H}_3, \mathcal{S}_2)$. Then $\langle\langle \mathcal{S}_1, \mathcal{H}_1, \mathcal{H}_2|\Omega\rangle, \mathcal{H}_3, \mathcal{S}_2\rangle$ and $\langle \overline{\mathcal{S}_1}, \overline{\mathcal{H}_1}, \langle \mathcal{H}_2, \mathcal{H}_3, \mathcal{S}_2|\Omega\rangle\rangle$ are defined and*

$$0 \in \langle\langle \mathcal{S}_1, \mathcal{H}_1, \mathcal{H}_2|\Omega\rangle, \mathcal{H}_3, \mathcal{S}_2\rangle + \langle \overline{\mathcal{S}_1}, \overline{\mathcal{H}_1}, \langle \mathcal{H}_2, \mathcal{H}_3, \mathcal{S}_2|\Omega\rangle\rangle.$$

Proof. Let $\mathcal{G}_1 = (\langle \mathcal{S}_1, \mathcal{H}_1, \mathcal{H}_2|\Omega\rangle, \mathcal{H}_3, \mathcal{S}_2)$ and $\mathcal{G}_2 = (\overline{\mathcal{S}_1}, \overline{\mathcal{H}_1}, \langle \mathcal{H}_2, \mathcal{H}_3, \mathcal{S}_2|\Omega\rangle)$. Define a function $\Omega^1 : P\mathcal{G}_1 \to \mathcal{A}$ and $\Omega^2 : P\mathcal{G}_2 \to \mathcal{A}$ by

$$\Omega^1(u,v) = \begin{cases} \sum_{w \in \mu(h(\mathcal{S}_1,\mathcal{H}_1,\mathcal{H}_2))} \overline{\Omega(u,w)}\Omega(w,v) & u \in h(\mathcal{G}_1)^{\min} \\ \Omega(u,v) & u \notin h(\mathcal{G}_1)^{\min} \end{cases}$$

and

$$\Omega^2(u,v) = \begin{cases} \sum_{w \in \mu(h(\mathcal{H}_2,\mathcal{H}_3,\mathcal{S}_2))} \overline{\Omega(u,w)}\Omega(w,v) & v \in h(\mathcal{G}_2)^{\max} \\ \overline{\Omega(u,v)} & v \notin h(\mathcal{G}_2)^{\max}. \end{cases}$$

If $u \notin h(\mathcal{G}_1)^{\min}$, then Ω^1 obviously satisfies the condition of partial defining systems on $\mathcal{G}_1$. For $u \in h(\mathcal{G}_1)^{\min}$, we have:

$$\begin{aligned} d\Omega^1(u,v) = & \sum_{\substack{u<x<w \\ w \in \mu(h(\mathcal{S}_1,\mathcal{H}_1,\mathcal{H}_2))}} \Omega(u,x)\overline{\Omega(x,w)}\Omega(w,v) \\ & + (-1)^{|\Omega(u,w)|} \sum_{\substack{w<y<v \\ w \in \mu(h(\mathcal{S}_1,\mathcal{H}_1,\mathcal{H}_2))}} \overline{\Omega(u,w)}\,\overline{\Omega(w,y)}\Omega(y,v) \\ = & \sum_{\substack{u<w<v \\ w \notin \mu(h(\mathcal{S}_1,\mathcal{H}_1,\mathcal{H}_2))}} \sum_{\substack{u<z<w \\ z \in \mu(h(\mathcal{S}_1,\mathcal{H}_1,\mathcal{H}_2))}} \overline{\overline{\Omega(u,z)}\Omega(z,w)}\Omega(w,v) \\ = & \sum_{u<w<v} \overline{\Omega^1(u,w)}\Omega^1(w,v) \end{aligned}$$

Then we have obtained that Ω^1 is a defining system on $\mathcal{G}_1$. Analogously, we obtain that Ω^2 is a defining system on $\mathcal{G}_2$. Define an element $\Delta_{uv} \in H_*(\mathcal{A})$ for $(u,v) \in h(\mathcal{G}_1)^{\min} \times h(\mathcal{G}_1)^{\max} = h(\mathcal{G}_2)^{\min} \times h(\mathcal{G}_2)^{\min}$ by

$$\Delta_{uv} = \sum_{w \in \mu(h(\mathcal{H}_2))} \overline{\Omega(u,w)}\Omega(w,v).$$

Then we have:

$$\begin{aligned} d\Delta_{uv} = & \sum_{w \in \mu(h(\mathcal{H}_2))} (d\overline{\Omega(u,w)})\Omega(w,v) + (-1)^{|\Omega(u,w)|}\overline{\Omega(u,w)}d\Omega(w,v)) \\ = & \sum_{w \in \mu(h(\mathcal{H}_2))} (\sum_{u<x<w} \Omega(u,x)\overline{\Omega(x,w)})\Omega(w,v) - \Omega(u,w) \sum_{w<y<v} \overline{\Omega(w,y)}\Omega(y,v)) \\ = & \sum_{\substack{u<x<v \\ x \notin \mu(h(\mathcal{H}_2,\mathcal{H}_3,\mathcal{S}_2))}} \overline{\overline{\Omega(u,x)}} \sum_{\substack{x<w<v \\ w \in \mu(h(\mathcal{H}_2,\mathcal{H}_3,\mathcal{S}_2))}} \overline{\Omega(x,w)})\Omega(w,v) \\ & + \sum_{\substack{u<y<v \\ y \notin \mu(h(\mathcal{S}_1,\mathcal{H}_1,\mathcal{H}_2))}} \sum_{\substack{u<w<y \\ w \in \mu(h(\mathcal{S}_1,\mathcal{H}_1,\mathcal{H}_2))}} \overline{\Omega(u,w)\overline{\overline{\Omega(w,y)}}})\Omega(y,v) \\ = & \Omega^2_{uv} + \Omega^1_{uv}. \end{aligned}$$

Thus the proof is completed. □

Corollary 5.1. *Let $\mathcal{S}$ be an $\mathcal{A}$-colored short hammock and let $\mathcal{H}_1, \mathcal{H}_2$ be $\mathcal{A}$-colored hammocks.*

1. *Let $\mathcal{S}, \mathcal{H}_1, \mathcal{H}_2$ form an $\mathcal{A}$-colored hammock $(\mathcal{S}, \mathcal{H}_1, \mathcal{H}_2)$ in which $\mathcal{H}_1$ is locally full. Suppose that $\langle(\mathcal{S}, \mathcal{H}_1)_{uv}\rangle = \{0\}$ for each vertices u, v such that u is not minimal. Then each defining system Ω on $(\mathcal{H}_1, \mathcal{H}_2)$ induces a defining system Ω' on $(\mathcal{S}, \mathcal{H}_1)$ such that $\langle\langle\mathcal{S}, \mathcal{H}_1|\Omega'\rangle, \mathcal{H}_2\rangle$ is defined and*

$$\langle\overline{\mathcal{S}}, \langle\mathcal{H}_1, \mathcal{H}_2|\Omega\rangle\rangle \subset -\langle\langle\mathcal{S}, \mathcal{H}_1|\Omega'\rangle, \mathcal{H}_2\rangle.$$

2. *Let $\mathcal{S}, \mathcal{H}_1, \mathcal{H}_2$ form an $\mathcal{A}$-colored hammock $(\mathcal{H}_1, \mathcal{H}_2, \mathcal{S})$ in which $\mathcal{H}_2$ is locally full. Suppose that $\langle(\mathcal{H}_2, \mathcal{S})_{uv}\rangle = \{0\}$ for each vertices u, v such that v is not maximal. Then each defining system Ω on $(\mathcal{H}_1, \mathcal{H}_2)$ induces a defining system Ω' on $(\mathcal{H}_2, \mathcal{S})$ such that $\langle\overline{\mathcal{H}_1}, \langle\mathcal{H}_2, \mathcal{S}|\Omega'\rangle\rangle$ is defined and*

$$\langle\langle\mathcal{H}_1, \mathcal{H}_2|\Omega\rangle, \mathcal{S}\rangle \subset -\langle\overline{\mathcal{H}_1}, \langle\mathcal{H}_2, \mathcal{S}|\Omega'\rangle\rangle.$$

Proof. We only prove 1 since 2 is quite analogously done. Note first that, since $\mathcal{H}_1$ is full in $(\mathcal{S}, \mathcal{H}_1, \mathcal{H}_2)$, so is $\mathcal{H}_2$. Then $(\mathcal{H}_1, \mathcal{H}_2)$ is also locally full in $(\mathcal{S}, \mathcal{H}_1, \mathcal{H}_2)$. It follows from Corollary 3.2 that $\Omega|_{\mathcal{H}_1}$ extends to a partial defining system Ω' on $(\mathcal{S}, \mathcal{H}_1, \mathcal{H}_2)$ over $P(\mathcal{S}, \mathcal{H}_1) \cup P(\mathcal{H}_1, \mathcal{H}_2)$. Then, as in the proof of Proposition 5.1, we have a defining system Ω'' on $(\langle\mathcal{S}, \mathcal{H}_1|\Omega'\rangle, \mathcal{H}_2)$ such that $\langle\overline{\mathcal{S}}, \langle\mathcal{H}_1, \mathcal{H}_2|\Omega\rangle\rangle = \langle\langle\mathcal{S}, \mathcal{H}_1|\Omega'\rangle, \mathcal{H}_2|\Omega''\rangle$, here the left hand side is always defined and its indeterminacy is 0. □

We abbreviate the formulae in Corollary 5.1 by

$$\langle\overline{\mathcal{S}}, \langle\mathcal{H}_1, \mathcal{H}_2\rangle\rangle \subset -\langle\langle\mathcal{S}, \mathcal{H}_1\rangle, \mathcal{H}_2\rangle,\ \langle\langle\mathcal{H}_1, \mathcal{H}_2\rangle, \mathcal{S}\rangle \subset -\langle\overline{\mathcal{H}_1}, \langle\mathcal{H}_2, \mathcal{S}\rangle\rangle.$$

We also have the following associativity of hammock Massey products.

Theorem 5.3. *Let $\mathcal{S}_i$ be an $\mathcal{A}$-colored short hammock for $i = 1, 2, 3$ and let $\mathcal{H}$ be an $\mathcal{A}$-colored quasi-hammock.*

1. *Let $\mathcal{S}_i, \mathcal{H}$ form an $\mathcal{A}$-colored hammock $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \mathcal{H})$ in which $(\mathcal{S}_1, \mathcal{S}_2)$ and $(\mathcal{S}_2, \mathcal{S}_3)$ are locally full. Then each defining system Ω on $(\langle\overline{\mathcal{S}_1}, \mathcal{S}_2\rangle, \mathcal{S}_3, \mathcal{H})$ induces a defining system Ω' on $(\mathcal{S}_1, \langle\mathcal{S}_2, \mathcal{S}_3\rangle, \mathcal{H})$ such that*

$$\langle\langle\overline{\mathcal{S}_1}, \mathcal{S}_2\rangle, \mathcal{S}_3, \mathcal{H}|\Omega\rangle + \langle\mathcal{S}_1, \langle\mathcal{S}_2, \mathcal{S}_3\rangle, \mathcal{H}|\Omega'\rangle = 0.$$

2. *Let $\mathcal{S}_i, \mathcal{H}$ form an $\mathcal{A}$-colored hammock $(\mathcal{H}, \mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ in which $(\mathcal{S}_1, \mathcal{S}_2)$ and $(\mathcal{S}_2, \mathcal{S}_3)$ are locally full. Then each defining system Ω on $(\mathcal{H}, \mathcal{S}_1, \langle\mathcal{S}_2, \mathcal{S}_3\rangle)$ induces a defining system Ω' on $(\mathcal{H}, \langle\overline{\mathcal{S}_1}, \mathcal{S}_2\rangle, \mathcal{S}_3)$ such that*

$$\langle\mathcal{H}, \mathcal{S}_1, \langle\mathcal{S}_2, \mathcal{S}_3\rangle|\Omega\rangle + \langle\mathcal{H}, \langle\overline{\mathcal{S}_1}, \mathcal{S}_2\rangle, \mathcal{S}_3|\Omega'\rangle = 0.$$

Proof. Since 2 is analogous, we only show 1. Fix a defining system Λ on $(\overline{\mathcal{S}_1}, \mathcal{S}_2)$ such that $\Lambda_{uv} = \Omega(u,v)$ for $(u,v) \in h(\mathcal{S}_1)^{\min} \times h(\mathcal{S}_2)^{\max}$. Let Ω be a defining system on $(\langle\overline{\mathcal{S}_1}, \mathcal{S}_2\rangle, \mathcal{S}_3, \mathcal{H})$. Define a function $\Omega' : P(\mathcal{S}_1, \langle\mathcal{S}_2, \mathcal{S}_3\rangle, \mathcal{H}) \to \mathcal{A}$ by:

$$\Omega'(u,v) = \begin{cases} \overline{\Lambda(u,v)} & v \in h(\mathcal{S}_1)^{\max} \\ \displaystyle\sum_{w \in V(h(\mathcal{S}_2)) \cap V(h(\mathcal{S}_3))} \overline{\Lambda(u,w)}\Omega(w,v) & u \in h(\mathcal{S}_1)^{\max} \\ -\Omega(u,v) & u \in h(\mathcal{S}_1)^{\min} \\ \Omega(u,v) & \text{otherwise} \end{cases}$$

Then we can easily see that Ω' is a defining system on $(\mathcal{S}_1, \langle\mathcal{S}_2, \mathcal{S}_3\rangle, \mathcal{H})$. Let $(u,v) \in h(\mathcal{S}_1, \langle\mathcal{S}_2, \mathcal{S}_3\rangle, \mathcal{H})^{\min} \times h(\mathcal{S}_1, \langle\mathcal{S}_2, \mathcal{S}_3\rangle, \mathcal{H})^{\max} = h(\langle\overline{\mathcal{S}_1}, \mathcal{S}_2\rangle, \mathcal{S}_3, \mathcal{H})^{\min} \times h(\langle\overline{\mathcal{S}_1}, \mathcal{S}_2\rangle, \mathcal{S}_3, \mathcal{H})^{\max}$. For $u \in h(\mathcal{S}_1)^{\min}$, we have:

$$\begin{aligned} \Omega'_{uv} &= \sum_{u<w<v} \overline{\Omega'(u,w)}\Omega'(w,v) \\ &= \sum_{\substack{u<w<v \\ w \in V(h(\mathcal{S}_2))}} \Lambda(u,w) \sum_{x \in h(\mathcal{S}_2)^{\max}} \overline{\Lambda(w,x)}\Omega(x,v) + \sum_{\substack{u<w<v \\ w \in h(\mathcal{S}_3)^{\max}}} -\overline{\Omega(u,w)}\Omega(w,v) \\ &= -\Omega_{uv} \end{aligned}$$

and the proof is completed. □

Generalizing the above theorem, we have:

Theorem 5.4. *Let $\mathcal{L}, \mathcal{U}$ be $\mathcal{A}$-colored quasi-hammocks and let $\mathcal{S}_1, \ldots, \mathcal{S}_{2k+1}$ be $\mathcal{A}$-colored short hammocks. Suppose that $\mathcal{L}, \mathcal{U}, \mathcal{S}_1, \ldots, \mathcal{S}_{2k+1}$ satisfy the following conditions.*

1. *$\mathcal{L}, \mathcal{U}, \mathcal{S}_1, \ldots, \mathcal{S}_{2k+1}$ form an $\mathcal{A}$-colored hammock $\mathcal{H} = (\mathcal{L}, \mathcal{S}_1, \ldots, \mathcal{S}_{2k+1}, \mathcal{U})$ in which $\mathcal{S}_1, \ldots, \mathcal{S}_{2k+1}$ are locally full.*
2. *There is a partial defining system Ω on $\mathcal{H}$ over $\bigcup_{q=1}^{k+1} P\mathcal{G}_q$, where $\mathcal{G}_q = (\mathcal{S}_q, \ldots, \mathcal{S}_{q+k})$ for $q = 1, \ldots, k+1$.*
3. *$\langle\overline{\mathcal{L}}, \overline{\mathcal{S}_1}, \ldots, \overline{\mathcal{S}_{q-1}}, \langle\mathcal{G}_q|\Omega\rangle, \mathcal{S}_{q+k+1}, \ldots, \mathcal{S}_{2k+1}, \mathcal{U}\rangle$ is strictly defined for $q = 1, \ldots, k+1$.*

Then we have

$$0 \in \sum_{q=1}^{k+1} \langle\overline{\mathcal{L}}, \overline{\mathcal{S}_1}, \ldots, \overline{\mathcal{S}_{q-1}}, \langle\mathcal{G}_q|\Omega\rangle, \mathcal{S}_{q+k+1}, \ldots, \mathcal{S}_{2k+1}, \mathcal{U}\rangle.$$

Proof. We abbreviate $(\overline{\mathcal{L}}, \overline{\mathcal{S}_1}, \ldots, \overline{\mathcal{S}_{q-1}}, \langle\mathcal{G}_q|\Omega\rangle, \mathcal{S}_{q+k+1}, \ldots, \mathcal{S}_{2k+1}, \mathcal{U})$ by $\mathcal{H}_q$. Since $\langle\mathcal{H}_1\rangle$ and $\langle\mathcal{H}_{k+1}\rangle$ are strictly defined, so are $\langle\overline{\mathcal{L}}, \overline{\mathcal{S}_1}, \ldots, \overline{\mathcal{S}_k}\rangle$ and $\langle\overline{\mathcal{S}_{k+2}}, \ldots, \overline{\mathcal{S}_{2k+1}}, \overline{\mathcal{U}}\rangle$ such that $\langle\overline{\mathcal{L}}, \overline{\mathcal{S}_1}, \ldots, \overline{\mathcal{S}_k}\rangle = \{0\}$ and $\langle\overline{\mathcal{S}_{k+2}}, \ldots, \overline{\mathcal{S}_{2k+1}}, \overline{\mathcal{U}}\rangle =$

$\{0\}$. Then it follows from Proposition 5.1 that $\langle \mathcal{E} \rangle$ and $\langle \mathcal{F} \rangle$ are strictly defined such that $\langle \mathcal{E} \rangle = \{0\}$ and $\langle \mathcal{F} \rangle = \{0\}$, where $\mathcal{E} = (\mathcal{L}, \mathcal{S}_1, \ldots, \mathcal{S}_k)$ and $\mathcal{F} = (\mathcal{S}_{k+2}, \ldots, \mathcal{S}_{2k+1}, \mathcal{U})$. Thus we can extend Ω to a partial defining system over $\{(u,v) \in V(\mathcal{E}) \times V(\mathcal{E}) | u < v\} \cup \{(u,v) \in V(\mathcal{F}) \times V(\mathcal{F}) | u < v\} \cup \bigcup_{q=1}^{k+1} P\mathcal{G}_q$ which we denote also by Ω.

Put V_i for $i = 0, \ldots, 2k+2$ as:

$$V_i = \begin{cases} V(h(\mathcal{L})) & i = 0 \\ V(h(\mathcal{S}_i)) \cap V(h(\mathcal{S}_{i+1})) & i = 1, \ldots, 2k \\ V(h(\mathcal{U})) \cap V(h(\mathcal{S}_{2k+1})) & i = 2k+1 \\ V(h(\mathcal{U})) \setminus V(h(\mathcal{S}_{2k+2})) & i = 2k+2 \end{cases}$$

Let $V(i,j) = V_i \cup \ldots \cup V_j$ for $i < j$. Define a function $\Omega^1 : D^1 \to \mathcal{A}$ for a subset $D^1 \subset P\mathcal{H}_1$ by:

$$\Omega^1(u,v) = \begin{cases} \overline{\Omega(u,v)} & v \in V_0 \\ \Omega(u,v) & u \in V(k+1, 2k+2) \\ \displaystyle\sum_{w \in \mu(h(\mathcal{S}_1, \ldots, \mathcal{S}_{k+1}))} \overline{\Omega(u,w)}\Omega(w,v) & v \in V(h(\mathcal{G}_1)) \cap V(h(\mathcal{S}_{k+2})) \end{cases}$$

It is easily seen that Ω^1 is a partial defining system on $\mathcal{H}_1$ over D^1 and then it extends to a defining system $\widetilde{\Omega}^1$ on $\mathcal{H}_1$ since $\langle \mathcal{H}_1 \rangle$ is strictly defined. Suppose that we have obtained a defining system $\widetilde{\Omega}^r : P\mathcal{H}_r \to \mathcal{A}$ for $r = 1, \ldots, q-1$. Define a function $\Omega^q : D^q \to \mathcal{A}$ for a subset $D^q \subset P\mathcal{H}_q$ by:

(1)
$$\Omega^q(u,v) = \begin{cases} \overline{\Omega(u,v)} & v \in V(0, q-1) \\ \Omega(u,v) & u \in V(q+k, 2k+2) \\ \displaystyle\sum_{w \in \mu(h(\mathcal{S}_q, \ldots, \mathcal{S}_{k+i+1}))} \overline{\Omega(u,w)}\Omega(w,v) & \\ \displaystyle\qquad - \sum_{r=i+1}^{q-1} \widetilde{\Omega}^r(u,v) & \begin{array}{l} v \in V(h(\mathcal{G}_q)) \cap V(h(\mathcal{S}_{q+k+1})) \\ u \in V_i, i = 0, \ldots, q-1 \end{array} \end{cases}$$

Let us verify that Ω^q is a partial defining system so that Ω^q extends to a defining system $\widetilde{\Omega}^q$ on $\mathcal{H}_q$. Put $v \in V_{q+k}$ and $u \in L$. Then we have:

$$
\begin{aligned}
d\Omega^q(u,v) = &\sum_{w\in V(q,q+k-1)} \Big(\sum_{x\in V(0,q-1)} \Omega(u,x)\overline{\Omega(x,w)}\Omega(w,v) \\
&\qquad - \sum_{y\in V(k,q+k-1)} \Omega(u,w)\overline{\Omega(w,y)}\Omega(y,v)\Big) \\
&\qquad\qquad - \sum_{\substack{r<q \\ z\in V(0,r-1)\cup V(r+k,q+k-1)}} \overline{\Omega^r(u,z)}\Omega^r(z,v) \\
= &\sum_{\substack{x\in V(0,q-1) \\ w\in V(q,q+k-1)}} \Omega(u,x)\overline{\Omega(x,w)}\Omega(w,v) \\
&\quad - \sum_{i=0}^{q-2}\sum_{z\in V_i}\sum_{i+1\le r<q} \overline{\Omega^r(u,z)}\Omega^r(z,v) \\
&\qquad - \sum_{\substack{w\in V(q,q+k-1) \\ y\in V(k,q+k-1)}} \Omega(u,w)\overline{\Omega(w,y)}\Omega(y,v) \\
&\qquad\qquad - \sum_{\substack{r<q \\ z\in V(r+k,q+k-1)}} \overline{\Omega^r(u,z)}\Omega^r(z,v) \\
= &\sum_{u<w<v} \overline{\Omega^q(u,w)}\Omega^q(w,v).
\end{aligned}
$$

By a quite analogous calculation, we can see that Ω^q is a partial defining system on $\mathcal{H}_q$ over D^q.

Finally, we define a function $\widetilde{\Omega}^{k+1} : P\mathcal{H}_{k+1} \to \mathcal{A}$ by

$$
\Omega^{k+1}(u,v) = \begin{cases}
\displaystyle\sum_{w\in\mu(h(\mathcal{S}_{k+1},\dots,\mathcal{S}_{k+i+1}))} \overline{\Omega(u,w)}\Omega(w,v) - \sum_{r=1}^{k}\widetilde{\Omega}^r(u,v) & v\in V_{2k+1}, u\in V_i \\ & i=0,\dots,k \\
\displaystyle -\sum_{r=1}^{k}\widetilde{\Omega}^r(u,v) & v\in V_{2k+2}
\end{cases}
$$

This is also seen to be a defining system on $\mathcal{H}_{k+1}$. By definition, we have

$$
\sum_{q=1}^{k+1}\widetilde{\Omega}^q_{uv} = 0
$$

for $(u,v)\in h(\mathcal{L},\mathcal{S}_1,\dots,\mathcal{S}_{2k+1},\mathcal{U})^{\min}\times h(\mathcal{L},\mathcal{S}_1,\dots,\mathcal{S}_{2k+1},\mathcal{U})^{\max}$ and the proof is completed. □

We can show the following associativity in hammock Massey products by constructing defining systems as in (1) and using Δ_{uv} as in the proof of Theorem 5.2.

Theorem 5.5. *Let $\mathcal{S}_1, \ldots, \mathcal{S}_{2k}$ be $\mathcal{A}$-colored short hammocks and let $\mathcal{G}$ be an $\mathcal{A}$-colored hammock. Suppose that $\mathcal{S}_1, \ldots, \mathcal{S}_{2k}, \mathcal{G}$ satisfy the following conditions.*

1. *$\mathcal{S}_1, \ldots, \mathcal{S}_{2k}, \mathcal{G}$ form an $\mathcal{A}$-colored hammock $\mathcal{H} = (\mathcal{S}_1, \ldots, \mathcal{S}_k, \mathcal{G}, \mathcal{S}_{k+1}, \ldots, \mathcal{S}_{2k})$ in which all $\mathcal{S}_1, \ldots, \mathcal{S}_{2k}, \mathcal{G}$ are locally full.*
2. *There is a partial defining system Ω on $\mathcal{H}$ over $\bigcup_{q=1}^{k+1} P\mathcal{G}_q$, where $\mathcal{G}_q = (\mathcal{S}_q, \ldots, \underline{\mathcal{S}_k, \mathcal{G}, \mathcal{S}_{k+1}, \ldots}, \mathcal{S}_{q+k-1})$ for $q = 1, \ldots, k+1$.*
3. *$\langle \overline{\mathcal{S}_1}, \ldots, \overline{\mathcal{S}_{q-1}}, \langle \mathcal{G}_q | \Omega \rangle, \mathcal{S}_{k+1}, \ldots, \mathcal{S}_{q+k-1} \rangle$ is strictly defined for $q = 1, \ldots, k+1$.*

Then we have

$$0 \in \sum_{q=1}^{k+1} \langle \overline{\mathcal{S}_1}, \ldots, \overline{\mathcal{S}_{q-1}}, \langle \mathcal{G}_q | \Omega \rangle, \mathcal{S}_{k+1}, \ldots, \mathcal{S}_{q+k-1} \rangle.$$

6. Jacobi identity

In this final section, we consider Jacobi identity in hammock Massey products. let us first recall from [3] an n-homotopy permutative algebra.

Definition 6.1. The DGA $\mathcal{A}$ is called n-homotopy permutative if there is a sequence of R-module maps $\{h_l : \bigotimes^l \mathcal{A} \to \mathcal{A}\}_{2 \le l \le n}$ in which h_l is of degree $1 - l$ and satisfies:

$$\begin{aligned} dh_l(u_1 \otimes \cdots \otimes u_l) = & \sum_{k=1}^{l} h_l(\overline{u_1} \otimes \cdots \otimes \overline{u_{k-1}} \otimes du_k \otimes u_{k+1} \otimes \cdots \otimes u_l) \\ & - \sum_{k=1}^{l-1} h_{l-1}(\overline{u_1} \otimes \cdots \otimes \overline{u_{k-1}} \otimes \overline{u_k} u_{k+1} \otimes u_{k+2} \otimes \cdots \otimes u_l) \\ & - (-1)^{(|u_1|+1)\sum_{k=2}^{l}(|u_k|+1)} h_{l-1}(\overline{u_2} \otimes \cdots \otimes \overline{u_{l-1}} \otimes \overline{u_l} u_1) \end{aligned}$$

for $u_1, \ldots, u_l \in \mathcal{A}$. If a DGA $\mathcal{A}$ admits an infinite sequence of the above R-module maps $\{h_l : \bigotimes^l \mathcal{A} \to \mathcal{A}\}_{n \ge 2}$, then we simply call $\mathcal{A}$ homotopy permutative.

Let $\mathcal{H}$ be an $\mathcal{A}$-colored simple hammock with $h(\mathcal{H})^{\min} = \{u\}$ and $h(\mathcal{H})^{\max} = \{v\}$. Then, for $u < w < v$, we have $\mathcal{A}$-colored simple hammocks $\mathcal{H}_{uw}$ and $\mathcal{H}_{wv}$, and, by Proposition 2.1, we obtain an $\mathcal{A}$-colored simple hammock

$$\mathcal{H}_w = (\mathcal{H}_{wv}, \mathcal{H}_{uw}).$$

Jacobi identity in hammock Massey products are described using $\mathcal{H}_w$.

In order to state Jacobi identity in hammock Massey products, we prepare some function which will be convenient for sign calculation. Put $s(w) = (\deg(u, w) + 2)(\deg(w, v) + 2) + 1$ for $u < w < v$ and $s(u) = 1$. Then for a defining system on $\mathcal{H}$, we have

$$[\overline{\Omega(u, w)}, \Omega(w, v)] = \overline{\Omega(u, w)}\Omega(w, v) - (-1)^{s(w)}\overline{\Omega(w, v)}\Omega(u, w)$$

and

$$s(w) = (|\Omega(u,w)|+1)(|\Omega(w,v)|+1)+1,$$

where $[-,-]$ stands for the commutator in $\mathcal{A}$.

Theorem 6.1. *Let the DGA $\mathcal{A}$ be homotopy permutative and let $\mathcal{H}$ be an $\mathcal{A}$-colored simple hammock with $h(\mathcal{H})^{\min} = \{u\}$ and $h(\mathcal{H})^{\max} = \{v\}$. Suppose that $\langle \mathcal{H} \rangle$ is defined and $\langle \mathcal{H}_w \rangle$ is strictly defined for each $u < w < v$. Then we have*

$$\langle \mathcal{H} \rangle \subset \sum_{u<w<v} (-1)^{s(w)} \langle \mathcal{H}_w \rangle.$$

Proof. Let Ω be a defining system on $\mathcal{H}$. We first extend Ω to a function $V(h(\mathcal{H})) \times V(h(\mathcal{H})) \to \mathcal{A}$ which restricts to a defining system on $\mathcal{H}_w$ for each $u < w < v$. Let $u < x < y < v$ such that the number of $V(h(\mathcal{H}_{ux}))$ and $V(h(\mathcal{H}_{yv}))$ are smallest among $u < x$ and $y < v$. Then we obtain an $\mathcal{A}$-colored hammock $(\mathcal{H}_{yv}, \mathcal{H}_{ux})$ which is an $\mathcal{A}$-colored sub-hammock of $\mathcal{H}_w$ for some $x < w < y$ so that $\langle \mathcal{H}_{yv}, \mathcal{H}_{ux} \rangle$ is strictly defined. Then we have a defining system on $(\mathcal{H}_{yv}, \mathcal{H}_{ux})$. Then by proceeding induction on the number of $V(h(\mathcal{H}_{ux}))$ and $V(h(\mathcal{H}_{yv}))$, we get the desired function on $V(h(\mathcal{H})) \times V(h(\mathcal{H}))$ minus the diagonal set. Thus, finally, we obtain the desired function Λ by putting $\Lambda(x,x) = 0$.

For $x, y \in V(h(\mathcal{H}))$, we have:

$$\begin{aligned}(-1)^{s(x)}[\overline{\Lambda(x,y)}, \Lambda(y,x)] &= (-1)^{s(x)}\overline{\Lambda(x,y)}\Lambda(y,x) \\ &\quad + (-1)^{s(x)+(|\Lambda(x,y)|+1)(|\Lambda(y,x)|+1)}\overline{\Lambda(y,x)}\Lambda(x,y) \\ &= (-1)^{s(x)}\overline{\Lambda(x,y)}\Lambda(y,x) + (-1)^{s(y)}\overline{\Lambda(y,x)}\Lambda(x,y)\end{aligned}$$

Then we obtain:

$$\begin{aligned}\sum_{u<w<v} (-1)^{s(w)} \langle \mathcal{H}_w | \Lambda \rangle &= \sum_{u<w<v} (-1)^{s(w)} \overline{\Lambda(w,v)}\Lambda(u,w) \\ &\quad + \sum_{u<x<y<v} (-1)^{s(x)} [\overline{\Lambda(x,y)}, \Lambda(y,x)] \\ &= \sum_{u<w<v} (-1)^{s(w)} \overline{\Lambda(u,w)}\Lambda(w,v) \\ &\quad + \sum_{u\le x<y<v} (-1)^{s(x)} [\overline{\Lambda(x,y)}, \Lambda(y,x)] \\ &= \sum_{u<w<v} (-1)^{s(w)} \overline{\Omega(u,w)}\Omega(w,v) \\ &\quad + \sum_{u\le x<y<v} (-1)^{s(x)} [\overline{\Lambda(x,y)}, \Lambda(y,x)] \\ &= \langle \mathcal{H} | \Omega \rangle + \sum_{u\le x<y<v} (-1)^{s(x)} [\overline{\Lambda(x,y)}, \Lambda(y,x)]\end{aligned}$$

Hence it only remains to show that $\sum_{u\le x<y<v}(-1)^{s(x)}[\overline{\Lambda(x,y)},\Lambda(y,x)]$ is a boundary. It follows from a straightforward calculation that

$$d(\sum_{l=2}^{\infty}\sum_{u\le v_0<\cdots<v_{l-1}<v}(-1)^{s(v_0)+1}h_l(\Lambda(v_0,v_1)\otimes\cdots\otimes\Lambda(v_{l-1},v_0))) \\ =\sum_{u\le x<y<v}(-1)^{s(x)}[\overline{\Lambda(x,y)},\Lambda(y,x)]$$

and then the proof is completed. □

e-mail: mizuno@herb.ocn.ne.jp

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