J. Math. Kyoto Univ. (JMKYAZ)
48-4 (2008), 895–906

On the image of code polynomials under theta map

By

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Abstract

The theta map sends code polynomials into the ring of Siegel modular forms of even weights. Explicit description of the image is known for $g \leq 3$ and the surjectivity of the theta map follows. Instead it is known that this map is not surjective for $g \geq 5$. In this paper we discuss the possibility of an embedding between the associated projective varieties. We prove that this is not possible for $g \geq 4$ and consequently we get the non surjectivity of the graded rings for the remaining case $g = 4$.

1. Introduction

One of the main theme in the theory of modular forms is to determine explicitly the structure of the graded rings of modular forms in terms of generators and relations. This study started systematically with Igusa about 50 years ago.

In several fundamental papers he determined the structure of the graded ring of Siegel modular forms of degree two, cf. [12], [13] and [15] using different methods.

First he applied his study [11] of moduli theory of the (hyperelliptic) curves of genus two in view of binary sextics, cf. [11]. The second method is based on, what he calls, a 'fundamental lemma' [14]. Using successively invariant theory of a finite group to the fundamental lemma, he determined the ring of modular forms of degree two, cf. [13]. In the third method [15], he defines the ρ homomorphism from a subring of the ring of modular forms to the projective invariants of binary forms. Then classical invariant theory is used to obtain the ring of modular forms.

Few years later in [4], Freitag used a more analytic method (see also [8]): he did a careful analysis of the vanishing locus of a distinguished modular form. This allowed him to reduce the problem to the study of graded ring related to lower dimensional varieties.

After this, only a few other cases were determined in the case of Siegel modular forms of degree two, because of the difficulty of the subject, cf. [9] and [6].

Received May 26, 2008

Tsuyumine, cf. [28], was the first one who determined the generators of the graded ring of Siegel modular forms of degree three. He got the result passing through a lot of difficulties. Essentially he used a mix between Igusa's third method and Freitag's method and deeply used Shioda's result about the structure of the ring of binary octavics, cf. [27].

Later Runge used Igusa's second method for determining the graded ring, in fact using the result of [23], he was able to determine the structure of a graded ring of Siegel modular forms of degree three related to a subgroup of finite index of the integral symplectic group. Then he applied Igusa's going down process.

Runge's method had also the advantage of relating the ring of Siegel modular forms with the ring R_3 of code polynomials of genus three. In fact he proved that the ring of modular forms is isomorphic to $R_3/\langle J^{(3)}\rangle$. Here $J^{(3)}$ is the difference of the code polynomials for the $e_8 \oplus e_8$ and d_{16}^+ codes.

Moreover he proved that this map exists for any genus g and has the image contained in the graded subring of modular forms of even weight. Also with this restriction, it can be easily shown that this map is not surjective for large g.

But there was always the hope that this map was surjective for other mall values of g. In fact, since the structure of the ring of code polynomials is easier to determine than the structure of the ring of modular forms, one would like to apply Runge's method to higher g.

However, a previous result of the second named author, cf. [22], implied that the map is not surjective when $g \geq 5$. Still it was open in the case $g = 4$. Moreover if one restrict the attention to code polynomials of degree divisible by 8 and hence to Siegel modular forms of weight divisible by 4, the negative result in [22] can be by passed.

However in [18] was observed that as a consequence of the results of [3] or [1] and [17] this map could not be sujective if $g \geq 6$. In fact dimension of the space of modular forms of weight 12 is greater than the dimension of space of weight enumerators of degree 24.

Meanwhile in genus 4 there were some partial results in [5] and in [18] leading toward a possible surjectivity.

In this note we show that the map

$$Th_2 : R_g^{(8)} \to M(\Gamma_g)^{(4)}$$

is not surjective if $g \geq 4$.

Here $M(\Gamma_g)$ stands for the graded ring of Siegel modular forms of degree g and the exponents (8) and (4) mean that we consider only code polynomials of degree divisible by 8 and modular forms of weight divisible by 4 that are related to theta series.

2. Notation and context

Let g be a positive integer. We denote by $\Gamma_g := Sp(g,\mathbb{Z})$ the integral symplectic group; it acts on the Siegel upper-half space $\mathcal{H}_g$ by

$$\sigma \cdot \tau := (A\tau + B)(C\tau + D)^{-1},$$

where $\sigma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g$. An element τ of $\mathcal{H}_g$ is called *reducible* if there exists $\sigma \in \Gamma_g$ such that

$$\sigma \cdot \tau = \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix}, \quad \tau_i \in \mathcal{H}_{g_i}, g_i > 0, \quad g_1 + g_2 = g;$$

otherwise we say that τ is irreducible.

Let k be a positive integer and Γ be a finite index subgroup. A multiplier system of weight $k/2$ for Γ is a map $v : \Gamma \to \mathbb{C}^*$, such that the map

$$\sigma \mapsto v(\sigma) \det(C\tau + D)^{k/2}$$

satisfies the cocycle condition for every $\sigma \in \Gamma$ and $\tau \in \mathcal{H}_g$ (note that the function $\det(C\tau + D)$ possesses a square root).

We shall write $f|_{k/2}\sigma$ for $\det(C\tau + D)^{-k/2} f(\sigma \cdot \tau)$.

With these notations, we say that a holomorphic function f defined on $\mathcal{H}_g$ is a modular form of weight $k/2$ with respect to Γ and v if

$$f|_{k/2}\sigma = v(\sigma) f \qquad \forall \sigma \in \Gamma,$$

and if additionally f is holomorphic at all cusps when $g = 1$.

We denote by $[\Gamma, k/2, v]$ the vector space of such functions. We shall consider the graded ring

$$M(\Gamma, v) := \bigoplus_{k=0}^{\infty} [\Gamma, k, v^k].$$

We omit the multiplier if it is trivial.

For $w \in \mathbb{C}$ write $\mathbf{e}(w) = e^{2\pi i w}$. For $\tau \in \mathcal{H}_g$ and column vectors $z \in \mathbb{C}^g, a, b \in \mathbb{Z}^g$, we define the theta function by

$$\theta\begin{bmatrix} a \\ b \end{bmatrix}(\tau, z) = \sum_{m \in \mathbb{Z}^g} \mathbf{e}\left(\frac{1}{2}(m + a)'\tau(m + a) + (m + a)'(z + b)\right),$$

where X' denotes the transpose of X.

For any $a \in \mathbb{Z}^g$, the function $\theta_r[a]$ defined by

$$\theta_r[a](\tau) = \theta\begin{bmatrix} a/r \\ 0 \end{bmatrix}(r\tau, 0)$$

is called an r-th order theta-constant. They depend only on a modulo r.

For any even positive integer r, we denote by $\Gamma_g(r, 2r)$ the subgroup of Γ_g of elements σ satisfying

$$\sigma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \equiv 1_{2g} \mod r,$$

$$(B)_0 \equiv (C)_0 \equiv 0 \mod 2r,$$

where $(X)_0$ means to take the vector determined by the diagonal coefficients of a square matrix X. If we drop the second condition, we get the principal congruence subgroup $\Gamma_g(r)$ of level r.

The functions $\theta_r[a]$ belong to $[\Gamma_g(r, 2r), 1/2, v_r]$ for a suitable multiplier v_r.

We denote by $\vec{\theta}_r = [\theta_r[a]]_{a \in \mathbb{Z}^g/r\mathbb{Z}^g}$ the vector of r-th order theta-constants.

We define r^g variables F_a for $a \in \mathbb{Z}^g/r\mathbb{Z}^g$. Let $\mathbb{C}[F_a : a \in \mathbb{Z}^g/r\mathbb{Z}^g]$ be the polynomial ring in these variables and $\mathbb{C}[F_a : a \in \mathbb{Z}^g/r\mathbb{Z}^g]^{(2)}$ the subring of even degree. For even r, there is a theta map

$$\mathrm{Th}_r : \mathbb{C}[F_a : a \in \mathbb{Z}^g/r\mathbb{Z}^g]^{(2)} \to M(\Gamma_g(r, 2r),\, \chi)$$

induced by sending F_a to $\theta_r[a]$ and the ring $M(\Gamma_g(r, 2r),\, \chi)$ is the integral closure of $\mathrm{Im}\,\mathrm{Th}_r$ inside its quotient field when r is greater than or equal to 4, cf. [14] and [16]. Since we are considering modular forms of integral weight, the multiplier χ is a character. It is trivial if and only if 4 divides r, otherwise χ^2 is trivial.

This theorem is called the 'fundamental lemma' of Igusa.

The second named author proved the same conclusion in the case $r = 2$, cf. [23] and [19] for the theta map

$$\mathrm{Th}_2 : \mathbb{C}[F_a : a \in \mathbb{F}_2^g]^{(2)} \to M(\Gamma_g(2, 4), \chi)$$

Also in this case χ^2 is trivial. Let $\Gamma_g^*(2, 4)$ be the kernel of χ. This is obviously a subgroup of index two in $\Gamma_g(2, 4)$, described by the condition $tr(A - 1_g) \equiv 0$ (mod 4), cf. [19].

To the map Th_2 is associated a projective map

$$th_2 : \overline{\mathcal{H}_g/\Gamma_g(2, 4)} \to \mathbb{P}^{2^g - 1}.$$

Here with the bar we denote the Satake compactification of the modular variety. This map has been studied in details in [23]. This paper concludes that the map is injective after proving generical injectivity. However, the argument there seems incomplete and at the moment we can only say that the map is generically injective. At any rate, Runge's results in [19] imply that when $g \leq 3$, the map is injective.

We know that the group Γ_g is generated by the elements $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $t(S) = \begin{pmatrix} 1 & S \\ 0 & 1 \end{pmatrix}$ for integral symmetric S. Moreover it acts on the r-th order theta-constants. On the second order we recall the action of the generators, cf. [19] for details. For these elements, we have

$$\vec{\theta}_2|_{\frac{1}{2}} t(S) = D_S \vec{\theta}_2 \text{ and } \vec{\theta}_2|_{\frac{1}{2}} J = \pm T_g \vec{\theta}_2,$$

where

$$D_S = \operatorname{diag}(i^{a' S a})_{a \in \mathbb{F}_2^g} \text{ and } T_g = \left(\frac{1+i}{2}\right)^g \left((-1)^{(a,b)}\right)_{a,b \in \mathbb{F}_2^g}.$$

The ± 1 comes from the choice of square root. The group

$$H_g = \langle T_g, D_S | \text{ integral symmetric } S \rangle \subseteq \mathrm{GL}(2^g, \mathbb{C})$$

is of finite order and the representation $\phi : \Gamma_g \to H_g / \pm 1$ defined by $\vec{\theta}_2 | \sigma = \pm \phi(\sigma) \vec{\theta}_2$ defines $\Gamma_g^*(2,4)$ as its kernel. Hence the map th_2 results to be $\Gamma_g / \Gamma_g(2,4)$ equivariant. Since the image is in the projective space, twisting by character produces nothing.

We see that an H_g-invariant polynomial goes to a level one Siegel form of even weight under the map Th_2. We denote by R_g the H_g-invariant subring of $\mathbb{C}[F_a : a \in \mathbb{F}_2^g]^{(2)}$ and R_g^m the vector space of H_g-invariant homogeneous polynomials of degree m, thus we have a theta map

$$\mathrm{Th}_2 : R_g \to M(\Gamma_g)$$

whose image is contained in $M(\Gamma_g)^{(2)}$, i.e., in the subring of modular forms of even weight.

We want to consider also the action of the group Γ_g on the fourth order theta-constants. It is useful to consider this action on different representatives that we are going to define.

A characteristic m is a column vector in $\mathbb{Z}^{2g}$, with m' and m'' as first and second entry vectors. We put

$$e(m) = (-1)^{(m', m'')}$$

and we say that m is even or odd according as $e(m) = 1$ or -1. Here we denoted by $(\cdot\,,\cdot)$ the standard scalar product.

We consider the theta function

$$\vartheta_m(\tau, z) = \theta \begin{bmatrix} m'/2 \\ m''/2 \end{bmatrix} (\tau, z).$$

This is also called even or odd according if m is even or odd. We recall from [10] the following formula:

$$\theta \begin{bmatrix} m'/4 \\ 0 \end{bmatrix} (4\tau, 4z) = \frac{1}{2^g} \sum_{m''} \mathbf{e}((1/2)(m', m'')) \vartheta_m(\tau, 2z).$$

Thus we can consider the theta-constants $\vartheta_m := \vartheta_m(\tau, 0)$ as the entries of the vector $\vec{\theta}_4$. We remind that in this case the entries at odd characteristics are 0.

On these entries the action of Γ_g is simpler, in fact it is monomial, cf. [10] or [12].

Moreover we recall the addition formula relating theta-constants of the second order with theta-constants of the fourth order:

$$\vartheta_m^2 = \sum_{a\in\mathbb{F}_2^g} (-1)^{(a,m'')}\theta_2[a+m']\theta_2[a].$$

3. The map th_2

In this section we shall consider with more details the map th_2. In particular we shall consider its injectivity on some special subloci.

We recall that in [26] has been proved that the map is injective along the hyperelliptic locus.

Here we are interested in the locus of the completely reducible periods, i.e. to the points τ that are Γ_g conjugated to points of the type

$$\begin{pmatrix} \lambda_1 & 0 & \dots & \dots & 0 \\ 0 & \lambda_2 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & 0 & \lambda_g \end{pmatrix} \qquad \lambda_i \in \mathcal{H}_1.$$

For doing this we need to recall and improve some results in [23] without going into details.

For $\sigma \in \Gamma_g$, the transformation formula

$$\vartheta_{\sigma\circ m}(\sigma\cdot\tau) = \kappa(\sigma)\det(C\tau+D)^{1/2}\chi_m(\sigma)\vartheta_m(\tau)$$

is known. For any characteristic m, the function χ_m becomes a character of the group $G' = \Gamma_g(2,4)/\Gamma_g(4,8)$. We set $G = G'/\pm 1$, then the group of characters $\hat{G}$ is spanned by the characters χ_m associated to the even characteristics.

For any point $\tau_0 \in \mathcal{H}_g/\Gamma_g(4,8)$ we consider the subgroup H_{τ_0} of $\hat{G}$ spanned by all characters $\chi_m\chi_n$ such that the products $\vartheta_m\vartheta_n$ do not vanish at τ_0. Moreover we set St_{τ_0} be the subgroup of G' generated by all σ fixing the point τ_0.

We recall from [23] the following

Theorem 3.1. *The following statements are equivalent*:
(i) *τ_0 is a reducible point.*
(ii) *St_{τ_0} is different from ± 1.*
(iii) *H_{τ_0} is a proper subgroup of $\hat{G}$.*

We need to mention some facts about this theorem. In the proof the author uses the theorem stating that the map θ_2 is injective. This, in general, is not necessary in fact theta-constants are used only to separate points in the same fibre of the covering map

$$\pi : \mathcal{H}_g/\Gamma_g(4,8) \to \mathcal{H}_g/\Gamma_g(2,4).$$

Moreover, since $St_{\tau_0}/\pm 1$ and H_{τ_0} are dual we have that the number of points in the fiber of $\pi(\tau_0)$ is equal to the order of H_{τ_0}.

Finally, cf.[25] also, we have the following

Corollary 3.1. *Let $\tau_0 \in \mathcal{H}_g/\Gamma_g(4,8)$. It is Γ_g equivalent to a point of the form τ that are Γ_g conjugated to points of the type*

$$\begin{pmatrix} \lambda_{i_1} & 0 & \dots & \dots & 0 \\ 0 & \lambda_{i_2} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & 0 & \lambda_{i_k} \end{pmatrix}$$

with all $\lambda_{i_j} \in \mathcal{H}_{g_j}$ irreducible and $g_1 + g_2 + \dots + g_k = g$ if and only if St_{τ_0} has order 2^k.

Let τ_0 be a completely reducible point, already in diagonal form. It is immediate to verify that St_{τ_0} is the subgroup of G' that is the image of $\sigma \in \Gamma_g(2,4)$ that are diagonal matrices, i.e. $B = C = 0$, $A = D = diag(\pm 1, \pm 1, \dots, \pm 1)$. Obviously it has order 2^g.

Proposition 3.1. *The map th_2 is injective along the completely reducible points.*

Proof. Assume that

$$th_2(\tau) = th_2(\tau')$$

for a completely reducible point τ and another point τ'. Here we can take τ in diagonal form. By addition formula, the same first order theta-constants vanish on τ and on τ'. Thus τ' is completely reducible and conjugate to τ by an element σ in the group $\Gamma_g(2)$ that fixes the characteristics of the first order theta-constants. But now $\Gamma_g(2)$ acts inducing different characters on the θ_m^2. Requiring that the character has to be the same, we have that $\sigma \in \Gamma_g(2,4)$. □

Another interesting fact about the completely reducible points is the following

Proposition 3.2. *Let τ_0 be a generic completely reducible point. If $\sigma \in \Gamma_g$ stabilizes τ, i.e.*

$$\sigma\tau_0 = \tau_0,$$

then $\sigma \in \Gamma_g(2,4)$.

Proof. Without any loss of generality we can assume τ_0 in diagonal form with $\tau_0 = x_0 + iy_0$, with x_0 and $y_0 > 0$ diagonal. If σ stabilizes τ_0 we have

$$A\tau_0 + B = \tau_0(C\tau_0 + D).$$

From this we get

$$Ax_0 + B = x_0(Cx_0 + D) - y_0Cy_0$$

and

$$Ay_0 = y_0(Cx_0 + D) + x_0Cy_0.$$

We can choose x_0 and y_0 in such form that for any $\sigma \in \Gamma_g$

$$Ax_0 - x_0(Cx_0 + D) + y_0Cy_0$$

is not an integral matrix, hence necessarily we have $B = 0$. Moreover from the first equation we can choose $y_0 >> 0$, so that $C = 0$, hence we have

$$Ax_0 = x_0D \quad \text{and} \quad Ay_0 = y_0D.$$

These conditions for generic diagonal matrices imply $A = D$ diagonal and integral. Thus we get the desired result. □

We conclude this section describing non-embeddability of $\mathcal{H}_g/\Gamma_g(2,4)$.

First we recall that in the paper [24] has been proved the following result.

Proposition 3.3. *When $g \geq 4$, the map th_2 is not an immersion at the completely reducible points.*

In fact the second author computed explicitly the dimension t_g of the tangent spaces at the generic points completely reducible points of $\mathcal{H}_g/\Gamma_g(2,4)$. Since there is a misprint in the formula, we reproduce it here:

$$t_g = g + \binom{g}{2} + \frac{1}{2}\sum_{h=3}^{g}\binom{g}{h}(h-1)!,$$

in which the third term is read as zero when $g = 1, 2$.

We recall the idea of the proof of this formula. Let K be the subgroup of $GL(g,\mathbb{Z})$ formed by the diagonal matrices and $S_g(\mathbb{C})$ be the set of symmetric matrices, thus a neighborhood of a generic completely reducible point $\tau_0 \in \mathcal{H}_g/\Gamma_g(2,4)$ looks like a neighborhood of $0 \in S_g(\mathbb{C})/K$.

Moreover the ideal $\mathfrak{m}_0 \subset \mathbb{C}[X_{i,j}]^K$ is generated by the monomials

$$X_{11}, X_{22}, \ldots, X_{gg}, \text{ and } X_{i_1,i_2}X_{i_2,i_3}\ldots X_{i_n,i_1}$$

with $1 \leq i_1 < i_2 < \cdots < i_n \leq g$.

These monomials are also a basis of $\mathfrak{m}_0/\mathfrak{m}_0^2$. Thus the dimension of the tangent space is the number of such monomials that is exactly t_g.

When $g \geq 4$, we have $t_g > 2^g - 1$ and hence we cannot have an immersion of a neighborhood of the point τ_0 into $\mathbb{P}^{2^g-1}$. We have thus proved that $\mathcal{H}_g/\Gamma_g(2,4)$ cannot be embedded in $\mathbb{P}^{2^g-1}$ when $g \geq 4$.

4. The map Θ_2

In this section we prove our main result. As we mentioned before, the transformation formula of theta-constants implies that the map θ_2 is $\Gamma_g/\Gamma_g(2,4)$ equivariant. Therefore we have a map

$$\mathrm{Th}_2 : R_g \to M(\Gamma_g)$$

whose image is contained in $M(\Gamma_g)^{(2)}$.

From Runge's results, cf. [19], [20] and [21], we have that the map is surjective onto $M(\Gamma_g)^{(2)}$ when $g \leq 3$.

In general a map Th_2 induces a projective map

$$\Theta_2 : \overline{\mathcal{H}_g/\Gamma_g} \to Proj(R_g).$$

As a consequence of Runge's results it is an immersion when $g \leq 3$. For higher g, we have the following

Theorem 4.1. *$\Theta_2 : \overline{\mathcal{H}_g/\Gamma_g} \to Proj(R_g)$ is not an embedding when $g \geq 4$*

Proof. We consider the action of $K_g = \Gamma_g/\Gamma_g(2,4)$ at a generic completely reducible point $\tau \in \mathcal{H}_g/\Gamma_g(2,4)$. The result of Proposition 4 implies that the group acts freely on τ. Let $x = \theta_2(\tau)$, since the map is equivariant the group acts freely also on $\mathbb{P}^{2^g-1}$.

Let $\pi : \mathcal{H}_g/\Gamma_g(2,4) \to \mathcal{H}_g/\Gamma_g$ and $\phi : \mathbb{P}^{2^g-1} \to \mathbb{P}^{2^g-1}/K_g$, the maps that make commutative the following diagram

$$\begin{array}{ccc} \mathcal{H}_g/\Gamma_g(2,4) & \longrightarrow & \mathbb{P}^{2^g-1} \\ \downarrow \pi & & \downarrow \phi \\ \mathcal{H}_g/\Gamma_g & \longrightarrow & \mathbb{P}^{2^g-1}/K_g. \end{array}$$

We set $\tau' := \pi(\tau)$ and $x' = \phi(x)$. Since the action of the group was free we have that the dimension of the tangent spaces at τ' and at x' are equal to the dimension of the tangent spaces at τ and x, respectively. Hence, in particular, they have different dimension, thus Θ_2 is not an immersion. □

As an immediate consequence, using basic fact from [7, page 92], Exercise 3.12, we have that

Corollary 4.1. *When $g \geq 4$, for any k the homomorphism*

$$\mathrm{Th}_2 : R_g^{(4k)} \to M(\Gamma_g)^{(2k)}$$

is never surjective.

We interpret this corollary in a restricted case, which is of another interest. To do this, we denote by $\mathbb{C}[\vartheta_\Lambda]$ the ring of theta series of all even unimodular lattices Λ where

$$\vartheta_\Lambda(\tau) = \sum_{v_1, v_2, \ldots, v_g \in \Lambda} \prod_{i,j} \mathbf{e}((v_i, v_j)\tau_{ij}/2).$$

Note that an even unimodular lattice exists if and only if the rank of a lattice is a multiple of 8 and that the weight of the corresponding theta series is a half of the rank. We recall that for n large enough, cf. [2] or [29], the space of modular forms $[\Gamma_g, n]$ is spanned by theta series provided that 4 divides n. Our final result in this paper is, specializing the previous corollary to the case $k = 2$,

Corollary 4.2. *When $g \geq 4$ the homomorphism*

$$\mathrm{Th}_2 : R_g^{(8)} \to \mathbb{C}[\vartheta_\Lambda]$$

is not surjective for infinitely many degrees.

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