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A counterexample to a conjecture of complete fan

By

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Abstract

If a Griffiths domain D is a symmetric Hermitian domain, the toroidal compactification of the quotient space $\Gamma\backslash D$, associated to a projective fan and a discrete subgroup Γ of $\mathrm{Aut}(D)$, was constructed by Mumford et al. Kazuya Kato and Sampei Usui studied extensions of $\Gamma\backslash D$ for a Griffiths domain D in general, and introduced a notion of "complete fan" as a generalization of a notion of projective fan. The existence of complete fans is expected. In this paper, we give an example of D which has no complete fan.

1. Introduction

Let D be a Griffiths domain, let Γ be a "neat" discrete subgroup of $\mathrm{Aut}(D)$, and let Σ be a fan consisting of rational nilpotent cones in $\mathrm{Lie}(\mathrm{Aut}(D))$ which is "strongly compatible" with Γ. Kazuya Kato and Sampei Usui [KU] introduced the notion of "polarized logarithmic Hodge structure" and enlarged the space $\Gamma\backslash D$ to the space $\Gamma\backslash D_\Sigma$ by adding the classes modulo Γ of nilpotent orbits in the directions of cones containe d in Σ as the boundary points. They proved that the space $\Gamma\backslash D_\Sigma$ is the fine moduli space of polarized logarithmic Hodge structures of type $\Phi := (w, (h^{p,q})_{p,q\in\mathbb{Z}}, H_{\mathbb{Z}}, \langle\ ,\ \rangle, \Gamma, \Sigma)$ ([KU] 4.2.1, Theorem B), and that $\Gamma\backslash D_\Sigma$ is a "logarithmic manifold" which is nearly a complex analytic manifold but has "slits" caused by "Griffiths transversality" condition at the boundary ([KU] 4.1.1, Theorem A).

In the classical situation, that is, D is a symmetric Hermitian domain, the toroidal projective compactification $\Gamma\backslash D_\Sigma$ of $\Gamma\backslash D$ was constructed with a sufficiently big fan Σ, called a projective fan, by A. Ash, D. Mumford, M. Rapoport and Y. S. Tai [AMRT].

For general D, Kato and Usui introduced in [KU] a "complete fan" as a generalization of a projective fan, and they gave a conjecture of the existence of such fans ([KU]12.6.3). As an example, they gave a concrete description of the space $\Gamma\backslash D_\Sigma$ for Hodge type $h^{2,0} = h^{0,2} = 2,\ h^{1,1} = 1$ and for $\Sigma = \Xi$;

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i.e., the fan consisting of all rational nilpotent cones whose rank are less than or equal to one in Lie(Aut(D)) in [KU] 12.2.2. In this case, the fan $\Sigma = \Xi$ is complete.

In the present work, we started to generalize the description of the above example, but in fact we encounter a counterexample to the conjecture of existence of complete fans. We show that D with $h^{2,0} = h^{1,1} = h^{0,2} = 2$ has no complete fans (Theorem 5.1). For the proof, we first show that the rank of any rational nilpotent cone, which appears in a nilpotent orbit, is less than or equal to two (Proposition 5.1). Next, assuming the existence of a complete fan Σ on D, we derive a contradiction: Σ has two different cones of rank two which have a common point as in each of their interiors.

The plan of this paper is as follows. In Section 2 to Section 4, we prepare notation and explain the background. In Section 5, we state the main result and prove it.

After the present work, a modified version of the conjecture about complete fan is added at the end of 12.7 in [KU].

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2. Polarized Hodge structures and Griffiths domains

In this section, we recall the definition of polarized Hodge structures and Griffiths domains (cf. [G], [KU]). Let $w \in \mathbb{Z}$, and let $(h^{p,q})_{p,q\in\mathbb{Z}}$ be a family of non-negative integers such that $h^{p,q} = 0$ unless $p + q = w$, $h^{p,q} \neq 0$ for only finitely many (p, q), and such that $h^{p,q} = h^{q,p}$ for all p, q.

Definition 2.1. A Hodge structure of weight w and of Hodge type $(h^{p,q})_{p,q\in\mathbb{Z}}$ is a pair $(H_{\mathbb{Z}}, F)$ consisting of a free $\mathbb{Z}$-module $H_{\mathbb{Z}}$ of finite rank and of a decreasing filtration F on $H_{\mathbb{C}} := \mathbb{C} \otimes_{\mathbb{Z}} H_{\mathbb{Z}}$ (that is, a family $(F^p)_{p\in\mathbb{Z}}$ of $\mathbb{C}$-subspaces of $H_{\mathbb{C}}$ such that $F^{p+1} \subset F^p$ for all p), which satisfies the following conditions.

(1) $\dim_{\mathbb{C}}(F^p/F^{p+1}) = h^{p,q}$ $(p + q = w)$.
(2) $H_{\mathbb{C}} = \bigoplus_{p+q=w}(F^p \cap \bar{F}^q)$.

Definition 2.2. A polarized Hodge structure of weight w and of Hodge type $(h^{p,q})_{p,q\ in\mathbb{Z}}$ is a triple $(H_{\mathbb{Z}}, \langle\ ,\ \rangle, F)$ consisting of a Hodge structure $(H_{\mathbb{Z}}, F)$ of weight w and of a non-degenerate $\mathbb{Q}$-bilinear form $\langle\ ,\ \rangle$ on $H_{\mathbb{Q}} = \mathbb{Q} \otimes_{\mathbb{Z}} H_{\mathbb{Z}}$, symmetric for even w and skew-symmetric for odd w, which satisfies the following two conditions.

(3) $\langle F^p, F^q\rangle = 0$ $(p + q > w)$.

(4) The Hermitian form

$$H_{\mathbb{C}} \times H_{\mathbb{C}} \to \mathbb{C}, \ \ (x, y) \mapsto \langle C_F(x), \bar{y} \rangle,$$

is positive definite.

Here $\langle \ , \ \rangle$ is regarded as the natural extension to $\mathbb{C}$-bilinear form, $^{-}$ is the complex conjugation with respect to $H_{\mathbb{Z}}$, and C_F is the Weil operator which is a $\mathbb{C}$-linear map and defined by $C_F(x) := i^{p-q}x$ for $x \in F^p \cap \bar{F}^q$ with $p+q = w$. The condition (3) is called the Riemann-Hodge first bilinear relation.

Let w and $(h^{p,q})_{p,q\in\mathbb{Z}}$ be as before. We fix a 4-tuple $\Phi_0 = (w, (h^{p,q})_{p,q\in\mathbb{Z}}, H_{\mathbb{Z}}, \langle \ , \ \rangle)$, where $H_{\mathbb{Z}}$ is a free $\mathbb{Z}$-module of rank $\Sigma_{p,q} h^{p,q}$, and $\langle \ , \ \rangle$ is a non-degenerate bilinear form on $H_{\mathbb{Q}} := \mathbb{Q} \otimes_{\mathbb{Z}} H_{\mathbb{Z}}$ which is symmetric if w is even and skew-symmetric if w is odd.

Definition 2.3. The Griffiths domain D of type Φ_0 is the set of all decreasing filtrations F on $H_{\mathbb{C}} := \mathbb{C} \otimes_{\mathbb{Z}} H_{\mathbb{Z}}$ such that the triple $(H_{\mathbb{Z}}, \langle \ , \ \rangle, F)$ is a polarized Hodge structure of weight w and of Hodge type $(h^{p,q})_{p,q\in\mathbb{Z}}$. The compact dual $\check{D}$ of D is defined to be the set of all decreasing filtrations on $H_{\mathbb{C}}$ which satisfies the above conditions (1) and (3).

Let

$$G_{\mathbb{Z}} := \mathrm{Aut}(H_{\mathbb{Z}}, \langle \ , \ \rangle),$$

and for $R = \mathbb{Q}, \mathbb{R}, \mathbb{C}$, let

$$\begin{aligned} H_R :=& R \otimes_{\mathbb{Z}} H_{\mathbb{Z}}, \ G_R := \mathrm{Aut}(H_R, \langle \ , \ \rangle), \\ \mathfrak{g}_R :=& \mathrm{Lie}(G_R) \\ =& \{N \in \mathrm{End}_R(H_R) \mid \langle Nx, y \rangle + \langle x, Ny \rangle = 0 \text{ for all } x, y \in H_R\}. \end{aligned}$$

3. Nilpotent orbits and polarized mixed Hodge structures

In this section, we recall the definition of nilpotent orbits and of polarized mixed Hodge structures, and their relation after [D], [S], [CKS], [KU].

We fix $\Phi_0 = (w, (h^{p,q})_{p,q\in\mathbb{Z}}, H_{\mathbb{Z}}, \langle \ , \ \rangle)$ as in Section 2.

Definition 3.1. Let $W = (W_l)_{l\in\mathbb{Z}}$ be an increasing filtration on $H_{\mathbb{Q}}$ consisting of subspaces W_l $(l \in \mathbb{Z})$ of $H_{\mathbb{Q}}$. Let $F \in \check{D}$. (W, F) is said to be a mixed Hodge structure, if it satisfies the following condition.

For each $l \in \mathbb{Z}$, the decreasing filtration $F\,\mathrm{Gr}_l^W = (F^p\,\mathrm{Gr}_l^W)_{p\in\mathbb{Z}}$ on $\mathrm{Gr}_{l,\mathbb{C}}^W := W_{l,\mathbb{C}}/W_{l-1,\mathbb{C}}$ is a Hodge structure of weight l. Here $F^p\,\mathrm{Gr}_l^W := F^p \cap W_{l,\mathbb{C}}/F^p \cap W_{l-1,\mathbb{C}}$.

Proposition 3.1 (cf. [D] Proposition 1.6.1)**.** *Let $N \in \mathfrak{g}_{\mathbb{Q}}$ be a nilpotent element. Then, there exists an increasing filtration $W(N)$ consisting of subspaces $W(N)_l$ $(l \in \mathbb{Z})$ of $H_{\mathbb{Q}}$ which satisfies the following conditions.*

If we assume that $N^k \neq 0, N^{k+1} = 0$ $(k \in \mathbb{N})$, then

(1) $0 = W(N)_{-k-1} \subset \cdots \subset W(N)_k = H_{\mathbb{Q}}$,
(2) $NW(N)_l \subset W(N)_{l-2}$ *for all* $l \in \mathbb{Z}$,
(3) $N^l : \mathrm{Gr}_l^{W(N)} \to \mathrm{Gr}_{-l}^{W(N)}$ *is an isomorphism* $(0 \leq l \leq k)$.

The increasing filtration $W(N)$ is called *monodromy weight filtration* associated to the nilpotent element $N \in \mathfrak{g}_{\mathbb{Q}}$.

Definition 3.2 ([S] (6.16))**.** Let N be a nilpotent element of $\mathfrak{g}_{\mathbb{Q}}$. A mixed Hodge structure (W, F) is of weight w and N-polarized if it satisfies the following conditions.

(1) $W = W(N)[-w]$,
(2) $NF^p \subset F^{p-1}$ for all $p \in \mathbb{Z}$.
(3) Let $P_{w+l} = \mathrm{Ker}(N^{l+1} : \mathrm{Gr}^W_{w+l} \to \mathrm{Gr}^W_{w-l-2})$. For the non-degenerate bilinear form

$$\langle\ , N^l\ \rangle : P_{w+l} \times P_{w+l} \to \mathbb{Q};\ (v, w) \mapsto \langle v, N^l w\rangle$$

on P_{w+l}, F induces a polarized Hodge structure of weight $w + l$ on P_{w+l}.

Definition 3.3 ([KU] 0.4.2, 1.3.1)**.** A subset σ of $\mathfrak{g}_{\mathbb{R}}$ is said to be a nilpotent cone, if the following conditions are satisfied.

(1) $\sigma = \mathbb{R}_{\geq 0}N_1 + \cdots + \mathbb{R}_{\geq 0}N_n$ for some $n \geq 1$ and for some $N_1, \ldots, N_n \in \sigma$.
(2) Any element of σ is nilpotent as an endomorphism of $H_{\mathbb{R}}$.
(3) $[N, N^{'}] = 0$ for any $N,\ N^{'} \in \sigma$ as endomorphisms of $H_{\mathbb{R}}$,
where $[N, N^{'}] := NN^{'} - N^{'}N$.

We recall some notion about nilpotent cones in [KU] 0.4.3, 1.3.2.

A nilpotent cone is said *rational*, if we can take $N_1, \ldots, N_n \in \mathfrak{g}_{\mathbb{Q}}$ in 3.3 (1).

For a nilpotent cone σ, a *face* of σ is a non-empty subset τ of σ which satisfies the following two conditions.

(1) If $x, y \in \tau$ and $a \in \mathbb{R}_{\geq 0}$, then $x + y,\ ax \in \tau$.
(2) If $x, y \in \sigma$ and $x + y \in \tau$, then $x, y \in \tau$.

Definition 3.4 ([KU] 0.4.4, 1.3.3)**.** A fan in $\mathfrak{g}_{\mathbb{Q}}$ is a non-empty set Σ of rational nilpotent cones in $\mathfrak{g}_{\mathbb{R}}$ satisfying the following three conditions:

(1) If $\sigma \in \Sigma$, any face of σ belongs to Σ.
(2) If $\sigma, \sigma^{'} \in \Sigma$, $\sigma \cap \sigma^{'}$ is a face of σ and of $\sigma^{'}$.
(3) Any $\sigma \in \Sigma$ is sharp. That is, $\sigma \cap (-\sigma) = \{0\}$.

Example 3.1 ([KU] 0.4.5, 1.3.11)**.** (i) Let

$$\Xi := \{(\mathbb{R}_{\geq 0})N \mid N \text{ is a nilpotent element of } \mathfrak{g}_{\mathbb{Q}}\}.$$

Then Ξ is a fan in $\mathfrak{g}_{\mathbb{Q}}$.

(ii) Let σ be a sharp rational nilpotent cone. Then the set of all faces of σ is a fan in $\mathfrak{g}_{\mathbb{Q}}$.

Let σ be a nilpotent cone in $\mathfrak{g}_{\mathbb{R}}$. For $R = \mathbb{R}, \mathbb{C}$, we denote by σ_R the R-linear span of $\sigma \subset \mathfrak{g}_{\mathbb{R}}$.

Definition 3.5 ([KU] 0.4.7, 1.3.7)**.** Let $\sigma = \Sigma_{1 \le j \le r}(\mathbb{R}_{\ge 0})N_j$ be a rational nilpotent cone. A subset Z of $\check{D}$ is said to be a σ-nilpotent orbit if there is $F \in \check{D}$ which satisfies $Z = \exp(\sigma_{\mathbb{C}})F$ and satisfies the following two conditions.

(1) $N_j F^p \subset F^{p-1}$ $(1 \le j \le r, p \in \mathbb{Z})$.
(2) $\exp(\sum_{1 \le j \le r} z_j N_j)F \in D$ if $z_j \in \mathbb{C}$ and $\mathrm{Im}(z_j) \gg 0$.

The conditions (1) and (2) are called *Griffiths transversality* and *positivity*, respectively.

We say that the pair (σ, F), consisting of a rational nilpotent cone $\sigma \subset \mathfrak{g}_{\mathbb{R}}$ and of $F \in \check{D}$, generates a nilpotent orbit if $Z = \exp(\sigma_{\mathbb{C}})F$ is a σ-nilpotent orbit.

Example 3.2. Let $w = 2$, $h^{2,0} = h^{1,1} = h^{0,2} = 2$, $h^{p,q} = 0$ otherwise, and $H_{\mathbb{Z}}$ be a free $\mathbb{Z}$-module with a basis $(e_j)_{1 \le j \le 6}$. Let $\langle\ ,\ \rangle : H_{\mathbb{Q}} \times H_{\mathbb{Q}} \to \mathbb{Q}$ be the $\mathbb{Q}$-bilinear form defined by

$$(\langle e_i, e_j \rangle)_{1 \le i,j \le 6} = \begin{pmatrix} -1_2 & O & O \\ O & E & O \\ O & O & E \end{pmatrix}, \text{ where } 1_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ E = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Let $H'_{\mathbb{Q}} := \bigoplus_{1 \le j \le 4} \mathbb{Q}e_j$. For $a \in H'_{\mathbb{Q}}$, let $N_a : H_{\mathbb{Q}} \to H_{\mathbb{Q}}$ be the nilpotent endomorphism given by

$$N_a(b) = -\langle a, b \rangle e_5 \ (b \in H'_{\mathbb{Q}}), \quad N_a(e_5) = 0, \quad N_a(e_6) = a.$$

Note that, for all $a, a' \in H_{\mathbb{Q}}$, $N_a, N_{a'} \in \mathfrak{g}_{\mathbb{Q}}$ and $[N_a, N_{a'}] = 0$. Let $F \in \check{D}$ be given by $F^2 = \mathbb{C}(ie_1 + e_2) \oplus \mathbb{C}e_6$, and $F^1 = (F^2)^{\perp}$. Let $\sigma = \mathbb{R}_{\ge 0}(-N_{e_3}) + \mathbb{R}_{\ge 0}N_{e_4}$. Then, (σ, F) generates a nilpotent orbit.

Proposition 3.2 ([S] (6.16) Theorem, [CKS] (3.13) Corollary)**.**
Assume that the weight of D is w. For a nilpotent element $N \in \mathfrak{g}_{\mathbb{Q}}$ and $F \in \check{D}$, the following conditions are equivalent.

(1) *$(W(N)[-w], F)$ is an N-polarized mixed Hodge structure.*
(2) *$(\mathbb{R}_{\ge 0}N, F)$ generates a nilpotent orbit.*

Proposition 3.3 ([CK] (1.7) Lemma)**.** *Let $\sigma \subset \mathfrak{g}_{\mathbb{R}}$ be a rational nilpotent cone and $F \in \check{D}$. If we assume that the weight of D is w and that the pair (σ, F) generates a nilpotent orbit, then*

(1) *For all $N \in (\mathrm{rel.\,int}\,\sigma) \cap \mathfrak{g}_{\mathbb{Q}}$, $(W(N)[-w], F)$ is an N-polarized mixed Hodge structure.*

(2) *For $N_1, N_2 \in (\mathrm{rel.\,int}\,\sigma) \cap \mathfrak{g}_{\mathbb{Q}}$, $W(N_1) = W(N_2)$.*
Here we denote the relative interior of σ by rel. int σ.

Definition 3.6 ([KU] 0.4.8, 1.3.8). Let Σ be a fan in $\mathfrak{g}_{\mathbb{Q}}$. As a set, we define D_Σ by

$$D_\Sigma := \{(\sigma, Z) \mid \sigma \in \Sigma, Z \subset \check{D} \text{ is a } \sigma\text{-nilpotent orbit}\}.$$

Note that we have the inclusion map

$$D \hookrightarrow D_\Sigma,\ F \mapsto (\{0\}, \{F\}).$$

Definition 3.7 ([KU] 0.4.10, 1.3.10). Let Σ be a fan in $\mathfrak{g}_{\mathbb{Q}}$ and let Γ be a subgroup of $G_{\mathbb{Z}}$.

(i) We say Γ is compatible with Σ if the following condition (1) is satisfied.

(1) If $\gamma \in \Gamma$ and $\sigma \in \Sigma$, then $\mathrm{Ad}(\gamma)(\sigma) \in \Sigma$. Here, $\mathrm{Ad}(\gamma)(\sigma) = \gamma\sigma\gamma^{-1}$. Note that, if Γ is compatible with Σ, Γ acts on D_Σ by

$$\gamma : (\sigma, Z) \mapsto (\mathrm{Ad}(\gamma)(\sigma), \gamma Z)\ (\gamma \in \Gamma).$$

(ii) We say Γ is strongly compatible with Σ if it is compatible with Σ and the following condition (2) is also satisfied. For $\sigma \in \Sigma$, define

$$\Gamma(\sigma) := \Gamma \cap \exp(\sigma).$$

(2) The cone σ is generated by $\log \Gamma(\sigma)$, that is, any element of σ can be written as a sum of c $\log(\gamma)$ ($c \in \mathbb{R}_{\geq 0},\ \gamma \in \Gamma(\sigma)$).

Assume that Γ is "neat" and strongly compatible with Σ. Γ $backslash D_\Sigma$ is a "logarithmic manifold" which is nearly a complex analytic manifold but has "slits" (see [KU]).

4. Complete fan

In this section, we recall the definition of a space D_{val} and the definition of a complete fan after [KU].

Definition 4.1 ([KU] Definition 5.3.1). We define

$$\mathcal{V} := \left\{ (A, V) \;\middle|\; \begin{array}{l} \text{A is a } \mathbb{Q}\text{-linear subspace of } \mathfrak{g}_{\mathbb{Q}} \text{ consisting of} \\ \text{mutually commutative nilpotent elements,} \\ V \text{ is a valuative submonoid of } A^* := \mathrm{Hom}_{\mathbb{Q}}(A, \mathbb{Q}) \\ \text{with } V \cap (-V) = \{0\} \end{array} \right\}.$$

Here a submonoid V of A^* is said to be a valuative submonoid, if $V \cup (-V) = A^*$.

For $(A, V) \in \mathcal{V}$, let $\mathcal{F}(A, V)$ be the set of all rational nilpotent cones $\sigma \subset \mathfrak{g}_{\mathbb{R}}$ satisfying the following (1) and (2).

(1) $\sigma_{\mathbb{R}} = A_{\mathbb{R}}$.

(2) Let $(\sigma \cap A)^\vee := \{h \in A^* \mid h(\sigma \cap A) \subset \mathbb{Q}_{\geq 0}\}$. Then $(\sigma \cap A)^\vee \subset V$.

Definition 4.2 ([KU] Definition 5.3.3)**.** (i) We define

$$\check{D}_{\mathrm{val}} := \left\{ (A,V,Z) \,\middle|\, \begin{array}{l} (A,V) \in \mathcal{V}, \\ Z \text{ is an } \exp(A_{\mathbb{C}})\text{-orbit in } \check{D} \end{array} \right\}.$$

(ii) We define

$$D_{\mathrm{val}} := \left\{ (A,V,Z) \,\middle|\, \begin{array}{l} (A,V,Z) \in \check{D}_{\mathrm{val}}, \\ \text{there exists } \sigma \in \mathcal{F}(A,V) \text{ such that} \\ Z \text{ is a } \sigma\text{-nilpotent orbit} \end{array} \right\}.$$

Definition 4.3. Let Σ be a fan in $\mathfrak{g}_{\mathbb{Q}}$. For $(A,V) \in \mathcal{V}$, we define

$$X_{A,V,\Sigma} := \{\sigma \in \Sigma \mid \sigma \cap A_{\mathbb{R}} \in \mathcal{F}(A,V)\}.$$

It is known that, if $X_{A,V,\Sigma}$ is not empty, then there exists the smallest element σ_0 of $X_{A,V,\Sigma}$ ([KU] Lemma 5.3.4).

Definition 4.4 ([KU] Definition 5.3.5)**.** For a fan Σ in $\mathfrak{g}_{\mathbb{Q}}$, we define

$$D_{\Sigma,\mathrm{val}} := \left\{ (A,V,Z) \,\middle|\, \begin{array}{l} (A,V,Z) \in \check{D}_{\mathrm{val}},\ X_{A,V,\Sigma} \text{ is not empty,} \\ \exp(\sigma_{0,\mathbb{C}})Z \text{ is a } \sigma_0\text{-nilpotent orbit} \end{array} \right\}.$$

Here σ_0 is just as above.

Definition 4.5 ([KU] Definition 12.6.1)**.** A fan Σ in $\mathfrak{g}_{\mathbb{Q}}$ is complete, if $D_{\mathrm{val}} = D_{\Sigma,\mathrm{val}}$.

In the case where a Griffiths domain D is a symmetric Hermitian domain, a fan Σ, used in the construction of the toroidal projective compactification $G_{\mathbb{Z}}\backslash D_{\Sigma}$ in [AMRT], is complete ([KU] 12.6.4). For general D, the existence of complete fans which are strongly compatible with $G_{\mathbb{Z}}$ was expected in [KU] conjecture 12.6.3. In the next section, we give a counterexample to that conjecture.

5. A special case

In this section, we state our main result whose proof will be given in Section 8. Let $w = 2$, and let $h^{p,q} = 2$ ($p+q = 2,\ p,q \geq 0$), and $h^{p,q} = 0$ otherwise. We consider about the existence of the complete fans in this case. Let $(e_j)_{1\leq j\leq 6}$ be a free basis of $H_{\mathbb{Z}}$ and $\langle\ ,\ \rangle : H_{\mathbb{Q}} \times H_{\mathbb{Q}} \to \mathbb{Q}$ be the bilinear form on $H_{\mathbb{Q}}$ given by

$$(\langle e_i, e_j\rangle)_{1\leq i,j\leq 6} = \begin{pmatrix} -1_2 & O & O \\ O & E & O \\ O & O & E \end{pmatrix}, \text{ where } 1_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ E = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Theorem 5.1. *There is no complete fan for D with* $w = 2,\ h^{p,q} = 2$ $(p+q = 2,\ p,q \geq 0)$, $h^{p,q} = 0$ *otherwise.*

6. The ranks of rational nilpotent cones in nilpotent orbits

We assume that we are in the situation of Section 5. Let $N \in \mathfrak{g}_{\mathbb{Q}}$ be a non zero nilpotent element. If we assume that there is an $\mathbb{R}_{\geq 0}N$-nilpotent orbit, then the following three cases occur.

(1) $N^2 = 0,\ \dim(\mathrm{Im}(N)) = 2.$
(2) $N^3 = 0,\ \dim(\mathrm{Im}(N^2)) = 1.$
(3) $N^3 = 0,\ \dim(\mathrm{Im}(N^2)) = 2.$

For a rational nilpotent cone σ, if we assume that there is a σ-nilpotent orbit, then the type of $N \in (\mathrm{rel.\,int}\,\sigma) \cap \mathfrak{g}_{\mathbb{Q}}$, which is one of the above three cases, is independent of the choice of N ([CK]). We estimate the rank of these σ. Here, the rank of σ is defined as the dimension of the vector space generated by σ over $\mathbb{R}$.

Proposition 6.1. *Let σ be a rational nilpotent cone in $\mathfrak{g}_{\mathbb{R}}$ and $F \in \check{D}$ and assume that the pair (σ, F) generates a nilpotent orbit.*
(i) *If* rel. int σ *contains a rational nilpotent element of type* (1), *the rank of σ is less than or equal to one.*
(ii) *If* rel. int σ *contains a rational nilpotent element of type* (2), *the rank of σ is less than or equal to two.*
(iii) *If* rel. int σ *contains a rational nilpotent element of type* (3), *the rank of σ is less than or equal to two.*

Proof. Let $N \in (\mathrm{rel.\,int}\,\sigma) \cap \mathfrak{g}_{\mathbb{Q}}$, and let $W = W(N)[-2]$. Then the pair (W, F) is an N-polarized mixed Hodge structure.

(i). Assume that N is of type (1). Since $W_l = 0$ $(l \leq 0)$, $W_1 = \mathrm{Im}\, N$, $W_2 = \mathrm{Ker}\, N$, $W_l = H_{\mathbb{Q}}$ $(l \geq 3)$, and $\dim W_1 = 2$, if we take a basis (x_1, x_2) of W_1, we have $\langle x_i, x_j \rangle = 0$ $(i, j = 1, 2)$ and $(\mathbb{Q}x_1)^{\perp} \neq (\mathbb{Q}x_2)^{\perp}$ for the polarization $\langle\ ,\ \rangle$. Therefore, we can choose two elements $y_1 \in (\mathbb{Q}x_1)^{\perp} \setminus (\mathbb{Q}x_2)^{\perp}$ and $y_2 \in (\mathbb{Q}x_2)^{\perp} \setminus (\mathbb{Q}x_1)^{\perp}$ such that $\langle x_1, y_2 \rangle = \langle x_2, y_1 \rangle = 1$. Since the images of y_1 and y_2 in Gr_3^W are linearly independent, they form a basis of Gr_3^W. Then since $\langle N(y_1 + y_2), y_1 + y_2 \rangle = \langle N y_i, y_i \rangle = 0$ $(i = 1, 2)$, N can be written as follows for some $c \in \mathbb{Q}$.

(1) $Ny_2 = -cx_2,\ Ny_1 = cx_1,\ NW_2 = 0.$

This implies that all two elements $N_1, N_2 \in (\mathrm{rel.\,int}\,\sigma) \cap \mathfrak{g}_{\mathbb{Q}}$ are linearly dependent. Thus we see that the rank of σ is less than or equal to one.

(ii). Assume that N is of type (2). Since $W_l = 0$ $(l \leq -1)$, $W_0 = W_1 = \mathrm{Im}\, N^2$, $W_2 = W_3 = \mathrm{Ker}\, N^2$, $W_l = H_{\mathbb{Q}}$ $(l \geq 4)$ and $\dim W_0 = 1$, we can take a base x of W_0 and $y \in W_4 \backslash W_3$ such that $\langle x, y \rangle = 1$. If we assume that the rank of σ is more than or equal to three, there are three linearly independent elements $N_1, N_2, N_3 \in (\mathrm{rel.\,int}\,\sigma) \cap \mathfrak{g}_{\mathbb{Q}}$. Since $\mathrm{Gr}^W_{4,\mathbb{C}} = F^2\,\mathrm{Gr}^W_{4,\mathbb{C}} \cap \overline{F^2\,\mathrm{Gr}^W_{4,\mathbb{C}}}$ and $N_i F^2 \subset F^1$ $(1 \leq i \leq 3)$, we have $N_i\,\mathrm{Gr}^W_{4,\mathbb{C}} \subset F^1\,\mathrm{Gr}^W_{2,\mathbb{C}} \cap \overline{F^1\,\mathrm{Gr}^W_{2,\mathbb{C}}}$ $(1 \leq i \leq 3)$. Moreover, since $\dim_{\mathbb{C}} F^1 \mathrm{Gr}^W_{2,\mathbb{C}} \cap \overline{F^1 \mathrm{Gr}^W_{2,\mathbb{C}}} = 2$, there exists $(c_1, c_2, c_3) \in \mathbb{Q}^3 \setminus \{(0,0,0)\}$ such that $(\sum_{1\le j\le 3} c_j N_j)y \in W_1$. Let $N' = \sum_{1\le j\le 3} c_j N_j$. Let $w \in W_3$ and $N'w = ax$ with $a \in \mathbb{Q}$. Then, since $N' \in \mathfrak{g}_{\mathbb{Q}}$ and $N'y \in W_1$, we have

(2) $a = \langle N'w, y\rangle = -\langle w, N'y\rangle = 0.$

On the other hand, since

(3) $\langle N'y, y\rangle = -\langle y, N'y\rangle = -\langle N'y, y\rangle,$

we see $\langle N'y, y\rangle = 0$, and hence $N'y = 0$. Thus $N' = 0$. This contradicts the assumption that N_1, N_2, N_3 are linearly independent.

Before the proof of Proposition 6.1 (iii), we prepare a lemma.

Lemma 6.1. *Let $N \in \mathfrak{g}_{\mathbb{Q}}$ be a nilpotent element of type* (3) *and $W = W(N)[-2]$. Then, the bilinear form on Gr^W_2 induced by $\langle\ ,\ \rangle$ is negative definite.*

Proof. Since $N^2 : \mathrm{Gr}^W_4 \to \mathrm{Gr}^W_0$ is an isomorphism and $\dim \mathrm{Gr}^W_2 = 2$, $N : \mathrm{Gr}^W_4 \to \mathrm{Gr}^W_2$ is an isomorphism. Since the polarization $\langle\ , N^2\ \rangle$ on Gr^W_4 is positive definite, we have for $0 \ne w \in \mathrm{Gr}^W_4$, $\langle Nw, Nw\rangle = -\langle w, N^2 w\rangle < 0$. The assertion follows. □

Proof of Proposition 6.1 (iii). Assume that N is of type (3). Since $W_l = 0$ $(l \le -1)$, $W_0 = W_1 = \mathrm{Im}\, N^2$, $W_2 = W_3 = \mathrm{Ker}\, N^2$, $W_l = H_{\mathbb{Q}}$ $(l \ge 4)$ and $\dim W_0 = 2$, if we take a basis (x_1, x_2) of W_0, we have $\langle x_i, x_j\rangle = 0$ $(i, j = 1, 2)$ and $(\mathbb{Q}x_1)^\perp \ne (\mathbb{Q}x_2)^\perp$ for the polarization $\langle\ ,\ \rangle$. Therefore, we can choose two elements $y_1 \in (\mathbb{Q}x_1)^\perp \setminus (\mathbb{Q}x_2)^\perp$ and $y_2 \in (\mathbb{Q}x_2)^\perp \setminus (\mathbb{Q}x_1)^\perp$ such that $\langle x_1, y_2\rangle = \langle x_2, y_1\rangle = 1$. Clearly the images of y_1 and y_2 in Gr^W_4 are linearly independent and hence form a basis of Gr^W_4. If we assume that the rank of σ is more than or equal to three, there are three linearly independent elements $N_1, N_2, N_3 \in (\mathrm{rel.\,int}\,\sigma) \cap \mathfrak{g}_{\mathbb{Q}}$. By the definition of nilpotent cone, $[N_i, N_j] = 0$ $(1 \le i, j \le 3)$ and hence

(4) $c_{i,j} := \langle N_j y_2, N_i y_1\rangle - \langle N_j y_1, N_i y_2\rangle = \langle [N_i, N_j] y_1, y_2\rangle = 0.$

We claim that there exists $(c_1, c_2, c_3) \in \mathbb{Q}^3 \setminus \{(0,0,0)\}$ such that $(\sum_{1\le j\le 3} c_j N_j)y_1$, $(\sum_{1\le i\le 3} c_j N_j)y_2 \in W_1$. In fact, since $\dim \mathrm{Gr}^W_2 = 2$, by a suitable linear transformation of N_1, N_2 and N_3, we may assume that $N_1 y_1 \in W_1$. By this fact and by $c_{1,i} = 0$ $(i = 2, 3)$, we have $\langle N_i y_1, N_1 y_2\rangle = \langle N_1 y_1, N_i y_2\rangle = 0$ $(i = 2, 3)$. If $N_1 y_2 \in W_1$, our claim is true for $c_1 = 1$, $c_2 = c_3 = 0$. Assume $N_1 y_2 \notin W_1$. By Lemma 6.1, the bilinear form on Gr^W_2 induced by $\langle\ ,\ \rangle$ is negative definite. Since $\dim \mathrm{Gr}^W_2 = 2$, the images of $N_2 y_1, N_3 y_1$ on Gr^W_2 are linearly dependent. Therefore, replacing N_2 with a suitable linear combination of N_2 and N_3, we may assume that $N_2 y_1 \in W_1$. Since the images of $N_i y_2$ $(1 \le i \le 3)$ in Gr^W_2 are linearly dependent, we can take $(c_1, c_2, c_3) \in \mathbb{Q}^3 \setminus \{(0,0,0)\}$ satisfying $(\sum_{1\le j\le 3} c_j N_j)y_2 \in W_1$. If $N_3 y_1 \in W_1$, since $N_i y_1 \in W_1$ $(i = 1, 2)$, we have $(\sum_{1\le j\le 3} c_j N_j)y_1 \in W_1$

and hence our claim is true for the above (c_1, c_2, c_3). Assume $N_3 y_1 \notin W_1$. Since $\langle N_3 y_1, N_i y_2\rangle = \langle N_i y_1, N_3 y_2\rangle = 0$ $(i = 1, 2)$ (because $c_{i,3} = 0$ and $N_i y_1 \in W_1$ $(i = 1, 2)$) and $\dim \mathrm{Gr}_2^W = 2$, the images of $N_1 y_2$ and $N_2 y_2$ in Gr_2^W are linearly dependent. Therefore, we can take $(c_1, c_2) \in \mathbb{Q}^2 \setminus \{(0, 0)\}$ satisfying $(c_1 N_1 + c_2 N_2) y_2 \in W_1$. Since $N_1 y_1, N_2 y_1 \in W_1$, we have $(c_1 N_1 + c_2 N_2) y_1 \in W_1$ and hence our claim is true for the above (c_1, c_2).

Let $N' = \sum_{1\le j\le 3} c_j N_j$. Let $w \in W_3$ and $N' w = a_1 x_1 + a_2 x_2$ with $a_1, a_2 \in \mathbb{Q}$. Then, since $N' \in \mathfrak{g}_{\mathbb{Q}}$ and $N' y_j \in W_1$ $(j = 1, 2)$, we have

$$(5)\ a_i = \langle N' w, y_j\rangle = -\langle w, N' y_j\rangle = 0 \quad (i, j = 1, 2,\ i \neq j).$$

Moreover, since $\langle N'(y_1 + y_2), y_1 + y_2\rangle = \langle N' y_j, y_j\rangle = 0$ $(j = 1, 2)$, N' can be written as follows for some $c \in \mathbb{Q}$.

$$(6)\ N' y_2 = -c x_2,\ N' y_1 = c x_1,\ N' W_3 = 0.$$

Since N_1, N_2, N_3 are linearly independent, we have $c \neq 0$. By $F^2 \mathrm{Gr}_4^W = W_{4,\mathbb{C}}/W_{3,\mathbb{C}}$, there exists $w_1, w_2 \in W_{3,\mathbb{C}}$ such that $y_i + w_i \in F^2$ $(i = 1, 2)$. Setting $v_i = y_i + w_i$ $(i = 1, 2)$, we have $\langle N' v_1, v_2\rangle = \langle N' y_1, y_2\rangle = c \neq 0$. But, this contradicts the fact that $N' F^2 \subset F^1$, since $N_i F^2 \subset F^1$ $(1 \le i \le 3)$. Thus we see that the rank of σ is less than or equal to two. □

7. Special nilpotent orbits of rank 1

In this section, we prepare some lemmas for the proof of Theorem 5.1. We keep the notation of Section 5. Let $H'_{\mathbb{Q}} = \bigoplus_{1\le j\le 4} \mathbb{Q} e_j$. For $a \in H'_{\mathbb{Q}}$, let N_a be the nilpotent operator defined by

$$(1)\ N_a(b) = -\langle a, b\rangle e_5 \quad (b \in H'_{\mathbb{Q}}),\ N_a(e_5) = 0,\ N_a(e_6) = a.$$

We note that, for all $a, a' \in H_{\mathbb{Q}}$, $[N_a, N_{a'}] = 0$.

For $a \in H'_{\mathbb{Q}}$ satisfying $\langle a, a\rangle \neq 0$, the weight filtration $W = W(N_a)[-2]$ on $H_{\mathbb{Q}}$ is given as follows.

$$(2)\ W_l = H_{\mathbb{Q}}\ (l \ge 4),\ W_3 = W_2 = H'_{\mathbb{Q}} \oplus \mathbb{Q} e_5,$$
$$W_1 = W_0 = \mathbb{Q} e_5,\ W_l = 0\ (l \le -1).$$

Lemma 7.1. *Let $a \in H'_{\mathbb{Q}}$. If we assume that there exists $F \in \check{D}$ such that $(\mathbb{R}_{\ge 0} N_a, F)$ generates a nilpotent orbit, then $-\langle a, a\rangle > 0$.*

Proof. We assume that $(\mathbb{R}_{\ge 0} N_a, F)$ generates a nilpotent orbit and let $W = W(N_a)[-2]$. Since (W, F) is an N_a-polarized mixed Hodge structure, we have $W_{4,\mathbb{C}}/W_{3,\mathbb{C}} = \mathbb{C}[e_6] = F^2 \mathrm{Gr}_4^W \cap \overline{F^2 \mathrm{Gr}_4^W}$, where, for $v \in H_{\mathbb{C}}$, $[v]$ is the image of v in $W_{4,\mathbb{C}}/W_{3,\mathbb{C}}$. Since the polarization $\langle\ , N_a^2\ \rangle$ on Gr_4^W is positive definite, we have $\langle e_6, N_a^2 e_6\rangle = -\langle a, a\rangle > 0$. □

Lemma 7.2. *Let $Q = \{w \in H'_{\mathbb{C}} \mid \langle w, w\rangle = 0,\ -\langle w, \bar{w}\rangle > 0\}$. For $w \in Q$, let $F_w \in \check{D}$ be given by $F_w^2 = \mathbb{C} w \oplus \mathbb{C} e_6$. Then, for all $a \in H'_{\mathbb{Q}}$ satisfying $-\langle a, a\rangle > 0$, the following two conditions are equivalent.*

(i) $(\mathbb{R}_{\geq 0}N_a,\ F_w)$ *generates a nilpotent orbit.*
(ii) $\langle a, w\rangle = 0$.

Proof. (i) $\Rightarrow$ (ii). Since (N_a, F_w) satisfies the Griffiths transversality, we have $N_a F_w^2 \subset F_w^1$, and hence $\langle a, w\rangle = 0$.

(ii) $\Rightarrow$ (i). By the condition (ii), the pair $(\mathbb{R}_{\geq 0}N_a, F_w)$ clearly satisfies the Griffiths transversality. We show that the pair $(\mathbb{R}_{\geq 0}N_a, F_w)$ satisfies the positivity. It is enough to show that for $y > 0$, $\exp(iyN_a)F_w \in D$. Let $v_1 = \exp(iyN_a)w$ and $v_2 = \exp(iyN_a)e_6$. Then, $v_1 = w,\ v_2 = e_6 + iya + y^2\langle a, a\rangle e_5/2$ by (ii). Let

(1) $A = -\begin{pmatrix} \langle v_1, \bar{v_1}\rangle & \langle v_1, \bar{v_2}\rangle \\ \langle v_2, \bar{v_1}\rangle & \langle v_2, \bar{v_2}\rangle \end{pmatrix}.$

Since $\det(A) = 2y^2\langle w, \bar{w}\rangle\langle a, a\rangle > 0$ and $\mathrm{tr}(A) = -\langle w, \bar{w}\rangle - 2y^2\langle a, a\rangle > 0$, we have $\exp(iyN_a)F_w \in D$. □

8. The proof of Theorem 5.1

In this section, we prove Theorem 5.1.

Let

(1) $f_1 := N_{e_3},\ f_2 := N_{e_4},\ A := \mathbb{Q}f_1 \oplus \mathbb{Q}f_2.$

Let V be the valuative submonoid of A^* defined by

(2) $$V = (\mathbb{Q}_{\geq 0}(f_1 + f_2)^* + \mathbb{Q}_{\geq 0}(-f_1 + f_2)^*) \cup (\mathbb{Q}_{>0}(-f_1 - f_2)^* + \mathbb{Q}_{>0}(-f_1 + f_2)^*),$$

where for $f \in A$, f^* is dual to f. Let $\sigma_0 = \mathbb{R}_{\geq 0}(-f_1) + \mathbb{R}_{\geq 0}f_2$, and define $F_0 \in \check{D}$ by

(3) $$F_0^2 = \mathbb{C}(ie_1 + e_2) \oplus \mathbb{C}e_6, \quad F_0^1 = (F_0^2)^{\perp}.$$

Let $Z = \exp(\sigma_{0,\mathbb{C}})F_0$. Then, as is clearly seen, Z is a σ_0-nilpotent orbit, hence $(A, V, Z) \in D_{\mathrm{val}}$ (see Section 4).

Assume that there exists a complete fan Σ in $\mathfrak{g}_{\mathbb{Q}}$. By the definition of complete fan, there exists the smallest element σ of $X_{A,V,\Sigma}$, and $\exp(\sigma_{\mathbb{C}})Z$ is a σ-nilpotent orbit (see Definition 4.3 and the note after it). By Proposition 6.1, the rank of σ is less than or equal to two. Since $\sigma \cap A_{\mathbb{R}} \in \mathcal{F}(A, V)$, we have $(\sigma \cap A_{\mathbb{R}})_{\mathbb{R}} = A_{\mathbb{R}}$. Therefore, $\sigma_{\mathbb{R}} = \sigma_{0,\mathbb{R}} = A_{\mathbb{R}}$, and hence $Z = \exp(\sigma_{\mathbb{C}})F_0$ is a σ-nilpotent orbit. Hence, by Lemma 7.1, $sf_1 + tf_2 \in (\mathrm{rel.\,int}\,\sigma) \cap A$ implies $st < 0$. It follows that $\sigma \subset \sigma_0 \cup (-\sigma_0)$. Since σ and σ_0 are sharp (Definition 3.4), we have $\sigma \subset \sigma_0$ or $\sigma \subset -\sigma_0$. Actually we have $\sigma \subset \sigma_0$ because $(\sigma \cap A)^{\vee} \subset V$ (Definition 4.1).

For $s, t \in \mathbb{Q}$, let $N_{s,t} := sf_1 + tf_2 \in (\mathrm{rel.\,int}\,\sigma) \cap A$, and let $F_{s,t} \in \check{D}$ be given by

(4) $F^2_{s,t} = \mathbb{C}((s-t)e_1 - i(s+t)e_2 - \sqrt{2}se_3 + \sqrt{2}te_4) \oplus \mathbb{C}e_6$.

Since the rank of σ is two, we can choose s and t satisfying $s+t \neq 0$. Let $Z(s,t) = \exp(\mathbb{C}N_{s,t})F_{s,t}$. Then, by Lemma 7.2, $Z(s,t)$ is an $\mathbb{R}_{\geq 0}N_{s,t}$-nilpotent orbit. Therefore, setting

(5) $A(s,t) := \mathbb{Q}N_{s,t}, \quad V(s,t) := \mathbb{Q}_{\geq 0}N^*_{s,t}$,

where $N^*_{s,t}$ denotes the base of $A(s,t)^*$ which is dual to $N_{s,t}$, we have $(A(s,t), V(s,t), Z(s,t)) \in D_{\mathrm{val}}$. Since Σ is a complete fan, this element belongs to $D_{\Sigma,\mathrm{val}}$. Let $\sigma(s,t)$ be the smallest element of $X_{A(s,t),V(s,t),\Sigma}$. We claim $\sigma(s,t) = \sigma$. In fact, since $(\sigma(s,t) \cap A(s,t))^\vee = V(s,t)$, we have $N_{s,t} \in \sigma(s,t)$. Since σ, $\sigma(s,t) \in \Sigma$, $N_{s,t} \in \mathrm{rel.\,int}\,\sigma$, and Σ is a fan, we have $\sigma(s,t) = \sigma$. Therefore, $\exp(\sigma(s,t)_{\mathbb{C}})Z(s,t) = \exp(\sigma_{\mathbb{C}})F_{s,t}$ is a σ-nilpotent orbit. In particular, $(\sigma, F_{s,t})$ satisfies the Griffiths transversality, that is, $f_iF^2_{s,t} \subset F^1_{s,t}$ $(i=1,2)$, and hence we have $s=t=0$. But this contradicts the choice of s and t. □

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References

[AMRT] A. Ash, D. Mumford, M. Rapoport and Y. S. Tai, *Smooth compactification of locally symmetric varieties*, Math. Sci. Press, Brookline, 1975.

[CK] E. Cattani and A. Kaplan, *Polarized mixed Hodge structures and the local monodromy of a variation of Hodge structure*, Invent. Math. **67** (1982), 101–115.

[CKS] E. Cattani, A. Kaplan and W. Schmid, *Degeneration of Hodge structures*, Ann. of Math. **123** (1986), 457–535.

[D] P. Deligne, *La conjecture de Weil, II*, Publ. Math. I.H.E.S. **52** (1980), 137–252.

[G] P. A. Griffiths, *Periods of integrals on algebraic manifolds. I. Construction and properties of modular varieties.* Amer. J. Math. **90** (1968), 568–626.

[KU] K. Kato and S. Usui, *Classifying spaces of degenerating polarized Hodge structures*, Ann. Math. Studies, Princeton Univ. Press, Princeton, 2008 (in press).

[S] W. Schmid, *Variation of Hodge structure: the singularities of the period mapping*, Invent. Math. **22** (1973), 211–319.