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Universal lifts of chain complexes over non-commutative parameter algebras

To the memory of Professor Masayoshi Nagata.

By

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Abstract

We define the notion of universal lift of a projective complex based on non-commutative parameter algebras, and prove its existence and uniqueness. We investigate the properties of parameter algebras for universal lifts.

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1. Introduction

In this paper, k always denotes a field and R is an arbitrary associative k-algebra. When we say an R-module, we always mean a left R-module unless

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otherwise stated.

From the view point of representation theory, the final goal of the theory of R-modules should be to construct the moduli consisting of the isomorphism classes of R-modules, by which we mean a geometric realization of the set of isomorphism classes. Generally speaking, it is however impossible to describe all the isomorphism classes of R-modules, even if we restrict ourselves to consider indecomposable ones. One should say that the construction of moduli for R-modules is hopeless.

But there is a way to observe the moduli from the local view point. Fixing an R-module M, and assuming there is a modulus containing M as a rational closed point, we can ask how it looks in the neighbourhood of the point, which is nothing but to consider the universal deformation of M. In such a context, the existence of formal local moduli is known ([2], [3], [6]).

To explain this, let $\mathcal{C}_k$ be the category of commutative artinian local k-algebras with residue field k and k-algebra homomorphisms. We consider the covariant functor

$$\mathcal{F}_M : \mathcal{C}_k \to (Sets),$$

which maps $A \in \mathcal{C}_k$ to the set of infinitesimal deformations of M along A, i.e.

$$\mathcal{F}_M(A) = \left\{ \begin{array}{l} (R,A)\text{-bimodules } X \text{ that are flat over } A \\ \text{and } X \otimes^L_A k \cong M \text{ as left } R\text{-modules} \end{array} \right\} / \cong,$$

where $\cong$ means (R,A)-bimodule isomorphism. Under these circumstances the following theorem is known to hold.

Theorem 1.1 (Schlessinger's Theorem 1968). *Suppose* $\mathrm{Ext}^1_R(M,M)$ *is of finite dimension as a k-vector space. Then the functor $\mathcal{F}_M$ is pro-representable. More precisely, there exist a commutative noetherian complete local k-alegbra Q with residue field k and an (R,Q)-bimodule U that is flat over Q such that there is an isomorphism*

$$\mathrm{Hom}_{k\text{-}alg}(Q, \quad) \cong \mathcal{F}_M$$

as functors on $\mathcal{C}_k$. The isomorphism is given in such a way that each $f \in \mathrm{Hom}_{k\text{-}alg}(Q,A)$ is mapped to $[U \otimes_Q {}_fA] \in \mathcal{F}_M(A)$ for $A \in \mathcal{C}_k$, where ${}_fA$ denotes the right A-module A regarded as a left Q-module through f.

In such a circumstance, we call U the universal family of deformations of M, and call Q (resp. Spec Q) the commutative parameter algebra (resp. the parameter space) of U.

One of the easiest examples is the deformation of Jordan canonical forms.

Example 1.1. Consider an $n \times n$ matrix which is of an irreducible Jordan canonical form:

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ : & : & : & : & : \\ : & : & : & : & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

Setting $R = k[x]$, we know that this is equivalent to consider the indecomposable R-module $M = k[x]/(x^n)$. In this case, we can take $Q = k[[t_0, \dots, t_{n-1}]]$ as the commutative parameter algebra, and $U = Q[x]/(x^n + t_{n-1}x^{n-1} + \cdots + t_0)$ as the universal family of deformations of M. If we consider this in a matrix form, we obtain a so-called Sylvester family of matrices.

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & 1 \\ -t_0 & -t_1 & -t_2 & \cdots & -t_{n-1} \end{pmatrix}$$

Under the setting of Theorem 1.1, since the bimodule U is flat as a right Q-module, the functor $U \otimes_Q^L - \; : D(Q) \to D(R)$ between derived categories is defined. Remark that $U \otimes_Q^L k = M$. Thus the functor induces a map between Yoneda algebras.

$$\rho^{\cdot} \; : \; \mathrm{Ext}^{\cdot}_Q(k, k) \to \mathrm{Ext}^{\cdot}_R(M, M).$$

Of most interest is the mapping

$$\rho^2 \; : \; \mathrm{Ext}^2_Q(k, k) \to \mathrm{Ext}^2_R(M, M),$$

which is often called the obstruction map. Our motivation of this paper starts with the observation that ρ^2 does not work well as a comparison map between cohomology modules. We show this by the above example. In fact, we see in Example 1.1 that

$$\begin{array}{l} \mathrm{Ext}^2_Q(k,k) = \langle \text{Koszul relations of degree 2 in the variables } t_i\text{'s} \rangle^* \\ \qquad\qquad \downarrow \;\; \rho^2 \\ \;\; \mathrm{Ext}^2_R(M,\ M) = (0). \end{array}$$

Compared with that $\mathrm{Ext}^2_R(M, M) = (0)$, the k-vector space $\mathrm{Ext}^2_Q(k, k)$ has dimension $n(n-1)/2$. This is one of the examples that shows that ρ^2 does not work well as a comparison map of cohomology modules. Here we should notice that the Koszul relations of degree 2 are derived from the commutativity relations of the variables $t_0, \dots, t_{n-1}$.

Thinking this phenomenon over, we get the idea that the parameters $t_0, \dots, t_{n-1}$ should be regarded as non-commutative variables. Now we propose the following idea.

Idea 1. Parameter algebras should be non-commutative.

If we simply generalize the arguments in the commutative setting, we will have difficulty in showing the flatness of the universal family of deformations over the non-commutative parameter algebra. The reason for this is that the local criterion of flatness does not necessarily hold for modules over non-commutative rings. Therefore, to avoid the argument about flatness, we also propose the following idea.

Idea 2. We should consider the deformation of chain complexes instead of modules.

The deformation of chain complexes is nothing but the lifting of complexes, which we mainly discuss in this paper. In such a way, we necessarily come to think of "the universal lifts of chain complexes over non-commutative parameter algebras".

Just to explain about the lifting of chain complexes, let us introduce several notation concerning chain complexes. When we say $\mathbb{F} = (F, d)$ is a chain complex (or simply a complex) of R-modules, we mean that $F = \oplus_{i\in\mathbb{Z}} F_i$ is a graded R-module and $d : F \to F[-1]$ is a graded homomorphism satisfying $d^2 = 0$. A projective complex $\mathbb{F} = (F, d)$ is just a complex where the underlying graded module F is a projective R-module. If $\mathbb{F} = (F, d)$ is a projective complex, then we define $\mathrm{Ext}^i_R(\mathbb{F}, \mathbb{F})$ to be the set of homotopy equivalence classes of chain homomorphisms on $\mathbb{F}$ of degree $-i$.

We introduce the category $\mathcal{A}_k$, whose objects are artinian local k-algebras with residue field k with k-algebra homomorphisms as morphisms. (Note that an object of $\mathcal{A}_k$ is not necessarily a commutative ring, but it is a finite dimensional k-algebra.) Now let $A \in \mathcal{A}_k$ and let $\mathbb{F} = (F, d)$ be a projective complex of R-modules. Then, $(F \otimes_k A, \Delta)$ is said to be a lift of $\mathbb{F}$ to A if it is a chain complex of $R \otimes_k A^{op}$-modules, and satisfies the equality $\Delta \otimes_A k = d$.

The aim of this paper is to construct the universal lift of a given projective complex $\mathbb{F} = (F, d)$ which dominates all the lifts of $\mathbb{F}$ to all non-commutative artinian k-algebras in $\mathcal{A}_k$, and to investigate the properties of its parameter algebra.

We should note that such a universal lift is no longer defined on an artinian algebra, but defined on a 'pro-artinian' local k-algebra. We call such a pro-artinian algebra a complete local k-algebra by an abuse of the terminology for commutative rings. The non-commutative formal power series ring $k\langle\langle t_1, \dots, t_r\rangle\rangle$ with non-commutative variables $t_1, \dots, t_r$ is an example of complete local k-algebra. This is actually complete and separated in the $(t_1, \dots, t_r)$-adic topology. And a complete local k-algebra is defined to be a residue ring of the non-commutative formal power series ring by a closed ideal. (See Definition 2.1 and Proposition 2.1.) In particular all artinian algebras in $\mathcal{A}_k$ are complete local k-algebras. But the difficulty here is that complete local k-algebras are not necessarily noetherian rings.

We can extend the notion of lifting to the lifting to complete local k-algebras. In fact, $(F\widehat{\otimes}_k A, \Delta_A)$ is said to be a lift of $\mathbb{F}$ to a complete local k-algebra A if it is a chain complex of $R\widehat{\otimes}_k A^{op}$-modules and the equality $\Delta_A \otimes_A k = d$ holds. (See Section 2.3 for the complete tensor product $\widehat{\otimes}$.)

To give a precise definition of universal lifts, let $\mathbb{F} = (F, d)$ be a projective complex of R-modules which we fix. Then we define a covariant functor $\mathcal{F} : \mathcal{A}_k \to (Sets)$ by setting as $\mathcal{F}(A)$ the set of chain-isomorphism classes of lifts of $\mathbb{F}$ to A for any $A \in \mathcal{A}_k$. If we have a complete local k-algebra P and a lift $\mathbb{L} = (F\widehat{\otimes}_k P, \Delta_P)$ of $\mathbb{F}$ to P, then we can define a natural transformation

$\phi_{\mathbb{L}} : \mathrm{Hom}_{k\text{-alg}}(P, -) \to \mathcal{F}$ of functors by setting $\phi_{\mathbb{L}}(f) = (F \otimes_k A, \Delta_P \otimes_P {}_f A)$ for $A \in \mathcal{A}_k$ and $f \in \mathrm{Hom}_{k\text{-alg}}(P, A)$, where ${}_f A$ denotes the right A-module A regarded as a left P-module through f.

A chain complex $\mathbb{L} = (F \widehat{\otimes}_k P, \Delta_P)$ is said to be a universal lift of $\mathbb{F}$, if $\phi_{\mathbb{L}}$ is an isomorphism of functors. In this case, we say that P is a parameter algebra.

The first main result of this paper is about the existence and the uniqueness of universal lifts, which we summarize as follows. (See Theorem 3.1 and Theorem 3.2.)

Theorem 1.2. *Let $\mathbb{F} = (F, d)$ be a projective complex of R-modules. We assume that it satisfies $r = \dim_k \mathrm{Ext}^1_R(\mathbb{F}, \mathbb{F}) < \infty$. Then the following statements hold true.*

(1) *There exists a universal lift $\mathbb{L}_0 = (F \widehat{\otimes}_k P_0, \Delta_0)$ of $\mathbb{F}$.*

(2) *A parameter algebra P_0 is unique up to k-algebra isomorphisms.*

(3) *Fixing a parameter algebra P_0, a universal lift $\mathbb{L}_0$ is unique up to chain isomorphisms of complexes of $R \widehat{\otimes}_k P_0^{op}$-modules.*

(4) *The parameter algebra has a description $P_0 \cong T/I$, where $T = k\langle\langle t_1, \dots, t_r \rangle\rangle$ is a non-commutative formal power series ring of r variables and I is a closed ideal which is contained in the square of the unique maximal ideal of T.*

We shall give a proof of this theorem in Section 3, where we need several new ideas to do so, because complete local k-algebras are not necessarily noetherian. We should remark that every complete local k-algebra can be a parameter algebra. In fact, for any complete local k-algebra P with maximal ideal $\mathfrak{m}_P$, P itself is the parameter algebra for the universal lift of a free resolution of the left P-module $k = P/\mathfrak{m}_P$. (See Theorem 3.3).

This theorem is essentially used in the proofs in Section 4, where we investigate the properties of parameter algebras by considering the comparison of cohomology modules. As one of the main results there, we can give a certain structure theorem for parameter algebras. In fact, assuming that $r = \dim_k \mathrm{Ext}^1_R(\mathbb{F}, \mathbb{F}) < \infty$ and $\ell = \dim_k \mathrm{Ext}^2_R(\mathbb{F}, \mathbb{F}) < \infty$ for a projective complex $\mathbb{F}$ of R-modules, we have a description of the parameter algebra P_0 as $P_0 \cong k\langle\langle t_1, \dots t_r \rangle\rangle / \overline{(f_1, \dots, f_\ell)}$. (See Theorem 4.3.) In particular, if $\dim_k \mathrm{Ext}^2_R(\mathbb{F}, \mathbb{F}) = 0$, then the parameter algebra equals a non-commutative formal power series ring.

Let P_0 be the parameter algebra of the universal lifts of $\mathbb{F}$ which is described as $P_0 = T/I_0$, where T is a non-commutative formal power series ring and I_0 is a closed ideal of T with $I_0 \subseteq \mathfrak{m}_T^2$. Then we prove in Theorem 4.5 that there is an isomorphism of k-vector spaces

$$\mathrm{Ext}^1_R(\mathbb{F}, \mathbb{F})^2 \cong \mathrm{Hom}_k(I_0 / I_0 \cap \mathfrak{m}_T^3, k),$$

where the left hand side means the k-subspace of $\mathrm{Ext}^2_R(\mathbb{F}, \mathbb{F})$ generated by all the products of two elements in $\mathrm{Ext}^1_R(\mathbb{F}, \mathbb{F})$. This isomorphism shows that $I_0 \subseteq \mathfrak{m}_T^3$ if and only if $\mathrm{Ext}^1_R(\mathbb{F}, \mathbb{F})^2 = 0$. (See Corollary 4.2.)

We can also regard such all observations as results of comparison of cohomology modules. For this, we assume that $\mathbb{F} = (F, d)$ is a right bounded projective complex of R-modules, and let $\mathbb{L}_0 = (F \widehat{\otimes}_k P_0, \Delta_0)$ be the universal lift of $\mathbb{F}$. For any integer n, we have a projective complex of $R \otimes_k (P_0/\mathfrak{m}_{P_0}^n)^{op}$-modules;

$$\mathbb{L}_0^{(n)} = (F \otimes_k P_0/\mathfrak{m}_{P_0}^n,\ \Delta_0 \otimes_{P_0} P_0/\mathfrak{m}_{P_0}^n),$$

which is a lift of $\mathbb{F}$ to $P_0/\mathfrak{m}_{P_0}^n$. Therefore we have a morphism of Yoneda algebras as before;

$$\mathrm{Ext}^{\cdot}_{P_0/\mathfrak{m}_{P_0}^n}(k, k) \to \mathrm{Ext}^{\cdot}_R(\mathbb{F}, \mathbb{F}).$$

Taking the direct limit, we finally get the k-algebra homomorphism

$$\rho^{\cdot} \ : \ \varinjlim \mathrm{Ext}^{\cdot}_{P_0/\mathfrak{m}_{P_0}^n}(k, k) \to \mathrm{Ext}^{\cdot}_R(\mathbb{F}, \mathbb{F}).$$

Our main problem is to see how the mapping ρ^i behaves for $i \geq 0$. One can easily observe that $\rho^0 \ : \ \varinjlim \mathrm{Hom}_{P_0/\mathfrak{m}_{P_0}^n}(k, k) = k \to \mathrm{End}_R(\mathbb{F})$ is a natural embedding and hence it is always an injection. Furthermore, by our construction of $\mathbb{L}_0$ in Theorem 1.2, we see that $\rho^1 \ : \ \varinjlim \mathrm{Ext}^1_{P_0/\mathfrak{m}_{P_0}^n}(k, k) = (\mathfrak{m}_{P_0}/\mathfrak{m}_{P_0}^2)^* \to \mathrm{Ext}^1_R(\mathbb{F}, \mathbb{F})$ is a bijection.

One of the main theorems of this paper is Theorem 4.6, in which we prove that $\rho^2 \ : \ \varinjlim \mathrm{Ext}^2_{P_0/\mathfrak{m}_{P_0}^n}(k, k) \to \mathrm{Ext}^2_R(\mathbb{F}, \mathbb{F})$ is always an injection. This actually realizes Idea 1. We should notice that this holds because we had extended the notion of parameter algebras to non-commutative rings.

2. Non-commutative complete local algebras

2.1. Definitions and properties

Throughout this paper, k always denotes a field. Let A be an associative k-algebra. By an ideal of A we always mean a two-sided ideal. When S is a subset of A, we denote by (S) the minimum ideal of A that contains S.

Definition 2.1. Let A be an associative local k-algebra with Jacobson radical $\mathfrak{m}_A$. We say that A is a complete local k-algebra if the following three conditions are satisfied.

(a) The natural inclusion $k \subset A$ induces an isomorphism $k \cong A/\mathfrak{m}_A$.

(b) The k-vector space $\mathfrak{m}_A/\mathfrak{m}_A^2$ is of finite dimension.

(c) A is complete and separated in the $\mathfrak{m}_A$-adic topology, i.e. the natural projections $A \to A/\mathfrak{m}_A^n$ $(n \in \mathbb{N})$ induce an isomorphism $A \cong \varprojlim A/\mathfrak{m}_A^n$.

For a complete local k-algebra A, we always denote by $\mathfrak{m}_A$ the Jacobson radical of A, and we regard A as a topological ring with $\mathfrak{m}_A$-adic topology.

Note that any artinian local k-algebra A with $A/\mathfrak{m}_A \cong k$ is a complete local k-algebra in our sense.

Example 2.1. Let $S = k\langle t_1, t_2, \ldots, t_r\rangle$ be a free k-algebra over variables $t_1, t_2, \ldots, t_r$, and let $J = (t_1, t_2, \ldots, t_r)$. We denote by T the J-adic completion of S, i.e.

$$T = \varprojlim S/J^n,$$

and we call T the **non-commutative formal power series ring**, which is denoted by $k\langle\langle t_1, t_2, \ldots, t_r\rangle\rangle$. Clearly from the definition, T is a complete local k-algebra with maximal ideal $\mathfrak{m}_T = (t_1, t_2, \ldots, t_r)$.

Note that each element of T has a unique expression as a formal infinite sum $\sum_\lambda c_\lambda m_\lambda$, where $c_\lambda \in k$ and the m_λ's are distinct monomials on $t_1, t_2, \ldots, t_r$.

Remark 1. Let $f : A \to B$ be a k-algebra homomorphism of complete local k-algebras. Then it is easy to see that f is a local homomorphism, i.e. $f(\mathfrak{m}_A) \subseteq \mathfrak{m}_B$. In particular, f is a continuous map.

Definition 2.2. Let A be a complete local k-algebra and let I be an ideal (resp. a left or right ideal). Then we denote the closure of I by $\overline{I}$, i.e. $\overline{I} = \bigcap_{n=0}^{\infty}(I + \mathfrak{m}_A^n)$. It is easy to see that $\overline{I}$ is also an ideal (resp. a left or right ideal). We say that I is a closed ideal (resp. a closed left or right ideal) if $I = \overline{I}$.

Remark 2. If A is a commutative complete local k-algebra, then it is well-known that A is noetherian and every ideal of A is closed (cf. [1]). But, in general, a non-commutative complete local k-algebra is not necessarily noetherian, and an ideal may not be closed.

For example, let $T = k\langle\langle x, y\rangle\rangle$ and let $I = (x)$. Since any element of I is a finite sum of elements of the form axb with $a, b \in T$, one can easily see that $\sum_{n=1}^{\infty} y^n x y^n$ belongs to $\overline{I}$, but not to I.

Remark 3. If I is a closed ideal of a complete local k-algebra. Then, I is complete and separated in the relative topology on I, i.e.

$$I = \varprojlim I/I \cap \mathfrak{m}_A^n.$$

Lemma 2.1. *Let A be a complete local k-algebra and let I be an ideal of A. Then, A/I is a complete local k-algebra if and only if I is a closed ideal.*

Proof. Note that the residue ring A/I is complete (but may not be separated) in $\mathfrak{m}_A$-adic topology. If I is a closed ideal, then A/I is separated, hence A/I is a complete local k-algebra. Conversely, if A/I is a complete local k-algebra, then the natural projection $f : A \to A/I$ is continuous and $\{0\} \subseteq A/I$ is closed. Therefore $I = f^{-1}(\{0\})$ is closed. $\square$

Lemma 2.2. *Let $f : A \to B$ be a k-algebra homomorphism of complete local k-algebras. Suppose that the induced mapping $\overline{f} : \mathfrak{m}_A/\mathfrak{m}_A^2 \to \mathfrak{m}_B/\mathfrak{m}_B^2$ is surjective. Then f is a surjective homomorphism.*

Proof. It is easy to see by induction on n that the induced mappings $\overline{f} : \mathfrak{m}_A^n/\mathfrak{m}_A^{n+1} \to \mathfrak{m}_B^n/\mathfrak{m}_B^{n+1}$ are surjective for all $n \geq 1$. Then, for a given $b \in B$, we can find $a_i \in \mathfrak{m}_A^i$ $(0 \leq i \leq n)$ such that $f(a_0 + a_1 + \cdots + a_n) - b \in \mathfrak{m}_B^{n+1}$ for $n \geq 0$. Thus, putting $a = \sum_{n=0}^{\infty} a_n$, we have $a \in A$ and $f(a) = b$, since f is continuous. □

Proposition 2.1. *Let A be a complete local k-algebra. Then, there are a non-commutative formal power series ring $T = k\langle\langle t_1, t_2, \ldots, t_r\rangle\rangle$ and a k-algebra homomorphism $f : T \to A$ such that the induced mapping $\overline{f} : \mathfrak{m}_T/\mathfrak{m}_T^2 \to \mathfrak{m}_A/\mathfrak{m}_A^2$ is bijective.*

In particular, A can be described as $A \cong T/I$, where I is a closed ideal of T and $I \subseteq \mathfrak{m}_T^2$.

Proof. Take $x_1, x_2, \ldots, x_r \in \mathfrak{m}_A \backslash \mathfrak{m}_A^2$ which give rise to a basis of the k-vector space $\mathfrak{m}_A/\mathfrak{m}_A^2$. Now define a k-algebra homomorphism $f : T = k\langle\langle t_1, t_2, \ldots, t_r\rangle\rangle \to A$ by $f(t_i) = x_i$ $(1 \leq i \leq r)$. Then it is obvious that f satisfies the desired conditions. □

Definition 2.3. We denote by $\widehat{\mathcal{A}}_k$ the category of complete local k-algebras and k-algebra homomorphisms. We also denote by $\mathcal{A}_k$ the category of artinian local k-algebras A with $A/\mathfrak{m}_A \cong k$ and k-algebra homomorphisms. Obviously, $\mathcal{A}_k$ is a full subcategory of $\widehat{\mathcal{A}}_k$.

Remark 4. Let A be a complete local k-algebra. Then $A/\mathfrak{m}_A^n \in \mathcal{A}_k$ for any $n \geq 1$ and by definition $A = \varprojlim A/\mathfrak{m}_A^n$. Conversely, let

$$\cdots \longrightarrow A_{n+1} \xrightarrow{f_{n+1}} A_n \xrightarrow{f_n} A_{n-1} \longrightarrow \cdots \xrightarrow{f_2} A_1$$

be a projective system in $\mathcal{A}_k$ such that each f_n induces an isomorphism $\mathfrak{m}_{A_n}/\mathfrak{m}_{A_n}^2 \cong \mathfrak{m}_{A_{n-1}}/\mathfrak{m}_{A_{n-1}}^2$. Then we have that $\varprojlim A_n \in \widehat{\mathcal{A}}_k$.

In fact, we see from Lemma 2.1 that each A_n is isomorphic to T/I_n for any $n \geq 1$, where $I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq I_{n+1} \supseteq \cdots$ are closed ideals of the non-commutative formal power series ring T . Then we have $\varprojlim A_n \cong T/\bigcap_{n=1}^{\infty} I_n$ and $\bigcap_{n=1}^{\infty} I_n$ is a closed ideal of T. Thus the claim follows from Lemma 2.1.

We remark here on the closedness of certain ideals in the non-commutative formal power series ring. First we note the following lemma.

Lemma 2.3. *Let I be a left ideal of $T = k\langle\langle t_1, t_2, \ldots, t_r\rangle\rangle$, and suppose that I is finitely generated as a left ideal. Then I is a free module as a left T-module.*

Proof. We note that $I/\mathfrak{m}_T I$ is a finite dimensional k-vector space. Hence we can take a finite number of elements $f_1, \ldots, f_n \in I$ which yield a base of the k-vector space $I/\mathfrak{m}_T I$. First we claim that I is generated by $f_1, \ldots, f_n$ as a left ideal.

To show this, let x be any element of I. Since $I = T\{f_1, \ldots, f_n\} + \mathfrak{m}_T I$, there are elements $a_{01}, \ldots, a_{0n} \in T$ such that $x - \sum_{i=1}^{n} a_{0i} f_i \in \mathfrak{m}_T I$. Then, apply the same argument to this element, we can find $a_{11}, \dots, a_{1n} \in \mathfrak{m}_T$ such that $x - \sum_{i=1}^n a_{0i} f_i - \sum_{i=1}^n a_{1i} f_i \in \mathfrak{m}_T^2 I$. Inductively, one can show that there are $a_{\ell 1}, \dots, a_{\ell n} \in \mathfrak{m}_T^\ell$ with $x - \sum_{i=1}^n (a_{0i} + a_{1i} + \cdots + a_{\ell i}) f_i \in \mathfrak{m}_T^{\ell+1} I$ for any $\ell \geq 1$. Now put $\alpha_i = \sum_{\ell=0}^\infty a_{\ell i}$ which are well-defined elements in T, and we have $x = \sum_{i=1}^n \alpha_i f_i$. Thus the set $\{f_1, \dots, f_n\}$ generates I as a left ideal.

Now we prove that $\{f_1, \dots, f_n\}$ is a free basis of I as a left T-module. To show this, let $\sum_{i=1}^n a_i f_i = 0$, where $a_i \in T$ $(1 \leq i \leq n)$. We have to show $a_i = 0$ for each i.

For this, we only have to prove, by induction on $\ell \geq 1$, that a_i $(1 \leq i \leq n)$ belong to $\mathfrak{m}_T^\ell$ for all $a_i \in T$ $(1 \leq i \leq n)$ which satisfy the equality $\sum_{i=1}^n a_i f_i = 0$.

Since $\{f_1, \dots, f_n\}$ is a k-base of $I/\mathfrak{m}_T I$, it is trivial that $a_i \in \mathfrak{m}_T$ $(1 \leq i \leq n)$. Hence the claim holds for $\ell = 1$. Now assume $a_i \in \mathfrak{m}_T^\ell$ $(1 \leq i \leq n)$ for $\ell \geq 1$. Then we may write $a_i = \sum_{j=1}^r t_j b_{ji}$ for some $b_{ji} \in \mathfrak{m}_T^{\ell-1}$. Thus we have

$$\sum_{j=1}^r t_j \left(\sum_{i=1}^n b_{ji} f_i \right) = 0,$$

in T. Since an element of T has a unique expression as a formal infinite sum of monomials with coefficients in k, it follows that $\sum_{i=1}^n b_{ji} f_i = 0$ for any j. Then, by the induction hypothesis, we have $b_{ji} \in \mathfrak{m}_T^\ell$, and hence $a_i = \sum_{j=1}^r t_j b_{ji} \in \mathfrak{m}_T^{\ell+1}$ as desired. □

The following lemma is known as Nagata's theorem for commutative formal power series ring, which is easily generalized to non-commutative ones.

Lemma 2.4. *Let $T = k\langle\langle t_1, t_2, \dots, t_r \rangle\rangle$ be a non-commutative formal power series ring. Suppose a descending sequence $\mathfrak{a}_1 \supset \mathfrak{a}_2 \supset \mathfrak{a}_3 \supset \cdots$ of left ideals of T satisfies the equality $\bigcap_{i=1}^\infty \mathfrak{a}_i = (0)$. Then the linear topology on T defined by $\{\mathfrak{a}_i \mid i = 1, 2, \dots\}$ is stronger than the $\mathfrak{m}_T$-adic topology.*

Proof. The proof given in [5, (30.1)] is valid for non-commutative case. □

Proposition 2.2. *Let I be a left ideal of a complete local k-algebra A. If one of the following conditions holds, then I is a closed left ideal in A.*

(a) A is a non-commutative formal power series ring $T = k\langle\langle t_1, t_2, \dots, t_r \rangle\rangle$ and I is finitely generated as a left ideal.

(b) I is of finite length as a left A-module, i.e. $\dim_k I < \infty$.

Proof. (a) By Lemma 2.3 we may write $I = T\{f_1, \dots, f_n\} = Tf_1 \oplus \cdots \oplus Tf_n$. We prove the lemma by induction on n. If $n = 0$, then it is trivially true.

Suppose $n > 0$ and set $J = T\{f_2, \dots, f_n\}$. (We understand $J = (0)$ if $n = 1$.) Note that we have a direct decomposition $I = Tf_1 \oplus J$ as a left T-module. Now we set

$$\mathfrak{a}_\ell = \{c \in T \mid cf_1 + g \in \mathfrak{m}_T^\ell \text{ for some } g \in J\},$$

for each $\ell > 0$. Note that $\mathfrak{a}_\ell$ is a left ideal, $\mathfrak{a}_\ell \supseteq \mathfrak{a}_{\ell+1}$ and $\mathfrak{m}_T^\ell \subseteq \mathfrak{a}_\ell$ for all ℓ. First of all, we claim that the following equality holds.

$$(*) \qquad \bigcap_{\ell=1}^{\infty} \mathfrak{a}_\ell = (0)$$

In fact, for any element $c \in \bigcap_{\ell=1}^{\infty} \mathfrak{a}_\ell$, there is an element $g_\ell \in J$ with $cf_1 + g_\ell \in \mathfrak{m}_T^\ell$ for each ℓ. Since $g_\ell - g_{\ell+1} \in \mathfrak{m}_T^\ell$, we see that $\{g_\ell\}$ forms a Cauchy sequence in the $\mathfrak{m}_T$- adic topology. Therefore we see that $cf_1 + \lim_{\ell\to\infty} g_\ell = 0$. Since J is a closed ideal by the induction hypothesis, we have $\lim_{\ell\to\infty} g_\ell \in J$, and thus we have $cf_1 \in J$. Then the direct decomposition $I = Tf_1 \oplus J$ forces $cf_1 = 0$, hence $c = 0$. This proves the equality $(*)$. Note from Lemma 2.4 that the ideals $\mathfrak{a}_\ell$ define the topology equivalent to the $\mathfrak{m}_T$-adic topology.

Now, to prove that I is closed, take an element $x \in \overline{I}$. We want to show $x \in I$. Take a sequence $\{a_\ell \mid \ell = 1, 2, \ldots\}$ in I which converges to x in the $\mathfrak{m}_T$-adic topology. We may assume that $a_\ell - a_{\ell+1} \in \mathfrak{m}_T^\ell$ for each ℓ. Each a_ℓ has a unique description $a_\ell = \sum_{i=1}^n b_{\ell,i} f_i$ for some $b_{\ell,i} \in T$. Thus $\sum_{i=1}^n (b_{\ell,i} - b_{\ell+1,i}) f_i \in \mathfrak{m}_T^\ell$. Therefore $b_{\ell,1} - b_{\ell+1,1} \in \mathfrak{a}_\ell$ for any ℓ. Then, by the fact we have shown above, we see that $\{b_{\ell,1} \mid i = 1, 2, \ldots\}$ is a Cauchy sequence in the $\mathfrak{m}_T$-adic topology. This is true for the sequences $\{b_{\ell,i} \mid \ell = 1, 2, \ldots\}$ for all i $(1 \leq i \leq n)$. Since T is complete in the $\mathfrak{m}_T$-adic topology, the sequence $\{b_{\ell,i} \mid \ell \geq 1\}$ converges to an element $c_i \in T$ for each i. Then, $x = \lim_{\ell\to\infty} a_\ell = \sum_{i=1}^n \lim_{\ell\to\infty} b_{\ell i} f_i = \sum_{i=1}^n c_i f_i \in I$ as desired.

(b) Since $\bigcap_{n=1}^{\infty} I \cap \mathfrak{m}_A^n \subseteq \bigcap_{n=1}^{\infty} \mathfrak{m}_A^n = (0)$, and since $\dim_k I < \infty$, there is an integer n_0 such that $I \cap \mathfrak{m}_A^n = (0)$ for $n \geq n_0$. Thus $I + \mathfrak{m}_A^n = I \oplus \mathfrak{m}_A^n$ for $n \geq n_0$. Therefore,

$$\overline{I} = \bigcap_{n=n_0}^{\infty} I + \mathfrak{m}_A^n = \bigcap_{n=n_0}^{\infty} I \oplus \mathfrak{m}_A^n = I \oplus \bigcap_{n=n_0}^{\infty} \mathfrak{m}_A^n = I.$$

□

Corollary 2.1. *Let $A \in \widehat{\mathcal{A}}_k$ and let I be an ideal of A. Suppose one of the conditions in the previous proposition holds. Then we have $A/I \in \widehat{\mathcal{A}}_k$.*

The Artin-Rees lemma for non-commutative formal power series ring holds in the following form.

Corollary 2.2. *Let I be a finitely generated left ideal of the non-commutative formal power series ring $T = k\langle\langle t_1, t_2, \ldots, t_r\rangle\rangle$. Then, the relative topology on I induced from T is equivalent to the $\mathfrak{m}_T$-adic topology on I. That is, for any $m \geq 1$, there is an integer $\ell \geq 1$ such that $\mathfrak{m}_T^\ell \cap I \subseteq \mathfrak{m}_T^m I$.*

Proof. In the proof of (a) in Proposition 2.2, we have shown that, for a given $m \geq 1$, there is an integer $\ell_1 \geq 1$ such that $\mathfrak{a}_{\ell_1} \subseteq \mathfrak{m}_T^m$. This shows that $c_1 f_1 + \cdots + c_n f_n \in \mathfrak{m}_T^{\ell_1}$ implies $c_1 \in \mathfrak{m}_T^m$. This is true for any i $(1 \leq i \leq n)$, that is, there is an integer $\ell_i > 0$ such that $c_1f_1 + \cdots + c_nf_n \in \mathfrak{m}_T^{\ell_i}$ implies $c_i \in \mathfrak{m}_T^m$. Now take ℓ so that $\ell > \ell_i$ for all i $(1 \le i \le n)$. Then we have that $c_1f_1 + \cdots + c_nf_n \in \mathfrak{m}_T^{\ell}$ implies $c_i \in \mathfrak{m}_T^m$ for all i. Hence $I \cap \mathfrak{m}_T^{\ell} \subseteq \mathfrak{m}^m I$. □

Definition 2.4. Let A be a complete local k-algebra and let S be a subset of A. Then we say that an ideal I **is analytically generated by** S if $I = \overline{(S)}$.

Proposition 2.3. *Let A be a complete local k-algebra and let S be a subset of a closed ideal I of A. Then, I is analytically generated by S if the image of S generates $I/\overline{\mathfrak{m}_A I + I\mathfrak{m}_A}$ as a k-vector space.*

Furthermore, if S is a finite subset, then the converse is also true.

Proof. Suppose that the image of S generates $I/\overline{\mathfrak{m}_A I + I\mathfrak{m}_A}$, and let n be an arbitrary natural number. Since $\overline{\mathfrak{m}_A I + I\mathfrak{m}_A} \subseteq \mathfrak{m}_A I + I\mathfrak{m}_A + (\mathfrak{m}_A^n \cap I) \subseteq I$, the set S generates $I/\mathfrak{m}_A I + I\mathfrak{m}_A + (\mathfrak{m}_A^n \cap I)$ as a k-vector space, hence

$$\begin{aligned} I &= (S) + \mathfrak{m}_A I + I\mathfrak{m}_A + (\mathfrak{m}_A^n \cap I) \\ &= (S) + \mathfrak{m}_A((S) + \mathfrak{m}_A I + I\mathfrak{m}_A) + ((S) + \mathfrak{m}_A I + I\mathfrak{m}_A)\mathfrak{m}_A + (\mathfrak{m}_A^n \cap I) \\ &= (S) + (\mathfrak{m}_A^2 I + \mathfrak{m}_A I\mathfrak{m}_A + I\mathfrak{m}_A^2) + (\mathfrak{m}_A^n \cap I) \\ &\cdots \\ &= (S) + \sum_{i+j=s} \mathfrak{m}_A^i I\mathfrak{m}_A^j + (\mathfrak{m}_A^n \cap I), \end{aligned}$$

for $1 \le s \le n$. Finally, putting $s = n$, we have that $I = (S) + (\mathfrak{m}_A^n \cap I)$. Since this equality holds for all $n \ge 1$, we have $I = \overline{(S)}$.

To prove the converse, we assume that I is analytically generated by a finite subset S. Then the equality $I = \bigcap_{n=1}^{\infty}((S) + \mathfrak{m}_A^n)$ holds. Since $I \subseteq (S) + \mathfrak{m}_A^n$, we have $I = (S) + (\mathfrak{m}_A^n \cap I)$ for all $n \ge 1$. Thus the image of S generates $I/\mathfrak{m}_A I + I\mathfrak{m}_A + (\mathfrak{m}_A^n \cap I)$ as a k-vector space, for all $n \ge 1$. In particular, $\dim_k I/\mathfrak{m}_A I + I\mathfrak{m}_A + (\mathfrak{m}_A^n \cap I) \le |S|$. Since $|S|$ is finite, there is an integer $n_0 > 0$ such that $I/\mathfrak{m}_A I + I\mathfrak{m}_A + (\mathfrak{m}_A^n \cap I) = I/\mathfrak{m}_A I + I\mathfrak{m}_A + (\mathfrak{m}_A^{n+1} \cap I)$ for all $n \ge n_0$. Thus, we have the equality $I/\overline{\mathfrak{m}_A I + I\mathfrak{m}_A} = I/\mathfrak{m}_A I + I\mathfrak{m}_A + (\mathfrak{m}_A^{n_0} \cap I)$, which is generated by S as a k-vector space. □

Corollary 2.3. *Let I be a closed ideal in a complete local k-algebra A. Then, the equality $I = \overline{\mathfrak{m}_A I + I\mathfrak{m}_A}$ implies $I = (0)$.*

Corollary 2.4. *Let I be a closed ideal in a complete local k-algebra A. Then, I is analytically generated by a finite number of elements of I if and only if* $\dim_k \left(I/\overline{\mathfrak{m}_A I + I\mathfrak{m}_A}\right) < \infty$.

Corollary 2.5. *Let I be a closed ideal in a complete local k-algebra A that is analytically generated by a finite number of elements. Then, the equality*

$$\overline{\mathfrak{m}_A I + I\mathfrak{m}_A} = \mathfrak{m}_A I + I\mathfrak{m}_A + (\mathfrak{m}_A^n \cap I)$$

holds for any large integer $n \gg 1$.

Proof. See the proof of Proposition 2.3. □

It is well-known that the category $\mathcal{A}_k$ admits the fiber products.

Lemma 2.5. *The category $\widehat{\mathcal{A}}_k$ admits the fiber products, that is, any diagram in $\widehat{\mathcal{A}}_k$*

$$\begin{tikzcd} & B \arrow[d, "g"] \\ A \arrow[r, "f"] & C \end{tikzcd}$$

can be embedded into a pull-back diagram

$$\begin{tikzcd} Q \arrow[r] \arrow[d] & B \arrow[d, "g"] \\ A \arrow[r, "f"] & C. \end{tikzcd}$$

Proof. For any integer n, we have a diagram in $\mathcal{A}_k$

$$\begin{tikzcd} & B/\mathfrak{m}_B^n \arrow[d, "g_n"] \\ A/\mathfrak{m}_A^n \arrow[r, "f_n"] & C/\mathfrak{m}_C^n \end{tikzcd},$$

from which we have a fiber product $Q_n := A/\mathfrak{m}_A^n \times_{C/\mathfrak{m}_C^n} B/\mathfrak{m}_B^n$ in the category $\mathcal{A}_k$. It is clear that $\{Q_n | \ n \geq 1\}$ forms a projective system in $\mathcal{A}_k$. Put $Q = \varprojlim Q_n$, and we have $Q \in \widehat{\mathcal{A}}_k$ by Remark 4. It is routine to show that Q is a fiber product in $\widehat{\mathcal{A}}_k$. □

Remark 5. In the setting of Lemma 2.5, the fiber product Q and its Jacobson radical $\mathfrak{m}_Q$ can be described in the following way :

$$Q = \{(a,b) \in A{\times}B \mid f(a) = g(b)\}, \quad \mathfrak{m}_Q = \{(a,b) \in A{\times}B \mid f(a) = f(b) \in \mathfrak{m}_C\}.$$

We denote the fiber product Q by $A \times_C B$.

2.2. Small extensions

Let A be a complete local k-algebra. We say that an element $\epsilon \neq 0$ in A is a socle element of A if $\mathfrak{m}_A\epsilon = \epsilon\mathfrak{m}_A = 0$. Note that an element ϵ of A is a socle element if and only if the ideal (ϵ) is a one-dimensional k-vector space. Note that if A is an artinian local k-algebra then there exists at least one socle element.

One should remark from Corollary 2.1 that, if ϵ is a socle element in a complete local k-algebra A, then $\overline{A} = A/(\epsilon)$ is also a complete local k-algebra.

Definition 2.5. A pair (A', ϵ) is called a small extension of a complete local k-algebra A if ϵ is a socle element of a complete local k-algebra A' and

$A'/(\epsilon) \cong A$ as a k-algebra. To describe the small extension (A', ϵ) of A, we often write it as a short exact sequence

$$0 \longrightarrow k \xrightarrow{\ \epsilon\ } A' \xrightarrow{\ \pi\ } A \longrightarrow 0,$$

where π is the natural projection.

Lemma 2.6. *Let (A', ϵ) be a small extension of a complete local k-algebra A.*

*(a) If $\epsilon \notin \mathfrak{m}_{A'}^2$, then there is a k-algebra homomorphism $\iota : A \to A'$ that is a right inverse of $\pi : A' \to A$. In this case, A' is isomorphic to $A[x]/(x^2, \mathfrak{m}_A x, x\mathfrak{m}_A)$ as a k-algebra, which we call a **trivial small extension** of A.*

*(b) If $\epsilon \in \mathfrak{m}_{A'}^2$, and if $A = T/I$ where I is a closed ideal of $T = k\langle\langle t_1, t_2 \ldots, t_r\rangle\rangle$ and $I \subseteq \mathfrak{m}_T^2$, then there is a closed ideal $J \subseteq I$ of T such that $A' \cong T/J$ and the length $\ell_T(I/J) = 1$. In this case, we say that (A', ϵ) is a **nontrivial small extension**.*

Proof. (a) Suppose $\epsilon \notin \mathfrak{m}_{A'}^2$. Then, since $(\epsilon) \cong k$, we have $(\epsilon) \cap \mathfrak{m}_{A'}^2 = (0)$. Thus we can take a k-subspace $\mathfrak{n}$ of $\mathfrak{m}_{A'}$ such that $\mathfrak{m}_{A'}^2 \subseteq \mathfrak{n}$ and $\mathfrak{m}_{A'} = (\epsilon) \oplus \mathfrak{n}$ as a k-vector space. Noting that $\mathfrak{n}^2 = \mathfrak{m}_{A'}^2$, we see that the k-subspace $k \oplus \mathfrak{n} \subseteq A'$ is actually a k-subalgebra and the restriction to $k \oplus \mathfrak{n}$ of $\pi : A' \to A$ yields an isomorphism $k \oplus \mathfrak{n} \cong A$.

(b) Suppose $\epsilon \in \mathfrak{m}_{A'}^2$. Then we have $\mathfrak{m}_{A'}/\mathfrak{m}_{A'}^2 \cong \mathfrak{m}_A/\mathfrak{m}_A^2$. It follows from Lemma 2.1 that there is a commutative diagram in $\widehat{\mathcal{A}}_k$

$$\begin{array}{ccc} T & = & T \\ {\scriptstyle f'}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ A' & \longrightarrow & A, \end{array}$$

where f and f' are surjective and $I = \mathrm{Ker}(f)$. It is easy to see that $J = \mathrm{Ker}(f')$ satisfies the desired conditions. □

Definition 2.6. Let $A \in \widehat{\mathcal{A}}_k$. For small extensions (A_1, ϵ_1) and (A_2, ϵ_2) of A, we say that (A_1, ϵ_1) and (A_2, ϵ_2) are equivalent, denoted by $(A_1, \epsilon_1) \sim (A_2, \epsilon_2)$, if there is a k- algebra isomorphism $f : A_1 \to A_2$ with $f(\epsilon_1) = \epsilon_2$. We denote by $\mathcal{T}(A)$ the set of equivalence classes of small extensions of A :

$$\mathcal{T}(A) = \{(A', \epsilon) \mid \epsilon \in A' \in \widehat{\mathcal{A}}_k,\ (\epsilon) \cong k,\ A'/(\epsilon) \cong A\}/\sim .$$

For a small extension (A', ϵ) we denote its equivalence class by $[A', \epsilon]$.

Note from Lemma 2.6 that trivial small extensions defines a unique element of $\mathcal{T}(A)$.

Lemma 2.7. *Let $A \in \widehat{\mathcal{A}}_k$. Then $\mathcal{T}(A)$ is an abelian group in which the zero element is the class of a trivial small extension.*

Proof. Let $[A_1, \epsilon_1]$ and $[A_2, \epsilon_2]$ be elements in $\mathcal{T}(A)$. Then we have the following commutative diagram by taking the fiber product.

$$
\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & k \arrow[r, equal] \arrow[d, "(0,\epsilon_2)"'] & k \arrow[d, "\epsilon_2"'] & \\
0 \arrow[r] & k \arrow[r, "(\epsilon_1,0)"] \arrow[d, equal] & A_1 \times_A A_2 \arrow[r] \arrow[d] & A_2 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & k \arrow[r, "\epsilon_1"] & A_1 \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}
$$

Put $B = A_1 \times_A A_2/(\epsilon_1, -\epsilon_2)$ and it follows from the exact sequence of the middle row in the diagram that there is an exact sequence

$$0 \longrightarrow k \xrightarrow{(\epsilon_1,0)} B \longrightarrow A \longrightarrow 0.$$

Note that, since $A_1 \times_A A_2$ is a complete local k-algebra by Lemma 2.5 and $(\epsilon_1, -\epsilon_2)$ is its socle element, it follows from Corollary 2.1 that B is a complete local k- algebra. Hence $(B, (\epsilon_1, 0))$ is a small extension of A. Note that $(\epsilon_1, 0) = (0, \epsilon_2)$ in B. Now we define the sum by

$$[A_1, \epsilon_1] + [A_2, \epsilon_2] = [B, (\epsilon_1, 0)].$$

Then it is routine to verify that $\mathcal{T}(A)$ is an abelian group by this definition of addition. Actually, the commutativity of sum is given by the isomorphism

$$A_1 \times_A A_2/(\epsilon_1, -\epsilon_2) \cong A_2 \times_A A_1/(\epsilon_2, -\epsilon_1), \quad (\epsilon_1, 0) \leftrightarrow (\epsilon_2, 0).$$

The associativity is induced by

$$\begin{aligned}&\{A_1 \times_A A_2/(\epsilon_1, -\epsilon_2)\} \times_A A_3/((\epsilon_1, 0), -\epsilon_3)\\ &\qquad\qquad \cong A_1 \times_A \{A_2 \times_A A_3/(\epsilon_2, -\epsilon_3)\}/(\epsilon_1, -(\epsilon_2, 0)).\end{aligned}$$

Let (A_0, ϵ_0) be a trivial small extension of A. Then we can show $A_1 \times_A A_0/(\epsilon_1, -\epsilon_0) \cong A_1$, which implies that $[A_0, \epsilon_0]$ is the zero element in $\mathcal{T}(A)$. Note that the inverse element is given in the following.

$$-[A_1, \epsilon_1] = [A_1, -\epsilon_1]$$

In fact, using the following lemma 2.8, one can show the isomorphism

$$A_1 \times_A A_1/(\epsilon_1, \epsilon_1) \cong A_1 \times_k D/(\epsilon_1, 0) \cong A_0.$$

$\square$

Lemma 2.8. *Let $A \in \widehat{\mathcal{A}}_k$ and let $D = k[\epsilon_0]/(\epsilon_0^2)$, which we call the ring of dual numbers over k. Then we have the following isomorphism of complete local k-algebras for any $(A_1, \epsilon_1) \in \mathcal{T}(A)$.*

$$A_1 \times_A A_1 \cong A_1 \times_k D$$

Proof. Define $f : A_1 \times_k D \to A_1 \times_A A_1$ by $f((a_1, \overline{a_1} + c\epsilon_0)) = (a_1, a_1 + c\epsilon_1)$, where $a_1 \in A_1$ and $c \in k$, and $\overline{a_1} \in k = A_1/\mathfrak{m}_{A_1}$ is the natural image of $a_1 \in A_1$. Then it is easy to see that f is an isomorphism of k-algebras. $\square$

Let $[A_1, \epsilon_1] \in \mathcal{T}(A)$ for $A \in \widehat{\mathcal{A}}_k$. We define the scalar product by an element $c \in k$ as follows :

$$c \cdot [A_1, \epsilon] = \begin{cases} [A_1, c^{-1}\epsilon] & (c \neq 0) \\ \text{the class of a trivial small extension} & (c = 0). \end{cases}$$

Lemma 2.9. *Let $A \in \widehat{\mathcal{A}}_k$. Then $\mathcal{T}(A)$ is a k-vector space by the above action of k.*

Proof. Let $c_1, c_2 \in k$ and $[A_1, \epsilon_1], [A_2, \epsilon_2] \in \mathcal{T}(A)$. It is obvious from the definition that $(c_1c_2) \cdot [A_1, \epsilon_1] = c_1(c_2 \cdot [A_1, \epsilon_1])$. When $c_1 \neq 0$, the identity $c_1 \cdot ([A_1, \epsilon_1] + [A_2, \epsilon_2]) = c_1 \cdot [A_1, \epsilon_1] + c_1 \cdot [A_2, \epsilon_2]$ follows from the isomorphism $A_1 \times_A A_2/(\epsilon_1, -\epsilon_2) \cong A_1 \times_A A_2/(c_1^{-1}\epsilon_1, -c_1^{-1}\epsilon_2)$. We have to verify the equality $(c_1 + c_2) \cdot [A_1, \epsilon_1] = c_1 \cdot [A_1, \epsilon_1] + c_2 \cdot [A_1, \epsilon_1]$. If one of c_1, c_2 and $c_1 + c_2$ is equal to zero, then it is easy to see the equality holds. We assume that $c_1 \neq 0, c_2 \neq 0$ and $c_1 + c_2 \neq 0$. In this case, we have from Lemma 2.8 the isomorphism

$$A_1 \times_A A_1/(c_1^{-1}\epsilon_1,\ -c_2^{-1}\epsilon_1) \cong A_1 \times_k D/(c_1^{-1}\epsilon_1, -(c_1^{-1} + c_2^{-1})\epsilon_0) \cong A_1,$$

and by this isomorphism $(c_1^{-1}\epsilon_1, 0)$ corresponds to $(c_1 + c_2)^{-1}\epsilon_1 \in A_1$. Hence, $[A_1 \times_A A_1/(c_1^{-1}\epsilon_1,\ -c_2^{-1}\epsilon_1),\ (c_1^{-1}\epsilon_1, 0)] = [A_1,\ (c_1 + c_2)^{-1}\epsilon_1]$. $\square$

Lemma 2.10. *Let $A_1, A_2 \in \widehat{\mathcal{A}}_k$ and let $f : A_1 \to A_2$ be a k-algebra homomorphism. Then f induces a k-linear map $f^* : \mathcal{T}(A_2) \to \mathcal{T}(A_1)$. Therefore, $\mathcal{T}$ is a contravariant functor from $\widehat{\mathcal{A}}_k$ to the category of k-vector spaces.*

Proof. For a given $[A_2', \epsilon_2'] \in \mathcal{T}(A_2)$, take a fiber product

$$\begin{tikzcd} 0 \arrow[r] & k \arrow[r, "\epsilon_1'"] \arrow[d, equal] & A_1' \arrow[r] \arrow[d] & A_1 \arrow[r] \arrow[d, "f"] & 0 \\ 0 \arrow[r] & k \arrow[r, "\epsilon_2'"] & A_2' \arrow[r] & A_2 \arrow[r] & 0, \end{tikzcd}$$

and we get a small extension (A_1', ϵ_1') of A_1. Now define $f^*([A_2', \epsilon_2']) = [A_1', \epsilon_1']$. It is not difficult to verify that f^* is a k-linear mapping. $\square$

Definition 2.7. Let I be a closed ideal of $T = k\langle\langle t_1, t_2, \ldots, t_r\rangle\rangle$. We always regard I as a topological T-bimodule by the relative topology induced from T. Therefore, the set $\{I \cap \mathfrak{m}_T^n \mid n = 1, 2, \ldots\}$ gives the fundamental open neighbourhoods of 0 in I. We also consider the unique simple T-bimodule k with discrete topology. We set

$$\mathrm{Hom}_{con}(I, k) = \{f : I \to k \mid f \text{ is a continuous } T\text{-bimodule homomorphism}\}.$$

Note that $f \in \mathrm{Hom}_{T\text{-bimod}}(I, k)$ belongs to $\mathrm{Hom}_{con}(I, k)$ if and only if $f(I \cap \mathfrak{m}_T^n) = 0$ for a large integer n. It is clear that $\mathrm{Hom}_{con}(I, k)$ is naturally a k-vector space.

Since $f(\mathfrak{m}_T I + I\mathfrak{m}_T) = 0$ for $f \in \mathrm{Hom}_{con}(I, k)$, such an f induces the continuous map $\overline{f} : I/\overline{\mathfrak{m}_T I + I\mathfrak{m}_T} \to k$. Hence,

$$\mathrm{Hom}_{con}(I, k) \cong \{\overline{f} : I/\overline{\mathfrak{m}_T I + I\mathfrak{m}_T} \to k \mid \overline{f} \text{ is a continuous } k\text{-linear map}\}.$$

Note, however, that the induced topology on $I/\overline{\mathfrak{m}_T I + I\mathfrak{m}_T}$ may not be discrete.

Let $A \in \widehat{\mathcal{A}}_k$, which we describe as $A = T/I$ where T is a non-commutative formal power series ring and $I \subseteq \mathfrak{m}_T^2$. Under such a circumstance, we define the mapping

$$\tau : \mathrm{Hom}_{con}(I, k) \to \mathcal{T}(A),$$

as follows: For $f \in \mathrm{Hom}_{con}(I, k)$, if $f = 0$, define $\tau(f)$ to be the class of a trivial small extension. If $f \neq 0$, then $I_f := \mathrm{Ker}(f) = f^{-1}(0)$ is a closed ideal of T and hence $A_f := T/I_f$ is a complete local k-algebra and we can take a unique element $\epsilon_f \in I/I_f \subseteq A_f$ with $f(\epsilon_f) = 1$. Since $I = I_f + (\epsilon_f)$, (A_f, ϵ_f) is a small extension of A. We define $\tau(f) = [A_f, \epsilon_f]$.

Proposition 2.4. *The mapping* $\tau : \mathrm{Hom}_{con}(I, k) \to \mathcal{T}(A)$ *is an isomorphism of* k*-vector spaces.*

Proof. First we show that τ is a k-linear mapping. To show that $\tau(cf) = c \cdot \tau(f)$ for $c \in k$ and $f \in \mathrm{Hom}_{con}(I, k)$, we may assume that $c \neq 0$. Then it is trivial that $I_f = I_{cf}$, hence $A_f = A_{cf}$ and $\epsilon_{cf} = c^{-1}\epsilon_f$. Thus it follows that $\tau(cf) = c \cdot \tau(f)$.

To show $\tau(f + g) = \tau(f) + \tau(g)$ for $f, g \in \mathrm{Hom}_{con}(I, k)$, we assume that $f \neq 0, g \neq 0$ and $f + g \neq 0$. (Otherwise, the equality is proved easily.) Suppose f and g are linearly dependent over k, hence $f = cg$ for some $c\ (\neq 0, -1) \in k$. In this case, we have $I_f = I_g = I_{f+g}$. Since $(f + g)(\epsilon_g) = c + 1$, we see $\epsilon_{f+g} = (c + 1)^{-1}\epsilon_g$. Therefore, $\tau(f + g) = [A_{f+g}, \epsilon_{f+g}] = [A_g, (c + 1)^{-1}\epsilon_g] = (c + 1) \cdot [A_g, \epsilon_g] = c \cdot [A_g, \epsilon_g] + [A_g, \epsilon_g] = c \cdot \tau(g) + \tau(g) = \tau(f) + \tau(g)$.

Now suppose f and g are linearly independent over k. In this case $I_f \neq I_g$, and hence $I_f + I_g = I$. It then follows from the obvious exact sequence

$$0 \longrightarrow T/I_f \cap I_g \xrightarrow{\ \phi\ } A_f \times A_g \longrightarrow A \longrightarrow 0$$

that we can take $e_f, e_g \in I$ whose images in $T/I_f \cap I_g$ are mapped respectively to $(\epsilon_f, 0), (0, \epsilon_g)$ by ϕ. Note that $I_f \cap I_g \subset I_{f+g}$. And note also that $f(e_f) =$

1, $f(e_g) = 0$, $g(e_f) = 0$ and $g(e_g) = 1$, hence $(f+g)(e_f - e_g) = 0$ and $(f+g)(e_f) = 1$. It is then easy to see that $I_{f+g} = (I_f \cap I_g) + (e_f - e_g)$. Since $A_f \times_A A_g \cong T/I_f \cap I_g$, we have from the definition that $\tau(f) + \tau(g) = [A_f, \epsilon_f] + [A_g, \epsilon_g] = [T/I_{f+g}, e_f] = \tau(f+g)$.

Now we have proved that τ is k-linear. Assume $f \neq 0$. Then, since $\epsilon_f \in I/I_f$ and $I \subseteq \mathfrak{m}_T^2$, we have $\epsilon_f \in \mathfrak{m}_{A_f}^2$. This implies $\tau(f) \neq 0$ by Lemma 2.6. Thus τ is injective. The surjectivity of τ is obvious from the definition of τ and $\mathcal{T}(A)$. □

2.3. Complete tensor products

In this section, let R be an associative algebra over a field k.

Definition 2.8. For a complete local k-algebra $A \in \widehat{\mathcal{A}}_k$, we define the complete tensor product $R \mathbin{\widehat{\otimes}}_k A$ as follows :

$$R \mathbin{\widehat{\otimes}}_k A = \varprojlim R \otimes_k A/\mathfrak{m}_A^n.$$

Note that $R \mathbin{\widehat{\otimes}}_k A$ is an associative k-algebra, since each mapping $R \otimes_k A/\mathfrak{m}_A^{n+1} \to R \otimes_k A/\mathfrak{m}_A^n$ is a k-algebra homomorphism for $n \geq 1$. Also note that, if $A \in \mathcal{A}_k$, then $R \mathbin{\widehat{\otimes}}_k A = R \otimes_k A$ is an ordinary tensor product of k-algebras.

Remark 6. In general, $R \otimes_k A$ is a subalgebra of $R \mathbin{\widehat{\otimes}}_k A$. However, they are distinct in general.

For example, let $R = k[x]$ and $T = k\langle\langle t \rangle\rangle = k[[t]]$ (with one variable). Then, we have $R \otimes_k T = k[[t]]\ [x] \subset R \mathbin{\widehat{\otimes}}_k T = k[x]\ [[t]]$, which are actually distinct.

Definition 2.9. Let M be a left R-module and let X be a right (resp. left) A-module, where $A \in \widehat{\mathcal{A}}_k$. Then note that, for each $n \geq 1$, $M \otimes_k X/X\mathfrak{m}_A^n$ (resp. $M \otimes_k X/\mathfrak{m}_A^n X$) is a left $R \otimes_k (A/\mathfrak{m}_A^n)^{op}$-module (resp. a left $R \otimes_k (A/\mathfrak{m}_A^n)$-module), i.e. a left module over R and a right (resp. left) module over $A/\mathfrak{m}_A^n$. We define the complete tensor product by

$$M \mathbin{\widehat{\otimes}}_k X = \varprojlim M \otimes_k X/X\mathfrak{m}_A^n \quad (\text{resp. } M \mathbin{\widehat{\otimes}}_k X = \varprojlim M \otimes_k X/\mathfrak{m}_A^n X),$$

which is a left $R \mathbin{\widehat{\otimes}}_k A^{op}$-module (resp. a left $R \mathbin{\widehat{\otimes}}_k A$-module) by the reason above.

We always consider $M \otimes_k X$ and $M \mathbin{\widehat{\otimes}}_k X$ with $\mathfrak{m}_A$-adic topology. In general, there is a natural mapping $M \otimes_k A \to M \mathbin{\widehat{\otimes}}_k A$, which is the completion map in $\mathfrak{m}_A$-adic topology.

Remark 7. (a) If M is of finite dimension as a k-vector space with a k-basis $\{e_1, \dots, e_\ell\}$, then we have

$$M \mathbin{\widehat{\otimes}}_k A = \varprojlim \left(\bigoplus_{i=1}^{\ell} e_i k \otimes_k A/\mathfrak{m}_A^n \right) = \bigoplus_{i=1}^{n} e_i \left(\varprojlim A/\mathfrak{m}_A^n \right) = \bigoplus_{i=1}^{n} e_i A$$

for any $A \in \widehat{\mathcal{A}}_k$. Thus $M \mathbin{\widehat{\otimes}_k} A$ is a free module as a right A-module if $\dim_k M < \infty$.

(b) Suppose M is of infinite dimension as a k-vector space with basis $\{e_\lambda \mid \lambda \in \Lambda\}$. In this case, we have

$$M \mathbin{\widehat{\otimes}_k} A = \varprojlim \left[\bigoplus_{\lambda} e_\lambda (A/\mathfrak{m}_A^n) \right].$$

Therefore, an element of $M \mathbin{\widehat{\otimes}_k} A$ is described to be a formal sum $\sum_\lambda e_\lambda x_\lambda$ ($x_\lambda \in A$) as an element of $\prod_\lambda e_\lambda A$. Note that $\sum_\lambda e_\lambda x_\lambda \in \prod_\lambda e_\lambda A$ belongs to $M \mathbin{\widehat{\otimes}_k} A$ if and only if

$$\sharp\{\lambda \in \Lambda \mid x_\lambda \notin \mathfrak{m}_A^n\} < \infty$$

for all $n \geq 1$.

Lemma 2.11. *Let $A \in \widehat{\mathcal{A}}_k$.*

(*a*) *Let X be a right A-module, and let*

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

be a short exact sequence of left R-modules. Then the complete tensor product by X induces the exact sequence of left $R \mathbin{\widehat{\otimes}_k} A^{op}$-modules

$$0 \longrightarrow L \mathbin{\widehat{\otimes}_k} X \longrightarrow M \mathbin{\widehat{\otimes}_k} X \longrightarrow N \mathbin{\widehat{\otimes}_k} X \longrightarrow 0.$$

(*b*) *Let M be a left R-module, and let*

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0,$$

be a short exact sequence of left A-modules. Then we have an exact sequence of left $R \mathbin{\widehat{\otimes}_k} A$-modules

$$0 \longrightarrow \varprojlim (M \otimes_k (X/X \cap \mathfrak{m}_A^n Y)) \longrightarrow M \mathbin{\widehat{\otimes}_k} Y \longrightarrow M \mathbin{\widehat{\otimes}_k} Z \longrightarrow 0.$$

In particular, if the relative topology on X induced from the $\mathfrak{m}_A$-adic topology on Y is equivalent to the $\mathfrak{m}_A$-adic topology on X, then we have an exact sequence of $R \mathbin{\widehat{\otimes}_k} A$-modules

$$0 \longrightarrow M \mathbin{\widehat{\otimes}_k} X \longrightarrow M \mathbin{\widehat{\otimes}_k} Y \longrightarrow M \mathbin{\widehat{\otimes}_k} Z \longrightarrow 0.$$

Proof. The proof is similar to the commutative complete case in [1], [4] or [5]. □

Let M be a left R-module and let X be a left A-module where $A \in \widehat{\mathcal{A}}_k$. Then, from the definition of complete tensor products, we see that there is a natural mapping

$$\gamma_{M,X} \ : \ (M \mathbin{\widehat{\otimes}_k} A) \otimes_A X \to M \mathbin{\widehat{\otimes}_k} X.$$

In fact, $\gamma_{M,X}$ is induced from the natural mappings

$$(M \mathbin{\widehat{\otimes}_k} A) \otimes_A X \to (M \otimes_k A/\mathfrak{m}_A^n) \otimes_A X = M \otimes_k X/\mathfrak{m}_A^n X.$$

Lemma 2.12. *Under the circumstances above, suppose that the left A-module X is finitely generated. Then, $\gamma_{M,X} : (M \mathbin{\widehat{\otimes}_k} A) \otimes_A X \to M \mathbin{\widehat{\otimes}_k} X$ is surjective for any left R-module M.*

Proof. By the assumption, there is a surjective homomorphism of left A-modules $f : F = \oplus_{i=1}^{\ell} Ae_i \to X$, where F is a free left A-module of finite rank ℓ. Remark that $\gamma_{M,F}$ is an isomorphism. Naturally we have a commutative diagram

$$\begin{array}{ccc} (M \mathbin{\widehat{\otimes}_k} A) \otimes_A F & \xrightarrow{1\otimes f} & (M \mathbin{\widehat{\otimes}_k} A) \otimes_A X \\ {\scriptstyle \gamma_{M,F}}\downarrow & & {\scriptstyle \gamma_{M,X}}\downarrow \\ M \mathbin{\widehat{\otimes}_k} F & \xrightarrow{1\widehat{\otimes} f} & M \mathbin{\widehat{\otimes}_k} X. \end{array}$$

Since the horizontal mappings in the diagram are surjective (see Lemma 2.11), and since $\gamma_{M,F}$ is an isomorphism, we see that $\gamma_{M,X}$ is surjective. $\square$

Proposition 2.5. *Let $T = k\langle\langle t_1, t_2, \ldots, t_r\rangle\rangle$ be a non-commutative formal power series ring, and let M be an arbitrary left R-module. Then $M \mathbin{\widehat{\otimes}_k} T$ is flat as a right T-module.*

Proof. To prove the flatness, it is enough to show the following.
(*) For any finitely generated left ideal $\mathfrak{a}$, the mapping $1 \otimes j : (M \mathbin{\widehat{\otimes}_k} T) \otimes_T \mathfrak{a} \to (M \mathbin{\widehat{\otimes}_k} T) \otimes_T T$ induced from the inclusion $j : \mathfrak{a} \to T$ is injective.

Note that there is a commutative diagram

$$\begin{array}{ccc} (M \mathbin{\widehat{\otimes}_k} T) \otimes_T \mathfrak{a} & \xrightarrow{1\otimes j} & (M \mathbin{\widehat{\otimes}_k} T) \otimes_T T \\ {\scriptstyle \gamma_{M,\mathfrak{a}}}\downarrow & & {\scriptstyle \gamma_{M,T}}\downarrow \\ M \mathbin{\widehat{\otimes}_k} \mathfrak{a} & \xrightarrow{1\widehat{\otimes} j} & M \mathbin{\widehat{\otimes}_k} T, \end{array}$$

where we should note that $\gamma_{M,\mathfrak{a}}$ is an isomorphism, since $\mathfrak{a}$ is a free module of finite rank by Lemma 2.3. Thus, to prove the proposition, it is sufficient to show that $1\widehat{\otimes} j : M \mathbin{\widehat{\otimes}_k} \mathfrak{a} \to M \mathbin{\widehat{\otimes}_k} T$ is injective. By virtue of Lemma 2.11, we only have to show that the relative topology on $\mathfrak{a}$ from T is equal to the $\mathfrak{m}_A$-adic topology. But this has been proved in Corollary 2.2. $\square$

Proposition 2.6. *Let $A \in \widehat{\mathcal{A}}_k$, and let M be an arbitrary left R-module. Suppose A is of the form $A = T/I$ where $T = k\langle\langle t_1, t_2, \ldots, t_r\rangle\rangle$ is a non-commutative formal power series ring and I is an ideal of T that is finitely generated as a left ideal. Then $M \mathbin{\widehat{\otimes}_k} A$ is flat as a right A-module.*

Proof. From the short exact sequence of left T-modules

$$0 \longrightarrow I \longrightarrow T \longrightarrow A \longrightarrow 0,$$

we have the commutative diagram

$$\begin{array}{ccccccccc} & & (M \mathbin{\widehat{\otimes}_k} T) \otimes_T I & \longrightarrow & (M \mathbin{\widehat{\otimes}_k} T) \otimes_T T & \longrightarrow & (M \mathbin{\widehat{\otimes}_k} T) \otimes_T A & \longrightarrow & 0, \\ & & {\scriptstyle \gamma_{M,I}}\downarrow & & {\scriptstyle \gamma_{M,T}}\downarrow & & {\scriptstyle \gamma_{M,A}}\downarrow & & \\ 0 & \longrightarrow & M \mathbin{\widehat{\otimes}_k} I & \longrightarrow & M \mathbin{\widehat{\otimes}_k} T & \longrightarrow & M \mathbin{\widehat{\otimes}_k} A & \longrightarrow & 0, \end{array}$$

where the both rows are exact sequences by Corollary 2.2 and Lemma 2.11. We already know that $\gamma_{M,I}$ and $\gamma_{M,T}$ are isomorphisms, since I is a free module of finite rank by Lemma 2.3. It follows that $\gamma_{M,A} : (M \widehat{\otimes}_k T) \otimes_T A \to M \widehat{\otimes}_k A$ is bijective, which is actually an isomorphism of left $R \widehat{\otimes}_k A^{op}$-modules. Since $(M \widehat{\otimes}_k T) \otimes_T A$ is flat as a right A-module by Proposition 2.5, we can conclude that $M \widehat{\otimes}_k A$ is also flat over A. □

Note, in general, $M \widehat{\otimes}_k A$ is not flat as a right A-module.

Example 2.2. Let $T = k\langle\langle x, y\rangle\rangle$ be the non-commutative formal power series ring of two variables and let us consider the closed ideals in T,

$$I = \overline{(xy^n x \mid n = 0, 1, 2, \dots)} \quad \subseteq \quad J = \overline{(x)}.$$

Note that an element of I (resp. J) is a formal infinite sum $\sum_\lambda c_\lambda m_\lambda$ with $c_\lambda \in k$ and with monomials m_λ involving x at least twice (resp. once).

Consider the mapping $\varphi : T/J \to T/I$ defined by right multiplication by x, i.e. $\varphi(a \mod J) = ax \mod I$. Note that φ is a well-defined homomorphism of left T-modules, and it is injective.

Now let A be the residue ring T/I and consider φ to be an injective homomorphism of left A-modules. Let $M = \bigoplus_{i=1}^{\infty} e_i k$ be a k-vector space of countably infinite dimension. Then we can show that the mapping

$$(M \widehat{\otimes}_k A) \otimes \varphi \ : \ (M \widehat{\otimes}_k A) \otimes_A T/J \longrightarrow (M \widehat{\otimes}_k A) \otimes_A T/I = M \widehat{\otimes}_k A$$

is not injective. In fact, an element $z = \sum_{i=1}^{\infty} e_i \otimes y^i x y^i \in M \widehat{\otimes}_k A$ is mapped to $\varphi(z) = \sum_{i=1}^{\infty} e_i \otimes y^i x y^i x$ by φ, which is zero in $M \widehat{\otimes}_k A$. However, z never belongs to $(M \widehat{\otimes}_k A)J$, because any element of $(M \widehat{\otimes}_k A)J$ is a finite sum of the form $\sum_i z_i j_i$ ($z_i \in M \widehat{\otimes}_k A, j_i \in J$) and z is never of this form.

We can conclude from this observation that $M \widehat{\otimes}_k A$ is not flat as a right A-module.

3. Universal lifts of chain complexes

3.1. Lifts to artinian local algebras

In this section k is a field and R is an associative k-algebra.

By a graded left R-module F, we just mean a direct sum $F = \bigoplus_{i\in\mathbb{Z}} F_i$ where each F_i is a left R-module. If F is a graded left R-module and if j is an integer, then the shifted graded left R-module $F[j]$ is defined to be $F[j]_i = F_{i+j}$ for any $i \in \mathbb{Z}$. A graded homomorphism $f : F \to G$ of graded left R-module is an R-homomorphism with $f(F_i) \subseteq G_i$ for any $i \in \mathbb{Z}$. If $f : F \to G$ is a graded homomorphism, we denote by f_i the restriction of f on F_i for each i. We refer to a graded homomorphism $F \to G[j]$ as a graded homomorphism of degree j.

By a chain complex of left R-modules or simply a complex over R, we mean a pair $\mathbb{F} = (F, d)$ where F is a graded left R-module and d is a graded homomorphism of degree -1 such that $d^2 = 0$. A complex $\mathbb{F} = (F, d)$ over R is described as

$$\cdots \longrightarrow F_{i+1} \xrightarrow{d_{i+1}} F_i \xrightarrow{d_i} F_{i-1} \longrightarrow \cdots .$$

We say that a complex $\mathbb{F} = (F, d)$ is a projective complex over R if the underlying graded left R-module F is projective.

Let $\mathbb{F} = (F, d)$ and $\mathbb{G} = (G, d')$ be chain complexes over R. A chain homomorphism $f : \mathbb{F} \to \mathbb{G}$ of degree j is a graded homomorphism $f : F \to G[j]$ satisfying $f \cdot d + (-1)^{j+1} d' \cdot f = 0$. A chain isomorphism $f : \mathbb{F} \to \mathbb{G}$ of complexes is a chain homomorphism of degree 0 that is bijective. If there is a chain isomorphism between $\mathbb{F}$ and $\mathbb{G}$, then we say that they are isomorphic as chain complexes over R and we denote it by $\mathbb{F} \cong \mathbb{G}$.

Now let $\mathbb{F}$ and $\mathbb{G}$ be projective complexes over R, and let $f, g : \mathbb{F} \to \mathbb{G}$ be chain homomorphisms of degree j. We say that f and g are homotopically equivalent, denoted by $f \sim g$, if there is a graded homomorphism $h : F \to G[j+1]$ such that $f = g + (h \cdot d + (-1)^j d' \cdot h)$. We denote the set of all the homotopy equivalence classes of chain homomorphisms of degree j by $\mathrm{Ext}_R^{-j}(\mathbb{F}, \mathbb{G})$, which is clearly equipped with structure of k-vector space.

For graded homomorphisms $f : F \to F[j]$ and $g : F \to F[\ell]$, we define a graded homomorphism $[f, g] : F \to F[j + \ell]$ by

$$[f, g] = f \cdot g + (-1)^{j+\ell} g \cdot f.$$

Note that $f : F \to F[j]$ is a chain homomorphism if and only if $[d, f] = 0$. Also note that $f \sim 0$ if and only if there is a graded homomorphism $g : F \to F[j+1]$ with $f = [d, g]$.

Let $\varphi : R \to S$ be a k-algebra homomorphism and let $\mathbb{F} = (F, d)$ be a projective complex over R. In this case, we denote by S_φ (resp. ${}_\varphi S$) the left (resp. right) S-module S with right (resp. left) R-module structure through φ. Then the chain complex $S_\varphi \otimes_R \mathbb{F}$ (resp. $\mathbb{F} \otimes_R {}_\varphi S$) of projective left (resp. right) S-modules is defined to be $(S_\varphi \otimes_R F,\ S_\varphi \otimes_R d)$ (resp. $(F \otimes_R {}_\varphi S,\ d \otimes_R {}_\varphi S)$).

Recall that we denote by $\mathcal{A}_k$ the category of artinian local k-algebras A with $A/\mathfrak{m}_A \cong k$ and k-algebra homomorphisms. If F is a graded projective (resp. free) left R-module and if $A \in \mathcal{A}_k$, then $F \otimes_k A$ is a graded projective (resp. free) left $R \otimes_k A^{op}$-module.

Definition 3.1. Let $\mathbb{F} = (F, d)$ be a projective complex over R and let $A \in \mathcal{A}_k$. We say that a projective complex $(F \otimes_k A, \Delta)$ over $R \otimes_k A^{op}$ is **a lifting chain complex of $\mathbb{F}$ to** A (or simply a **lift of $\mathbb{F}$ to** A) if it satisfies the equality $(F \otimes_k A, \Delta) \otimes_A k = \mathbb{F}$.

To be more general, let $\varphi : A \to B$ be a morphism in $\mathcal{A}_k$. A projective complex $(F \otimes_k A, \Delta_A)$ over $R \otimes_k A^{op}$ is said to be a **lift** of a projective complex $(F \otimes_k B, \Delta_B)$ over $R \otimes_k B^{op}$ if it satisfies the equality $(F \otimes_k A, \Delta_A) \otimes_A ({}_\varphi B) = (F \otimes_k B, \Delta_B)$. And a projective complex $(F \otimes_k B, \Delta_B)$ over $R \otimes_k B^{op}$ is said to be **liftable** to A if there is a lift of $(F \otimes_k B, \Delta_B)$ to A.

The aim of this section is to construct a universal one among those lifts of a given projective complex $\mathbb{F} = (F, d)$ over R. For this, in the rest of this paper, $\mathbb{F} = (F, d)$ always denotes a fixed projective complex over R.

Lemma 3.1. *Let $A \in \mathcal{A}_k$. Then, since A is of finite dimension as a k-vector space, we may take a k-basis $\{1\} \cup \{x_1, \dots, x_r\} \cup \{y_j | \ 1 \le j \le s\}$ of A*

so that $\{x_1 \dots, x_r\}$ *yields a* k*-basis of* $\mathfrak{m}_A/\mathfrak{m}_A^2$ *and* $\{y_j |\ 1 \leq j \leq s\}$ *is a* k*-basis of* $\mathfrak{m}_A^2$.

(*a*) *For any graded homomorphism* $\Delta : F \otimes_k A \to F \otimes_k A[n]$ *of left* $R \otimes_k A^{op}$*-modules, there uniquely exist graded homomorphisms* $f, g_i, h_j : F \to F[n]$ $(1 \leq i \leq r,\ 1 \leq j \leq s)$ *of left* R*-modules such that*

$$\Delta = f \otimes 1 + \sum_{i=1}^{r} g_i \otimes x_i + \sum_{j=1}^{s} h_j \otimes y_j.$$

(*b*) *Let* $(F \otimes_k A, \Delta)$ *be a lift of* $\mathbb{F} = (F, d)$ *to* A*. Then* Δ *has a description as in* (*a*)*, where* $f = d$ *and each* $g_i : F \to F[-1]$ $(1 \leq i \leq r)$ *is a chain homomorphism.*

Proof. (a) For any $z \in F_\ell$, we can uniquely write $\Delta(z) = z_0 \otimes 1 + \sum_{i=1}^{r} z_i \otimes x_i + \sum_{j=1}^{s} w_j \otimes y_j$ for some $z_0, z_i, w_j \in F_{\ell+n}$ $(1 \leq i \leq r,\ 1 \leq j \leq s)$. Then, define $f, g_i, h_j : F \to F[n]$ by $f(z) = z_0,\ g_i(z) = z_i$ and $h_j(z) = w_j$ and it is easy to see that they are graded homomorphisms of left R-modules. Note that

$$\Delta(z \otimes a) = f(z) \otimes a + \sum_{i=1}^{r} g_i(z) \otimes x_i a + \sum_{j=1}^{s} h_j(z) \otimes y_j a,$$

for any $z \otimes a \in F \otimes_k A$.

(b) Since $d = \Delta \otimes_A k = \Delta \otimes_A A/\mathfrak{m}_A$, we have $f = d$. Similarly, we have $\Delta \otimes_A A/\mathfrak{m}_A^2 = d \otimes 1 + \sum_{i=1}^{r} g_i \otimes x_i$ as a graded homomorphism $F \otimes_k A/\mathfrak{m}_A^2 \to F \otimes_k A/\mathfrak{m}_A^2[-1]$. Since $(\Delta \otimes_A A/\mathfrak{m}_A^2)^2 = 0$ and since $(d \otimes 1)^2 = 0$, it follows that $\sum_{i=1}^{r} dg_i \otimes x_i + g_i d \otimes x_i = \sum_{i=1}^{r} [d, g_i] \otimes x_i = 0$ as a graded homomorphism on $F \otimes_k A/\mathfrak{m}_A^2$. Hence we have $[d, g_i] = 0$ for all i. □

Corollary 3.1. *Let* $A \in \mathcal{A}_k$ *and suppose* $\mathfrak{m}_A^2 = 0$*. Then, for any lift* $(F \otimes_k A, \Delta)$ *of* $\mathbb{F} = (F, d)$ *to* A*, the differentiation* Δ *is given by*

$$\Delta = d \otimes 1 + \sum_{i=1}^{r} g_i \otimes x_i,$$

where $\{x_1, \dots, x_r\}$ *is a* k*-basis of* $\mathfrak{m}_A$ *and each* $g_i : F \to F[-1]$ *is a chain homomorphism* $(1 \leq i \leq r)$.

Lemma 3.2. *Let* $\varphi : A \to B$ *be a surjective morphism in* $\mathcal{A}_k$ *and let* $(F \otimes_k B, \Delta)$ *be a lifting chain complex of* $\mathbb{F} = (F, d)$ *to* B.

(*a*) *Any graded homomorphism* $\alpha : F \otimes_k B \to F \otimes_k B$ *of graded* $R \otimes_k B^{op}$*-modules is liftable to a graded homomorphism* $F \otimes_k A \to F \otimes_k A$ *of graded* $R \otimes_k A^{op}$*-modules. That is, there is a graded homomorphism* $\beta : F \otimes_k A \to F \otimes_k A$ *with* $\beta \otimes_A {}_{\varphi}B = \alpha$.

(*b*) *If* α *is an isomorphism in* (*a*)*, then* β *is also an isomorphism.*

Proof. (a) Since $F \otimes_k A$ is a left projective $R \otimes_k A^{op}$-module, and since $1 \otimes \varphi : F \otimes_k A \to F \otimes_k B$ is a surjective homomorphism, one can find a left

$R \otimes_k A^{op}$-homomorphism β which makes the following diagram commutative.

$$\begin{array}{ccc} F \otimes_k A & \xrightarrow{\beta} & F \otimes_k A \\ {\scriptstyle 1\otimes\varphi}\downarrow & & {\scriptstyle 1\otimes\varphi}\downarrow \\ F \otimes_k B & \xrightarrow{\alpha} & F \otimes_k B \end{array}$$

(b) To prove that β is an isomorphism, we may assume that $\varphi : A \to B$ is a small extension, because any surjective morphism in $\mathcal{A}_k$ is a composition of a finite sequence of small extensions. So we may have a short exact sequence

$$0 \longrightarrow k \xrightarrow{\epsilon} A \longrightarrow B \longrightarrow 0.$$

Hence, we have a commutative diagram of left $R \otimes_k A^{op}$-modules

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F & \xrightarrow{\epsilon} & F \otimes_k A & \longrightarrow & F \otimes_k B & \longrightarrow & 0 \\ & & {\scriptstyle \alpha\otimes k}\downarrow & & {\scriptstyle \beta}\downarrow & & {\scriptstyle \alpha}\downarrow & & \\ 0 & \longrightarrow & F & \xrightarrow{\epsilon} & F \otimes_k A & \longrightarrow & F \otimes_k B & \longrightarrow & 0. \end{array}$$

Since α is an isomorphism, it is clear that so is β. $\square$

Corollary 3.2. *Let $\varphi : A \to B$ be a surjective morphism in $\mathcal{A}_k$ as in the lemma. Suppose we have two chain complexes $(F \otimes_k B, \Delta_1)$ and $(F \otimes_k B, \Delta_2)$ which are lifts of $\mathbb{F}$ to B and are isomorphic to each other as chain complexes over $R \otimes_k B^{op}$. If $(F \otimes_k B, \Delta_1)$ is liftable to A, then so is $(F \otimes_k B, \Delta_2)$.*

Proof. By the assumption, there is a graded isomorphism $\alpha : F \otimes_k B \to F \otimes_k B$ such that $\Delta_2 = \alpha\Delta_1\alpha^{-1}$. Let $(F \otimes_k A, \Delta_1')$ be a lift of $(F \otimes_k B, \Delta_1)$ to A. By Lemma 3.2, α is lifted to an isomorphism $\beta : F \otimes_k A \to F \otimes_k A$. Then it is easy to see that $(F \otimes_k A, \beta\Delta_1'\beta^{-1})$ is a lift of $(F \otimes_k B, \Delta_2)$ to A. $\square$

Lemma 3.3. *Let (A', ϵ) be a small extension of $A \in \mathcal{A}_k$, and let $(F \otimes_k A, \Delta)$ be a lift of $\mathbb{F}$ to A. Suppose that chain complexes $(F \otimes_k A', \Delta_1)$ and $(F \otimes_k A', \Delta_2)$ are lifts of $(F \otimes_k A, \Delta)$ to A'.*

(*a*) *Then there is a chain homomorphism $h : F \to F[-1]$ such that $\Delta_2 = \Delta_1 + h \otimes \epsilon$.*

(*b*) *The following two conditions are equivalent.*

1. *The equivalence class $[h] \in \mathrm{Ext}^1_R(\mathbb{F}, \mathbb{F})$ is zero.*
2. *There is an isomorphism $\varphi : (F \otimes_k A', \Delta_1) \to (F \otimes_k A', \Delta_2)$ of chain complexes over $R \otimes_k A'^{op}$ such that $\varphi \otimes_{A'} A$ is the identity mapping on $F \otimes_k A$.*

Proof.

(a) We can take a k-basis of $\mathfrak{m}_{A'}$ containing ϵ as a member. Then, both Δ_1 and Δ_2 have the descriptions as in Lemma 3.1. Since $\Delta_1 \otimes_{A'} A = \Delta = \Delta_2 \otimes_{A'} A$, the difference $\Delta_2 - \Delta_1$ has a description $h \otimes \epsilon$. We have to show that h is a chain map. Since $\Delta_1^2 = \Delta_2^2 = 0$ and $\epsilon^2 = 0$, we have

$$0 = \Delta_2^2 = (\Delta_1 + h \otimes \epsilon)^2 = \Delta_1 \cdot (h \otimes \epsilon) + (h \otimes \epsilon) \cdot \Delta_1.$$

Note that $\Delta_1 \cdot (1 \otimes \epsilon) = d \otimes \epsilon = (1 \otimes \epsilon) \cdot \Delta_1$, since Δ_1 has a description

$$\Delta_1 = d \otimes 1 + \sum_i g_i \otimes x_i + \sum_j h_j \otimes y_j,$$

as in Lemma 3.1 and $\epsilon x_i = x_i \epsilon = 0$ so on. Therefore, we have $(dh + hd) \otimes \epsilon = 0$, hence $[d, h] = 0$.

(b) $[(1) \Rightarrow (2)]$: If $[h] = 0$, then there is a graded homomorphism $g : F \to F$ of degree 0 such that $h = [d, g]$. Define a mapping $\varphi : F \otimes_k A' \to F \otimes_k A'$ by $\varphi = 1 \otimes 1 + g \otimes \epsilon$, which maps $x \otimes a \in F \otimes_k A'$ to $x \otimes a + g(x) \otimes \epsilon a$. Then it is easy to see that φ is an automorphism of a graded left $R \otimes_k A'^{op}$-module, and the inverse is given by $\varphi^{-1} = 1 \otimes 1 - g \otimes \epsilon$. Then, we have the following equalities.

$$\begin{aligned}
\varphi^{-1}\Delta_1\varphi &= (1 \otimes 1 - g \otimes \epsilon)\Delta_1(1 \otimes 1 + g \otimes \epsilon) \\
&= \Delta_1 + \Delta_1(g \otimes \epsilon) - (g \otimes \epsilon)\Delta_1 \\
&= \Delta_1 + (dg \otimes \epsilon - gd \otimes \epsilon) \\
&= \Delta_1 + [d, g] \otimes \epsilon = \Delta_2
\end{aligned}$$

Therefore, φ satisfies the conditions in (2).

$[(2) \Rightarrow (1)]$: By Lemma 3.1, we have a description $\varphi = 1 \otimes 1 + g \otimes \epsilon$ and $\varphi^{-1} = 1 \otimes 1 - g \otimes \epsilon$ for some graded homomorphism $g : F \to F$ of degree 0. Hence, by the same computation as above, we have

$$\Delta_2 = \varphi^{-1}\Delta_1\varphi = \Delta_1 + [d, g] \otimes \epsilon.$$

Therefore, $h = [d, g] \sim 0$. □

Proposition 3.1. *Let*

$$\begin{tikzcd}
B \arrow[r] \arrow[d] & A_1 \arrow[d, "a_1"'] \\
A_2 \arrow[r, "a_2"] & A
\end{tikzcd}$$

be a diagram of a fiber product in $\mathcal{A}_k$ with a_2 being a surjective map.

(*a*) *Let $\varphi_1 : F \otimes_k A_1 \to F \otimes_k A_1[j]$ and $\varphi_2 : F \otimes_k A_2 \to F \otimes_k A_2[j]$ be graded homomorphisms of degree j such that $\varphi \otimes_{A_1} A = \varphi_2 \otimes_{A_2} A (= \varphi)$. Then there is a graded homomorphism $\Phi : F \otimes B \to F \otimes_k B[j]$ of degree j with $\Phi \otimes_B A_i = \varphi_i$ for $i = 1, 2$.*

(*b*) *Let $(F \otimes_k A_1, \Delta_1)$ and $(F \otimes_k A_2, \Delta_2)$ be lifts of a chain complex $(F \otimes_k A, \Delta)$. Then there is a chain complex $(F \otimes_k B, \Delta_B)$ which is a lift of both of $(F \otimes_k A_1, \Delta_1)$ and $(F \otimes_k A_2, \Delta_2)$.*

(*c*) *Let $(F \otimes_k A_1, \Delta_1)$ and $(F \otimes_k A_2, \Delta_2)$ be chain complexes such that there is an isomorphism*

$$(F \otimes_k A_1, \Delta_1) \otimes_{A_1} A \cong (F \otimes_k A_2, \Delta_2) \otimes_{A_2} A$$

of chain comlexes over $R \otimes_k A^{op}$. Then there is a chain complex $(F \otimes_k B, \Delta_B)$ which satisfies $(F \otimes_k B, \Delta_B) \otimes_B A_i \cong (F \otimes_k A_i, \Delta_i)$ for $i = 1, 2$.

Proof. (a) Since there is commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F \otimes_k B & \longrightarrow & (F \otimes_k A_1) \times (F \otimes_k A_2) & \longrightarrow & F \otimes_k A & \longrightarrow & k \\ & & & & \Big\downarrow{\scriptstyle \varphi_1 \times \varphi_2} & & \Big\downarrow{\scriptstyle \varphi} & & \\ 0 & \longrightarrow & F \otimes_k B[j] & \longrightarrow & (F \otimes_k A_1)[j] \times (F \otimes_k A_2)[j] & \longrightarrow & F \otimes_k A[j] & \longrightarrow & k, \end{array}$$

it induces a mapping $\Phi : F \otimes_k B \to F \otimes_k B[j]$.

(b) Just apply (a) to Δ_1 and Δ_2, and we get a graded homomorphism $\Delta_B : F \otimes_k B \to F \otimes_k B[-1]$. It is clear that $\Delta_B^2 = 0$.

(c) By definition of isomorphisms of chain complexes, there is a graded isomorphism $\alpha : F \otimes_k A \to F \otimes_k A$ of graded $R \otimes_k A^{op}$-modules, such that $\Delta_1 \otimes_{A_1} A = \alpha \cdot (\Delta_2 \otimes_{A_2} A) \cdot \alpha^{-1}$. Since a_2 is surjective, there is a graded isomorphism $\beta : F \otimes_k A_2 \to F \otimes_k A_2$ which lifts α, by Lemma 3.2. Put $\Delta_2' = \beta \cdot \Delta_2 \cdot \beta^{-1}$ and apply (b) to the chain complexes $(F \otimes_k A_1, \Delta_1)$ and $(F \otimes_k A_2, \Delta_2')$, and we obtain a chain complex $(F \otimes_k B, \Delta_B)$ which is a lift of the both of them. $\square$

3.2. Construction of maximal lifts

As in the previous section, let R be an associative algebra over a field k and let $\mathbb{F} = (F, d)$ be a projective complex over R. In the rest of the paper we always assume that

$$r = \dim_k \operatorname{Ext}^1_R(\mathbb{F}, \mathbb{F}) < \infty. \tag{3.1}$$

Under this assumption, we take chain homomorphism $t_i^* : F \to F[-1]$ $(1 \leq i \leq r)$ whose equivalence classes $\{[t_1^*], \ldots, [t_r^*]\}$ is a k-basis of $\operatorname{Ext}^1_R(\mathbb{F}, \mathbb{F})$. We take variables $t_1, \ldots, t_r$ corresponding to this basis, and consider the non-commutative formal power series ring $T = k\langle\langle t_1, t_2, \ldots, t_r \rangle\rangle$. Now define $\delta : F \otimes_k T/\mathfrak{m}_T^2 \to F \otimes_k T/\mathfrak{m}_T^2$ by

$$\delta = d \otimes 1 + \sum_{i=1}^{r} t_i^* \otimes t_i. \tag{3.2}$$

It follows from Corollary 3.1 that $(F \otimes_k T/\mathfrak{m}_T^2, \delta)$ is a lift of $\mathbb{F}$ to $T/\mathfrak{m}_T^2$.

Definition 3.2. Let I be a closed ideal of T. We define the complete tensor product of a graded projective left R-module with T/I as follows:

$$F \mathbin{\widehat{\otimes}_k} T/I := \textstyle\bigoplus_{i \in \mathbb{Z}} (F_i \mathbin{\widehat{\otimes}_k} T/I).$$

Now let I be a closed ideal of T and let $(F \mathbin{\widehat{\otimes}_k} T/I,\ \Delta)$ be a chain complex. If $I' \supseteq I$ be another closed ideal of T, then there is a natural projection $T/I \to T/I'$, which induces, by Lemma 2.11, a surjective homomorphism

$$F \mathbin{\widehat{\otimes}_k} T/I \to F \mathbin{\widehat{\otimes}_k} T/I'.$$

Thus we have a surjective homomorphism of left $R \mathbin{\widehat{\otimes}_k} (T/I)^{op}$-modules

$$\pi : (F \mathbin{\widehat{\otimes}_k} T/I) \otimes_{T/I} T/I' \to F \mathbin{\widehat{\otimes}_k} T/I'.$$

Note that π may not be an isomorphism.

For each $n \geq 1$, Δ induces a graded homomorphism $\Delta_n : F \otimes_k (T/I + \mathfrak{m}_T^n) \to F \otimes_k (T/I + \mathfrak{m}_T^n)[-1]$ and it holds that $\Delta = \varprojlim \Delta_n$. Since $I \subseteq I'$, each Δ_n induces a graded homomorphism $\Delta'_n : F \otimes_k T/(I' + \mathfrak{m}_T^n) \to F \otimes_k (T/I' + \mathfrak{m}_T^n)[-1]$, and we obtain a graded homomorphism $\Delta' = \varprojlim \Delta'_n : F \mathbin{\widehat{\otimes}}_k T/I' \to F \mathbin{\widehat{\otimes}}_k T/I'[-1]$. By an abuse of notation, we denote this mapping Δ' by $\Delta \otimes_{T/I} T/I'$.

Definition 3.3. Let $I_1 \subseteq I_2 \subseteq \mathfrak{m}_T^2$ be closed ideals of T. A chain complex $(F \mathbin{\widehat{\otimes}}_k T/I_1, \Delta_1)$ is called a lift of a chain complex $(F \mathbin{\widehat{\otimes}}_k T/I_2, \Delta_2)$ if $\Delta_2 = \Delta_1 \otimes_{T/I_1} T/I_2$.

Definition 3.4. Let $T/I \in \widehat{\mathcal{A}}_k$ where $T = k\langle\langle t_1, t_2, \dots, t_r\rangle\rangle$ is a non-commutative formal power series ring and $I \subseteq \mathfrak{m}_T^2$. And let $(F \mathbin{\widehat{\otimes}}_k T/I, \Delta)$ be any chain complex which is a lift of $\mathbb{F}$. Now we consider the following set of lifting chain complexes of $(F \mathbin{\widehat{\otimes}}_k T/I,\ \Delta)$:

$$\begin{aligned}\mathcal{I}(I,\Delta) = \{(T/I',\ \Delta') \mid\ & I' \text{ is a closed ideal of } T \text{ with } \ I' \subseteq I \text{ and} \\ & (F \mathbin{\widehat{\otimes}}_k T/I',\ \Delta') \text{ is a lifting chain complex of } \ (F \mathbin{\widehat{\otimes}}_k T/I,\ \Delta)\}\end{aligned}$$

We define an order relation on the set $\mathcal{I}(I,\Delta)$ as follows:

$$(T/I_1,\ \Delta_1) > (T/I_2,\ \Delta_2) \iff I_1 \subseteq I_2 \ \text{ and } \ \ \Delta_1 \otimes_{T/I_1} T/I_2 = \Delta_2$$

Lemma 3.4. *The ordered set $\mathcal{I}(I,\Delta)$ is an inductively ordered set. In particular, there exists a maximal element in $\mathcal{I}(I,\Delta)$.*

Definition 3.5. If $(T/I_0,\ \Delta_0)$ is a maximal element in $\mathcal{I}(I,\Delta)$ as in the lemma, then we say that the chain complex $(F \mathbin{\widehat{\otimes}}_k (T/I_0), \Delta_0)$ is **a maximal lift of** $(F \mathbin{\widehat{\otimes}}_k T/I,\ \Delta)$.

Proof. Let $\{(T/I_\lambda, \Delta_\lambda) \mid \lambda \in \Lambda\}$ be a totally ordered subset of $\mathcal{I}(I,\Delta)$. Note that $J = \bigcap_{\lambda\in\Lambda} I_\lambda$ is a closed ideal of T and that $\varprojlim T/I_\lambda = T/J$. Hence $(F \mathbin{\widehat{\otimes}}_k T/J, \Delta_J) = \varprojlim_\lambda (F \mathbin{\widehat{\otimes}}_k T/I_\lambda, \Delta_\lambda)$ is a lifting chain complex of $(F \mathbin{\widehat{\otimes}}_k T/I, \Delta)$. Therefore, $(T/J, \Delta_J) \in \mathcal{I}(I,\Delta)$ and $(T/J, \Delta_J) > (T/I_\lambda, \Delta_\lambda)$ for any $\lambda \in \Lambda$. Thus $\mathcal{I}(I,\Delta)$ is an inductively ordered set. The existence of maximal element of $\mathcal{I}(I,\Delta)$ follows from Zorn's lemma. $\square$

We should remark the following

Lemma 3.5. *If $(T/I_1,\ \Delta_1)$ is not a maximal element in $\mathcal{I}(I,\Delta)$, then there is a nontrivial small extension T/I_2 of T/I_1 such that $(T/I_2, \Delta_2) \in \mathcal{I}(I,\Delta)$ is strictly bigger than $(T/I_1,\ \Delta_1)$, for some Δ_2.*

Proof. Take a strictly bigger element $(T/I'_1,\ \Delta'_1) > (T/I_1,\ \Delta_1)$ in $\mathcal{I}(I,\Delta)$. Since $I'_1 \subsetneq I_1$ are closed ideals, there is an integer n with $I'_1 + (\mathfrak{m}_T^n \cap I) \neq I_1$. In fact, if not, we will have $I_1 \subseteq I'_1 + \mathfrak{m}_T^n$ for any n, because the right hand side is the closed ideal containing $I_1' + (\mathfrak{m}_T^n \cap I)$. Then we shall have $I_1 \subseteq \overline{I_1'} = I_1'$, a contradiction.

Now, since $A = T/\overline{I_1' + (\mathfrak{m}_T^n \cap I)}$ is a complete local k-algebra and since the image J_1 of I_1 in A is an ideal of A of finite length, we can find a closed ideal J_2 of A contained in J_1 with $\mathrm{length}(J_1/J_2) = 1$. See Proposition 2.2. Taking the inverse image of J_2 in T, we have a closed ideal I_2 of T contained in I_1 and $\mathrm{length}(I_1/I_2) = 1$. Finally set $\Delta_2 = \Delta_1' \otimes_{T/I_1'} T/I_2$, and we easily see that $(T/I_2, \Delta_2)$ meets the requirements. □

The following is an easy consequence of Lemma 3.2.

Lemma 3.6. *Let $\varphi : A \to B$ be a surjective morphism in $\widehat{\mathcal{A}}_k$ where* $\mathrm{Ker}(\varphi)$ *is of finite length, and let $(F \widehat{\otimes}_k B, \Delta)$ be a lifting chain complex of $\mathbb{F}$.*

(*a*) *Any graded homomorphism $\alpha : F \widehat{\otimes}_k B \to F \widehat{\otimes}_k B$ is liftable to a graded homomorphism $F \widehat{\otimes}_k A \to F \widehat{\otimes}_k A$. That is, there is a graded homomorphism $\beta : F \widehat{\otimes}_k A \to F \widehat{\otimes}_k B$ with $\beta \otimes_{A} {}_{\varphi}B = \alpha$.*

(*b*) *If α is an isomorphism in* (*a*), *then β is also an isomorphism.*

Proof. (a) By induction on the length of $\mathrm{Ker}(\varphi)$, we may assume that $A \to B$ is a small extension. In this case, it is easily seen that the following diagram is a pull-back diagram of right A-modules for any integer n which satisfies $\mathrm{Ker}(\varphi) \cap \mathfrak{m}_A^n = (0)$.

$$\begin{array}{ccc} A/\mathfrak{m}_A^{n+1} & \xrightarrow{\varphi_{n+1}} & B/\mathfrak{m}_B^{n+1} \\ \downarrow & & \downarrow \\ A/\mathfrak{m}_A^n & \xrightarrow{\varphi_n} & B/\mathfrak{m}_B^n, \end{array}$$

where φ_n is the induced mapping by φ and the vertical arrows are natural projections. Thus the diagram

$$\begin{array}{ccc} F \otimes_k A/\mathfrak{m}_A^{n+1} & \xrightarrow{1\otimes\varphi_{n+1}} & F \otimes_k B/\mathfrak{m}_B^{n+1} \\ {\scriptstyle p_n}\downarrow & & {\scriptstyle q_n}\downarrow \\ F \otimes_k A/\mathfrak{m}_A^n & \xrightarrow{1\otimes\varphi_n} & F \otimes_k B/\mathfrak{m}_B^n, \end{array}$$

is a pull-back diagram of $R \widehat{\otimes}_k A^{op}$-modules. Denote by α_n the mapping $\alpha \otimes_k B/\mathfrak{m}_B^n : F \otimes_k B/\mathfrak{m}_B^n \to F \otimes_k B/\mathfrak{m}_B^n$. If we have an $R \otimes_k A/\mathfrak{m}_A^n$-homomorphism $\beta_n : F \otimes_k A/\mathfrak{m}_A^n \to F \otimes_k A/\mathfrak{m}_A^n$ with $(1 \otimes \varphi_n) \cdot \beta_n \cdot p_n = q_n \cdot \alpha_{n+1} \cdot (1 \otimes \varphi_{n+1})$, then it follows that there uniquely exists $\beta_{n+1} : F \otimes_k A/\mathfrak{m}_A^{n+1} \to F \otimes_k A/\mathfrak{m}_A^{n+1}$ such that $(1 \otimes \varphi_{n+1}) \cdot \beta_{n+1} = \alpha_{n+1} \cdot (1 \otimes \varphi_{n+1})$ and $p_n \cdot \beta_{n+1} = \beta_n \cdot p_n$. Therefore, by induction, we have such β_n for all $n \geq 1$. Then, setting $\beta = \varprojlim \beta_n$, we see that β is a lift of the mapping α.

(b) In the proof above, if α is an isomorphism, then each β_n is also an isomorphism by Lemma 3.2, hence so is β. □

Lemma 3.7. *Let* $(F \widehat{\otimes}_k T/I_0,\ \Delta_0)$ *be a maximal lift of* $(F \widehat{\otimes}_k T/I,\ \Delta)$. *Then, any chain complex* $(F \widehat{\otimes}_k T/I_0,\ \Delta_1)$ *which is isomorphic to* $(F \widehat{\otimes}_k T/I_0,\ \Delta_0)$ *as a complex over* $R \widehat{\otimes}_k (T/I_0)^{op}$ *is also a maximal lift of* $(F \widehat{\otimes}_k T/I,\ \Delta)$.

Proof. There is a graded isomorphism $\alpha : F \widehat{\otimes}_k T/I_0 \to F \widehat{\otimes}_k T/I_0$ with $\Delta_1 = \alpha\Delta_0\alpha^{-1}$. If $(F \widehat{\otimes}_k T/I_0,\ \Delta_1)$ is not a maximal lift, then by Lemma 3.5 there is a nontrivial small extension T/I_2 of T/I_0 such that $(T/I_2, \Delta_2) > (T/I_0, \Delta_1)$ in $\mathcal{I}(I, \Delta)$. We can lift the isomorphism α to $\beta : F \widehat{\otimes}_k T/I_2 \to F \widehat{\otimes}_k T/I_2$ by Lemma 3.6. Then it is easy to see that $(T/I_2, \beta^{-1}\Delta_2\beta) > (T/I_0, \Delta_0)$ in $\mathcal{I}(I, \Delta)$, and it contradicts to the assumption. $\square$

Lemma 3.8. *Let* $T = k\langle\langle t_1, t_2, \ldots, t_r\rangle\rangle$ *be a non-commutative formal power series ring.*

(*a*) *For any given* $f_i \in \mathfrak{m}_T^2$ $(1 \le i \le r)$, *we define a* k*-algebra homomorphism* $\varphi : T \to T$ *by* $\varphi(t_i) = t_i + f_i$ $(1 \le i \le r)$. *Then,* φ *is an automorphism of* T *such that it induces the identity mapping on* $T/\mathfrak{m}_T^2$.

(*b*) *Any* k*-algebra automorphism of* T *which induces the identity on* $T/\mathfrak{m}_T^2$ *is given as in* (*a*).

(*c*) *Let* $I_1 \subseteq I_2 \subseteq \mathfrak{m}_T^2$ *be closed ideals of* T *and let* $\psi : T/I_1 \to T/I_2$ *be any* k*-algebra homomorphism that induces the identity on* $T/\mathfrak{m}_T^2$. *Then there is a* k*-algebra automorphism* $\varphi : T \to T$ *with* $\varphi(I_1) \subseteq I_2$ *and the induced mapping* $\overline{\varphi} : T/I_1 \to T/I_2$ *equals* ψ.

Proof. (a) It is obvious that φ induces the identity on $\mathfrak{m}_T/\mathfrak{m}_T^2$. Hence it follows from Lemma 2.2 that $\varphi : T \to T$ is a surjective k-algebra homomorphism. In particular, every induced mapping $\varphi_n : T/\mathfrak{m}_T^n \to T/\mathfrak{m}_T^n$ is surjective as well. Comparing the lengths we conclude that each φ_n is bijective. Hence $\varphi = \varprojlim_n \varphi_n$ is an automorphism.

(b) Trivial.

(c) By the assumption, we can choose $f_i \in \mathfrak{m}_T^2$ $(1 \le i \le r)$ so that $\psi(t_i \pmod{I_1}) = t_i + f_i \pmod{I_2}$ $(1 \le i \le r)$. Now define an automorphism $\varphi : T \to T$ by $\varphi(t_i) = t_i + f_i$ $(1 \le i \le r)$, and it is easy to see that φ satisfies the desired condition. $\square$

Now, as in the beginning of this section, we consider the lifting chain complex $(F \otimes_k T/\mathfrak{m}_T^2,\ \delta)$ with $\delta = d \otimes 1 + \sum_{i=1}^r t_i^* \otimes t_i$ as in Equation (3.2), where $t_i^* : F \to F[-1]$ $(1 \le i \le r)$ are chain homomorphisms whose equivalence classes $[t_1^*], \ldots, [t_r^*]$ form a k-basis of $\mathrm{Ext}_R^1(\mathbb{F}, \mathbb{F})$.

Theorem 3.1. *A maximal lift of* $(F \otimes_k T/\mathfrak{m}_T^2,\ \delta)$ *is unique up to* k*-algebra automorphisms and chain isomorphisms. I.e., if we have two maximal elements* $(T/I_0, \Delta_0)$ *and* $(T/I_1, \Delta_1)$ *in* $\mathcal{I}(\mathfrak{m}_T^2, \delta)$, *then there exists a* k*-algebra automorphism* $\varphi : T \to T$ *such that* φ *induces a* k*-algebra isomorphism* $\overline{\varphi} : T/I_0 \to T/I_1$ *and* $(F \widehat{\otimes}_k (T/I_0), \Delta_0) \otimes_{T/I_0} {}_{\overline{\varphi}}(T/I_1)$ *is isomorphic to* $(F \widehat{\otimes}_k (T/I_1), \Delta_1)$ *as a complex over* $R \widehat{\otimes}_k (T/I_1)^{op}$.

Proof. (1) First of all we note from Remark 4 that there is an infinite descending sequence of ideals of T; $L_0 = \mathfrak{m}_T^2 \supset L_1 \supset L_2 \supset L_3 \supset \cdots \supset I_1$ such that length$(L_n/L_{n+1}) = 1$ for all $n \geq 1$ and $T/I_1 = \varprojlim T/L_n$. Note that each $T/L_{n+1} \to T/L_n$ is a small extension in $\mathcal{A}_k$. Note also that $F \mathbin{\widehat{\otimes}_k} T/I_1 = \varprojlim F \otimes_k T/L_n$.

(2) By induction on n, we shall construct a k-algebra homomorphism

$$\overline{\varphi_n} : T/I_0 \to T/L_n,$$

and an automorphism of a graded $R \otimes_k (T/L_n)^{op}$-module

$$\alpha_n : F \otimes_k (T/L_n) \to F \otimes_k (T/L_n),$$

which satisfy the following four conditions.

(0) $\overline{\varphi_0} : T/I_0 \to T/L_0 = T/\mathfrak{m}_T^2$ is a natural projection and $\alpha_0 = 1$.

(i) The following diagram is commutative:

$$\begin{tikzcd} T/I_0 \arrow[d, "\overline{\varphi_n}"'] \arrow[dr, "\overline{\varphi_{n-1}}"] & \\ T/L_n \arrow[r] & T/L_{n-1}, \end{tikzcd}$$

where the horizontal map is a natural projection.

(ii)

$$\alpha_n \otimes_{T/L_n} (T/L_{n-1}) = \alpha_{n-1}$$

(iii)

$$\Delta_0 \otimes_{T/I_0} {}_{\overline{\varphi_n}}(T/L_n) = \alpha_n \cdot \big(\Delta_1 \otimes_{T/I_1} (T/L_n)\big) \cdot \alpha_n^{-1}$$

(3) Suppose we obtain such $\overline{\varphi_n}$ and α_n for $n \geq 0$ satisfying the above conditions. Then, by (i) and (ii), we have a k-algebra homomorphism

$$\overline{\varphi} = \varprojlim \overline{\varphi_n} \; : \; T/I_0 \to T/I_1$$

and an automorphism of graded $R \mathbin{\widehat{\otimes}_k} (T/I_1)^{op}$-modules

$$\alpha = \varprojlim \alpha_n \; : \; F \mathbin{\widehat{\otimes}_k} T/I_1 \to F \mathbin{\widehat{\otimes}_k} T/I_1.$$

And it follows from (iii) that

$$(*) \qquad \Delta_0 \otimes_{T/I_0} {}_{\overline{\varphi}}(T/I_1) = \alpha \cdot \Delta_1 \cdot \alpha^{-1}.$$

Therefore we have the isomorphism

$$(F \mathbin{\widehat{\otimes}_k} (T/I_0), \;\; \Delta_0) \otimes_{T/I_0} {}_{\overline{\varphi}}(T/I_1) \cong (F \mathbin{\widehat{\otimes}_k} (T/I_1), \;\; \Delta_1),$$

as a chain complex of left $R \mathbin{\widehat{\otimes}_k} (T/I_1)^{op}$-modules.

Now we prove that $\overline{\varphi}$ is an isomorphism. Since $\overline{\varphi}$ induces the identity mapping on $T/\mathfrak{m}_T^2$ by the condition (0), we can apply Lemma 3.8 to get a k-algebra automorphism $\varphi : T \to T$ with $\varphi(I_0) \subseteq I_1$ and φ induces $\overline{\varphi} : T/I_0 \to T/I_1$. Here suppose $\varphi(I_0) \subsetneqq I_1$. Then we would have from $(*)$ that $(F \mathbin{\widehat{\otimes}}_k (T/I_1),\ \alpha\Delta_1\alpha^{-1})$ were liftable to $(F \mathbin{\widehat{\otimes}}_k (T/I_0),\ \Delta_0) \otimes_{T/I_0\ \varphi} (T/\varphi(I_0))$. This is a contradiction, because $(F \mathbin{\widehat{\otimes}}_k (T/I_1),\ \alpha\Delta_1\alpha^{-1})$ is a maximal lift by Lemma 3.7. Thus we have shown $\varphi(I_0) = I_1$ and hence $\overline{\varphi} : T/I_0 \to T/I_1$ is an isomorphism.

In such a way, we have verified that the theorem is proved once we have $\overline{\varphi_n}$ and α_n for $n \geq 0$ satisfying the conditions in (2).

(4) Now we shall construct $\overline{\varphi_n}$ and α_n by induction on n. To do this, assume we already have $\overline{\varphi_n}$ and α_n satisfying the conditions in (2) for an integer $n \geq 0$.

We take an element $\epsilon \in L_n$ which gives a socle element of T/L_{n+1} so that $L_n = L_{n+1} + (\epsilon)$, and hence we have a small extension

$$0 \longrightarrow k \xrightarrow{\ \epsilon\ } T/L_{n+1} \longrightarrow T/L_n \longrightarrow 0.$$

By Lemma 3.8 there is a k-algebra automorphism $\varphi_n : T \to T$ such that $\varphi_n(I_0) \subseteq L_n$ and $\overline{\varphi_n}$ is induced from φ_n.

(5) Under the circumstances as in (4), we claim that

$$\varphi_n(I_0) \subseteq L_{n+1}.$$

On the contrary, assume that $\varphi_n(I_0) \not\subseteq L_{n+1}$. Then, since $\varphi_n(I_0) \subseteq L_n$, we have $L_n = L_{n+1} + \varphi_n(I_0)$. Therefore, there is a fiber product diagram

$$\begin{array}{ccc} T/L_{n+1} \cap \varphi_n(I_0) & \longrightarrow & T/\varphi_n(I_0) \\ \big\downarrow & & \big\downarrow \\ T/L_{n+1} & \longrightarrow & T/L_n. \end{array}$$

Now let β_{n+1} be any lift of α_n to T/L_{n+1}, i.e.

$$\begin{array}{ccc} F \otimes_k T/L_{n+1} & \xrightarrow{\beta_{n+1}} & F \otimes_k T/L_{n+1} \\ \big\downarrow & & \big\downarrow \\ F \otimes_k T/L_n & \xrightarrow{\alpha_n} & F \otimes_k T/L_n, \end{array}$$

where β_{n+1} is also an isomorphism of graded left $R \otimes_k (T/L_{n+1})^{op}$-modules by Lemma 3.2. Then, the chain complexes

$$(F \otimes_k (T/L_{n+1}),\ \ \beta_{n+1} \cdot \big(\Delta_1 \otimes_{T/I_1} (T/L_{n+1})\big) \cdot \beta_{n+1}^{-1})$$

and

$$(**) \qquad (F \mathbin{\widehat{\otimes}}_k T/\varphi_n(I_0),\ \ \Delta_0 \otimes_{T/I_0\ \ \varphi_n} (T/\varphi_n(I_0))$$

are lifts of $(F\otimes_k(T/L_n),\ \alpha_n\cdot\big(\Delta_1\otimes_{T/I_1}(T/L_n)\big)\cdot\alpha_n^{-1})$ to T/L_{n+1} and $T/\varphi_n(I_0)$ respectively. And by Lemma 3.1, they are liftable to $T/L_{n+1}\cap\varphi_n(I_0)$. Note that $(F\ \widehat{\otimes}_k\ (T/I_0),\ \Delta_0)$, as well as the chain complex $(**)$, is a maximal lift of (F,d). Hence it never be liftable to a nontrivial extension ring. Thus it follows that $L_{n+1}\cap\varphi_n(I_0)=\varphi_n(I_0)$, therefore $\varphi_n(I_0)\subseteq L_{n+1}$ as claimed above.

(6) By the claim (5), the k-algebra automorphism φ_n induces a map $\overline{\varphi_n}: T/I_0\to T/L_{n+1}$. Then, the chain complexes

$$(F\otimes_k(T/L_{n+1}),\quad \beta_{n+1}\cdot\big(\Delta_1\otimes_{T/I_1}(T/L_{n+1})\big)\cdot\beta_{n+1}^{-1})$$

and

$$(F\otimes_k(T/L_{n+1}),\quad \Delta_0\otimes_{T/I_0}\ {}_{\overline{\varphi_n}}(T/L_{n+1}))$$

are lifts of $(F\otimes_k(T/L_n),\ \ \alpha_n\cdot\big(\Delta_1\otimes_{T/I_1}(T/L_n)\big)\cdot\alpha_n^{-1})$. Therefore, by Lemma 3.3, there is a chain homomorphism $h: F\to F[-1]$ with

$$(***)\quad \beta_{n+1}\cdot\big(\Delta_1\otimes_{T/I_1}(T/L_{n+1})\big)\cdot\beta_{n+1}^{-1}=\Delta_0\otimes_{T/I_0}\ {}_{\overline{\varphi_n}}(T/L_{n+1})+h\otimes\epsilon.$$

Since the classes of $t_1^*,\dots,t_r^*$ form a k-basis of $\mathrm{Ext}^1_R(\mathbb{F},\mathbb{F})$, we may describe h as

$$h=\sum_{i=1}^{r}c_it_i^*+[d,H],$$

for some $c_i\in k$ and a graded homomorphism $H: F\to F$ of degree 0. Now we define a k-algebra automorphism

$$\varphi_{n+1}: T\to T\qquad \text{by}\qquad \varphi_{n+1}(t_i)=\varphi_n(t_i)+c_i\epsilon$$

for $1\le i\le r$. Then φ_{n+1} is well-defined, because $\epsilon\in L_n\subseteq\mathfrak{m}_T^2$. Note that, for $1\le i,j\le r$, we have $\varphi_{n+1}(t_it_j)=\varphi_n(t_it_j)+c_i(\epsilon\varphi_n(t_j)+\varphi_n(t_i)\epsilon)+c_ic_j\epsilon^2\equiv\varphi_n(t_it_j)\pmod{L_{n+1}}$. Thus, we see $\varphi_{n+1}(x)\equiv\varphi_n(x)\pmod{L_{n+1}}$ for all $x\in\mathfrak{m}_T^2$. Therefore by the claim (5) and by the fact that $I_0\subseteq\mathfrak{m}_T^2$, we have $\varphi_{n+1}(I_0)\subseteq L_{n+1}$, hence φ_{n+1} induces the k-algebra map $\overline{\varphi_{n+1}}: T/I_0\to T/L_{n+1}$ and the diagram

$$\begin{tikzcd} T/I_0 \arrow[d, "\overline{\varphi_{n+1}}"'] \arrow[dr, "\overline{\varphi_n}"] & \\ T/L_{n+1} \arrow[r] & T/L_n, \end{tikzcd}$$

is commutative.

By the definition of φ_{n+1}, it follows that

$$\Delta_0\otimes_{T/I_0}\ {}_{\overline{\varphi_{n+1}}}(T/L_{n+1})=\Delta_0\otimes_{T/I_0}\ {}_{\overline{\varphi_n}}(T/L_{n+1})+\sum_{i=1}^{r}t_i^*\otimes c_i\epsilon,$$

thus we see from $(***)$ that

$$\beta_{n+1}\cdot\big(\Delta_1\otimes_{T/I_1}(T/L_n)\big)\cdot\beta_{n+1}^{-1}=\Delta_0\otimes_{T/I_0}\ {}_{\overline{\varphi_{n+1}}}(T/L_{n+1})+[d,H].$$

Now define, as in the proof of Lemma 3.3, an automorpshim α_{n+1} by

$$\alpha_{n+1} = (1 - H \otimes \epsilon) \cdot \beta_{n+1}.$$

Then we have that

$$\alpha_{n+1} \cdot \big(\Delta_1 \otimes_{T/I_1} (T/L_n)\big) \cdot \alpha_{n+1}^{-1} = \Delta_0 \otimes_{T/I_0} {}_{\overline{\varphi_{n+1}}}(T/L_{n+1}),$$

and also

$$\alpha_{n+1} \otimes_{T/L_{n+1}} T/L_n = \beta_{n+1} \otimes_{T/L_{n+1}} T/L_n = \alpha_n.$$

Therefore, we have obtained $\overline{\varphi_{n+1}}$ and α_{n+1} satisfing the conditions (i), (ii) and (iii), and the proof is completed by induction. □

3.3. Universal lifts

As in the previous section, let $\mathbb{F} = (F, d)$ be a projective complex over R that satisfies the fundamental assumption

$$r = \dim_k \operatorname{Ext}^1_R(\mathbb{F}, \mathbb{F}) < \infty.$$

We define a functor $\mathcal{F} : \mathcal{A}_k \to (Sets)$ as follows: For any $A \in \mathcal{A}_k$, we set

$$\mathcal{F}(A) = \{(F \otimes_k A,\ \Delta) \mid \text{it is a lifting chain complex of } \mathbb{F} \text{ to } A\}/\cong,$$

where $\cong$ denotes the isomorphism as chain complexes over $R \otimes_k A^{op}$. If $f : A \to B$ is a morphism in $\mathcal{A}_k$, then we define a mapping $\mathcal{F}(f) : \mathcal{F}(A) \to \mathcal{F}(B)$ by

$$\mathcal{F}(f)((F \otimes_k A,\ \Delta)) = (F \otimes_k A,\ \Delta) \otimes_A {}_f B.$$

Note that $\mathcal{F}$ is a covariant functor.

Definition 3.6. Let $P \in \widehat{\mathcal{A}}_k$ and let $\mathbb{L} = (F \mathbin{\widehat{\otimes}_k} P, \Delta)$ be a lifting chain complex of $\mathbb{F}$ to P.

(a) We define a morphism between functors on $\mathcal{A}_k$;

$$\phi_{\mathbb{L}} \ : \ \operatorname{Hom}_{k\text{-alg}}(P, \quad) \to \mathcal{F},$$

by

$$\phi_{\mathbb{L}}(f) = (F \mathbin{\widehat{\otimes}_k} P,\ \Delta) \otimes_P {}_f A$$

for $f \in \operatorname{Hom}_{k\text{-alg}}(P, A)$ with $A \in \mathcal{A}_k$.

(b) We say that the chain complex $\mathbb{L}$ is **a universal lift of** $\mathbb{F}$ if the morphism $\phi_{\mathbb{L}}$ is an isomorphism. Thus, in this case, the functor $\mathcal{F}$ on the category $\mathcal{A}_k$ is pro-representable by $P \in \widehat{\mathcal{A}}_k$. If $\mathbb{L} = (F \mathbin{\widehat{\otimes}_k} P, \Delta)$ is a universal lift of $\mathbb{F}$, then P is called **a parameter algebra of the universal lift of** $\mathbb{F}$.

Lemma 3.9. *If there is a universal lift of* $\mathbb{F}$*, then any parameter algebras of any universal lifts are isomorphic each other as* k*-algebras.*

Proof. In fact, if P_1 and P_2 are such parameter algebras, then we have an isomorphism $\mathrm{Hom}_{k\text{-alg}}(P_1, \ \ ) \cong \mathrm{Hom}_{k\text{-alg}}(P_2, \ \ )$ as functors on $\mathcal{A}_k$. Then, it is easy to see that there are isomorphisms $P_1/\mathfrak{m}_{P_1}^n \cong P_2/\mathfrak{m}_{P_2}^n$ as k-algebras for any $n \geq 1$, which are compatible with the projections $P_1/\mathfrak{m}_{P_1}^{n+1} \to P_1/\mathfrak{m}_{P_1}^n$ and $P_2/\mathfrak{m}_{P_2}^{n+1} \to P_2/\mathfrak{m}_{P_2}^n$. Hence $P_1 \cong P_2$. $\square$

Theorem 3.2. *The following two conditions are equivalent for a lifting chain complex $\mathbb{L}_0 = (F \mathbin{\widehat{\otimes}_k} P_0, \Delta_0)$ of $\mathbb{F}$ where $P_0 = T/I_0 \in \widehat{\mathcal{A}}_k$ with $I_0 \subseteq \mathfrak{m}_T^2$.*
(a) $\mathbb{L}_0$ is a universal lift of $\mathbb{F} = (F, d)$.
(b) $\mathbb{L}_0$ is a maximal lift of $(F \otimes_k T/\mathfrak{m}_T^2, \delta)$.
In particular, there always exists a universal lift of $\mathbb{F}$, and it is unique up to k-algebra automorphisms and chain isomorphisms.

Proof.
[(b) $\Rightarrow$ (a)] : Let $\mathbb{L}_0$ be a maximal lift of $(F \otimes_k T/\mathfrak{m}_T^2, \delta)$. To simplify the notation we write $\phi_{\mathbb{L}_0}$ as ϕ. We would like to prove that

$$\phi(A) \ : \ \mathrm{Hom}_{k\text{-alg}}(P_0, A) \to \mathcal{F}(A)$$

is a bijection for any $A \in \mathcal{A}_k$. We prove this by induction on the length of A. If $\mathrm{length}(A) = 1$, then $A = k$ and $\phi(k)$ is clealy bijective.

[The surjectivity of $\phi(A)$]: Take a socle element $\epsilon \in A$, and we have a small extension

$$0 \longrightarrow k \xrightarrow{\ \epsilon\ } A \xrightarrow{\ \pi\ } \overline{A} \longrightarrow 0$$

where $\overline{A} = A/(\epsilon)$. By the induction hypothesis, $\phi(\overline{A})$ is bijective.

$$\begin{array}{ccc} \mathrm{Hom}_{k\text{-alg}}(P_0, A) & \xrightarrow{\phi(A)} & \mathcal{F}(A) \\ {\scriptstyle \pi_*}\downarrow & & \downarrow{\scriptstyle \mathcal{F}(\pi)} \\ \mathrm{Hom}_{k\text{-alg}}(P_0, \overline{A}) & \xrightarrow{\phi(\overline{A})} & \mathcal{F}(\overline{A}) \end{array}$$

To prove the surjectivity of $\phi(A)$, let $(F \otimes_k A, \Delta)$ be any element of $\mathcal{F}(A)$. Since $\phi(\overline{A})$ is surjective, there is $g \in \mathrm{Hom}_{k\text{-alg}}(P_0, \overline{A})$ such that

$$(F \otimes_k A, \Delta) \otimes_A {}_{\pi}\overline{A} \cong (F \mathbin{\widehat{\otimes}_k} P_0, \Delta_0) \otimes_{P_0} {}_{g}\overline{A}.$$

Hence, it follows from Lemma 3.2 that there is an isomorphism of graded modules $\alpha : F \otimes_k A \to F \otimes_k A$ such that

$$(F \otimes_k A, \alpha\Delta\alpha^{-1}) \otimes_A {}_{\pi}\overline{A} = (F \mathbin{\widehat{\otimes}_k} P_0, \Delta_0) \otimes_{P_0} {}_{g}\overline{A}.$$

Now taking the fiber product

$$\begin{array}{ccc} P & \longrightarrow & P_0 \\ \downarrow & & \downarrow{\scriptstyle g} \\ A & \xrightarrow{\ \pi\ } & \overline{A}, \end{array}$$

we see from the above equality that the complex $(F \widehat{\otimes}_k P_0, \Delta_0) \otimes_{P_0} {}_g\overline{A}$ is liftable to A, and it follows from Proposition 3.1 that the chain complex $(F \widehat{\otimes}_k P_0, \Delta_0)$ is liftable to P. If the extension $P \to P_0$ is nontrivial, then it contradicts to that $(F \widehat{\otimes}_k P_0, \Delta_0)$ is a maximal lift. Hence $P \to P_0$ is a trivial small extension, and $P \to P_0$ has a right inverse in $\widehat{\mathcal{A}}_k$. In particular, the k-algebra map $g : P_0 \to \overline{A}$ can be lifted to the k-algebra map $f : P_0 \to A$, i.e. $\pi \cdot f = g$. Then, note that both $(F \otimes_k A, \Delta_0 \otimes_{P_0} {}_f A)$ and $(F \otimes_k A, \alpha\Delta\alpha^{-1})$ are lifts of $(F \otimes_k \overline{A}, \Delta_0 \otimes_{P_0} {}_g\overline{A})$. Hence, by Lemma 3.3, we have

$$\Delta_0 \otimes_{P_0} {}_f A = \alpha\Delta\alpha^{-1} + h \otimes \epsilon$$

for some chain homomorphism $h : F \to F[-1]$ of degree -1. Then we may write

$$[h] = \sum_{i=1}^{r} c_i [t_i^*] \qquad (c_i \in k),$$

as an element of $\mathrm{Ext}^1_R(\mathbb{F}, \mathbb{F})$. Now define a k-algebra map $\widetilde{\varphi} : T \to A$ by

$$\widetilde{\varphi}(t_i) = f(t_i) - c_i \epsilon \qquad (1 \le i \le r).$$

It can be easily verified that $\widetilde{\varphi}(t_i t_j) = f(t_i t_j)$ for any i, j. Since $I_0 \subseteq \mathfrak{m}_T^2$, we have $\widetilde{\varphi}(I_0) = f(I_0) = 0$, thus $\widetilde{\varphi}$ induces the k-algebra map $\varphi : P_0 \to A$ and $\varphi|_{\mathfrak{m}_{P_0}^2} = f|_{\mathfrak{m}_{P_0}^2}$. Then, by the choice of φ, we see that

$$\Delta_0 \otimes_{P_0} {}_\varphi A = \Delta_0 \otimes_{P_0} {}_f A - \sum_{i=1}^{r} c_i t_i^* \otimes \epsilon = \alpha\Delta\alpha^{-1} + (h - \sum_{i=1}^{r} c_i t_i^*) \otimes \epsilon.$$

Thus it follows from Lemma 3.3(b) that

$$(F \widehat{\otimes}_k P_0, \Delta_0) \otimes_{P_0} {}_\varphi A \cong (F \otimes_k A, \alpha\Delta\alpha^{-1}) \cong (F \otimes_k A, \Delta).$$

This proves the surjectivity of $\phi(A)$.

[The injectivity of $\phi(A)$]: Let φ_1, φ_2 be k-algebra homomorphisms $P_0 \to A$ with $(F \widehat{\otimes}_k P_0, \Delta_0) \otimes_{P_0} {}_{\varphi_1} A \cong (F \widehat{\otimes}_k P_0, \Delta_0) \otimes_{P_0} {}_{\varphi_2} A$. We want to show $\varphi_1 = \varphi_2$. Take a socle element $\epsilon \in A$ and we consider the small extension

$$0 \longrightarrow k \xrightarrow{\ \epsilon\ } A \xrightarrow{\ \pi\ } \overline{A} \longrightarrow 0.$$

Then, by the induction hypothesis, we have $\pi \cdot \varphi_1 = \pi \cdot \varphi_2$. Now consider the mapping $\psi = \varphi_1 - \varphi_2 : P_0 \to A$, and we see that the image of ψ is contained in $k\epsilon$. Note that $\psi(1) = 0$ and that $\psi(xy) = \varphi_1(x)\psi(y) - \psi(x)\varphi_2(y) = 0$ if $x, y \in \mathfrak{m}_{P_0}$, since $\mathfrak{m}_{P_0}\epsilon = \epsilon\mathfrak{m}_{P_0} = 0$. Therefore $\psi(\mathfrak{m}_{P_0}^2) = 0$, and we have $\varphi_1|_{\mathfrak{m}_{P_0}^2} = \varphi_2|_{\mathfrak{m}_{P_0}^2}$. Since $\varphi_1 = \varphi_2 + \psi$, it follows

$$\Delta_0 \otimes_{P_0} {}_{\varphi_1} A = \Delta_0 \otimes_{P_0} {}_{\varphi_2} A + \sum_{i=1}^{r} t_i^* \otimes \psi(t_i).$$

Denoting $\psi(t_i) = c_i\epsilon$ with $c_i \in k$ for $1 \le i \le r$, we have the equality

$$\Delta_0 \otimes_{P_0\ \varphi_1} A = \Delta_0 \otimes_{P_0\ \varphi_2} A + \sum_{i=1}^{r} c_i t_i^* \otimes \epsilon.$$

Then it follows from Lemma 3.3(b) that $\sum_{i=1}^{r} c_i[t_i^*] = 0$ as an element of $\mathrm{Ext}^1_R(\mathbb{F},\mathbb{F})$. Since $\{[t_1^*],\dots,[t_r^*]\}$ is a k-basis of $\mathrm{Ext}^1_R(\mathbb{F},\mathbb{F})$, we have $c_i = 0$ for all i, hence $\psi = 0$.

[(a) $\Rightarrow$ (b)] : Suppose that $\mathbb{L}_0 = (F \widehat{\otimes}_k P_0, \Delta_0)$ is a universal lift of $\mathbb{F}$. Take any maximal lift $(F \widehat{\otimes}_k T/I_1, \Delta_1)$ of $(F \otimes_k T/\mathfrak{m}_T^2, \delta)$, and since it is a universal lift by the implication (b) $\Rightarrow$ (a), Lemma 3.9 forces P_0 to be isomorphic to T/I_1. Thus we may assume that $P_0 = T/I_1$. Lemma 3.4 implies that we can take a maximal element (I_2, Δ_2) in $\mathcal{I}(I_1, \Delta_0)$, which is in fact a maximal lift of $(F \otimes_k T/\mathfrak{m}_T^2, \delta)$. Then, again by the implication (b) $\Rightarrow$ (a), $(F \widehat{\otimes}_k T/I_2, \Delta_2)$ is also a universal lift of $\mathbb{F}$, and hence T/I_2 is isomorphic to T/I_1 by Lemma 3.9. Since $I_2 \subseteq I_1$, the following lemma forces $I_2 = I_1$. This implies that $(F \widehat{\otimes}_k P_0, \Delta_0) = (F \widehat{\otimes}_k T/I_2, \Delta_2)$, which is a maximal lift as desired. □

Lemma 3.10. *Let T be a non-commutative formal power series ring, and let $I_2 \subseteq I_1$ be closed ideals of T. Suppose T/I_1 is isomorphic to T/I_2 as a k-algebra. Then $I_1 = I_2$.*

Proof. The isomorphism $T/I_1 \cong T/I_2$ induces isomorphisms $T/I_1 + \mathfrak{m}_T^n \cong T/I_2 + \mathfrak{m}_T^n$ for any integer n. Since $I_2 + \mathfrak{m}_T^n \subseteq I_1 + \mathfrak{m}_T^n$, comparing the lengths, we have the equality $I_2 + \mathfrak{m}_T^n = I_1 + \mathfrak{m}_T^n$ for each n. Thus $I_2 = \bigcap_n I_2 + \mathfrak{m}_T^n = \bigcap_n I_1 + \mathfrak{m}_T^n = I_1$. □

3.4. Every complete local algebra is a parameter algebra

Lemma 3.11. *Let $A' \to A$ be a surjective morphism in $\mathcal{A}_k$. Suppose the following two conditions hold.*

(*a*) *$\mathbb{L} = (F \otimes_k A, \Delta)$ is a left $R \otimes_k A^{op}$-free resolution of a left $R \otimes_k A^{op}$-module M, and $\mathbb{L}$ is a lift of a free complex $\mathbb{F} = (F, d)$ over R.*

(*b*) *There is a left $R \otimes_k A'$-module M' such that M' is flat as a right A'-module and $M' \otimes_{A'} A \cong M$ as left $R \otimes_k A^{op}$-modules.*

Then there is a lifting chain complex $\mathbb{L}' = (F \otimes_k A', \Delta')$ of $\mathbb{L}$ that is a left $R \otimes_k A'^{op}$-free resolution of M'.

Proof. We may write $A = A'/I'$ where I' is an ideal of A'. Take a set of generators $\{x_\lambda \mid \lambda \in \Lambda\}$ of M as a left $R \otimes_k A^{op}$-module. Since $M \cong M'/M'I'$ as $R \otimes_k A'^{op}$-modules, we can take a subset $\{x'_\lambda \mid \lambda \in \Lambda\}$ of M' that is an inverse image of $\{x_\lambda\}$. Then the equality $M' = R\{x'_\lambda\}A' + M'I'$ holds. Since I' is a nilpotent ideal, we have $M' = R\{x'_\lambda\}A'$.

By this argument we can show that every surjective homomorphism $F_0 \otimes_k A \to M$ of left $R \otimes_k A^{op}$- modules can be lifted to a surjective homomorphism $F_0 \otimes_k A' \to M'$ of left $R \otimes_k A'^{op}$-modules. Now take the kernels of these surjective maps and we have exact sequences

$$0 \longrightarrow M_1 \longrightarrow F_0 \otimes_k A \longrightarrow M \longrightarrow 0,$$

$$0 \longrightarrow M_1' \longrightarrow F_0 \otimes_k A' \longrightarrow M' \longrightarrow 0.$$

Notice that M_1' is flat as a right A'-module, since $F_0 \otimes_k A'$ and M' are flat. Thus the isomorphism $M' \otimes_{A'} A \cong M$ implies $M_1' \otimes_{A'} A \cong M_1$. Then by the same manner as above, the surjective homomorphism $F_1 \otimes_k A \to M_1$ is liftable to a surjective homomorphism $F_1 \otimes_k A' \to M_1'$. In such a way, by induction, we can construct a chain complex with the underlying graded module $F \otimes_k A'$, which is a lift of $(F \otimes_k A, \Delta)$. □

Theorem 3.3. *Let R be a complete local k-algebra, i.e. $R \in \widehat{\mathcal{A}}_k$, and let $\mathbb{F} = (F, d)$ be a left R-free resolution of the residue field $k = R/\mathfrak{m}_R$. Then there is a universal lift of $\mathbb{F}$ that is an acyclic complex of the form $(F \widehat{\otimes}_k R, \Delta)$ with the homology $H_0(F \widehat{\otimes}_k R, \Delta) = R$. In particular, R is the parameter algebra of the universal lift of $\mathbb{F}$.*

Proof. Note that the obvious exact sequence $0 \to \mathfrak{m}_R \to R \to k \to 0$ implies

$$\mathrm{Ext}^1_R(\mathbb{F}, \mathbb{F}) = \mathrm{Ext}^1_R(k, k) \cong \mathrm{Hom}_k(\mathfrak{m}_R/\mathfrak{m}_R^2, k).$$

Thus if we denote $R = T/I$ where $T = k\langle\langle t_1, t_2, \dots, t_r\rangle\rangle$ and $I \subseteq \mathfrak{m}_T^2$, we can take the dual bais $\{t_1^*, \dots, t_r^*\}$ as a basis of $\mathrm{Ext}^1_R(\mathbb{F}, \mathbb{F})$. Set $\delta = d \otimes 1 + \sum_{i=1}^r t_i^* \otimes t_i$, and we see that $(F \otimes_k T/\mathfrak{m}_T^2,\ \delta)$ is a lift of $\mathbb{F}$ as in the beginning of Section 3.2. By the exact sequence of chain complexes

$$0 \longrightarrow (F \otimes_k \mathfrak{m}_T/\mathfrak{m}_T^2,\ d \otimes 1) \longrightarrow (F \otimes_k T/\mathfrak{m}_T^2,\ \delta) \longrightarrow (F, d) \longrightarrow 0,$$

and by the acyclicity of (F, d), we easily see that $(F \otimes_k T/\mathfrak{m}_T^2,\ \delta)$ is acyclic as well, and $H_0(F \otimes_k T/\mathfrak{m}_T^2,\ \delta) \cong T/\mathfrak{m}_T^2 = R/\mathfrak{m}_R^2$ as an $R \otimes_k (R/\mathfrak{m}_R^2)^{op}$-module.

Starting from $\Delta_2 = \delta$, and using Lemma 3.11, we can inductively construct a sequence of chain complexes $(F \otimes_k R/\mathfrak{m}_R^n,\ \Delta_n)$ for $n \geq 2$ satisfying the equalities $(F \otimes_k R/\mathfrak{m}_R^{n+1},\ \Delta_{n+1}) \otimes_{R/\mathfrak{m}_R^{n+1}} R/\mathfrak{m}_R^n = (F \otimes_k R/\mathfrak{m}_R^n,\ \Delta_n)$ and $H_0(F \otimes_k R/\mathfrak{m}_R^n,\ \Delta_n) = R/\mathfrak{m}_R^n$. Now let $\Delta = \varprojlim \Delta_n$, and we have a lifting chain complex $(F \widehat{\otimes}_k R,\ \Delta)$ of $(F \otimes_k T/\mathfrak{m}_T^2,\ \delta)$.

First, we claim that $(F \widehat{\otimes}_k R,\ \Delta)$ is acyclic and $H_0(F \widehat{\otimes}_k R,\ \Delta) \cong R$ as an $R \widehat{\otimes}_k R^{op}$-module. For this, let Ω_n^i be the kernel of $\Delta_{n,i} : F_i \otimes_k R/\mathfrak{m}_R^n \to F_{i-1} \otimes_k R/\mathfrak{m}_R^n$ for any $n, i \geq 0$ where we understand that $\Omega_n^0 = R/\mathfrak{m}_R^n$. By the proof of Lemma 3.11, we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega_{n+1}^i & \longrightarrow & F_i \otimes_k R/\mathfrak{m}_R^{n+1} & \longrightarrow & \Omega_{n+1}^{i-1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega_n^i & \longrightarrow & F_i \otimes_k R/\mathfrak{m}_R^n & \longrightarrow & \Omega_n^{i-1} & \longrightarrow & 0,
\end{array}$$

where the vertical arrows are surjective. This implies the exact sequence

$$0 \longrightarrow \varprojlim \Omega_n^i \longrightarrow F_i \widehat{\otimes}_k R \longrightarrow \varprojlim \Omega_n^{i-1} \longrightarrow 0,$$

and hence the complex $(F \widehat{\otimes}_k R,\ \Delta)$ is acyclic and $H_0(F \widehat{\otimes}_k R,\ \Delta) = \varprojlim R/\mathfrak{m}_R^n = R$ as desired.

Now we prove that $(F \widehat{\otimes}_k R,\ \Delta)$ is a maximal lift of $(F \otimes_k R/\mathfrak{m}_R^2,\ \delta)$. Suppose that it is not a maximal lift. Then there will be a nontrivial small extension

$$0 \longrightarrow k \xrightarrow{\ \epsilon\ } R' \xrightarrow{\ p\ } R \longrightarrow 0,$$

of R so that the complex $(F \widehat{\otimes}_k R,\ \Delta)$ is liftable to a chain complex $(F \widehat{\otimes}_k R',\ \Delta')$. The exact sequence

$$0 \longrightarrow (F, d) \xrightarrow{1\otimes\epsilon} (F \widehat{\otimes}_k R',\ \Delta') \longrightarrow (F \widehat{\otimes}_k R,\ \Delta) \longrightarrow 0$$

forces $(F \widehat{\otimes}_k R',\ \Delta')$ to be acyclic as well, and taking the homologies we have an exact sequence

$$0 \longrightarrow k \xrightarrow{\ \epsilon\ } H_0' \xrightarrow{\ \pi\ } R \longrightarrow 0,$$

where $H_0' = H_0(F \widehat{\otimes}_k R', \Delta')$ and π is a homomorphism of $R \widehat{\otimes}_k R'^{op}$-modules. Take an element $x_0 \in H_0'$ with $\pi(x_0) = 1$, and obviously we have $H_0' = x_0 R' + H_0'\epsilon$. Since $\epsilon^2 = 0$, it follows that $H_0' = x_0 R'$.

We claim that H_0' is a free module as a right R'-module. To prove this, assume $x_0 a' = 0$ for $a' \in R'$, and we want to show $a' = 0$. Suppose $a' \neq 0$. Since $0 = \pi(x_0 a') = 1_R \cdot a' = p(a')$, we see $a' \in \epsilon k$, hence $a' = \epsilon c$ for some $c \neq 0 \in k$. Then we have $x_0 \epsilon = 0$, and the right R'-module $H_0' = x_0 R'$ is in fact a right R-module. Hence $0 \to k \to H_0' \to R \to 0$ is an exact sequence of right R-modules. Therefore the sequence splits and $H_0' \cong k \oplus R$ as a right R-(hence R'-)module. This contradicts that H_0' is generated by a single element as a right R'-module.

Now we have shown H_0' is a free right R'-module. Since H_0' is a left R-module as well, for any $a \in R$, we find a unique element $a' \in R'$ with $a \cdot x_0 = x_0 \cdot a'$. Now define a map $f : R \to R'$ by $f(a) = a'$. Since $(ab)x_0 = a(x_0 f(b)) = (a x_0) f(b) = x_0(f(a)f(b))$, we can see that f is a k-algebra homomorphism. Since we have an equality $a = \pi(a x_0) = \pi(x_0 f(a)) = 1_R \cdot f(a) = p(f(a))$ for any $a \in R$, we see $p \cdot f = 1$. This contradicts that (R', ϵ) is a nontrivial extension, and the proof is completed. □

Remark 8. If R is left noetherian, then we can take as each F_i a finitely generated left R-free module, and $F_i \widehat{\otimes}_k R$ is a left $R \widehat{\otimes}_k R^{op}$-free module. In this case the acyclic complex $(F \widehat{\otimes}_k R, \Delta)$ in the theorem is a free resolution of R as a left $R \widehat{\otimes}_k R^{op}$-module. However, in general, notice that $(F \widehat{\otimes}_k R, \Delta)$ may not be a free complex of left $R \widehat{\otimes}_k R^{op}$- modules.

3.5. Deformation of modules

Let R be a k-algebra as before. In this section, we consider the case where the complex $\mathbb{F} = (F, d)$ is a free resolution of a left R-module M. Of course, in this setting, we have the equality $\mathrm{Ext}^1_R(M, M) = \mathrm{Ext}^1_R(\mathbb{F}, \mathbb{F})$ and it is assumed to be of finite dimension as before.

For any $A \in \mathcal{A}_k$, a left $R \otimes_k A^{op}$-module X is said to be a **flat deformation of** M **along** A if X is a flat module as a right A-module, and there is an isomorphism $X \otimes_A k \cong M$ as left R- modules. And two flat deformations X and Y of M along A are said to be isomorphic if they are isomorphic as $R \otimes_k A^{op}$-modules. We consider the following two functors $\mathcal{A}_k \to (Sets)$:

$$\mathcal{F}(A) = \text{the set of isomorphism classes of lifting chain complexes of } \mathbb{F},$$
$$\mathcal{F}_M(A) = \text{the set of isomorphism classes of flat deformations of } M.$$

Theorem 3.4. *We have an isomorphism $\mathcal{F} \cong \mathcal{F}_M$ as functors on $\mathcal{A}_k$. In particular, the functor $\mathcal{F}_M$ is pro-representable as is $\mathcal{F}$, i.e. there is an isomorphism $\mathcal{F}_M \cong \mathrm{Hom}_{k\text{-}alg}(P_0, \quad)$ of functors on $\mathcal{A}_k$, where P_0 is the parameter algebra of the universal lift of $\mathbb{F}$.*

Proof. Let $A \in \mathcal{A}_k$ and let $(F \otimes_k A, \Delta)$ be a lift of $\mathbb{F}$ to A. Notice that, by induction on the length of A, one can easily prove that $(F \otimes_k A, \Delta)$ is acyclic, as $\mathbb{F}$ is acyclic. Therefore it gives a left $R \otimes_k A^{op}$-free resolution of $H_0(F \otimes_k A, \Delta)$. Since $(F \otimes_k A, \Delta) \otimes_A k = \mathbb{F}$ is acyclic, we have $\mathrm{Tor}_i^{A^{op}}(H_0(F \otimes_k A, \Delta), k) = 0$ for $i > 0$. Since A is an artinian local algebra, this implies that $H_0(F \otimes_k A, \Delta)$ is flat as a right A- module. We should note that $H_0(F \otimes_k A, \Delta) \otimes_A k = H_0(\mathbb{F}) = M$. Hence $H_0(F \otimes_k A, \Delta)$ is a flat deformation of M along A. Thus we obtain a well-defined mapping $H_0 : \mathcal{F}(A) \to \mathcal{F}_M(A)$ by taking the 0-th homology. It is trivial that the map is injective, and Lemma 3.11 implies it is surjective. $\square$

4. Properties of parameter algebras

4.1. Obstruction maps

As before $\mathbb{F} = (F, d)$ is a projective complex over a k-algebra R. Let $P \in \widehat{\mathcal{A}}_k$ and let $\mathbb{L} = (F \widehat{\otimes}_k P, \ \Delta)$ be a lift of $\mathbb{F}$ to P, which we fix in this section. We aim at constructing the obstruction map

$$\alpha_{\mathbb{L}} : \mathcal{T}(P) \to \mathrm{Ext}_R^2(\mathbb{F}, \mathbb{F}),$$

which will enable us to compare the cohomology modules between P and R.

4.1. [To define $\alpha_{\mathbb{L}}$]:

Now, suppose we are given a class of small extension $[P', \epsilon] \in \mathcal{T}(P)$. Lemma 3.6 forces $\Delta : F \widehat{\otimes}_k P \to F \widehat{\otimes}_k P[-1]$ to be lifted to $\Delta' : F \widehat{\otimes}_k P' \to F \widehat{\otimes}_k P'[-1]$. Note that Δ' is just a lift as a graded homomorphism, and it may not holds that ${\Delta'}^2 = 0$. Recall from Lemma 2.11 that we then have a commutative diagram of graded left $R \widehat{\otimes}_k P'^{op}$-modules with exact rows

$$\begin{tikzcd}
0 \arrow[r] & F \arrow[r, "1\otimes\epsilon"] \arrow[d, "d"] & F \widehat{\otimes}_k P' \arrow[r, "1\otimes\pi"] \arrow[d, "\Delta'"] & F \widehat{\otimes}_k P \arrow[r] \arrow[d, "\Delta"] & 0 \\
0 \arrow[r] & F[-1] \arrow[r, "1\otimes\epsilon"] & F \widehat{\otimes}_k P'[-1] \arrow[r, "1\otimes\pi"] & F \widehat{\otimes}_k P[-1] \arrow[r] & 0.
\end{tikzcd}$$

Since $d^2 = 0$ and $\Delta^2 = 0$, we have the following commutative diagram.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F & \xrightarrow{1\otimes\epsilon} & F \mathbin{\widehat{\otimes}_k} P' & \xrightarrow{1\otimes\pi} & F \mathbin{\widehat{\otimes}_k} P & \longrightarrow & 0 \\
 & & {\scriptstyle 0}\downarrow & & {\scriptstyle {\Delta'}^2}\downarrow & & {\scriptstyle 0}\downarrow & & \\
0 & \longrightarrow & F[-2] & \xrightarrow{1\otimes\epsilon} & F \mathbin{\widehat{\otimes}_k} P'[-2] & \xrightarrow{1\otimes\pi} & F \mathbin{\widehat{\otimes}_k} P[-2] & \longrightarrow & 0
\end{array}$$

By chasing the diagram, we see that there is a graded left R-module homomorphism $\sigma : F \to F[-2]$ with ${\Delta'}^2 = \sigma \otimes \epsilon$, i.e. ${\Delta'}^2(\sum x \otimes a) = \sum \sigma(x) \otimes \epsilon a$ for any formal infinite sum $\sum x \otimes a \in F \mathbin{\widehat{\otimes}_k} P'$.

First we claim that

(i) $\sigma : F \to F[-2]$ is a chain map.

In fact, it holds that

$$\sigma \cdot d \otimes \epsilon = (\sigma \otimes \epsilon)\Delta' = {\Delta'}^3 = \Delta'(\sigma \otimes \epsilon) = d \cdot \sigma \otimes \epsilon,$$

hence it follows that $[d, \sigma] = d \cdot \sigma - \sigma \cdot d = 0$. Thus the graded homomorphism σ is a chain map of degree -2, therefore it defines an element $[\sigma] \in \mathrm{Ext}^2_R(\mathbb{F}, \mathbb{F})$.

Next we claim that

(ii) the class $[\sigma]$ does not depend on a choice of a lifting map Δ'.

In fact, if Δ' and Δ'' are two lifting maps of Δ, then we have a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F & \xrightarrow{1\otimes\epsilon} & F \mathbin{\widehat{\otimes}_k} P' & \xrightarrow{1\otimes\pi} & F \mathbin{\widehat{\otimes}_k} P & \longrightarrow & 0 \\
 & & {\scriptstyle 0}\downarrow & & {\scriptstyle \Delta'-\Delta''}\downarrow & & {\scriptstyle 0}\downarrow & & \\
0 & \longrightarrow & F[-1] & \xrightarrow{1\otimes\epsilon} & F \mathbin{\widehat{\otimes}_k} P'[-1] & \xrightarrow{1\otimes\pi} & F \mathbin{\widehat{\otimes}_k} P[-1] & \longrightarrow & 0,
\end{array}$$

from which we can see the existence of graded homomorphism $\tau : F \to F[-1]$ with $\Delta' - \Delta'' = \tau \otimes \epsilon$. Then, we have an equality

$${\Delta'}^2 = (\Delta'' + \tau \otimes \epsilon)^2 = {\Delta''}^2 + (d\tau + \tau d) \otimes \epsilon,$$

hence, setting ${\Delta'}^2 = \sigma' \otimes \epsilon$ and ${\Delta''}^2 = \sigma'' \otimes \epsilon$, we have $\sigma' - \sigma'' = d\tau + \tau d$, i.e. $[\sigma'] = [\sigma'']$.

Now we can define a mapping

$$\alpha_{\mathbb{L}} : \mathcal{T}(P) \to \mathrm{Ext}^2_R(\mathbb{F}, \mathbb{F})$$

by sending $[P', \epsilon]$ to the class $[\sigma]$.

By (i) and (ii) above, $\alpha_{\mathbb{L}}$ is a well-defined mapping. Furthermore, we can show the following.

Lemma 4.1. *The mapping $\alpha_{\mathbb{L}} : \mathcal{T}(P) \to \mathrm{Ext}^2_R(\mathbb{F}, \mathbb{F})$ is k-linear.*

Proof. To prove the equality

$$\alpha_{\mathbb{L}}(c_1[P_1,\epsilon_1]+c_2[P_2,\epsilon_2]) = c_1\alpha_{\mathbb{L}}([P_1,\epsilon_1]) + c_2\alpha_{\mathbb{L}}([P_2,\epsilon_2])$$

for $c_i \in k$ and $[P_i,\epsilon_i] \in \mathcal{T}(P)$ $(i=1,2)$, let us consider the pull-back diagram

$$\begin{array}{ccc} F \mathbin{\widehat{\otimes}_k} (P_1 \times_P P_2) & \longrightarrow & F \mathbin{\widehat{\otimes}_k} P_1 \\ \downarrow & & \downarrow \\ F \mathbin{\widehat{\otimes}_k} P_2 & \longrightarrow & F \mathbin{\widehat{\otimes}_k} P, \end{array}$$

and take lifting graded homomorphisms $\Delta_i : F \mathbin{\widehat{\otimes}_k} P_i \to F \mathbin{\widehat{\otimes}_k} P_i[-1]$ of Δ for $i=1,2$. We may assume that $c_i \neq 0$ for $i=1,2$. Since there is a commutative diagram with exact rows by Lemma 2.11 ;

$$\begin{array}{ccccc} 0 \to F \mathbin{\widehat{\otimes}_k} (P_1 \times_P P_2) & \longrightarrow & (F \mathbin{\widehat{\otimes}_k} P_1) \oplus (F \mathbin{\widehat{\otimes}_k} P_2) & \longrightarrow & F \mathbin{\widehat{\otimes}_k} P \to 0 \\ & & {\scriptstyle (\Delta_1,\Delta_2)}\downarrow & & {\scriptstyle \Delta}\downarrow \\ 0 \to F \mathbin{\widehat{\otimes}_k} (P_1 \times_P P_2)[-1] & \longrightarrow & F \mathbin{\widehat{\otimes}_k} P_1[-1] \oplus F \mathbin{\widehat{\otimes}_k} P_2[-1] & \longrightarrow & F \mathbin{\widehat{\otimes}_k} P[-1] \to 0, \end{array}$$

there is a naturally induced mapping $\widetilde{\Delta} : F \mathbin{\widehat{\otimes}_k} (P_1 \times_P P_2) \to F \mathbin{\widehat{\otimes}_k} (P_1 \times_P P_2)[-1]$ which is a lifting map of both of Δ_1 and Δ_2. Let us take a chain homomorphism σ_i of $\mathbb{F}$ so that $\Delta_i^2 = \sigma_i \otimes \epsilon_i$ for $i=1,2$. Then it can be seen that $\widetilde{\Delta}^2 = \sigma_1 \otimes (\epsilon_1, 0) + \sigma_2 \otimes (0,\epsilon_2)$. Recalling the definition of the sum $[Q,\epsilon] = c_1[P_1,\epsilon_1]+c_2[P_2,\epsilon_2]$, we have $Q = P_1 \times_P P_2/(c_1^{-1}\epsilon_1, -c_2^{-1}\epsilon_2)$ and ϵ is the class of $(c_1^{-1}\epsilon_1, 0)$. Thus, setting $\Delta' = \widetilde{\Delta} \otimes_{P_1 \times_P P_2} Q$, we have the mapping $\Delta' : F \mathbin{\widehat{\otimes}_k} Q \to F \mathbin{\widehat{\otimes}_k} Q[-1]$ which is a lifting map of Δ to Q, and we easily see that ${\Delta'}^2 = (c_1\sigma_1 + c_2\sigma_2) \otimes \epsilon$. Consequently, we have $\alpha_{\mathbb{L}}([Q,\epsilon]) = c_1[\sigma_1] + c_2[\sigma_2] = c_1\alpha_{\mathbb{L}}([P_1,\epsilon_1]) + c_2\alpha_{\mathbb{L}}([P_2,\epsilon_2])$. □

Lemma 4.2. *Let $f : P_1 \to P_2$ be a k-algebra map in $\widehat{\mathcal{A}}_k$, and let $\mathbb{L}_2 = (F \mathbin{\widehat{\otimes}_k} P_2, \Delta_2)$ be a lift of $\mathbb{F}$ to P_2. Suppose there exists a lift $\mathbb{L}_1 = (F \mathbin{\widehat{\otimes}_k} P_1, \Delta_1)$ of $\mathbb{L}_2$ to P_1, i.e. $\mathbb{L}_1 \otimes_{P_1\,f} P_2 = \mathbb{L}_2$. Then there is a commutative diagram*

$$\begin{array}{ccc} \mathcal{T}(P_2) & \xrightarrow{\quad f^* \quad} & \mathcal{T}(P_1) \\ & {\scriptstyle \alpha_{\mathbb{L}_2}} \searrow \quad \swarrow {\scriptstyle \alpha_{\mathbb{L}_1}} & \\ & \mathrm{Ext}_R^2(\mathbb{F},\mathbb{F}). & \end{array}$$

Proof. Set $[P_1',\epsilon_1] = f^*([P_2',\epsilon_2])$ for $[P_2',\epsilon_2] \in \mathcal{T}(P_2)$. From the definition, there is a pull-back diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & k & \xrightarrow{\epsilon_1} & P_1' & \longrightarrow & P_1 & \longrightarrow & 0 \\ & & \| & & {\scriptstyle f'}\downarrow & & {\scriptstyle f}\downarrow & & \\ 0 & \longrightarrow & k & \xrightarrow{\epsilon_2} & P_2' & \longrightarrow & P_2 & \longrightarrow & 0. \end{array}$$

Let Δ'_1 be any lift of Δ_1 onto $F \widehat{\otimes}_k P'_1$. Then $\Delta'_2 := \Delta'_1 \otimes_{P'_1\ f'} P'_2$ is a lift of $\Delta_2 = \Delta_1 \otimes_{P_1\ f} P_2$ onto $F \widehat{\otimes}_k P'_2$. Now write ${\Delta'}_1^2 = \sigma \otimes \epsilon_1$ so that $\alpha_{\mathbb{L}_1}([P'_1, \epsilon_1]) = [\sigma]$. Then, we have ${\Delta'}_2^2 = {\Delta'}_1^2 \otimes_{P'_1\ f'} P'_2 = \sigma \otimes \epsilon_2$, hence $\alpha_{\mathbb{L}_2}([P'_2, \epsilon]) = [\sigma]$. This shows that $\alpha_{\mathbb{L}_2} = \alpha_{\mathbb{L}_1} \cdot f^*$. $\square$

Theorem 4.2. *Let $\mathbb{L}_0 = (F \widehat{\otimes}_k P_0, \Delta_0)$ be the universal lift of $\mathbb{F} = (F, d)$ with parameter algebra P_0. Then, the k-linear mapping*

$$\alpha_{\mathbb{L}_0} : \mathcal{T}(P_0) \to \mathrm{Ext}^2_R(\mathbb{F}, \mathbb{F})$$

is an injection.

Proof. Let $[P_1, \epsilon_1] \in \mathcal{T}(P_0)$ be a nontrivial small extension. We only have to show $\alpha_{\mathbb{L}_0}([P_1, \epsilon_1]) \neq 0$. Suppose $\alpha_{\mathbb{L}_0}([P_1, \epsilon_1]) = 0$. Then, for any lifting map $\Delta_1 : F \widehat{\otimes}_k P_1 \to F \widehat{\otimes}_k P_1[-1]$ of Δ_0 to P_1, we have $\Delta_1^2 = \sigma \otimes \epsilon_1$, where $\sigma = [d, h]$ for some graded homomorphism $h : F \to F[-1]$. Now putting $\Delta'_1 = \Delta_1 - h \otimes \epsilon_1$, one can see that ${\Delta'_1}^2 = \Delta_1^2 - [d, h] \otimes \epsilon_1 = 0$. Therefore, $(F \widehat{\otimes}_k P_1, \Delta'_1)$ is a lifting chain complex of $(F \widehat{\otimes}_k P_0, \Delta_0)$. This is a contradiction, because $(F \widehat{\otimes}_k P_0, \Delta_0)$ is a maximal lift. See Theorem 3.2. $\square$

Corollary 4.1. *Suppose $\mathrm{Ext}^2_R(\mathbb{F}, \mathbb{F}) = 0$. Then the parameter algebra P_0 of the universal lift of $\mathbb{F}$ is isomorphic to the non-commutative formal power series ring $k\langle\langle t_1, t_2, \ldots, t_r \rangle\rangle$.*

Proof. Under the assumption, we have $\mathcal{T}(P_0) = 0$ by Theorem 4.2. Therefore if we describe $P_0 = T/I_0$ where T is a formal power series ring and $I \subseteq \mathfrak{m}_T^2$ is a closed ideal, then Proposition 2.4 forces $\mathrm{Hom}_{con}(I, k) = 0$. Thus we only have to show the following lemma. $\square$

Lemma 4.3. *Let I be a closed ideal of a non-commutative formal power series ring T. If $\mathrm{Hom}_{con}(I, k) = 0$, then $I = 0$.*

Proof. Suppose $I \neq 0$. Then, by Corollary 2.3, we have $I \neq \overline{\mathfrak{m}_T I + I\mathfrak{m}_T}$. Therefore, $I \neq \mathfrak{m}_T I + I \mathfrak{m}_T + (\mathfrak{m}_T^n \cap I)$ for a large integer n. Since $\mathrm{Hom}_{con}(I, k)$ contains every k-linear map $I/\mathfrak{m}_T I + I\mathfrak{m}_T + (\mathfrak{m}_T^n \cap I) \to k$, we have $\mathrm{Hom}_{con}(I, k) \neq 0$. $\square$

This corollary can be generalized to the following theorem.

Theorem 4.3. *Let $P_0 = T/I_0$ be the parameter algebra of the universal lift of $\mathbb{F} = (F, d)$, where T is a non-commutative formal power series ring and $I_0 \subseteq \mathfrak{m}_T^2$ is a closed ideal. Suppose $\ell = \dim_k \mathrm{Ext}^2_R(\mathbb{F}, \mathbb{F})$ is finite. Then, the ideal I_0 is analytically generated by at most ℓ elements.*

Proof. Combining Theorems 2.4 and 4.2, we have an injective k-linear map $\mathrm{Hom}_{con}(I_0, k) \to \mathrm{Ext}^2_R(\mathbb{F}, \mathbb{F})$. In particular, $\mathrm{Hom}_{con}(I_0, k)$ is a k-vector space of finite dimension. Since we have the equality $\mathrm{Hom}_{con}(I_0, k) = \bigcup_{n=1}^{\infty} \mathrm{Hom}_k(I_0/\mathfrak{m}_T I_0 + I_0 \mathfrak{m}_T + (\mathfrak{m}_T^n \cap I),\ k)$ by definition, there is an integer n_0 such that $\mathrm{Hom}_{con}(I_0, k) = \mathrm{Hom}_k(I_0/\mathfrak{m}_T I_0 + I_0\mathfrak{m}_T + (\mathfrak{m}_T^n \cap I),\ k)$ for $n \geq n_0$. Hence we have the equalities

$$\mathfrak{m}_T I_0 + I_0\mathfrak{m}_T + (\mathfrak{m}_T^n \cap I) = \mathfrak{m}_T I_0 + I_0\mathfrak{m}_T + (\mathfrak{m}_T^{n+1} \cap I) = \cdots = \overline{\mathfrak{m}_T I_0 + I_0\mathfrak{m}_T}.$$

Thus it follows that $\mathrm{Hom}_{con}(I_0, k) = \mathrm{Hom}_k(I_0/\overline{\mathfrak{m}_T I_0 + I_0\mathfrak{m}_T},\ k)$. Since this is of dimension at most ℓ, we have $\dim_k(I_0/\overline{\mathfrak{m}_T I_0 + I_0\mathfrak{m}_T}) \leq \ell$. Therefore, by virtue of Proposition 2.3, I_0 is analytically generated by at most ℓ elements. □

4.2. Universal lifts based on commutative algebras

Remark 9. Let $T = k\langle\langle t_1, t_2, \ldots, t_r\rangle\rangle$ be a non-commutative formal power series ring. We denote by C the commutator ideal which is a two-sided ideal generated by the commutators $t_it_j - t_jt_i$ $(1 \leq i < j \leq r)$, i.e.

$$C = (\{t_it_j - t_jt_i \mid 1 \leq i < j \leq r\}).$$

Note that C is not a closed ideal if $r \geq 2$. It is however easy to see that $T/\overline{C}$ is isomorphic to the commutative formal power series ring $k[[t_1, \ldots, t_r]]$.

Remark 10. Let I be an ideal of T that contains $\overline{C}$. Then, I is a closed ideal and there are a finite number of elements $f_1, \ldots, f_\ell \in I$ with the equality $I = (f_1, \ldots, f_\ell) + \overline{C}$.

In fact, it is well known that any ideal of $T/\overline{C} = k[[t_1, \ldots, t_r]]$ is closed. Since the natural projection $\pi : T \to T/\overline{C}$ is continuous, $I = \pi^{-1}(I/\overline{C})$ is closed as well. Since $T/\overline{C}$ is noetherian, we can find finite elements $f_1, \ldots, f_\ell$ which generate the ideal $I/\overline{C}$. Then we have the equality $I = (f_1, \ldots, f_\ell) + \overline{C}$.

Recall from Section 3.3 that $\mathcal{F} : \mathcal{A}_k \to (Sets)$ is a covariant functor such that $\mathcal{F}(A)$ is the set of isomorphism classes of lifting chain complexes of $\mathbb{F}$ to A, for any $A \in \mathcal{A}_k$. We consider here the restriction of $\mathcal{F}$ to commutative artinian algebras. For this end, we denote by $\mathcal{C}_k$ the category of commutative artinian local k-algebras A with $A/\mathfrak{m}_A \cong k$ and k-algebra homomorphisms. Note that $\mathcal{C}_k$ is a full subcategory of $\mathcal{A}_k$.

Theorem 4.4. *Let $\mathbb{L}_0 = (F \widehat{\otimes}_k P_0, \Delta_0)$ be the universal lift of $\mathbb{F}$ with parameter algebra $P_0 = T/I_0$ where T is a non-commutative formal power series ring and $I \subseteq \mathfrak{m}_T^2$. We set $Q_0 = T/I_0 + \overline{C}$ which is a commutative noetherian complete local k-algebra. Then, the restricted functor $\mathcal{F}|_{\mathcal{C}_k} : \mathcal{C}_k \to (Sets)$ is pro-represented by Q_0, i.e. there is an isomorphism $\mathcal{F}|_{\mathcal{C}_k} \cong \mathrm{Hom}_{k\text{-}alg}(Q_0,\quad)$ as functors on $\mathcal{C}_k$.*

Proof. Note that if $A \in \mathcal{C}_k$, then $\mathrm{Hom}_{k\text{-alg}}(P_0, A) = \mathrm{Hom}_{k\text{-alg}}(Q_0, A)$. The theorem follows from this observation. □

Definition 4.1. We call Q_0 in the theorem a **commutative parameter algebra of the universal lift of** $\mathbb{F}$. And we call $\mathbb{L}_0 \otimes_{P_0} Q_0$ the universal lift of $\mathbb{F}$ based on commutative parameter algebra.

Remark 11. If $\mathbb{F}$ is a projective resolution of a left R-module M, then the universal lift of $\mathbb{F}$ based on commutative parameter algebra is nothing but the universal deformation of M whose existence is mentioned in Theorem 1.1. (See also Proposition 3.4.)

The commutative parameter algebra Q_0 is of the form $k[[t_1, \dots, t_r]]/\mathfrak{a}$ where $\mathfrak{a} = I_0 + \overline{C}/\overline{C} \subseteq T/\overline{C} = k[[t_1, \dots, t_r]]$.

Proposition 4.1. *Let $Q_0 = k[[t_1, \dots, t_r]]/\mathfrak{a}$ be a commutative parameter algebra of the universal lift of $\mathbb{F}$. Then, the minimal number of generators of $\mathfrak{a}$ is at most $\dim_k \mathrm{Ext}^2_R(\mathbb{F}, \mathbb{F})$.*

Proof. It suffices to argue when $\ell = \mathrm{Ext}^2_R(\mathbb{F}, \mathbb{F})$ is finite. Then, by Theorem 4.3, there are $f_1, \dots, f_\ell \in I_0$ satisfying the equality $I_0 = \overline{(f_1, \dots, f_\ell)}$. Thus, by virtue of Remark 10, we have the equalities

$$I_0 + \overline{C} = \overline{I_0 + C} = \overline{C + (f_1, \dots, f_\ell)} = \overline{C} + (f_1, \dots, f_\ell).$$

Therefore $\mathfrak{a} = \overline{I_0 + C}/\overline{C}$ is generated by the images of $f_1, \dots, f_\ell$ in $T/\overline{C} = k[[t_1, \dots, t_r]]$. $\square$

4.3. Yoneda products

Let $\mathbb{F} = (F, d)$ be a projective complex over R with $r = \dim_k \mathrm{Ext}^1_R(\mathbb{F}, \mathbb{F})$ being finite as before. Then, as in the beginning of Section 3.2, we may consider the lifting chain complex $\mathbb{L} = (F \mathbin{\widehat{\otimes}_k} T/\mathfrak{m}_T^2, \delta)$ with $\delta = d \otimes 1 + \sum_{i=1}^r t_i^* \otimes t_i$, where $T = k\langle\langle t_1, t_2, \dots, t_r \rangle\rangle$ is a non-commutative formal power series ring and $t_1^*, \dots, t_r^*$ are chain homomorphisms which form a k-basis of $\mathrm{Ext}^1_R(\mathbb{F}, \mathbb{F})$. Since $\mathbb{L}$ is a lift of $\mathbb{F}$, we have the k-linear map

$$\alpha_{\mathbb{L}} : \mathcal{T}(T/\mathfrak{m}_T^2) \to \mathrm{Ext}^2_R(\mathbb{F}, \mathbb{F}).$$

by 4.1. See also Lemma 4.1.

Note that $\mathrm{Ext}^{\cdot}_R(\mathbb{F}, \mathbb{F}) = \bigoplus_{i=-\infty}^{\infty} \mathrm{Ext}^i_R(\mathbb{F}, \mathbb{F})$ is an algebra, called Yoneda algebra, whose multiplication is given by Yoneda product. In fact, if $f : F \to F[-i]$ and $g : F \to F[-j]$ are chain homomorphisms of degree $-i$ and $-j$ respectively, then the composite $f \cdot g : F \to F[-i-j]$ is a chain homomorphism of degree $-i-j$, and the product in $\mathrm{Ext}^{\cdot}_R(\mathbb{F}, \mathbb{F})$ is given by $[f][g] = [f \cdot g]$. In the following lemma, $\mathrm{Ext}^1_R(\mathbb{F}, \mathbb{F})^2$ denotes the k-subspace of $\mathrm{Ext}^2_R(\mathbb{F}, \mathbb{F})$ generated by all the products of two elements in $\mathrm{Ext}^1_R(\mathbb{F}, \mathbb{F})$.

Lemma 4.4. *Under the circumstances above, the image of $\alpha_{\mathbb{L}}$ is exactly $\mathrm{Ext}^1_R(\mathbb{F}, \mathbb{F})^2$, i.e., $\alpha_{\mathbb{L}}(\mathcal{T}(T/\mathfrak{m}_T^2)) = \mathrm{Ext}^1_R(\mathbb{F}, \mathbb{F})^2$.*

Proof. Recall from Proposition 2.4 that there is an isomorphism of k-vector spaces $\tau : \mathrm{Hom}_k(\mathfrak{m}_T^2/\mathfrak{m}_T^3, k) = \mathrm{Hom}_{con}(\mathfrak{m}_T^2, k) \to \mathcal{T}(T/\mathfrak{m}_T^2)$. Suppose $\tau(f) = [T/I, \epsilon]$ for $f \neq 0 \in \mathrm{Hom}_k(\mathfrak{m}_T^2/\mathfrak{m}_T^3, k)$. Then, by definition of τ, we have $f(\epsilon) = 1$ and I is the kernel of the composition mapping $\mathfrak{m}_T^2 \to \mathfrak{m}_T^2/\mathfrak{m}_T^3$ with $f : \mathfrak{m}_T^2/\mathfrak{m}_T^3 \to k$. Note, in this case, that T/I has $\{1, \overline{t_1}, \dots, \overline{t_r}, \epsilon\}$ as a

k-basis, where $\overline{t_i}$ denotes the image of t_i in T/I. Also note that if we denotes $f(\overline{t_i t_j}) = c_{ij} \in k$, then f is completely determined by these c_{ij}'s.

We can take the following map Δ as a lifting graded homomorphism of δ on $F \otimes_k T/I$.

$$\Delta = d \otimes 1 + \sum_{i=1}^{r} t_i^* \otimes \overline{t_i} + 0 \otimes \epsilon.$$

Then, we have $\Delta^2 = \sum_{i,j=1}^{r} t_i^* t_j^* \otimes \overline{t_i t_j}$. Since $\overline{t_i t_j} = f(\overline{t_i t_j})\epsilon = c_{ij}\epsilon$, it follows that $\Delta^2 = \sum_{i,j=1}^{r} c_{ij} t_i^* t_j^* \otimes \epsilon$. Therefore, from the definition of $\alpha_{\mathbb{L}}$, we see $\alpha_{\mathbb{L}}([T/I, \epsilon]) = \sum_{i,j=1}^{r} c_{ij}[t_i^*][t_j^*]$, which is in fact an element of $\mathrm{Ext}_R^1(\mathbb{F}, \mathbb{F})^2$. Since we can take any elements of k as c_{ij}, the image of $\alpha_{\mathbb{L}}$ is exactly $\mathrm{Ext}_R^1(\mathbb{F}, \mathbb{F})^2$. $\square$

Theorem 4.5. *Let $\mathbb{L}_0 = (F \widehat{\otimes}_k P_0, \Delta_0)$ be the universal lift of $\mathbb{F}$ with the parameter algebra $P_0 = T/I_0$, where $T = k\langle\langle t_1, t_2, \ldots, t_r \rangle\rangle$ is a non-commutative formal power series ring and $I_0 \subseteq \mathfrak{m}_T^2$. Then the image of the injective map $\alpha_{\mathbb{L}_0} : \mathcal{T}(P_0) \to \mathrm{Ext}_R^2(\mathbb{F}, \mathbb{F})$ contains $\mathrm{Ext}_R^1(\mathbb{F}, \mathbb{F})^2$. And there is an isomorphism of k-vector spaces*

$$\mathrm{Ext}_R^1(\mathbb{F}, \mathbb{F})^2 \cong \mathrm{Hom}_k(I_0/I_0 \cap \mathfrak{m}_T^3, k).$$

Proof. Let $\mathbb{L}$ be the lifting chain complex $(F \otimes_k T/\mathfrak{m}_T^2, \delta)$ of $\mathbb{F}$, where $\delta = d \otimes 1 + \sum_{i=1}^{r} t_i^* \otimes t_i$ as above. And let $\mathbb{L}_0 = (F \widehat{\otimes}_k T/I_0, \Delta_0)$ be the universal lift of $\mathbb{F}$. We denote by q the natural injection $I_0 \to \mathfrak{m}_T^2$ and by p the projection $T/I_0 \to T/\mathfrak{m}_T^2$.

Combining all the results in 2.4, 4.2 and 4.4, we have the following commutative diagram.

$$\begin{array}{cccccc} & \mathrm{Hom}_{con}(\mathfrak{m}_T^2, k) & \xrightarrow[\cong]{\tau} & \mathcal{T}(T/\mathfrak{m}_T^2) & \xrightarrow{\alpha_{\mathbb{L}}} & \mathrm{Ext}_R^1(\mathbb{F}, \mathbb{F})^2 \\ (4.1) & {\scriptstyle q^*}\downarrow & & {\scriptstyle p^*}\downarrow & & {\scriptstyle \iota}\downarrow \\ & \mathrm{Hom}_{con}(I_0, k) & \xrightarrow[\cong]{\tau} & \mathcal{T}(T/I_0) & \xrightarrow{\alpha_{\mathbb{L}_0}} & \mathrm{Ext}_R^2(\mathbb{F}, \mathbb{F}), \end{array}$$

where ι is a natural injection. Note from Theorem 4.2 and Lemma 4.4 that $\alpha_{\mathbb{L}_0}$ is injective, and $\alpha_{\mathbb{L}}$ is surjective. Therefore $\alpha_{\mathbb{L}_0}(\mathcal{T}(P_0))$ contains $\mathrm{Ext}_R^1(\mathbb{F}, \mathbb{F})^2$ and we have an isomorphism of k-vector spaces

$$q^*(\mathrm{Hom}_{con}(\mathfrak{m}_T^2, k)) \cong \mathrm{Ext}_R^1(\mathbb{F}, \mathbb{F})^2.$$

Note that $\mathrm{Hom}_{con}(\mathfrak{m}_T^2, k) = \mathrm{Hom}_k(\mathfrak{m}_T^2/\mathfrak{m}_T^3, k)$. Hence we may describe as follows:

$$\begin{aligned} \mathrm{Ker}(q^*) &= \{f \in \mathrm{Hom}_k(\mathfrak{m}_T^2/\mathfrak{m}_T^3,\ k) \mid f(I_0 + \mathfrak{m}_T^3/\mathfrak{m}_T^3) = 0\} \\ &= \mathrm{Hom}_k(\mathfrak{m}_T^2/I_0 + \mathfrak{m}_T^3,\ k). \end{aligned}$$

Thus, from the obvious exact sequence

$$0 \longrightarrow I_0/I_0 \cap \mathfrak{m}_T^3 \longrightarrow \mathfrak{m}_T^2/\mathfrak{m}_T^3 \longrightarrow \mathfrak{m}_T^2/I_0 + \mathfrak{m}_T^3 \longrightarrow 0,$$

we finally have

$$\begin{aligned}\operatorname{Ext}^1_R(\mathbb{F},\mathbb{F})^2 &\cong q^*(\operatorname{Hom}_{con}(\mathfrak{m}_T^2,k))\\ &\cong \operatorname{Hom}_k(\mathfrak{m}_T^2/\mathfrak{m}_T^3,k)/\operatorname{Hom}_k(\mathfrak{m}_T^2/I_0+\mathfrak{m}_T^3,\ k)\\ &\cong \operatorname{Hom}_k(I_0/I_0\cap\mathfrak{m}_T^3,k).\end{aligned}$$

□

Note in the theorem that $I_0/I_0\cap\mathfrak{m}_T^3$ is a finite dimensional k-vector space, since it is a subspace of $\mathfrak{m}_T^2/\mathfrak{m}_T^3$. As a direct consequence of the theorem we have the following corollary.

Corollary 4.2. *Let $P_0=T/I_0$ be the parameter algebra of the universal lift of $\mathbb{F}$, where $T=k\langle\langle t_1,t_2,\dots,t_r\rangle\rangle$ is a non-commutative formal power series ring and $I_0\subseteq\mathfrak{m}_T^2$. Then, $I_0\subseteq\mathfrak{m}_T^3$ if and only if $\operatorname{Ext}^1_R(\mathbb{F},\mathbb{F})^2=0$.*

Proposition 4.2. *Let $\mathbb{L}_0=(F\mathbin{\widehat{\otimes}_k}P_0,\Delta_0)$ be the universal lift of $\mathbb{F}$ with the parameter algebra $P_0=T/I_0$, where $T=k\langle\langle t_1,t_2,\dots,t_r\rangle\rangle$ is a non-commutative formal power series ring and $I_0\subseteq\mathfrak{m}_T^2$. Then the following two conditions are equivalent.*

(a) *The image of the mapping $\alpha_{\mathbb{L}_0}:\mathcal{T}(P_0)\to\operatorname{Ext}^2_R(\mathbb{F},\mathbb{F})$ is exactly $\operatorname{Ext}^1_R(\mathbb{F},\mathbb{F})^2$.*

(b) *There exist elements $f_1,\dots,f_\ell\in I_0$ which analytically generate the ideal I_0 such that they give rise to linearly independent elements in $\mathfrak{m}_T^2/\mathfrak{m}_T^3$.*

Proof. By the commutative diagram (4.1) in the proof of Theorem 4.5, we see that the condition (a) is equivalent to that the k-linear mapping $q^*:\operatorname{Hom}_{con}(\mathfrak{m}_T^2,k)\to\operatorname{Hom}_{con}(I_0,k)$ is surjective, where $q:I_0\to\mathfrak{m}_T^2$ is a natural injection.

To prove the implication $(a)\Rightarrow(b)$, suppose q^* is surjective. Since $\operatorname{Hom}_{con}(\mathfrak{m}_T^2,k)=\operatorname{Hom}_k(\mathfrak{m}_T/\mathfrak{m}_T^2,k)$ is a finite dimensional k-vector space, so is

$$\operatorname{Hom}_{con}(I_0,\ k)=\bigcup_{n\geq 1}\operatorname{Hom}_k(I_0/\mathfrak{m}_TI_0+I_0\mathfrak{m}_T+(I_0\cap\mathfrak{m}_T^n),\ k).$$

Hence there is an integer $n_0\geq 1$ such that

$$\mathfrak{m}_TI_0+I_0\mathfrak{m}_T+(I_0\cap\mathfrak{m}_T^{n_0})=\mathfrak{m}_TI_0+I_0\mathfrak{m}_T+(I_0\cap\mathfrak{m}_T^{n_0+1})=\cdots=\overline{\mathfrak{m}_TI_0+I_0\mathfrak{m}_T}.$$

Therefore q^* induces a surjective mapping $\operatorname{Hom}_k(\mathfrak{m}_T^2/\mathfrak{m}_T^3,k)\to\operatorname{Hom}_k(I_0/\overline{\mathfrak{m}_TI_0+I_0\mathfrak{m}_T},\ k)$, hence the natural mapping $I_0/\overline{\mathfrak{m}_TI_0+I_0\mathfrak{m}_T}\to\mathfrak{m}_T^2/\mathfrak{m}_T^3$ is injective. Now let us take the elements $f_1,\dots,f_r\in I_0$ whose images in $I_0/\overline{\mathfrak{m}_TI_0+I_0\mathfrak{m}_T}$ form a k-basis. Then, the images of $f_1,\dots,f_r$ in $\mathfrak{m}_T^2/\mathfrak{m}_T^3$ are linearly independent and it follows from Proposition 2.3 that I_0 is analytically generated by $f_1,\dots,f_r$.

To prove $(b)\Rightarrow(a)$, suppose we have elements $f_1,\dots,f_\ell\in I_0$ such that they give rise to linearly independent elements in $\mathfrak{m}_T^2/\mathfrak{m}_T^3$ and $I_0=\overline{(f_1,\dots,f_r)}$.

By Proposition 2.3 we may assume that the images of $f_1, \dots, f_r$ in $I_0/\overline{\mathfrak{m}_T I_0 + I_0 \mathfrak{m}_T}$ form a k-basis. Note that $I_0 \cap \mathfrak{m}_T^3$ is a closed ideal of T containing $\mathfrak{m}_T I_0 + I_0 \mathfrak{m}_T$, hence we have an inclusion relation $\overline{\mathfrak{m}_T I_0 + I_0 \mathfrak{m}_T} \subseteq I_0 \cap \mathfrak{m}_T^3 \subseteq \mathfrak{m}_T^3$. Therefore we obtain the natural map

$$I_0/\overline{\mathfrak{m}_T I_0 + I_0 \mathfrak{m}_T} \xrightarrow{g} I_0/I_0 \cap \mathfrak{m}_T^3 \subseteq \mathfrak{m}_T^2/\mathfrak{m}_T^3.$$

Since g maps the k-basis of $I_0/\overline{\mathfrak{m}_T I_0 + I_0 \mathfrak{m}_T}$ to a set of linearly independent elements in $\mathfrak{m}_T^2/\mathfrak{m}_T^3$, we have the injectivity of g. In particular, the equality $\overline{\mathfrak{m}_T I_0 + I_0 \mathfrak{m}_T} = I_0 \cap \mathfrak{m}_T^3$ holds. Hence it follows that

$$\mathfrak{m}_T I_0 + I_0 \mathfrak{m}_T + (I_0 \cap \mathfrak{m}_T^n) = I_0 \cap \mathfrak{m}_T^3,$$

for all $n \geq 3$. Thus we have the equality $\mathrm{Hom}_{con}(I_0,\ k) = \mathrm{Hom}_k(I_0/I_0 \cap \mathfrak{m}_T^3,\ k)$. Therefore the map $q^* : \mathrm{Hom}_{con}(\mathfrak{m}_T^2, k) \to \mathrm{Hom}_{con}(I_0, k)$ is the same as the k-dual of the natural injection $I_0/I_0 \cap \mathfrak{m}_T^3 \subseteq \mathfrak{m}_T^2/\mathfrak{m}_T^3$. The surjectivity of q^* is now obvious. $\square$

4.4. Comparison of cohomology

As in the previous sections, $\mathbb{F} = (F, d)$ denotes a projective complex over R, where R is an associative k-algebra. We assume that $r = \dim_k \mathrm{Ext}_R^1(\mathbb{F}, \mathbb{F})$ is finite as before. Adding to this assumption, we assume in the rest of the paper that the complex $\mathbb{F}$ is right bounded, i.e. there is an integer s such that $F_i = 0$ for $i < s$.

We also denotes by $\mathbb{L}_0 = (F \mathbin{\widehat{\otimes}_k} P_0, \Delta_0)$ the universal lift of $\mathbb{F}$ with the parameter algebra P_0. Note that $\mathbb{L}_0$ may not be projective as a right P_0-module. They are even non-flat as seen in Example 2.2.

For any integer $n \geq 1$, we set

$$\mathbb{L}_0^{(n)} = (F \otimes_k P_0/\mathfrak{m}_{P_0}^n,\ \Delta^{(n)}) = \mathbb{L} \otimes_{P_0} P_0/\mathfrak{m}_{P_0}^n.$$

In fact, each $\mathbb{L}_0^{(n)}$ is a right bounded complex of projective left $R \otimes_k (P_0/\mathfrak{m}_{P_0}^n)^{op}$-modules, in particular, it is a free right $P_0/\mathfrak{m}_{P_0}^n$-module.

For any associative k-algebra R, we denote by $D_+(R)$ the derived category consisting of right bounded complexes over R. Then, tensoring the chain complex $\mathbb{L}_0^{(n)}$ yields the functor ρ_n between the derived categories:

$$\rho_n : D_+(P_0/\mathfrak{m}_{P_0}^n) \to D_+(R),$$

which is defined by $\rho_n(X) = \mathbb{L}_0^{(n)} \otimes_{P_0/\mathfrak{m}_{P_0}^n} X$. This is well-defined, since $\mathbb{L}_0^{(n)}$ is a right bounded complex of projective left $R \otimes_k (P_0/\mathfrak{m}_{P_0}^n)^{op}$-modules.

Note that the natural projection $P_0/\mathfrak{m}_{P_0}^{n+1} \to P_0/\mathfrak{m}_{P_0}^n$ induces a natural functor $D_+(P_0/\mathfrak{m}_{P_0}^n) \to D_+(P_0/\mathfrak{m}_{P_0}^{n+1})$. And it is easy to see the diagram

$$\begin{tikzcd} D_+(P_0/\mathfrak{m}_{P_0}^n) \arrow[r, "\rho_n"] \arrow[d] & D_+(R) \\ D_+(P_0/\mathfrak{m}_{P_0}^{n+1}) \arrow[ur, "\rho_{n+1}"'] & \end{tikzcd} \tag{4.2}$$

is commutative.

Note that $\mathbb{L}_0^{(n)} \otimes_{P_0/\mathfrak{m}_{P_0}^n} k = \mathbb{F}$, hence we have $\rho_n(k) = \mathbb{F}$ for each $n \geq 1$. It follows that the functor ρ_n induces the map

$$\rho_n^i : \mathrm{Ext}^i_{P_0/\mathfrak{m}_{P_0}^n}(k,k) \to \mathrm{Ext}^i_R(\mathbb{F},\mathbb{F})$$

for all integers i. Then the commutative diagram (4.2) forces the commutativity of the following diagram.

$$\begin{tikzcd} \mathrm{Ext}^i_{P_0/\mathfrak{m}_{P_0}^n}(k,k) \arrow[r, "\rho_n^i"] \arrow[d] & \mathrm{Ext}^i_R(\mathbb{F},\mathbb{F}) \\ \mathrm{Ext}^i_{P_0/\mathfrak{m}_{P_0}^{n+1}}(k,k) \arrow[ur, "\rho_{n+1}^i"'] & \end{tikzcd} \tag{4.3}$$

Definition 4.2. From the commutative diagram (4.3), we can define the inductive limit $\rho_\infty^i = \varinjlim_n \rho_n^i$ for each i ;

$$\rho_\infty^i : \varinjlim_n \mathrm{Ext}^i_{P_0/\mathfrak{m}_{P_0}^n}(k,k) \to \mathrm{Ext}^i_R(\mathbb{F},\mathbb{F}).$$

The aim of this section is to show that ρ_∞^i is an injective map for $i = 0, 1, 2$.

Note that $\varinjlim_n \mathrm{Ext}^{\cdot}_{P_0/\mathfrak{m}_{P_0}^n}(k,k) = \bigoplus_{i \geq 0} \varinjlim_n \mathrm{Ext}^i_{P_0/\mathfrak{m}_{P_0}^n}(k,k)$, as well as $\mathrm{Ext}^{\cdot}_R(\mathbb{F},\mathbb{F})$, has a structure of algebra by Yoneda product, and

$$\rho_\infty^{\cdot} : \varinjlim_n \mathrm{Ext}^{\cdot}_{P_0/\mathfrak{m}_{P_0}^n}(k,k) \to \mathrm{Ext}^{\cdot}_R(\mathbb{F},\mathbb{F})$$

is an algebra map.

First, consider the case $i = 0$. Since $\mathrm{Ext}^0_{P_0/\mathfrak{m}_{P_0}^n}(k,k) = k$ and $\rho_n^0 : k \to \mathrm{Ext}^0_R(\mathbb{F},\mathbb{F})$ is a natural injection for any $n \geq 1$, we easily see the following lemma holds.

Lemma 4.5. *The mapping $\rho_\infty^0 : k \to \mathrm{Ext}^0_R(\mathbb{F},\mathbb{F})$ is a natural injection.*

To argue for the case $i = 1$, we should notice that $\mathrm{Ext}^1_{P_0/\mathfrak{m}_{P_0}^n}(k,k) \cong \mathrm{Hom}_k(\mathfrak{m}_{P_0}/\mathfrak{m}_{P_0}^2, k)$ for all $n \geq 2$ and the natural maps $\mathrm{Ext}^1_{P_0/\mathfrak{m}_{P_0}^n}(k,k) \to \mathrm{Ext}^1_{P_0/\mathfrak{m}_{P_0}^{n+1}}(k,k)$ coincide with the identity map on $\mathrm{Hom}_k(\mathfrak{m}_{P_0}/\mathfrak{m}_{P_0}^2, k)$. Hence we have

$$\rho_\infty^1 : \mathrm{Hom}_k(\mathfrak{m}_{P_0}/\mathfrak{m}_{P_0}^2, k) \to \mathrm{Ext}^1_R(\mathbb{F},\mathbb{F}).$$

We can prove this is actually an isomorphism.

Lemma 4.6. *The mapping $\rho_\infty^1 : \mathrm{Hom}_k(\mathfrak{m}_{P_0}/\mathfrak{m}_{P_0}^2, k) \to \mathrm{Ext}^1_R(\mathbb{F},\mathbb{F})$ is an isomorphism.*

Proof. By the observation above, we only have to prove that ρ_2^1 : $\mathrm{Hom}_k(\mathfrak{m}_{P_0}/\mathfrak{m}_{P_0}^2, k) \to \mathrm{Ext}_R^1(\mathbb{F}, \mathbb{F})$ is an isomorphism. Let us denote $P_0 = T/I_0$ where $T = k\langle\langle t_1, t_2, \dots, t_r\rangle\rangle$ and $I_0 \subseteq \mathfrak{m}_T^2$. Recall that $\mathrm{Ext}_R^1(\mathbb{F}, \mathbb{F})$ has a k-basis $\{[t_1^*], [t_2^*], \dots, [t_r^*]\}$. And, by definition, we have $\mathbb{L}_0^{(2)} = (F \otimes_k T/\mathfrak{m}_T^2,\ \delta)$ with $\delta = d \otimes 1 + \sum_{i=1}^r t_i^* \otimes \overline{t_i}$, where $\overline{t_i}$ denotes the image of t_i in $\mathfrak{m}_T/\mathfrak{m}_T^2$. Therefore, it is easy to see that the mapping ρ_2^1 is defined by

$$\rho_2^1(f) = \sum_{i=1}^r f(\overline{t_i})[t_i^*],$$

for $f \in \mathrm{Hom}_k(\mathfrak{m}_T/\mathfrak{m}_T^2, k)$. The lemma follows from this. $\square$

Now we proceed to the case $i = 2$. The goal here is to prove the following theorem.

Theorem 4.6. *There is an isomorphism $\beta : \mathcal{T}(P_0) \to \varinjlim \mathrm{Ext}^2_{P_0/\mathfrak{m}_{P_0}^n}(k, k)$ which makes the following diagram commutative.*

$$\begin{tikzcd} \mathcal{T}(P_0) \arrow[r, "\alpha_{\mathbb{L}_0}"] \arrow[d, "\beta"'] & \mathrm{Ext}_R^2(\mathbb{F}, \mathbb{F}) \\ \varinjlim_n \mathrm{Ext}^2_{P_0/\mathfrak{m}_{P_0}^n}(k, k) \arrow[ur, "\rho_\infty^2"'] & \end{tikzcd}$$

In particular, ρ_∞^2 is an injective map as is $\alpha_{\mathbb{L}_0}$.

To prove this, let A be an arbitrary artinian local k-algebra in $\mathcal{A}_k$, and let $\mathbb{G}^A = (G, d^A)$ be a free resolution of the left A-module $k = A/\mathfrak{m}_A$. Then, by Theorem 3.3, there is a universal lift of $\mathbb{G}^A$ of the form $\mathbb{G}_0^A = (G \otimes_k A, \Delta_0^A)$ whose parameter algebra is A. Since $\mathbb{G}_0^A$ is a lift of $\mathbb{G}^A$, we have the k-linear map

$$\alpha_{\mathbb{G}_0^A} : \mathcal{T}(A) \to \mathrm{Ext}_A^2(\mathbb{G}^A, \mathbb{G}^A) = \mathrm{Ext}_A^2(k, k)$$

which is defined in 4.1.

Lemma 4.7. *Let $A \in \mathcal{A}_k$ as above. Then the map $\alpha_{\mathbb{G}_0^A}$ is an isomorphism.*

Proof. Since A is a parameter algebra of the universal lift $\mathbb{G}_0^A$ of $\mathbb{G}$, Theorem 4.2 implies that $\alpha_{\mathbb{G}_0^A}$ is injective. Thus, to show this is an isomorphism, it is enough to show that $\dim_k \mathcal{T}(A) = \dim_k \mathrm{Ext}_A^2(k, k)$. Let us describe $A = T/I$ where T is the non-commutative formal power series ring and $I \subseteq \mathfrak{m}_T^2$. Note that the ideal I is open and closed in T, since A is artinian. Therefore, by Proposition 2.4, we have $\mathcal{T}(A) \cong \mathrm{Hom}_{con}(I, k) = \mathrm{Hom}_k(I/I\mathfrak{m}_A + \mathfrak{m}_A I, k)$. On the other hand, by the following lemma, we know that $\mathrm{Ext}_A^2(k, k) \cong \mathrm{Hom}_k(I/I\mathfrak{m}_T + \mathfrak{m}_T I, k) \cong \mathcal{T}(A)$. Hence $\dim_k \mathrm{Ext}_A^2(k, k) = \dim_k \mathcal{T}(A)$. $\square$

Lemma 4.8. *Let $P = T/I \in \widehat{\mathcal{A}}_k$ where $T = k\langle\langle t_1, t_2, \ldots, t_r\rangle\rangle$ is a non-commutative formal power series ring and $I \subseteq \mathfrak{m}_T^2$. Then there is an isomorphism* $\mathrm{Ext}_P^2(k,k) \cong \mathrm{Hom}_k(I/I\mathfrak{m}_T + \mathfrak{m}_T I, k) \cong \mathrm{Hom}_{T\text{-}bimod}(I,k)$.

Proof. By virtue of Lemma 2.3, there is a minimal free resolution of k as a left T-module of the form

$$0 \longrightarrow T^r \longrightarrow T \longrightarrow k \longrightarrow 0.$$

Therefore, tensoring P over T, we have an exact sequence of left P-modules

$$0 \longrightarrow \mathrm{Tor}_1^T(P,k) \longrightarrow P^r \longrightarrow P \longrightarrow k \longrightarrow 0.$$

Note that, by the exact sequence of right T-modules $0 \to I \to T \to P \to 0$, we have $\mathrm{Tor}_1^T(P,k) \cong I/I\mathfrak{m}_T$ which is an isomorphism of left P-modules. Therefore we have $\mathrm{Ext}_P^2(k,k) \cong \mathrm{Hom}_P(I/I\mathfrak{m}_T, k) = \mathrm{Hom}_k(I/I\mathfrak{m}_T + \mathfrak{m}_T I, k)$. □

One can show that the isomorphism $\alpha_{\mathbb{G}_0^A}$ does not depend on the choice of free resolution $\mathbb{G}^A$ and its lift $\mathbb{G}_0^A$. This follows from the following more general lemma.

Lemma 4.9. *Let $\mathbb{F}^{(1)} = (F^{(1)}, d^{(1)})$ and $\mathbb{F}^{(2)} = (F^{(2)}, d^{(2)})$ be right bounded projective complexes over R. For $A \in \mathcal{A}_k$, let $\mathbb{G}^{(i)} = (F^{(i)} \otimes_k A, \Delta^{(i)})$ be a lift of $\mathbb{F}^{(i)}$ to A for $i = 1, 2$. Suppose that there is a quasi-isomorphism $q : \mathbb{G}^{(1)} \to \mathbb{G}^{(2)}$ of chain complexes over $R \otimes_k A^{op}$. Then, there is a commutative diagram*:

$$\begin{array}{ccc} \mathcal{T}(A) & \xrightarrow{\alpha_{\mathbb{G}^{(1)}}} & \mathrm{Ext}_R^2(\mathbb{F}^{(1)}, \mathbb{F}^{(1)}) \\ {\scriptstyle \alpha_{\mathbb{G}^{(2)}}}\Big\downarrow & & {\scriptstyle q_*}\Big\downarrow \\ \mathrm{Ext}_R^2(\mathbb{F}^{(2)}, \mathbb{F}^{(2)}) & \xrightarrow{q^*} & \mathrm{Ext}_R^2(\mathbb{F}^{(1)}, \mathbb{F}^{(2)}) \end{array}$$

Proof. Let $[A', \epsilon] \in \mathcal{T}(A)$ and let $\widetilde{\Delta}^{(i)}$ be a lifting homomorphism $F^{(i)} \otimes_k A' \to F^{(i)} \otimes_k A'[-1]$ of $\Delta^{(i)}$ for $i = 1, 2$. Then, by the definition of $\alpha_{\mathbb{G}^{(i)}}$, we have the description $(\widetilde{\Delta}^{(i)})^2 = h^{(i)} \otimes_k \epsilon$ with $h^{(i)} : F^{(i)} \to F^{(i)}[-2]$ being a chain homomorphism, and the equality $\alpha_{\mathbb{G}^{(i)}}([A', \epsilon]) = [h^{(i)}]$ holds. Now take a lifting map $q' : F^{(1)} \otimes_k A' \to F^{(2)} \otimes_k A'$ of a graded homomorphism q and we have the commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F^{(1)} & \xrightarrow{1\otimes\epsilon} & F^{(1)} \otimes_k A' & \longrightarrow & F^{(1)} \otimes_k A & \longrightarrow & 0 \\ & & {\scriptstyle q\otimes_A k}\Big\downarrow & & {\scriptstyle q'}\Big\downarrow & & {\scriptstyle q}\Big\downarrow & & \\ 0 & \longrightarrow & F^{(2)} & \xrightarrow{1\otimes\epsilon} & F^{(2)} \otimes_k A' & \longrightarrow & F^{(2)} \otimes_k A & \longrightarrow & 0 \end{array}$$

Since $\Delta^{(2)} q = q \Delta^{(1)}$, we have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F^{(1)} & \xrightarrow{1\otimes\epsilon} & F^{(1)} \otimes_k A' & \longrightarrow & F^{(1)} \otimes_k A & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle 0} & & \Big\downarrow{\scriptstyle \widetilde{\Delta}^{(2)} q' - q' \widetilde{\Delta}^{(1)}} & & \Big\downarrow{\scriptstyle 0} & & \\ 0 & \longrightarrow & F^{(2)}[-1] & \xrightarrow{1\otimes\epsilon} & F^{(2)} \otimes_k A'[-1] & \longrightarrow & F^{(2)} \otimes_k A[-1] & \longrightarrow & 0 \end{array}$$

Hence there is a graded homomorphism $p : F^{(1)} \to F^{(2)}[-1]$ with the equality

$$\widetilde{\Delta}^{(2)} q' - q' \widetilde{\Delta}^{(1)} = p \otimes \epsilon.$$

Multiplying $\widetilde{\Delta}^{(2)}$ (resp. $\widetilde{\Delta}^{(1)}$) from the left (resp. right), we have equalities

$$d^{(2)} p \otimes \epsilon = \widetilde{\Delta}^{(2)}(p \otimes \epsilon) = (h^{(2)} \otimes \epsilon) q' - \widetilde{\Delta}^{(2)} q' \widetilde{\Delta}^{(1)} = h^{(2)} q \otimes \epsilon - \widetilde{\Delta}^{(2)} q' \widetilde{\Delta}^{(1)},$$

and

$$p d^{(1)} \otimes \epsilon = (p \otimes \epsilon) \widetilde{\Delta}^{(1)} = \widetilde{\Delta}^{(2)} q' \widetilde{\Delta}^{(1)} - q'(h^{(1)} \otimes \epsilon) = \widetilde{\Delta}^{(2)} q' \widetilde{\Delta}^{(1)} - q h^{(1)} \otimes \epsilon.$$

Consequently, the equality

$$h^{(2)} q - q h^{(1)} = d^{(2)} p + p d^{(1)}$$

holds. It follows that the equality $q_*([h^{(1)}]) = [q h^{(1)}] = [h^{(2)} q] = q^*([h^{(2)}])$ holds as an elements of $\mathrm{Ext}^2_R(\mathbb{F}^{(1)}, \mathbb{F}^{(2)})$. $\square$

Lemma 4.10. *Let $A \in \mathcal{A}_k$ and let $\mathbb{G}^A = (G, d^A)$ be a free resolution of the left A-module $k = A/\mathfrak{m}_A$. We take a universal lift of $\mathbb{G}^A$ of the form $\mathbb{G}^A_0 = (G \otimes_k A, \Delta^A_0)$ as above. Furthermore, suppose that there is a lifting chain complex $\mathbb{L} = (F \otimes_k A, \Delta)$ of $\mathbb{F} = (F, d)$. Then we have the following commutative diagram*

$$\begin{tikzcd} \mathcal{T}(A) \arrow[r, "\alpha_{\mathbb{L}}"] \arrow[d, "\alpha_{\mathbb{G}^A_0}"'] & \mathrm{Ext}^2_R(\mathbb{F}, \mathbb{F}) \\ \mathrm{Ext}^2_A(k, k), \arrow[ur, "\rho^2_{\mathbb{L}}"'] & \end{tikzcd}$$

where $\rho^2_{\mathbb{L}}$ is the map induced by the functor $\mathbb{L} \otimes_A - : D_+(A) \to D_+(R)$.

Proof. Consider the tensor product of chain complexes

$$\mathbb{X} := \mathbb{L} \otimes_A \mathbb{G}^A_0 = ((F \otimes_k A) \otimes_A (G \otimes_k A),\ d_X),$$

where $d_X = \Delta \otimes 1 + 1 \otimes \Delta^A_0$. Notice from Theorem 3.3 that $\mathbb{G}^A_0$ is a complex of free $A \otimes_k A^{op}$-modules and it is quasi-isomorphic to A as a chain complex of $A \otimes A^{op}$-modules. Therefore the chain complex $\mathbb{X}$ is quasi-isomorphic to $\mathbb{L}$ as a chain complex of $R \otimes_k A^{op}$-modules. By virtue of Lemma 4.9, it is sufficient to prove the commutativity of the following diagram.

$$\begin{tikzcd} \mathcal{T}(A) \arrow[r, "\alpha_{\mathbb{X}}"] \arrow[d, "\alpha_{\mathbb{G}^A_0}"'] & \mathrm{Ext}^2_R(\mathbb{X} \otimes_A k, \mathbb{X} \otimes_A k) \\ \mathrm{Ext}^2_A(k, k) \arrow[ur, "\rho^2_{\mathbb{X}}"'] & \end{tikzcd}$$

For this, let $[A', \epsilon] \in \mathcal{T}(A)$ and take a lifting map $\Gamma : G \otimes_k A' \to G \otimes_k A'[-1]$ of Δ^A_0. By definition of $\alpha_{\mathbb{G}^A_0}$, we have $\Gamma^2 = h \otimes \epsilon$ for some $h : G \to G[-2]$ and

$\alpha_{\mathbb{G}_0^A}([A',\epsilon]) = [h]$. Therefore $\rho_{\mathbb{X}}^2(\alpha_{\mathbb{G}_0^A}([A',\epsilon]))$ is represented by a chain map $1\otimes h$ on $\mathbb{X}\otimes_A k = \mathbb{F}\otimes_A \mathbb{G}^A$. On the other hand, $d'_X = \Delta\otimes 1 + 1\otimes\Gamma$ is a lifting map of d_X, and we have the equality ${d'_X}^2 = 1\otimes h\otimes\epsilon$. Hence it follows from the definition that $\alpha_{\mathbb{X}}([A',\epsilon]) = [1\otimes h]$ as well. Hence we have $\rho_{\mathbb{X}}^2\cdot\alpha_{\mathbb{G}_0^A} = \alpha_{\mathbb{X}}$. $\square$

Let $P\in\widehat{\mathcal{A}}_k$ be any complete local k-algebra. Then, induced from the natural projections $p_n : P/\mathfrak{m}_P^{n+1}\to P/\mathfrak{m}_P^n$ and $\pi_n : P\to P/\mathfrak{m}_P^n$ for each $n\ge 1$, there are natural mappings $p_n^* : \mathcal{T}(P/\mathfrak{m}_P^n)\to\mathcal{T}(P/\mathfrak{m}_P^{n+1})$ and $\pi_n^* : \mathcal{T}(P/\mathfrak{m}_P^n)\to\mathcal{T}(P)$. See Lemma 2.10. By the functorial property of $\mathcal{T}$, it is clear that the diagram

$$\begin{tikzcd} \mathcal{T}(P/\mathfrak{m}_P^n) \arrow[r, "\pi_n^*"] \arrow[d, "p_n^*"'] & \mathcal{T}(P) \\ \mathcal{T}(P/\mathfrak{m}_P^{n+1}) \arrow[ur, "\pi_{n+1}^*"'] & \end{tikzcd}$$

is commutative for each $n\ge 1$. Thus it induces the map

$$\gamma_P := \varinjlim \pi_n^* \ : \ \varinjlim \mathcal{T}(P/\mathfrak{m}_P^n)\to\mathcal{T}(P).$$

Lemma 4.11. *The map γ_P is an isomorphism for any $P\in\widehat{\mathcal{A}}_k$.*

Proof. First we show that each $\pi_n^* : \mathcal{T}(P/\mathfrak{m}_P^n)\to\mathcal{T}(P)$ is injective for $n\ge 2$, hence so is γ_P. For this, let $[A,\epsilon]\in\mathcal{T}(P/\mathfrak{m}_P^n)$. Then take a fiber product and we have the following commutative diagram with exact rows and columns.

$$\begin{tikzcd} & & 0 \arrow[d] & 0 \arrow[d] & \\ & & \mathfrak{m}_{A'}^n \arrow[r, equal] \arrow[d] & \mathfrak{m}_P^n \arrow[d] & \\ 0 \arrow[r] & k \arrow[r, "\epsilon'"] \arrow[d, equal] & A' \arrow[r] \arrow[d] & P \arrow[r] \arrow[d, "\pi_n"'] & 0 \\ 0 \arrow[r] & k \arrow[r, "\epsilon"] & A \arrow[r] \arrow[d] & P/\mathfrak{m}_P^n \arrow[r] \arrow[d] & 0 \\ & & 0 & 0 & \end{tikzcd}$$

By definition $\pi^*([A,\epsilon]) = [A',\epsilon']$. Suppose $[A',\epsilon'] = 0$ in $\mathcal{T}(P)$. Then, since the small extension (A',ϵ') is a trivial one, we have $\epsilon'\notin\mathfrak{m}_{A'}^2$ by Lemma 2.6. Then by the diagram above, we see $\epsilon\notin\mathfrak{m}_A^2$ as well. Hence $[A,\epsilon] = 0$ in $\mathcal{T}(P/\mathfrak{m}_P^n)$ again by Lemma 2.6.

Now we prove $\gamma_P : \varinjlim\mathcal{T}(P/\mathfrak{m}_P^n)\to\mathcal{T}(P)$ is surjective. For this, let $[A',\epsilon']$ be any element of $\mathcal{T}(P)$. Since $\bigcap_{n=1}^{\infty}\mathfrak{m}_{A'}^n = (0)$ and since (ϵ') is of finite length, there is an integer $n_0 \geq 1$ such that $(\epsilon') \cap \mathfrak{m}_{A'}^n = (0)$ for $n \geq n_0$. For such any n, we set $A_n = A'/\mathfrak{m}_{A'}^n$ and $\epsilon_n = \epsilon' \mod \mathfrak{m}_{A'}^n$. And it is easy to see that $[A_n, \epsilon_n] \in \mathcal{T}(P/\mathfrak{m}_P^n)$ and $\pi_n^*([A_n, \epsilon_n]) = [A', \epsilon']$ for $n \geq n_0$. The surjectivity of γ_P follows from this. □

Proof of Theorem 4.6.

Let $\mathbb{L}_0$ be a universal lift of $\mathbb{F}$ with the parameter algebra P_0 as in the setting of the theorem. We denote $\mathbb{L}_0^{(n)} = \mathbb{L} \otimes_{P_0} P_0/\mathfrak{m}_{P_0}^n$ and $\mathbb{G}_0^{(n)} = \mathbb{G}_0 \otimes_{P_0} P_0/\mathfrak{m}_{P_0}^n$, where $\mathbb{G}_0$ is the universal lift of a free left P_0-module k. Then, from Lemma 4.10, we have a commutative diagram

$$\begin{tikzcd} \mathcal{T}(P_0/\mathfrak{m}_{P_0}^n) \arrow[r, "\alpha_{\mathbb{L}_0^{(n)}}"] \arrow[d, "\alpha_{\mathbb{G}_0^{(n)}}"'] & \mathrm{Ext}_R^2(\mathbb{F}, \mathbb{F}) \\ \mathrm{Ext}^2_{P_0/\mathfrak{m}_{P_0}^n}(k,k). \arrow[ur, "\rho^2_{\mathbb{L}_0^{(n)}}"'] & \end{tikzcd}$$

Note from Lemma 4.7 that $\alpha_{\mathbb{G}_0^{(n)}}$ is an isomorphism. Now taking the inductive limit and setting $\beta = \varinjlim \alpha_{\mathbb{G}_0^{(n)}}$, we have a commutative diagram by Lemma 4.11 ;

$$\begin{tikzcd} \mathcal{T}(P_0) \arrow[r, "\varinjlim \alpha_{\mathbb{L}_0^{(n)}}"] \arrow[d, "\beta"'] & \mathrm{Ext}_R^2(\mathbb{F}, \mathbb{F}) \\ \varinjlim \mathrm{Ext}^2_{P_0/\mathfrak{m}_{P_0}^n}(k,k), \arrow[ur, "\varinjlim \rho^2_{\mathbb{L}_0^{(n)}}"'] & \end{tikzcd}$$

where β is an isomorphism as well. It is easy to see from the definition that $\varinjlim \alpha_{\mathbb{L}_0^{(n)}} = \alpha_{\mathbb{L}_0}$ and $\varinjlim \rho^2_{\mathbb{L}_0^{(n)}} = \rho^2_\infty$. □

We should note that there is a natural mapping

$$\nu : \varinjlim_n \mathrm{Ext}^2_{P/\mathfrak{m}_P^n}(k,k) \to \mathrm{Ext}^2_P(k,k).$$

However, the mapping ν is not an isomorphism in general. In fact, we can show the following proposition.

Proposition 4.3. *Let $P = T/I$ be a complete local k-algebra where T is a non-commutative formal power series ring and $I \subseteq \mathfrak{m}_T^2$. Then the natural map ν is always injective. It is an isomorphism if and only if the ideal $\mathfrak{m}_T I + I\mathfrak{m}_T$ is closed and* $\dim_k I/\mathfrak{m}_T I + I\mathfrak{m}_T$ *is finite.*

Proof. By Lemma 4.8, we know that $\mathrm{Ext}^2_P(k,k) \cong \mathrm{Hom}_{T\text{-bimod}}(I,k)$. On the other hand, it follows from Theorem 4.6 and Proposition 2.4 that

$$\varinjlim_n \mathrm{Ext}^2_{P/\mathfrak{m}_P^n}(k,k) \overset{\beta}{\cong} \mathcal{T}(P) \cong \mathrm{Hom}_{con}(I,k). \tag{4.4}$$

Through these isomorphisms, it can be seen that ν coincides with the natural map $\mathrm{Hom}_{con}(I,k) \to \mathrm{Hom}_{T\text{-bimod}}(I,k)$, which is of course an injection.

Suppose that $\mathfrak{m}_T I + I\mathfrak{m}_T$ is closed with $\dim_k I/\mathfrak{m}_T I + I\mathfrak{m}_T < \infty$. Then, by Corollary 2.5, the inclusion $\mathfrak{m}_T^n \cap I \subseteq \mathfrak{m}_T I + I\mathfrak{m}_T$ holds for large $n \gg 1$. Therefore we have $\mathrm{Hom}_{con}(I,k) = \mathrm{Hom}_{T\text{-bimod}}(I,k)$. See Definition 2.7.

On the contrary, assume $\mathrm{Hom}_{con}(I,k) = \mathrm{Hom}_{T\text{-bimod}}(I,k)$. If $\dim_k I/\mathfrak{m}_T I + I\mathfrak{m}_T = \infty$, then $\mathrm{Hom}_{T\text{-bimod}}(I,k) = \mathrm{Hom}_k(I/\mathfrak{m}_T I + I\mathfrak{m}_T, k)$ has uncountable dimension as a k-vector space. On th other hand, by the equality (4.4), $\mathrm{Hom}_{con}(I,k)$ has countable dimension, as it is an inductive limit of finite dimensional k-vector spaces. By this contradiction, we can conclude that $\dim_k I/\mathfrak{m}_T I + I\mathfrak{m}_T < \infty$. Then, since $\dim_k I/\overline{\mathfrak{m}_T I + I\mathfrak{m}_T} < \infty$, it follows

$$\mathrm{Hom}_{con}(I,k) = \mathrm{Hom}_k(I/\overline{\mathfrak{m}_T I + I\mathfrak{m}_T}, k).$$

Since this equals $\mathrm{Hom}_{T\text{-bimod}}(I,k) = \mathrm{Hom}_k(I/\mathfrak{m}_T I + I\mathfrak{m}_T, k)$, we see the equality $\overline{\mathfrak{m}_T I + I\mathfrak{m}_T} = \mathfrak{m}_T I + I\mathfrak{m}_T$. □

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