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# Fundamental groups of symmetric sextics

By

Alex Degtyarev

**Abstract**

We study the moduli spaces and compute the fundamental groups of plane sextics of torus type with at least two type $\mathbf{E}_6$ singular points. As a simple application, we compute the fundamental groups of 125 other sextics, most of which are new.

## 1. Introduction

### 1.1. Principal results

Recall that a plane sextic $B$ is said to be of *torus type* if its equation can be represented in the form $p^3 + q^2 = 0$, where $p$ and $q$ are certain homogeneous polynomials of degree 2 and 3, respectively. Alternatively, $B \subset \mathbb{P}^2$ is of torus type if and only if it is the ramification locus of a projection to $\mathbb{P}^2$ of a cubic surface in $\mathbb{P}^3$. A representation of the equation in the form $p^3 + q^2 = 0$ (up to the obvious equivalence) is called a *torus structure* of $B$. A singular point $P$ of $B$ is called *inner* (*outer*) with respect to a torus structure $(p, q)$ if $P$ does (respectively, does not) belong to the intersection of the conic $\{p = 0\}$ and the cubic $\{q = 0\}$. The sextic $B$ is called *tame* if all its singular points are inner. Note that, according to [5], each sextic $B$ considered in this paper has a unique torus structure; hence, we can speak about inner and outer singular points of $B$. For the reader's convenience, when listing the set of singularities of a sextic of torus type, we indicate the inner singularities by enclosing them in parentheses.

Apparently, it was O. Zariski [19] who first understood the importance of sextics of torus type. Since then, they have been a subject of intensive study. For details and further information, we refer to M. Oka, D. T. Pho [14], [15] (topology, sets of singularities, moduli, fundamental groups), H. Tokunaga [18] (algebro-geometric approach), and A. Degtyarev [5].

In recent paper [8], we described the moduli spaces and calculated the fundamental groups of all sextics of torus type of weight 8 and 9 (in a sense, those with the largest fundamental groups). The approach used in [8], reduc-

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ing sextics to maximal trigonal curves, was also helpful in the study of some other sextics with nonabelian groups (see [7]), and then, in [9], we classified all irreducible sextics for which this approach should work. The purpose of this paper is to treat one of the classes that appeared in [9]: sextics with at least two type $\mathbf{E}_6$ singular points; they are reduced to trigonal curves with the set of singularities $\mathbf{E}_6 \oplus \mathbf{A}_2$. Our principal results are Theorems 1.1.1 and 1.1.3 below.

Table 1. Sextics with two type $\mathbf{E}_6$ singular points

| | |
|---|---|
| $^*\,(3\mathbf{E}_6) \oplus \mathbf{A}_1$ | $^*\,(2\mathbf{E}_6 \oplus 2\mathbf{A}_2) \oplus \mathbf{A}_3$ |
| $(3\mathbf{E}_6)$ | $^*\,(2\mathbf{E}_6 \oplus 2\mathbf{A}_2) \oplus \mathbf{A}_2$ |
| $^*\,(2\mathbf{E}_6 \oplus \mathbf{A}_5) \oplus \mathbf{A}_2$ | $^*\,(2\mathbf{E}_6 \oplus 2\mathbf{A}_2) \oplus 2\mathbf{A}_1$ |
| $(2\mathbf{E}_6 \oplus \mathbf{A}_5) \oplus \mathbf{A}_1$ | $(2\mathbf{E}_6 \oplus 2\mathbf{A}_2) \oplus \mathbf{A}_1$ |
| $(2\mathbf{E}_6 \oplus \mathbf{A}_5)$ | $(2\mathbf{E}_6 \oplus 2\mathbf{A}_2)$ |

**Theorem 1.1.1.** *Any sextic of torus type with at least two type* $\mathbf{E}_6$ *singular points has one of the sets of singularities listed in Table* 1*. With the exception of* $(2\mathbf{E}_6 \oplus \mathbf{A}_5) \oplus \mathbf{A}_2$*, the moduli space of sextics of torus type realizing each set of singularities in the table is rational* (*in particular, it is nonempty and connected*)*; the moduli space of sextics with the exceptional set of singularities* $(2\mathbf{E}_6 \oplus \mathbf{A}_5) \oplus \mathbf{A}_2$ *consists of two isolated points, both of torus type.*

Note that we do not assume *a priori* that the curves are irreducible or have simple singularities only. Both assertions hold automatically for any sextic with at least two type $\mathbf{E}_6$ singular points, see the beginning of Section 2.7.

Theorem 1.1.1 is proved in Section 2.7. The two classes of sextics realizing the set of singularities $(2\mathbf{E}_6 \oplus \mathbf{A}_5) \oplus \mathbf{A}_2$ were first discovered in Oka, Pho [14]. The sets of singularities that can be realized by sextics of torus type are also listed in [14]. Note that the list given by Table 1 can also be obtained from the results of J.-G. Yang [20], using the characterization of irreducible sextics of torus type found in [5]. The deformation classification can be obtained using [4].

**Remark 1.1.2.** A simple calculation using [4] or [20] and the characterization of irreducible sextics of torus type found in [5] shows that the sets of singularities marked with a $^*$ in Table 1 are realized by sextics of torus type only. Each of the remaining five sets of singularities is also realized by a single deformation family of sextics not of torus type, see A. Özgüner [16] for details. Furthermore, Table 1 lists all sets of singularities of plane sextics, both of and not of torus type, containing at least two type $\mathbf{E}_6$ points.

**Theorem 1.1.3.** *Let* $B$ *be a sextic of torus type whose set of singularities* $\Sigma$ *is one of those listed in Table* 1*. Then the fundamental group* $\pi_1 := \pi_1(\mathbb{P}^2 \smallsetminus B)$ *is as follows*:

1. *if* $\Sigma = (2\mathbf{E}_6 \oplus 2\mathbf{A}_2) \oplus \mathbf{A}_3$*, then* $\pi_1$ *is the group* $G_3$ *given by* (4.3.7);
2. *if* $\Sigma = (3\mathbf{E}_6) \oplus \mathbf{A}_1$ *or* $(2\mathbf{E}_6 \oplus 2\mathbf{A}_2) \oplus 2\mathbf{A}_1$*, then* $\pi_1 = G_0 := \mathbb{B}_4/\sigma_2\sigma_1^2\sigma_2\sigma_3^2$;
3. *if* $\Sigma = (2\mathbf{E}_6 \oplus \mathbf{A}_5) \oplus \mathbf{A}_2$*, then, depending on the family,* $\pi_1$ *is one of the groups* $G_2'$*,* $G_2''$ *given by* (4.4.10) *and* (4.5.4)*, respectively*;
4. *otherwise,* $\pi_1 = \mathbb{B}_3/(\sigma_1\sigma_2)^3$.

(*Here,* $\{\sigma_1, \ldots, \sigma_{n-1}\}$ *is a canonical basis for the braid group* $\mathbb{B}_n$ *on* $n$ *strings.*)

The fundamental groups are calculated in §4. An alternative presentation of the groups $G_2'$, $G_2''$ mentioned in 1.1.3(3) is found in C. Eyral, M. Oka [10], where it is conjectured that the two groups are not isomorphic. We suggest to attack this problem studying the relation between $G_2'$, $G_2''$ and the local fundamental group at the type $\mathbf{A}_5$ singular point, *cf*. Proposition 4.6.1 and Conjecture 4.6.2. The group of a sextic of torus type with the set of singularities $(2\mathbf{E}_6 \oplus \mathbf{A}_5) \oplus \mathbf{A}_1$, see 1.1.3(4), is also found in [10]; the group of a sextic with the set of singularities $(3\mathbf{E}_6) \oplus \mathbf{A}_1$, see 1.1.3(2), as well as the groups of the three tame sextics listed in Table 1 (the sets of singularities $(3\mathbf{E}_6)$, $(2\mathbf{E}_6 \oplus \mathbf{A}_5)$, and $(2\mathbf{E}_6 \oplus 2\mathbf{A}_2)$) are found in Oka, Pho [15].

With the possible exception of $G_2'$, $G_2''$, all groups listed in Theorem 1.1.3 are 'geometrically' distinct in the sense of the following theorem.

**Theorem 1.1.4.** *All epimorphisms*

$$G_3 \twoheadrightarrow G_0 \twoheadrightarrow \mathbb{B}_3/(\sigma_1\sigma_2)^3, \qquad G_2', G_2'' \twoheadrightarrow \mathbb{B}_3/(\sigma_1\sigma_2)^3$$

*induced by the respective perturbations of the curves* (cf. *O. Zariski* [19]) *are proper,* i.e.*, they are not isomorphisms.*

This theorem is proved in Section 4.8. Some of the statements follow from the previous results by Eyral, Oka [10] and Oka, Pho [15].

As a further application of Theorem 1.1.3, we use the presentations obtained and the results of [8] to compute the fundamental groups of eight sextics of torus type and 117 sextics not of torus type that are not covered by M. V. Nori's theorem [13], see Theorems 5.2.1 and 5.3.1. As for most sets of singularities the connectedness of the moduli space has not been established (although expected), we state these results in the form of existence.

### 1.2. Contents of the paper

In §2, we use the results of [9] and construct the *trigonal models* of sextics in question, which are pairs $(\bar{B}, \bar{L})$, where $\bar{B}$ is a (fixed) trigonal curve in the Hirzebruch surface $\Sigma_2$ and $\bar{L}$ is a (variable) section. We study the conditions on $\bar{L}$ resulting in a particular set of singularities of the sextic. As a consequence, we obtain explicit equations of the sextics and rational parameterizations of the moduli spaces. Theorem 1.1.1 is proved here.

In §3, we present the classical Zariski–van Kampen method [12] in a form suitable for curves on Hirzebruch surfaces. The contents of this section is a formal account of a few observations found in [7] and [6].

In §4, we apply the classical Zariski–van Kampen theorem to the trigonal models constructed above and obtain presentations of the fundamental groups.

The main advantage of this approach (replacing sextics with their trigonal models) is the fact that the number of points to keep track of reduces from 6 to 4, which simplifies the computation of the braid monodromy. As a first application, we show that all groups can be generated by loops in a small neighborhood of (any) type $\mathbf{E}_6$ singular point of the curve.

In §5, we study perturbations of sextics considered in §§2 and 4. We confine ourselves to a few simple cases when the perturbed group is easily found by simple local analysis. This gives 117 new (compared to [8]) sextics with abelian fundamental group and 8 sextics of torus type. More complicated perturbations are not necessary, as the resulting sextics are not new, see Remark 5.3.2.

## 2. The trigonal model

### 2.1. Trigonal curves

Recall that the *Hirzebruch surface* $\Sigma_2$ is a geometrically ruled rational surface with an exceptional section $E$ of self-intersection $(-2)$. A *trigonal curve* is a reduced curve $\bar{B} \subset \Sigma_2$ disjoint from $E$ and intersecting each generic fiber of $\Sigma_2$ at three points. A *singular fiber* (sometimes referred to as *vertical tangent*) of a trigonal curve $\bar{B}$ is a fiber of $\Sigma_2$ that is not transversal to $\bar{B}$. The double covering $X$ of $\Sigma_2$ ramified at $\bar{B}+E$ is an elliptic surface, and the singular fibers of $\bar{B}$ are the projections of those of $X$. For this reason, to describe the topological types of singular fibers of $\bar{B}$, we use (one of) the standard notation for the types of singular elliptic fibers, referring to the corresponding extended Dynkin diagrams. The types are as follows:

- $\tilde{\mathbf{A}}_0^*$: a simple vertical tangent;
- $\tilde{\mathbf{A}}_0^{**}$: a vertical inflection tangent;
- $\tilde{\mathbf{A}}_1^*$: a node of $\bar{B}$ with one of the branches vertical;
- $\tilde{\mathbf{A}}_2^*$: a cusp of $\bar{B}$ with vertical tangent;
- $\tilde{\mathbf{A}}_p$, $\tilde{\mathbf{D}}_q$, $\tilde{\mathbf{E}}_6$, $\tilde{\mathbf{E}}_7$, $\tilde{\mathbf{E}}_8$: a simple singular point of $\bar{B}$ of the same type with minimal possible local intersection index with the fiber.

For the relation to Kodaira's classification of singular elliptic fibers and further details and references, see [6]. In the present paper, we merely use the notation.

The (*functional*) *j-invariant* $j = j_{\bar{B}} \colon \mathbb{P}^1 \to \mathbb{P}^1$ of a trigonal curve $\bar{B} \subset \Sigma_2$ is defined as the analytic continuation of the function sending a point $b$ in the base $\mathbb{P}^1$ of $\Sigma_2$ representing a nonsingular fiber $F$ of $\bar{B}$ to the $j$-invariant (divided by $12^3$) of the elliptic curve covering $F$ and ramified at $F \cap (\bar{B}+E)$. The curve $\bar{B}$ is called *isotrivial* if $j_{\bar{B}} = \text{const}$. Such curves can easily be enumerated, see, *e.g.*, [6]. The curve $\bar{B}$ is called *maximal* if it has the following properties:

- $\bar{B}$ has no singular fibers of type $\mathbf{D}_4$;
- $j = j_{\bar{B}}$ has no critical values other than 0, 1, and $\infty$;
- each point in the pull-back $j^{-1}(0)$ has ramification index at most 3;
- each point in the pull-back $j^{-1}(1)$ has ramification index at most 2.

The maximality of a non-isotrivial trigonal curve $\bar{B} \subset \Sigma_2$ can easily be detected by applying the Riemann–Hurwitz formula to the map $j_{\bar{B}} \colon \mathbb{P}^1 \to \mathbb{P}^1$;

it depends only on the (combinatorial) set of singular fibers of $\bar{B}$, see [6] for details. The classification of such curves reduces to a combinatorial problem; a partial classification of maximal trigonal curves in $\Sigma_2$ is found in [9]. An important property of maximal trigonal curves is their rigidity, see [6]: any small deformation of such a curve $\bar{B}$ is isomorphic to $\bar{B}$. For this reason, we do not need to keep parameters in the equations below.

### 2.2. The trigonal curve $\bar{B}$

Let $B$ be an *irreducible* sextic of torus type with *simple singularities only* and with at least two type $\mathbf{E}_6$ singular point. (Below, we show that the emphasized properties hold automatically, see 2.7.) Clearly, the set of inner singularities of $B$ can only be $(3\mathbf{E}_6)$, $(2\mathbf{E}_6 \oplus \mathbf{A}_5)$, or $(2\mathbf{E}_6 \oplus 2\mathbf{A}_2)$. Hence, according to [9], $B$ has an involutive symmetry (*i.e.*, projective automorphism) $c$ stable under equisingular deformations. Let $L_c$ and $O_c$ be, respectively, the fixed line and the isolated fixed point of $c$. One has $O_c \notin B$. Denote by $\mathbb{P}^2(O_c)$ the blow-up of $\mathbb{P}^2$ at $O_c$. Then, the quotient $\mathbb{P}^2(O_c)/c$ is the Hirzebruch surface $\Sigma_2$ and the projection $B/c$ is a trigonal curve $\bar{B} \subset \Sigma_2$ with the set of singularities $\mathbf{E}_6 \oplus \mathbf{A}_2$. The double covering $\mathbb{P}^2(O_c) \to \Sigma_2$ is ramified at $E$ and a generic section $\bar{L} \subset \Sigma_2$ (the image $L_c/c$) disjoint from $E$ and not passing through the type $\mathbf{E}_6$ singular point of $\bar{B}$ (as otherwise the two type $\mathbf{E}_6$ singular points of $B$ would merge to a single non-simple singularity).

Conversely, given a trigonal curve $\bar{B} \subset \Sigma_2$ with the set of singularities $\mathbf{E}_6 \oplus \mathbf{A}_2$ and a section $\bar{L} \subset \Sigma_2$ disjoint from $E$ and not passing through the type $\mathbf{E}_6$ singular point of $\bar{B}$, the pull-back of $\bar{B}$ in the double covering of $\Sigma_2/E$ ramified at $E/E$ and $\bar{L}$ is a sextic $B \subset \mathbb{P}^2$ with at least two type $\mathbf{E}_6$ singular points. Below we show that $B$ is necessarily of torus type, see (2.3.5).

### 2.3. Equations

Any trigonal curve $\bar{B} \subset \Sigma_2$ with the set of singularities $\mathbf{E}_6 \oplus \mathbf{A}_2$ is either isotrivial or maximal (see [9] for precise definitions); in particular, such curves are rigid, *i.e.*, within each of the two families, any two curves are isomorphic in $\Sigma_2$. A curve $\bar{B}$ can be obtained by an elementary transformation from a cuspidal cubic $C \subset \Sigma_1 = \mathbb{P}^2(O)$: the blow-up center $O$ should be chosen on the inflection tangent to $C$, and the elementary transformation should contract this tangent.

In appropriate affine coordinates $(x, y)$ in $\Sigma_2$ any trigonal curve $\bar{B}$ as above can be given by an equation of the form

$$f_r(x, y) := y^3 + r^2y^2 + 2rxy + x^2 = 0, \tag{2.3.1}$$

where $r \in \mathbb{C}$ is a parameter. If $r = 0$, the curve is isotrivial, its $j$-invariant being $j \equiv 0$. Otherwise, the automorphism $(x, y) \mapsto (r^3x, r^2y)$ of $\Sigma_2$ converts the curve to $f_1(x, y) = 0$. Below, in all plots and numeric evaluation, we use the value $r = 3$.

The $y$-discriminant of the polynomial $f_r$ given by (2.3.1) is $-x^3(27x-4r^3)$. Thus, if $r \neq 0$, the curve has three singular fibers, of types $\tilde{\mathbf{A}}_2$, $\tilde{\mathbf{A}}_0^*$ (vertical tangent), and $\tilde{\mathbf{E}}_6$ over $x = 0$, $4r^3/27$, and $\infty$, respectively. In the isotrivial case

$r = 0$, there are two singular fibers, of types $\tilde{\mathbf{A}}_2^*$ and $\tilde{\mathbf{E}}_6$, over $x = 0$ and $\infty$, respectively.

The curve $\bar{B}$ is rational; it can be parameterized by

$$x = x_t := rt^2 + t^3, \qquad y = y_t := -t^2. \tag{2.3.2}$$

The vertical tangency point of $\bar{B}$ corresponds to the value $t = -2r/3$.

Consider a section $\bar{L}$ of $\Sigma_2$ given by

$$y = s(x) := ax^2 + bx + c, \quad a \neq 0. \tag{2.3.3}$$

(The assumption $a \neq 0$ is due to the fact that $\bar{L}$ should not pass through the type $\mathbf{E}_6$ singular point of $\bar{B}$.) Let $B \subset \mathbb{P}^2$ be the pull-back of $\bar{B}$ under the double covering of $\Sigma_2/E$ ramified at $E/E$ and $\bar{L}$. It is a plane sextic which, in appropriate affine coordinates $(x, y)$ in $\mathbb{P}^2$, is given by the equation

$$f_r(x, y^2 + s(x)) = 0. \tag{2.3.4}$$

Obviously, $B$ is of torus type, the torus structure being

$$f_r(x, \bar{y}) = \bar{y}^3 + (r\bar{y} + x)^2, \quad \bar{y} = y^2 + s(x). \tag{2.3.5}$$

According to [5], this is the only torus structure on $B$. The inner singularities of $B$ are two type $\mathbf{E}_6$ points over the type $\mathbf{E}_6$ point of $\bar{B}$ and two cusps or one type $\mathbf{A}_5$ or $\mathbf{E}_6$ point over the cusp of $\bar{B}$. (There is only one point if $\bar{L}$ passes through the cusp of $\bar{B}$; this point is of type $\mathbf{E}_6$ if $\bar{L}$ is tangent to $\bar{B}$ at the cusp.) The outer singularities of $B$ arise from the tangency of $\bar{L}$ and $\bar{B}$: each point of $p$-fold intersection, $p > 1$, of $\bar{L}$ and $\bar{B}$ smooth for $\bar{B}$ gives rise to a type $\mathbf{A}_{p-1}$ outer singularity of $B$. For detail, see [7].

In the rest of this section, we discuss various degenerations of the pair $(\bar{B}, \bar{L})$ and parameterize the corresponding triples $(a, b, c)$. For convenience, each time we mention parenthetically the set of singularities of the sextic $B$ arising from $(\bar{B}, \bar{L})$.

### 2.4. Tangents and double tangents

Equating the values and the derivatives of $s(x_t(t))$ and $y_t(t)$, one concludes that a section $\bar{L}$ as in (2.3.3) is tangent to $\bar{B}$ at a point $(x_t(t), y_t(t))$, $t \neq 0$, $-2r/3$, (the set of singularities $(2\mathbf{E}_6 \oplus 2\mathbf{A}_2) \oplus \mathbf{A}_1$) if and only if

$$b = -2t^2(t + r)a - \frac{2}{3t + 2r}, \quad c = t^4(t + r)^2 a - \frac{t^3}{3t + 2r}. \tag{2.4.1}$$

Double tangents are described by the following lemma.

**Lemma 2.4.2.** *There exists a section $\bar{L}$ tangent to the curve $\bar{B}$ at two distinct points $(x_t(t_1), y_t(t_1))$ and $(x_t(t_2), y_t(t_2))$, $t_1 \neq t_2$, if and only if $t_1 + t_2 = -r/3$ and neither $t_1$ nor $t_2$ is $0$, $-r/6$, or $-2r/3$.*

*Proof.* Substituting $t = t_1$ and $t = t_2$ to (2.4.1), equating the resulting values of $b$ and $c$, solving both equations for $a$, and equating the results, one obtains $(t_1 - t_2)^2(3t_1 + 3t_2 + r) = 0$; now, the statement is immediate. $\square$

Thus, a section $\bar{L}$ as in (2.3.3) is double tangent to $\bar{B}$ (the set of singularities $(2\mathbf{E}_6 \oplus 2\mathbf{A}_2) \oplus 2\mathbf{A}_1$) if and only if, for some $t \neq 0, -r/6, -2r/3$, one has

$$\begin{aligned} a &= -\frac{27}{(3t-r)^2(3t+2r)^2}, \\ b &= \frac{2r(27t^2+9rt-2r^2)}{(3t-r)^2(3t+2r)^2}, \\ c &= -\frac{2t^3(3t+r)^3}{(3t-r)^2(3t+2r)^2}. \end{aligned} \tag{2.4.3}$$

A point of quadruple intersection of $\bar{L}$ and $\bar{B}$ can be obtained from Lemma 2.4.2 letting $t_1 = t_2$. (Alternatively, one can equate the derivatives of order 0 to 3 of $s(x_t(t))$ and $y_t(t)$.) As a result, $(x_t(t), y_t(t))$ is a point of quadruple intersection of $\bar{L}$ and $\bar{B}$ (the set of singularities $(2\mathbf{E}_6 \oplus 2\mathbf{A}_2) \oplus \mathbf{A}_3$) if and only if

$$t = -\frac{r}{6}, \qquad (a, b, c) = \Big(-\frac{16}{3r^4}, -\frac{88}{81r}, \frac{r^2}{4374}\Big). \tag{2.4.4}$$

All points of intersection of this section $\bar{L}$ and $\bar{B}$ are:

- transversal intersection at $t = \Big(-\frac{2}{3} + \frac{\sqrt{2}}{2}\Big)r$, $x = \Big(-\frac{19}{54} + \frac{\sqrt{2}}{4}\Big)r^3 \approx .0459$;
- transversal intersection at $t = \Big(-\frac{2}{3} - \frac{\sqrt{2}}{2}\Big)r$, $x = \Big(-\frac{19}{54} - \frac{\sqrt{2}}{4}\Big)r^3 \approx -19.1$;
- quadruple intersection at $t = -\frac{r}{6}$, $x = \frac{5r^3}{216} = .625$.

The curve $\bar{B}$ and the section $\bar{L}$ given by (2.4.4) are plotted in Figure 1 (in black and grey, respectively). The section is above the curve over $x = 0$; it intersects the topmost branch over $x \approx .0459$ and is tangent to the middle branch over $x = .625$.

### 2.5. Sections through the cusp

A section $\bar{L}$ as in (2.3.3) passes through the cusp of $\bar{B}$ (the set of singularities $(2\mathbf{E}_6 \oplus \mathbf{A}_5)$) if and only if $c = 0$; it is tangent to $\bar{B}$ at the cusp (the set of singularities $(3\mathbf{E}_6)$) if and only if, in addition, $b = -1/r$.

A section tangent to $\bar{B}$ at a point $(x_t(t), y_t(t))$, see (2.4.1), passes through the cusp of $\bar{B}$ (the set of singularities $(2\mathbf{E}_6 \oplus \mathbf{A}_5) \oplus \mathbf{A}_1$) if and only if

$$a = \frac{1}{t(t+r)^2(3t+2r)}, \quad b = -\frac{2(2t+r)}{(t+r)(3t+2r)}, \quad c = 0, \tag{2.5.1}$$

$t \neq 0, -r, -2r/3$. (Note that the value $t = -r$ corresponds to the smooth point of $\bar{B}$ in the same vertical fiber as the cusp.) Such a section is tangent

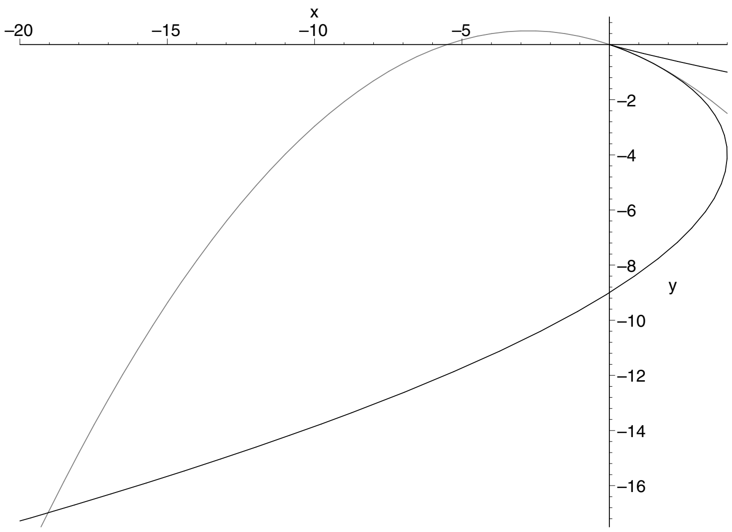


Figure 1. The set of singularities $(2\mathbf{E}_6 \oplus 2\mathbf{A}_2) \oplus \mathbf{A}_3$

to $\bar{B}$ at the cusp (the set of singularities $(3\mathbf{E}_6) \oplus \mathbf{A}_1$) if and only if

$$t = -\frac{r}{3}, \qquad (a, b, c) = \Bigl(-\frac{27}{4r^4}, -\frac{1}{r}, 0\Bigr). \tag{2.5.2}$$

The points of intersection of the latter section $\bar{L}$ and $\bar{B}$ are:

- the cusp of $\bar{B}$ at $t = 0$, $x = 0$;
- transversal intersection at $t = -\dfrac{4r}{3}$, $x = -\dfrac{16r^3}{27} = -16$;
- tangency at $t = -\dfrac{r}{3}$, $x = \dfrac{2r^3}{27} = 2$.

The section $\bar{L}$ given by (2.5.2) looks similar to that shown in Figure 1. (Near the cusp of $\bar{B}$, the two curves are too close to be distinguished visually.) Between $x = 0$ and $x = 2$, the section lies between the topmost and middle branches of $\bar{B}$.

### 2.6. Inflection tangents

Equating the derivatives of $s(x_t(t))$ and $y_t(t)$ of order 0, 1, and 2, one can see that a section $\bar{L}$ as in (2.3.3) is inflection tangent to $\bar{B}$ at a point $(x_t(t), y_t(t))$, $t \neq 0$, $-2r/3$, (the set of singularities $(2\mathbf{E}_6 \oplus 2\mathbf{A}_2) \oplus \mathbf{A}_2$) if and only if

(2.6.1)
$$a = \frac{3}{t(3t + 2r)^3}, \quad b = -\frac{2(12t^2 + 15rt + 4r^2)}{(3t + 2r)^3}, \quad c = -\frac{t^3(6t^2 + 6rt + r^2)}{(3t + 2r)^3}.$$

Such a section passes through the cusp of $\bar{B}$ (giving rise to the set of singularities $(2\mathbf{E}_6 \oplus \mathbf{A}_5) \oplus \mathbf{A}_2$) if and only if $t = (-3 \pm \sqrt{3})r/6$. Thus, we obtain *two families*, which are Galois conjugate over $\mathbb{Q}[\sqrt{3}]$, *cf*. [14]. For one of the families, one has

$$t=\Big(-\frac{1}{2}+\frac{\sqrt{3}}{6}\Big)r,\quad (a,b,c)=\Big(\frac{12(3-2\sqrt{3})}{r^4},-\frac{4(2-\sqrt{3})}{r},0\Big), \tag{2.6.2}$$

and the points of intersection of $\bar{L}$ and $\bar{B}$ are:

- the cusp of $\bar{B}$ at $t=0$, $x=0$;
- transversal intersection at $t=\Big(-\frac{1}{2}-\frac{\sqrt{3}}{2}\Big)r$, $x=\Big(-\frac{1}{4}-\frac{\sqrt{3}}{4}\Big)r^3\approx -18.4$;
- inflection tangency at $t=\Big(-\frac{1}{2}+\frac{\sqrt{3}}{6}\Big)r$, $x=\Big(\frac{1}{12}-\frac{\sqrt{3}}{36}\Big)r^3\approx .951$.

This section looks similar to that shown in Figure 1; between $x=0$ and $x\approx .951$, the section is just below the middle branch of the curve.

For the other family, one has

$$t=\Big(-\frac{1}{2}-\frac{\sqrt{3}}{6}\Big)r,\quad (a,b,c)=\Big(\frac{12(3+2\sqrt{3})}{r^4},-\frac{4(2+\sqrt{3})}{r},0\Big), \tag{2.6.3}$$

and the points of intersection of $\bar{L}$ and $\bar{B}$ are:

- the cusp of $\bar{B}$ at $t=0$, $x=0$;
- transversal intersection at $t=\Big(-\frac{1}{2}+\frac{\sqrt{3}}{2}\Big)r$, $x=\Big(-\frac{1}{4}+\frac{\sqrt{3}}{4}\Big)r^3\approx 4.94$;
- inflection tangency at $t=\Big(-\frac{1}{2}-\frac{\sqrt{3}}{6}\Big)r$, $x=\Big(\frac{1}{12}+\frac{\sqrt{3}}{36}\Big)r^3\approx 3.55$.

The curve $\bar{B}$ and the section $\bar{L}$ given by (2.6.3) are plotted in Figure 2, in black and grey, respectively.

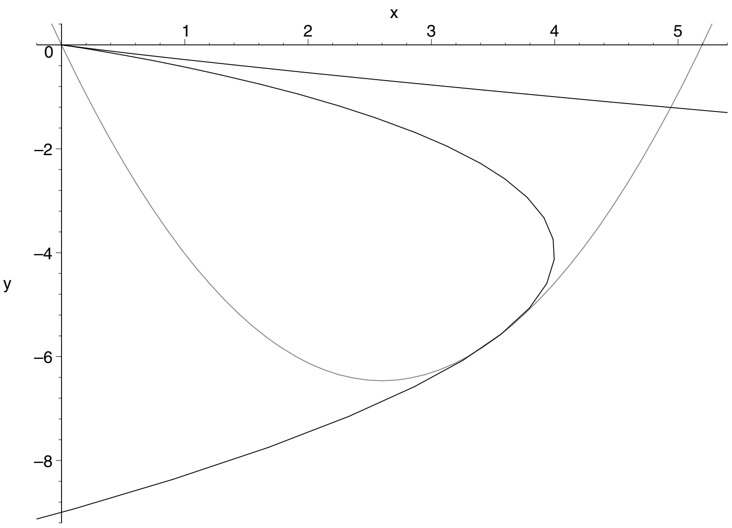


Figure 2. The set of singularities $(2\mathbf{E}_6\oplus\mathbf{A}_5)\oplus\mathbf{A}_2$, the family (2.6.3)

## 2.7. Proof of Theorem 1.1.1

First, note that any sextic with two type $\mathbf{E}_6$ singular points is irreducible and has simple singularities only. The first statement follows from the fact that an irreducible curve of degree 4 or 5 (respectively, $\leqslant 3$) may have at most one (respectively, none) type $\mathbf{E}_6$ singular point, and the second one, from the fact that a type $\mathbf{E}_6$ (respectively, non-simple) singular point takes 3 (respectively, $\geqslant 6$) off the genus, whereas the genus of a nonsingular sextic is 10. Thus, we can apply the results of [9] enumerating stable symmetries of curves.

For a set of singularities $\Sigma \supset 2\mathbf{E}_6$, consider the moduli space $\mathcal{M}(\Sigma)$ of sextics $B$ of torus type with the set of singularities $\Sigma$ and the moduli space $\tilde{\mathcal{M}}(\Sigma)$ of pairs $(B, c)$, where $B$ is a sextic as above and $c$ is a stable involution of $B$. Due to [9], the forgetful map $\tilde{\mathcal{M}}(\Sigma) \to \mathcal{M}(\Sigma)$ is generically finite-to-one and onto.

As explained in Sections 2.2 and 2.3, the space $\tilde{\mathcal{M}}(\Sigma)$ can be identified with the moduli space of pairs $(\bar{B}, \bar{L})$, where $\bar{B} \subset \Sigma_2$ is a trigonal curve given by (2.3.1) and $\bar{L}$ is a section of $\Sigma_2$ in a certain prescribed position with respect to $\bar{B}$. The spaces of pairs $(\bar{B}, \bar{L})$ are described in Sections 2.4–2.6, and for each $\Sigma \neq (2\mathbf{E}_6 \oplus \mathbf{A}_5) \oplus \mathbf{A}_2$, an explicit rational parameterization is found. (Strictly speaking, in order to pass to the moduli, we need to fix a value of $r$, say, $r = 3$. This results in a Zariski open subset of the moduli space. The portion corresponding to $r = 0$ has positive codimension as the isotrivial curve $f_0 = 0$ has 1-dimensional group $\mathbb{C}^*$ of symmetries.) Hence, the space $\tilde{\mathcal{M}}(\Sigma)$ is rational and, if $\dim \mathcal{M}(\Sigma) \leqslant 2$, so is $\mathcal{M}(\Sigma)$. The only case when $\dim \mathcal{M}(\Sigma) \geqslant 3$ is $\Sigma = (2\mathbf{E}_6 \oplus 2\mathbf{A}_2)$. In this case, each curve $B$ has a unique stable involution, see [9], and the map $\tilde{\mathcal{M}}(\Sigma) \to \mathcal{M}(\Sigma)$ is generically one-to-one; hence, $\mathcal{M}(\Sigma)$ is still rational.

In the exceptional case $\Sigma = (2\mathbf{E}_6 \oplus \mathbf{A}_5) \oplus \mathbf{A}_2$, the space $\mathcal{M}(\Sigma) = \tilde{\mathcal{M}}(\Sigma)$ consists of two points. The fact that any sextic with this set of singularities is of torus type follows immediately from [4]. □

**Remark 2.7.1.** The only sets of singularities containing $2\mathbf{E}_6$ where the curves have more than one (in fact, three) stable involutions are $(3\mathbf{E}_6)$ and $(3\mathbf{E}_6) \oplus \mathbf{A}_1$, see [9]. In both cases, the group of stable symmetries can be identified with the group $\mathbb{S}_3$ of permutations of the three type $\mathbf{E}_6$ points. It follows that all three involutions are conjugate by stable symmetries; hence, the map $\tilde{\mathcal{M}}(\Sigma) \to \mathcal{M}(\Sigma)$ is still one-to-one.

# 3. Van Kampen's method in Hirzebruch surfaces

In this section, we give a formal and detailed exposition of a few observations outlined in [7]. Keeping in mind future applications, we treat the general case of a Hirzebruch surface $\Sigma_k$, $k \geqslant 1$, and a *d-gonal curve* $C \subset \Sigma_k$, see Definition 3.1.1.

Certainly, the essence of this approach is due to van Kampen [12]; we merely introduce a few restrictions to the objects used in the construction which make the choices involved slightly more canonical and easier to handle.

By no means do we assert that the restrictions are necessary for the approach to work in general.

### 3.1. Preliminary definitions

Fix a Hirzebruch surface $\Sigma_k$, $k \geqslant 1$. Denote by $p\colon \Sigma_k \to \mathbb{P}^1$ the ruling, and let $E \subset \Sigma_k$ be the exceptional section, $E^2 = -k$. Given a point $b$ in the base $\mathbb{P}^1$, we denote by $F_b$ the fiber $p^{-1}(b)$. Let $F_b^\circ$ be the 'open fiber' $F_b \smallsetminus E$. Observe that $F_b^\circ$ is a dimension 1 affine space over $\mathbb{C}$; hence, one can speak about lines, circles, convexity, convex hulls, *etc.* in $F_b^\circ$. (Thus, strictly speaking, the notation $F_b^\circ$ means slightly more than just the set theoretical difference $F_b \smallsetminus E$: we always consider $F_b^\circ$ with its canonical affine structure.) Define the *convex hull* $\operatorname{conv} C$ of a subset $C \subset \Sigma_k \smallsetminus E$ as the union of its fiberwise convex hulls:

$$\operatorname{conv} C = \bigcup_{b \in \mathbb{P}^1} \operatorname{conv}(C \cap F_b^\circ).$$

**Definition 3.1.1.** Let $d \geqslant 1$ be an integer. A *d-gonal curve* (or *degree d curve*) on $\Sigma_k$ is a reduced algebraic curve $C \in |dE + dkF|$ disjoint from the exceptional section $E$. (Here, $F$ is any fiber of $\Sigma_k$.) A *singular fiber* of a $d$-gonal curve $C$ is a fiber of $\Sigma_k$ that intersects $C$ at fewer than $d$ points. (With a certain abuse of the language, the points in the base $\mathbb{P}^1$ whose pull-backs are singular fibers will also be referred to as singular fibers of $C$.)

**Remark 3.1.2.** Recall that the complement $\Sigma_k \smallsetminus E$ can be covered by two affine charts, with coordinates $(x, y)$ and $(x', y')$ and transition function $x' = 1/x$, $y' = y/x^k$. In the coordinates $(x, y)$, any $d$-gonal curve $C$ is given by an equation of the form

$$f(x,y) = \sum_{i=0}^{d} y^i q_i(x) = 0, \quad \deg q_i = k(d-i), \quad q_d = \text{const} \neq 0,$$

and the singular fibers of $C$ are those of the form $F_x$, where $x$ is a root of the $y$-discriminant $D_y$ of $f$. (The fiber $F_\infty$ over $x = \infty$ is singular for $C$ if and only if $\deg D_y < kd(d-1)$.)

### 3.2. Proper sections and braid monodromy

Fix a $d$-gonal curve $C \subset \Sigma_k$. The term 'section' below stands for a continuous section of (an appropriate restriction of) the fibration $p\colon \Sigma_k \to \mathbb{P}^1$.

**Definition 3.2.1.** Let $\Delta \subset \mathbb{P}^1$ be a closed (topological) disk. A partial section $s\colon \Delta \to \Sigma_k$ of $p$ is called *proper* if its image is disjoint from both $E$ and $\operatorname{conv} C$.

**Lemma 3.2.2.** *Any disk $\Delta \subset \mathbb{P}^1$ admits a proper section $s\colon \Delta \to \Sigma_k$. Any two proper sections over $\Delta$ are homotopic in the class of proper sections; furthermore, any homotopy over a fixed point $b \in \Delta$ extends to a homotopy over $\Delta$.*

*Proof.* The restriction $p'$ of $p$ to $\Sigma_k \smallsetminus (E \cup \operatorname{conv} C)$ is a locally trivial fibration with a typical fiber $F'$ homeomorphic to a punctured open disk. Since $\Delta$ is contractible, $p'$ is trivial over $\Delta$ and, after trivializing, sections over $\Delta$ can be identified with maps $\Delta \to F'$. Such maps do exist, and any two such maps are homotopic, again due to the fact that $\Delta$ is contractible. $\square$

Pick a closed disk $\Delta \subset \mathbb{P}^1$ as above and denote $\Delta^\sharp = \Delta \smallsetminus \{b_1, \ldots, b_l\}$, where $b_1, \ldots, b_l$ are the singular fibers of $C$ that belong to $\Delta$. Fix a point $b \in \Delta^\sharp$. The restriction $p^\sharp \colon p^{-1}(\Delta^\sharp) \smallsetminus (C \cup E) \to \Delta^\sharp$ is a locally trivial fibration with a typical fiber $F_b^\circ \smallsetminus C$, and any proper section $s\colon \Delta \to \Sigma_k$ restricts to a section of $p^\sharp$. Hence, given a proper section $s$, one can define the group $\pi_F := \pi_1(F_b^\circ \smallsetminus C, s(b))$ and the *braid monodromy* $m\colon \pi_1(\Delta^\sharp, b) \to \operatorname{Aut} \pi_F$. Informally, for a loop $\sigma\colon [0,1] \to \Delta^\sharp$, the automorphism $m([\sigma])$ of $\pi_F$ is obtained by dragging the fiber $F_b$ along $\sigma(t)$ while keeping the base point on $s(\sigma(t))$. (Formally, it is obtained by trivializing the fibration $\sigma^* p^\sharp$.)

It is essential that, in this paper, we reserve the term 'braid monodromy' for the homomorphism $m$ constructed using a *proper* section $s$. Under this convention, the following lemma is an immediate consequence of Lemma 3.2.2 and the obvious fact that the braid monodromy is homotopy invariant.

**Lemma 3.2.3.** *The braid monodromy $m\colon \pi_1(\Delta^\sharp, b) \to \operatorname{Aut} \pi_F$ is well defined and independent of the choice of a proper section over $\Delta$ passing through $s(b)$.* $\square$

**Remark 3.2.4.** More generally, given a path

$$\tilde\sigma\colon [0,1] \to p^{-1}(\Delta^\sharp) \smallsetminus (\operatorname{conv} C \cup E),$$

one can use Lemma 3.2.2 to conclude that the braid monodromy commutes with the translation isomorphism

$$T_\sigma\colon \pi_1(\Delta^\sharp, \sigma(0)) \to \pi_1(\Delta^\sharp, \sigma(1))$$

(where $\sigma = p \circ \tilde\sigma\colon [0,1] \to \Delta^\sharp$) and the isomorphism

$$\operatorname{Aut} \pi_1(F^\circ_{\sigma(0)} \smallsetminus C, \tilde\sigma(0)) \to \operatorname{Aut} \pi_1(F^\circ_{\sigma(1)} \smallsetminus C, \tilde\sigma(1))$$

induced by the translation $T_{\tilde\sigma}$ along $\tilde\sigma$.

**Remark 3.2.5.** For most computations, we will take for $s$ a 'constant section' constructed as follows: pick an affine coordinate system $(x, y)$, see Remark 3.1.2, so that the point $x = \infty$ does *not* belong to $\Delta$, and let $s$ be the section $x \mapsto c = \text{const}$, $|c| \gg 0$. (In other words, the graph of $s$ is the 1-gonal curve $\{y = c\} \subset \Sigma_k$.) Since the intersection $p^{-1}(\Delta) \cap \operatorname{conv} C \subset \Sigma_k \smallsetminus E$ is compact, such a section is indeed proper whenever $|c|$ is sufficiently large.

**Remark 3.2.6.** Another consequence of Lemma 3.2.3 is the fact that, for any nested pair of disks $\Delta_1 \subset \Delta_2$, the braid monodromy commutes with the inclusion homomorphism $\pi_1(\Delta_1^\sharp) \to \pi_1(\Delta_2^\sharp)$. Indeed, one can construct both monodromies using a proper section over $\Delta_2$ and restricting it to $\Delta_1$ when necessary.

Pick a basis $\zeta_1, \ldots, \zeta_d$ for $\pi_F$ and a basis $\sigma_1, \ldots, \sigma_l$ for $\pi_1(\Delta^\sharp, b)$. Denote $m_i = m(\sigma_i)$, $i = 1, \ldots, l$. The following statement is the essence of Zariski–van Kampen's method for computing the fundamental group of a plane algebraic curve, see [12] for the proof and further details.

**Theorem 3.2.7.** *Let $\Delta \subset \mathbb{P}^1$ be a closed disk as above, and assume that the boundary $\partial\Delta$ is free of singular fibers of $C$. Then one has*

$$\pi_1(p^{-1}(\Delta) \smallsetminus (C \cup E), s(b)) = \langle \zeta_1, \ldots, \zeta_d \mid m_i = \mathrm{id},\ i = 1, \ldots, l \rangle,$$

*where each* braid relation $m_i = \mathrm{id}$ *should be understood as a $d$-tuple of relations $\zeta_j = m_i(\zeta_j)$, $j = 1, \ldots, d$.* □

### 3.3. The monodromy at infinity

Let $b \in \Delta^\sharp \subset \Delta \subset \mathbb{P}^1$ be as in Section 3.2. Denote by $\rho_b \in \pi_F$ the 'counterclockwise' generator of the abelian subgroup $\mathbb{Z} \cong \pi_1(F_b^\circ \smallsetminus \operatorname{conv} C)$ of $\pi_F$. (In other words, $\rho_b$ is the class of a large circle in $F_b^\circ$ encompassing $\operatorname{conv} C \cap F_b^\circ$. If $\zeta_1, \ldots, \zeta_d$ is a 'standard basis' for $\pi_F$, *cf*. Figure 3, left, then $\rho_b = \zeta_1 \cdot \ldots \cdot \zeta_d$.) Clearly, $\rho_b$ is invariant under the braid monodromy and, properly understood, it is preserved by the translation homomorphism along any path in $p^{-1}(\Delta^\sharp) \smallsetminus (\operatorname{conv} C \cup E)$. (Indeed, as explained in the proof of Lemma 3.2.2, the fibration $p^{-1}(\Delta) \smallsetminus (\operatorname{conv} C \cup E) \to \Delta$ is trivial, hence 1-simple.) Thus, there is a canonical identification of the elements $\rho_{b'}$, $\rho_{b''}$ in the fibers over any two points $b', b'' \in \Delta^\sharp$; for this reason, we will omit the subscript $b$ in the sequel.

Assume that the boundary $\partial\Delta$ is free of singular fibers of $C$. Then, connecting $\partial\Delta$ with the base point $b$ by a path in $\Delta^\sharp$ and traversing it in the counterclockwise direction (with respect to the canonical complex orientation of $\Delta$), one obtains a certain element $[\partial\Delta] \in \pi_1(\Delta^\sharp, b)$ (which depends on the choice of the path above).

**Proposition 3.3.1.** *In the notation above, assume that the interior of $\Delta$ contains all singular fibers of $C$. Then, for any $\zeta \in \pi_F$, one has*

$$m([\partial\Delta])(\zeta) = \rho^k \zeta \rho^{-k}.$$

*(In particular, $m([\partial\Delta])$ does not depend on the choices in the definition of the class $[\partial\Delta]$.)*

*Proof.* Due to the homotopy invariance of the braid monodromy (and the invariance of $\rho$), one can replace $\Delta$ with any larger disk and assume that the base point $b$ is in the boundary. Consider affine charts $(x, y)$ and $(x', y')$, see Remark 3.1.2, such that the fiber $\{x = \infty\} = \{x' = 0\}$ does not belong to $\Delta$ (and hence is nonsingular for $C$), and replace $\Delta$ with the disk $\{|x| \leqslant 1/\epsilon\}$ for some positive $\epsilon \ll 1$. About $x' = 0$, the curve $C$ has $d$ analytic branches of the form $y' = c_i + x'\varphi_i(x')$, where $c_i$ are pairwise distinct constants and $\varphi_i$ are analytic functions, $i = 1, \ldots, d$. Restricting these expressions to the circle $x' = \epsilon \exp(-2\pi t)$, $t \in [0, 1]$, and passing to $x = 1/x'$ and $y = y'x^k$, one obtains

$y = c_i\epsilon^{-k}\exp(2k\pi t) + O(\epsilon^{-k+1})$, $i = 1, \ldots, d$. Thus, from the point of view of a trivialization of the ruling over $\Delta$ (*e.g.*, the one given by $y$), the parameter $\epsilon$ can be chosen so small that the $d$ branches move along $d$ pairwise disjoint concentric circles (not quite round), each branch making $k$ turns in the counterclockwise direction. On the other hand, one can assume that the base point remains in a constant section $y = c = \text{const}$ with $|c| \gg \epsilon^{-k}\max|c_i|$, see Remark 3.2.5. The resulting braid is the conjugation by $\rho^{-k}$. ☐

### 3.4. The relation at infinity

We are ready to state the principal result of this section. Fix a $d$-gonal curve $C \subset \Sigma_k$ and choose a closed disk $\Delta \subset \mathbb{P}^1$ satisfying the following conditions:

1. $\Delta$ contains all but at most one singular fibers of $C$;
2. none of the singular fibers of $C$ is in the boundary $\partial\Delta$.

As in Section 3.2, pick a base point $b \in \Delta^\sharp$, a basis $\zeta_1, \ldots, \zeta_d$ for the group $\pi_F$ over $b$, and a basis $\sigma_1, \ldots, \sigma_l$ for the group $\pi_1(\Delta^\sharp, b)$. Let $m_i = m(\sigma_i)$, $i = 1, \ldots, l$, where $m\colon \pi_1(\Delta^\sharp, b) \to \operatorname{Aut}\pi_F$ is the braid monodromy.

**Theorem 3.4.1.** *Under the assumptions* (1), (2) *above, one has*

$$\pi_1(\Sigma_k \smallsetminus (C \cup E)) = \bigl\langle \zeta_1, \ldots, \zeta_d \bigm| m_i = \mathrm{id},\ i = 1, \ldots, l,\quad \rho^k = 1 \bigr\rangle,$$

*where each braid relation $m_i = \mathrm{id}$ should be understood as a $d$-tuple of relations $\zeta_j = m_i(\zeta_j)$, $j = 1, \ldots, d$, and $\rho \in \pi_F$ is the element introduced in Section* 3.3.

The relation $\rho^k = 1$ in Theorem 3.4.1 is called the *relation at infinity.* If $k = 1$, it coincides with the well known relation $\rho = 1$ for the group of a plane curve.

*Proof.* First, consider the case when $\Delta$ contains *all* singular fibers of $C$. As in the proof of Proposition 3.3.1, one can replace $\Delta$ with any larger disk, *e.g.*, with the one given by $\{|x| \leqslant 1/\epsilon\}$, where $(x, y)$ are affine coordinates such that the point $x = \infty$ is not in $\Delta$ and $\epsilon$ is a sufficiently small positive real number. Furthermore, one can take for $s$ a constant section $x \mapsto \epsilon^{-k}c = \text{const}$, $|c| \gg 0$, see Remark 3.2.5, and choose the base point $b$ in the boundary $\partial\Delta$. The fundamental group $\pi_1(p^{-1}(\Delta) \smallsetminus (C \cup E))$ is given by Theorem 3.2.7, and the patching of the nonsingular fiber $\{x = \infty\} = \{x' = 0\}$ results in the additional relation $[\partial\Gamma] = 1$, where $\Gamma$ is the disk $\{y' = c,\ |x'| \leqslant \epsilon\}$. (Here, $x' = 1/x$ and $y' = y/x^k$ are the affine coordinates in the complementary chart, see Remark 3.1.2. We assume that the constant $|c|$ is so large that $\Gamma \cap \operatorname{conv} C = \varnothing$.) Restricting to the boundary $x' = \epsilon\exp(-2\pi t)$, $t \in [0, 1]$, and passing back to $(x, y)$, one finds that the loop $\partial\Gamma$ is given by $x = \epsilon^{-1}\exp(2\pi t)$, $y = \epsilon^{-k}c\exp(2k\pi t)$; it is homotopic to $\rho^k \cdot [s(\partial\Delta)]$. Since the loop $s(\partial\Delta)$ is contractible (along the image of $s$), the extra relation is $\rho^k = 1$, as stated.

Now, assume that one singular fiber of $C$ is not in $\Delta$. Extend $\Delta$ to a larger disk $\Delta' \supset \Delta$ containing the missing singular fiber (and extend the braid monodromy, see Remark 3.2.6). For $\Delta'$, the theorem has already been proved, and the resulting presentation of the group differs from the one given by $\Delta$ by an extra relation $m_{l+1} = \mathrm{id}$. However, under an appropriate choice of the additional generator $\sigma_{l+1}$, one has $[\partial\Delta'] = [\partial\Delta] \cdot \sigma_{l+1}$. Clearly, $m([\partial\Delta])$ is a word in $m_1, \ldots, m_l$ and, in view of Proposition 3.3.1, the monodromy $m([\partial\Delta'])$ is the conjugation by $\rho^{-k}$. Hence, in the presence of the relation at infinity $\rho^k = 1$, the additional relation $m_{l+1} = \mathrm{id}$ is a consequence of the other braid relations, and the statement follows. $\square$

## 4. The fundamental group

### 4.1. Preliminaries

Fix a sextic $B$, pick a stable involutive symmetry $c$ of $B$, see §2, and let $\bar{B}, \bar{L} \subset \Sigma_2 = \mathbb{P}^2(O_c)/c$ be the projections of $B$ and $L_c$, respectively. We start with applying Theorem 3.4.1 to the 4-gonal curve $\bar{B} + \bar{L}$ and computing the group $\bar{\pi}_1 := \pi_1(\Sigma_2 \smallsetminus (\bar{B} \cup \bar{L} \cup E))$.

In order to visualize the braid monodromy, we will consider the standard *real structure* (*i.e.*, anti-holomorphic involution) $\mathrm{conj}\colon (x, y) \mapsto (\bar{x}, \bar{y})$ on $\Sigma_2$, where bar stands for the complex conjugation. A reduced algebraic curve $C$ in $\Sigma_2$ is said to be *real* (with respect to conj) if it is conj-invariant (as a set). Alternatively, $C$ is real if and only if, in the coordinates $(x, y)$, it can be given by a polynomial with real coefficients. In particular, the curve $\bar{B}$ given by (2.3.1) is real. Given a real curve $C \subset \Sigma_2$, one can speak about its *real part* $C_{\mathbb{R}}$ (*i.e.*, the set of points of $C$ fixed by conj), which is a codimension 1 subset in the real part of $\Sigma_2$.

To use Theorem 3.4.1, we take for $\Delta$ a closed regular neighborhood of the smallest segment of the real axis $\mathbb{P}^1_{\mathbb{R}}$ containing all singular fibers of $\bar{B} + \bar{L}$ except the one of type $\tilde{\mathbf{E}}_6$ at infinity, see the shaded area in Figure 3, right. Recall that singular are the fiber $\{x = 0\}$ through the cusp, the vertical tangent $\{x = 4\}$, and the fibers through the points of intersection of $\bar{B}$ and $\bar{L}$. (As in §2, we use the value $r = 3$ for the numeric evaluation.) We only consider the four extremal sections $\bar{L}$ given by (2.4.4), (2.5.2), (2.6.2), and (2.6.3). In each case, all singular fibers are real; they are listed in §2.

To compute the braid monodromy, we use a constant real section $s\colon \Delta \to \Sigma_2$ given by $x \mapsto \mathrm{const} \gg 0$, see Remark 3.2.5, and the base point $b = (\epsilon, 0) \in \Delta$, where $\epsilon > 0$ is sufficiently small. The basis $\sigma_1, \ldots, \sigma_l$ for the group $\pi_1(\Delta^\sharp, b)$ is chosen as shown in Figure 3, right: each $\sigma_i$ is a small loop about a singular fiber connected to $b$ by a real segment, circumventing the interfering singular fibers in the counterclockwise direction. Let $F = F_b$ be the base fiber, and choose a basis $\alpha$, $\beta$, $\gamma$, $\delta$ for the group $\pi_F = \pi_1(F^\circ \smallsetminus (\bar{B} \cup \bar{L}), s(b))$ as shown in Figure 3, left. (Note that, in all cases considered below, all points of the intersection $F \cap (\bar{B} \cup \bar{L})$ are real.) The following notation convention is important for the sequel.

**Remark 4.1.1.** We use a double notation for the elements of the basis for $\pi_F$. On the one hand, to be consistent with Theorem 3.4.1, we denote them $\zeta_1, \ldots, \zeta_4$, numbering the loops consecutively according to the decreasing of the

$y$-coordinate of the point. Then the element $\rho \in \pi_F$ introduced in Section 3.3 is given by $\rho = \zeta_1\zeta_2\zeta_3\zeta_4$, and the relation at infinity in Theorem 3.4.1 turns to $(\zeta_1\zeta_2\zeta_3\zeta_4)^2 = 1$. On the other hand, to make the formulas more readable, we denote the basis elements by $\alpha$, $\beta$, $\gamma$, and $\delta$. The first three elements are numbered consecutively, whereas $\delta$ plays a very special rôle in the passage to the group $\pi_1(\mathbb{P}^2 \smallsetminus B)$, see Lemma 4.1.2 below: we always assume that $\delta$ is the element represented by a loop about the point $F \cap \bar{L}$. Thus, the position of $\delta$ in the sequence $(\alpha, \beta, \gamma, \delta)$ may change; this position is important for the expression for $\rho$ and hence for the relation at infinity.

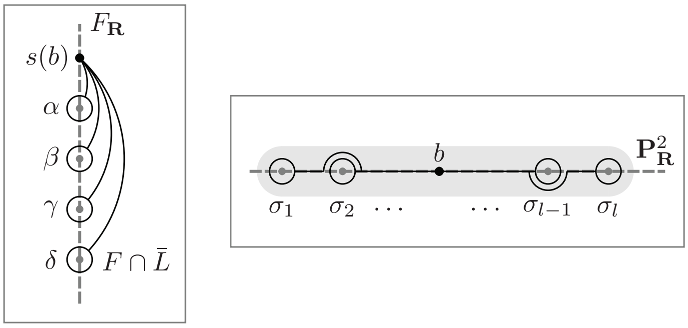


Figure 3. The basis $\alpha$, $\beta$, $\gamma$, $\delta$ and the loops $\sigma_i$

The passage from a presentation of $\bar{\pi}_1$ to the that of the group $\pi_1 := \pi_1(\mathbb{P}^2 \smallsetminus B)$ is given by the following lemma.

**Lemma 4.1.2.** *If $\bar{\pi}_1$ is given by $\langle \alpha, \beta, \gamma, \delta \mid R_j = 1,\ j = 1, \ldots, s\rangle$, then*

$$\pi_1 = \langle \alpha, \bar{\alpha}, \beta, \bar{\beta}, \gamma, \bar{\gamma} \mid R'_j = \bar{R}'_j = 1,\ j = 1, \ldots, s\rangle,$$

*where bar stands for the conjugation by $\delta$, $\bar{w} = \delta^{-1}w\delta$, each relation $R'_j$ is obtained from $R_j$, $j = 1, \ldots, s$, by letting $\delta^2 = 1$ and expressing the result in terms of the generators $\alpha$, $\bar{\alpha}$, $\ldots$, and $\bar{R}'_j = \delta^{-1}R'_j\delta$, $j = 1, \ldots, s$. (In other words, $\bar{R}'_j$ is obtained from $R'_j$ by interchanging $\alpha \leftrightarrow \bar{\alpha}$, $\beta \leftrightarrow \bar{\beta}$, and $\gamma \leftrightarrow \bar{\gamma}$.)*

*Proof.* The projection $\mathbb{P}^2 \smallsetminus (B \cup O_c) \to \Sigma_2 \smallsetminus (\bar{B} \cup E)$ is a double covering ramified at $\bar{L}$. Hence, one has

$$\pi_1 = \pi_1(\mathbb{P}^2 \smallsetminus (B \cup O_c)) = \operatorname{Ker}[\kappa\colon \bar{\pi}_1/\delta^2 \to \mathbb{Z}_2],$$

where $\kappa\colon \alpha, \beta, \gamma \mapsto 0$ and $\kappa\colon \delta \mapsto 1$. (Note that the compactification of the double covering above is *not* ramified at $\bar{B}$.) Lift $\kappa$ to a homomorphism $\tilde{\kappa}\colon \langle \alpha, \beta, \gamma, \delta\rangle \to \mathbb{Z}_2$. The two cosets modulo $\operatorname{Ker}\tilde{\kappa}$ are represented by 1 and $\delta$, and the standard calculation shows that $\operatorname{Ker}\tilde{\kappa}$ is the free group generated by $\alpha, \bar{\alpha}, \beta, \bar{\beta}, \gamma, \bar{\gamma}, \delta^2$. The kernel $N$ of the epimorphism $\operatorname{Ker}\tilde{\kappa} \twoheadrightarrow \pi_1$ is normally generated *in* $\langle \alpha, \beta, \gamma, \delta\rangle$ by $\delta^2$ and $R'_j$, $j = 1, \ldots, s$. Hence, one can remove the generator $\delta^2$ from the presentation. Besides, since the conjugation by $\delta$ is not an inner automorphism of $\operatorname{Ker}\tilde\kappa$, one should add the conjugates $\bar R'_j = \delta^{-1}R'_j\delta$ to obtain a set normally generating $N$ in $\operatorname{Ker}\tilde\kappa$. The resulting presentation of $\pi_1$ is the one stated in the lemma. $\square$

**Remark 4.1.3.** Note that $\bar{\ }\colon w \mapsto \bar w = \delta w\delta$ is an involutive automorphism of $\pi_1$. Hence, whenever a relation $R = 1$ holds in $\pi_1$, the relation $\bar R = 1$ also holds.

### 4.2. The set of singularities $(3\mathbf{E}_6) \oplus \mathbf{A}_1$

Take for $\bar L$ the section given by (2.5.2). The pair $(\bar B, \bar L)$ looks as shown in Figure 1, and the singular fibers are listed in 2.5. The generators $\zeta_1 = \alpha$, $\zeta_2 = \delta$, $\zeta_3 = \beta$, $\zeta_4 = \gamma$ for $\bar\pi_1$ are subject to the relations

| | |
|---|---|
| $(\delta\beta)^2 = (\beta\delta)^2$ | (the tangency point $x = 2$), |
| $(\delta\beta)\beta(\delta\beta)^{-1} = \gamma$ | (the vertical tangent $x = 4$), |
| $[\delta, \alpha\delta\beta\alpha] = 1, \quad \alpha\delta\beta\alpha = \beta\alpha\delta\beta$ | (the cusp $x = 0$), |
| $[\delta, (\alpha\delta\beta)\gamma(\alpha\delta\beta)^{-1}] = 1$ | (the transversal intersection $x = -16$), |
| $(\alpha\delta\beta\gamma)^2 = 1$ | (the relation at infinity). |

Letting $\delta^2 = 1$ and passing to $\alpha$, $\bar\alpha$, $\beta$, $\bar\beta$, $\gamma$, $\bar\gamma$, see Lemma 4.1.2, one can rewrite these relations in the following form:

$$[\beta, \bar\beta] = 1, \tag{4.2.1}$$

$$\gamma = \bar\beta, \quad \bar\gamma = \beta, \tag{4.2.2}$$

$$\alpha\bar\beta\bar\alpha = \bar\alpha\beta\alpha = \beta\alpha\bar\beta = \bar\beta\bar\alpha\beta, \tag{4.2.3}$$

$$\alpha\beta\alpha^{-1} = \bar\alpha\bar\beta\bar\alpha^{-1}, \tag{4.2.4}$$

$$\alpha\bar\beta\beta\bar\alpha\beta\bar\beta = 1. \tag{4.2.5}$$

(In (4.2.4) and (4.2.5), we eliminate $\gamma$ using (4.2.2).) Now, one can use the last relation in (4.2.3) to eliminate $\bar\alpha$: one has $\bar\alpha = \bar\beta^{-1}\beta\alpha\bar\beta\beta^{-1}$. Substituting this expression to $\alpha\bar\beta\bar\alpha = \beta\alpha\bar\beta$ and $\bar\alpha\beta\alpha = \beta\alpha\bar\beta$ in (4.2.3) and using (4.2.1), one obtains, respectively, the braid relations $\alpha\beta\alpha = \beta\alpha\beta$ and $\alpha\bar\beta\alpha = \bar\beta\alpha\bar\beta$. Conjugating by $\delta$, one also has $\bar\alpha\beta\bar\alpha = \beta\bar\alpha\beta$ and $\bar\alpha\bar\beta\bar\alpha = \bar\beta\bar\alpha\bar\beta$. Then, (4.2.4) turns to $\beta^{-1}\alpha\beta = \bar\beta^{-1}\bar\alpha\bar\beta$ and, eliminating $\bar\alpha$, one obtains $[\alpha, \bar\beta^2\beta^{-2}] = 1$. Finally, eliminating $\bar\alpha$ from the last relation (4.2.5), one gets $\alpha\beta^2\alpha\bar\beta^2 = 1$. Thus, the map $\beta \mapsto \sigma_1$, $\alpha \mapsto \sigma_2$, $\bar\beta \mapsto \sigma_3$ establishes an isomorphism

$$\pi_1(\mathbb{P}^2 \smallsetminus B) = \mathbb{B}_4/\langle[\sigma_2, \sigma_1^2\sigma_3^{-2}],\ \sigma_2\sigma_1^2\sigma_2\sigma_3^2\rangle.$$

It remains to notice that, in the presence of the second relation in the presentation above, the first one turns into $[\sigma_2, \sigma_1^2\sigma_2\sigma_1^2\sigma_2] = 1$, or $[\sigma_2, (\sigma_1\sigma_2)^3] = 1$, which holds automatically. Thus, one has

$$\pi_1(\mathbb{P}^2 \smallsetminus B) = \mathbb{B}_4/\sigma_2\sigma_1^2\sigma_2\sigma_3^2. \tag{4.2.6}$$

**Corollary 4.2.7.** *Let $D$ be a Milnor ball about a type $\mathbf{E}_6$ singular point of $B$. Then the inclusion homomorphism $\pi_1(D \smallsetminus B) \to \pi_1(\mathbb{P}^2 \smallsetminus B)$ is onto.*

*Proof.* Since any pair of type $\mathbf{E}_6$ singular points can be permuted by a stable symmetry of $B$, see [9], it suffices to prove the statement for the type $\mathbf{E}_6$ point resulting from the cusp of $\bar{B}$. In this case, the statement follows from (4.2.2), as $\alpha$, $\bar{\alpha}$, $\beta$, and $\bar{\beta}$ are all in the image of $\pi_1(D \smallsetminus B)$. $\square$

### 4.3. The set of singularities $(2\mathbf{E}_6 \oplus 2\mathbf{A}_2) \oplus \mathbf{A}_3$

Take for $\bar{L}$ the section given by (2.4.4). The pair $(\bar{B}, \bar{L})$ is plotted in Figure 1, and the singular fibers are listed in 2.4. The generators $\zeta_1 = \delta$, $\zeta_2 = \alpha$, $\zeta_3 = \beta$, $\zeta_4 = \gamma$ for $\bar{\pi}_1$ are subject to the relations

$$
\begin{array}{ll}
[\delta, \alpha] = 1 & \text{(the transversal intersection } x \approx .0459\text{)},\\
\alpha\beta\alpha = \beta\alpha\beta & \text{(the cusp } x = 0\text{)},\\
[\delta, \beta\alpha^{-1}\gamma\alpha\beta^{-1}] = 1 & \text{(the transversal intersection } x \approx -19.1\text{)},\\
(\delta\beta)^4 = (\beta\delta)^4 & \text{(the tangency point } x = .625\text{)},\\
(\delta\beta)^2\beta(\delta\beta)^{-2} = \gamma & \text{(the vertical tangent } x = 4\text{)},\\
(\delta\alpha\beta\gamma)^2 = 1 & \text{(the relation at infinity)}.
\end{array}
$$

(The third relation is simplified using $[\delta, \alpha] = 1$.) Letting $\delta^2 = 1$ and passing to $\alpha = \bar{\alpha}$, $\beta$, $\bar{\beta}$, $\gamma$, $\bar{\gamma}$, see Lemma 4.1.2, one can rewrite these relations as follows:

$$\alpha = \bar{\alpha}, \tag{4.3.1}$$

$$\alpha\beta\alpha = \beta\alpha\beta, \quad \alpha\bar{\beta}\alpha = \bar{\beta}\alpha\bar{\beta}, \tag{4.3.2}$$

$$\beta\alpha^{-1}\bar{\beta}\beta\bar{\beta}^{-1}\alpha\beta^{-1} = \bar{\beta}\alpha^{-1}\beta\bar{\beta}\beta^{-1}\alpha\bar{\beta}^{-1}, \tag{4.3.3}$$

$$(\bar{\beta}\beta)^2 = (\beta\bar{\beta})^2, \tag{4.3.4}$$

$$\bar{\beta}\beta\bar{\beta}^{-1} = \gamma, \quad \beta\bar{\beta}\beta^{-1} = \bar{\gamma}, \tag{4.3.5}$$

$$\alpha\bar{\beta}\beta\bar{\beta}\beta^{-1}\alpha\beta\bar{\beta}\beta\bar{\beta}^{-1} = 1. \tag{4.3.6}$$

(We use (4.3.1) and (4.3.5) to eliminate $\bar{\alpha}$, $\gamma$, and $\bar{\gamma}$ in the other relations.) Thus,

$$\pi_1(\mathbb{P}^2 \smallsetminus B) = G_3 := \big\langle \alpha, \beta, \bar{\beta} \bigm| (4.3.2)\text{–}(4.3.4),\ (4.3.6) \big\rangle. \tag{4.3.7}$$

The following statement is a consequence of the monodromy computation.

**Lemma 4.3.8.** *Let $F'$ be the fiber $\{x = \mathrm{const} \ll 0\}$ and let $\alpha_1$, $\beta_1$, $\gamma_1$, $\delta_1$ be the basis in $F'$ shown in Figure 4, left. Then, considering $\alpha_1$, $\beta_1$, and $\gamma_1$ as elements of $\bar{\pi}_1$, one has $\alpha_1 = \bar{\beta}$, $\beta_1 = \beta^{-1}\alpha\beta$, and $\gamma_1 = \gamma$.* $\square$

**Corollary 4.3.9.** *Let $D$ be a Milnor ball about a type $\mathbf{E}_6$ singular point of $B$. Then the inclusion homomorphism $\pi_1(D \smallsetminus B) \to \pi_1(\mathbb{P}^2 \smallsetminus B)$ is onto.*

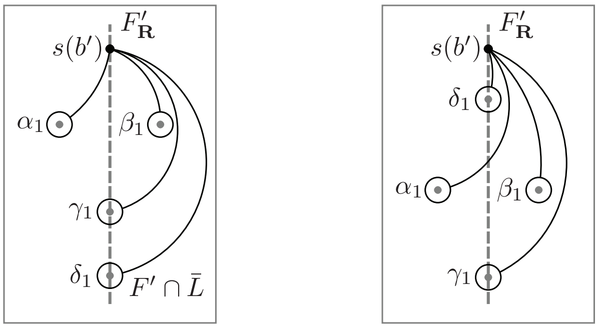


Figure 4. Generators in $F' = \{x = b' = \text{const} \ll 0\}$

*Proof.* In view of (4.3.5), one has $\beta = \alpha_1^{-1}\gamma_1\alpha_1$. Then $\alpha = \beta\beta_1\beta^{-1}$; hence, the elements $\alpha_1$, $\beta_1$, and $\gamma_1$ generate the group. On the other hand, $\alpha_1$, $\beta_1$, $\gamma_1$ are in the image of $\pi_1(D \smallsetminus B)$. □

### 4.4. The set of singularities $(2\mathbf{E}_6 \oplus \mathbf{A}_5) \oplus \mathbf{A}_2$: the first family

Take for $\bar{L}$ the section given by (2.6.2). The pair $(\bar{B}, \bar{L})$ looks as shown in Figure 1, and the singular fibers are listed in 2.6. The generators $\zeta_1 = \alpha$, $\zeta_2 = \beta$, $\zeta_3 = \delta$, $\zeta_4 = \gamma$ satisfy the following relations:

$$
\begin{aligned}
&[\delta, \alpha\beta] = 1, \quad \delta\alpha\beta\alpha = \beta\alpha\beta\delta && \text{(the cusp } x = 0\text{)},\\
&(\beta\delta)^3 = (\delta\beta)^3 && \text{(the tangency point } x \approx .951\text{)},\\
&(\beta\delta)\beta(\beta\delta)^{-1} = \gamma && \text{(the vertical tangent } x = 4\text{)},\\
&[\delta, \alpha^{-1}\gamma\alpha] = 1 && \text{(the transversal intersection } x \approx -18.4\text{)},\\
&(\alpha\beta\delta\gamma)^2 = 1 && \text{(the relation at infinity)}.
\end{aligned}
$$

Letting $\delta^2 = 1$ and passing to $\alpha$, $\bar{\alpha}$, $\beta$, $\bar{\beta}$, $\gamma$, $\bar{\gamma}$, see Lemma 4.1.2, one obtains

$$\alpha\beta = \bar{\alpha}\bar{\beta}, \quad \bar{\alpha}\bar{\beta}\bar{\alpha} = \beta\alpha\beta, \quad \alpha\beta\alpha = \bar{\beta}\bar{\alpha}\bar{\beta}, \tag{4.4.1}$$

$$\bar{\beta}\beta\bar{\beta} = \beta\bar{\beta}\beta, \tag{4.4.2}$$

$$\beta\bar{\beta}\beta^{-1} = \gamma, \quad \bar{\beta}\beta\bar{\beta}^{-1} = \bar{\gamma}, \tag{4.4.3}$$

$$\alpha^{-1}\gamma\alpha = \bar{\alpha}^{-1}\bar{\gamma}\bar{\alpha}, \tag{4.4.4}$$

$$\alpha\beta\bar{\gamma}\alpha\beta\gamma = 1. \tag{4.4.5}$$

The cusp relations (4.4.1) can be rewritten in the form

$$\bar{\alpha} = (\alpha\beta)^{-1}\beta(\alpha\beta), \quad \bar{\beta} = (\alpha\beta)\alpha(\alpha\beta)^{-1}, \quad (\alpha\beta)^3 = (\beta\alpha)^3, \tag{4.4.6}$$

or, in terms of $\bar{\alpha}$, $\bar{\beta}$, in the form

$$\alpha = (\bar{\alpha}\bar{\beta})^{-1}\bar{\beta}(\bar{\alpha}\bar{\beta}), \quad \beta = (\bar{\alpha}\bar{\beta})\bar{\alpha}(\bar{\alpha}\bar{\beta})^{-1}, \quad (\bar{\alpha}\bar{\beta})^3 = (\bar{\beta}\bar{\alpha})^3. \tag{4.4.7}$$

Geometrically, one has $\pi_1(D \smallsetminus B) = \langle \alpha, \beta \mid (\alpha\beta)^3 = (\beta\alpha)^3 \rangle$, where $D$ is a Milnor ball around the type $\mathbf{A}_5$ singular point.

Writing (4.4.5) as $\alpha\beta\bar{\gamma}\bar{\alpha}\bar{\beta}\gamma = 1$ and eliminating $\gamma$ and $\bar{\gamma}$ using (4.4.3) and (4.4.2), we can rewrite this relation in the form

$$\alpha\bar{\beta}\beta\bar{\alpha}\beta\bar{\beta} = 1. \tag{4.4.8}$$

Eliminating $\gamma$ and $\bar{\gamma}$ from (4.4.4), we obtain

$$\alpha^{-1}\beta\bar{\beta}\beta^{-1}\alpha = \bar{\alpha}^{-1}\bar{\beta}\beta\bar{\beta}^{-1}\bar{\alpha}. \tag{4.4.9}$$

Thus, we have

$$\pi_1(\mathbb{P}^2 \smallsetminus B) = G_2' := \langle \alpha, \beta \mid (\alpha\beta)^3 = (\beta\alpha)^3, \text{(4.4.2)}, \text{(4.4.8)}, \text{(4.4.9)} \rangle, \tag{4.4.10}$$

where $\bar{\alpha}$ and $\bar{\beta}$ are the words given by (4.4.6). I could not find any substantial simplification of this presentation. An alternative presentation of $G_2'$ (as well as of the group $G_2''$ introduced in (4.5.4) below) is given in Eyral, Oka [10].

As a part of computing the braid monodromy, we get the following lemma.

**Lemma 4.4.11.** *Let $F'$ be the fiber $\{x = \text{const} \ll 0\}$ and let $\alpha_1$, $\beta_1$, $\gamma_1$, $\delta_1$ be the basis in $F'$ shown in Figure 4, left. Then, considering $\alpha_1$, $\beta_1$, and $\gamma_1$ as elements of $\bar{\pi}_1$, one has $\alpha_1 = \beta$, $\beta_1 = \bar{\beta}^{-1}\bar{\alpha}\bar{\beta}$, and $\gamma_1 = \gamma$.* ☐

**Corollary 4.4.12.** *Let $D$ be a Milnor ball about a type $\mathbf{E}_6$ singular point of $B$. Then the inclusion homomorphism $\pi_1(D \smallsetminus B) \to \pi_1(\mathbb{P}^2 \smallsetminus B)$ is onto.*

*Proof.* Due to (4.4.3), one has $\bar{\beta} = \alpha_1^{-1}\gamma_1\alpha_1$. Then $\bar{\alpha} = \bar{\beta}\beta_1\bar{\beta}^{-1}$ and, in view of (4.4.7) and (4.4.3), $\bar{\alpha}$ and $\bar{\beta}$ generate the group. ☐

### 4.5. The set of singularities $(2\mathbf{E}_6 \oplus \mathbf{A}_5) \oplus \mathbf{A}_2$: the second family

Now, let $\bar{L}$ be the section given by (2.6.3). The pair $(\bar{B}, \bar{L})$ is plotted in Figure 2, and the singular fibers are listed in 2.6. The generators for $\pi_F$ are $\zeta_1 = \alpha$, $\zeta_2 = \beta$, $\zeta_3 = \delta$, $\zeta_4 = \gamma$, and the relations are:

| | |
|---|---|
| $[\delta, \alpha\beta] = 1, \quad \delta\alpha\beta\alpha = \beta\alpha\beta\delta$ | (the cusp $x = 0$), |
| $(\gamma\delta)^3 = (\delta\gamma)^3$ | (the tangency point $x \approx 3.55$), |
| $(\delta\gamma\delta)\gamma(\delta\gamma\delta)^{-1} = \beta$ | (the vertical tangent $x = 4$), |
| $[\delta, \gamma\alpha\gamma^{-1}] = 1$ | (the transversal intersection $x \approx 4.94$), |
| $(\alpha\beta\delta\gamma)^2 = 1$ | (the relation at infinity). |

Let $\delta^2 = 1$ and pass to the generators $\alpha$, $\bar{\alpha}$, $\beta$, $\bar{\beta}$, $\gamma$, $\bar{\gamma}$, see Lemma 4.1.2. Then, in addition to the cusp relations (4.4.6) (or (4.4.1)) and relation at infinity (4.4.5), we obtain

$$\gamma\bar{\gamma}\gamma = \bar{\gamma}\gamma\bar{\gamma}, \tag{4.5.1}$$

$$\bar{\gamma}\gamma\bar{\gamma}^{-1} = \beta, \quad \gamma\bar{\gamma}\gamma^{-1} = \bar{\beta} \tag{4.5.2}$$

$$\gamma\alpha\gamma^{-1} = \bar{\gamma}\bar{\alpha}\bar{\gamma}^{-1}. \tag{4.5.3}$$

Thus,

(4.5.4)
$$\pi_1(\mathbb{P}^2 \smallsetminus B) = G_2'' := \bigl\langle \alpha, \beta, \gamma, \bar\gamma \bigm| (\alpha\beta)^3 = (\beta\alpha)^3,\ (4.4.5),\ (4.5.1)\text{–}(4.5.3) \bigr\rangle,$$

where $\bar\alpha$ and $\bar\beta$ are the words given by (4.4.6). Note that one can eliminate either $\bar\gamma$, using (4.4.5), or $\beta$, using (4.5.2).

Extending the braid monodromy beyond the cusp of $B$ (to the negative values of $x$), we obtain the following statement.

**Lemma 4.5.5.** *Let $F'$ be the fiber $\{x = \mathrm{const} \ll 0\}$ and let $\alpha_1$, $\beta_1$, $\gamma_1$, $\delta_1$ be the basis in $F'$ shown in Figure 4, right. Then, considering $\alpha_1$, $\beta_1$, and $\gamma_1$ as elements of $\bar\pi_1$, one has $\alpha_1 = \bar\beta$, $\beta_1 = \bar\beta^{-1}\bar\alpha\bar\beta$, and $\gamma_1 = \gamma$.* □

**Corollary 4.5.6.** *Let $D$ be a Milnor ball about a type $\mathbf{E}_6$ singular point of $B$. Then the inclusion homomorphism $\pi_1(D \smallsetminus B) \to \pi_1(\mathbb{P}^2 \smallsetminus B)$ is onto.*

*Proof.* In view of (4.4.7) and (4.4.5), the elements $\bar\alpha = \alpha_1\beta_1\alpha_1^{-1}$, $\bar\beta = \alpha_1$, and $\gamma = \gamma_1$ generate the group. □

### 4.6. Comparing the two groups

Let $B'$ and $B''$ be the sextics considered in 4.4 and 4.5, respectively, so that their fundamental groups are $G_2'$ and $G_2''$. As explained in Eyral, Oka [10], the profinite completions of $G_2'$ and $G_2''$ are isomorphic (as the two curves are conjugate over an algebraic number field). Whether $G_2'$ and $G_2''$ themselves are isomorphic is still an open question. Below, we suggest an attempt to distinguish the two groups geometrically.

**Proposition 4.6.1.** *Let $D$ be a Milnor ball about the type $\mathbf{A}_5$ singular point of $B'$. Then the inclusion homomorphism $\pi_1(D \smallsetminus B') \to \pi_1(\mathbb{P}^2 \smallsetminus B')$ is onto.*

*Proof.* According to (4.4.10), the group $\pi_1(\mathbb{P}^2 \smallsetminus B') = G_2'$ is generated by $\alpha$ and $\beta$, which are both in the image of $\pi_1(D \smallsetminus B')$. □

**Conjecture 4.6.2.** *Let $D$ be a Milnor ball about the type $\mathbf{A}_5$ singular point of $B''$. Then the image of the inclusion homomorphism $\pi_1(D \smallsetminus B'') \to \pi_1(\mathbb{P}^2 \smallsetminus B'')$ does not contain $\gamma$ or $\bar\gamma$.*

**Remark 4.6.3.** If true, Conjecture 4.6.2 together with Proposition 4.6.1 would provide a topological distinction between the pairs $(\mathbb{P}^2, B')$ and $(\mathbb{P}^2, B'')$. Note that, according to [4], the two pairs are not diffeomorphic.

### 4.7. Other symmetric sets of singularities

The set of singularities $(3\mathbf{E}_6)$ is obtained by perturbing $\bar L$ in Section 4.2 to a section tangent to $\bar B$ at the cusp and transversal to $\bar B$ otherwise. This procedure replaces (4.2.1) with $\bar\beta = \beta$ or, alternatively, introduces a relation $\sigma_3 = \sigma_1$ in (4.2.6). The resulting group is $\mathbb{B}_3/(\sigma_1\sigma_2)^3$.

The sets of singularities of the form $(2\mathbf{E}_6 \oplus 2\mathbf{A}_2) \oplus \dots$ are obtained by perturbing $\bar{L}$ in Section 4.3. If $\bar{L}$ is perturbed to a double tangent (the set of singularities $(2\mathbf{E}_6 \oplus 2\mathbf{A}_2) \oplus 2\mathbf{A}_1$), relation (4.3.4) is replaced with $[\beta, \bar{\beta}] = 1$. Then, (4.3.6) turns to $\alpha\bar{\beta}^2\alpha\beta^2 = 1$, and (4.3.3) turns to

$$\beta\underline{\alpha^{-1}\beta\alpha}\beta^{-1} = \bar{\beta}\underline{\alpha^{-1}\bar{\beta}\alpha}\bar{\beta}^{-1}.$$

Replacing the underlined expressions using the braid relations (4.3.2) converts this relation to $\beta^2\alpha\beta^{-2} = \bar{\beta}^2\alpha\bar{\beta}^{-2}$, *i.e.*, $[\alpha, \bar{\beta}^2\beta^{-2}] = 1$. As explained in 4.2, the map $\beta \mapsto \sigma_1$, $\alpha \mapsto \sigma_2$, $\bar{\beta} \mapsto \sigma_3$ establishes an isomorphism $\pi_1(\mathbb{P}^2 \smallsetminus B) = \mathbb{B}_4/\sigma_2\sigma_1^2\sigma_2\sigma_3^2$.

Any other perturbation of $\bar{L}$ produces an extra point of its transversal intersection with $\bar{B}$, replacing (4.3.4) with $\beta = \bar{\beta}$. The resulting group is $\mathbb{B}_3/(\sigma_1\sigma_2)^3$.

Finally, the sets of singularities $(2\mathbf{E}_6 \oplus \mathbf{A}_5) \oplus \mathbf{A}_1$ and $(2\mathbf{E}_6 \oplus \mathbf{A}_5)$ are obtained by perturbing the inflection tangency point of $\bar{L}$ and $\bar{B}$ in Section 4.4. This procedure replaces (4.4.2) with $\bar{\beta} = \beta$. Then, from the first relation in (4.4.1) one has $\bar{\alpha} = \alpha$, relation (4.4.3) results in $\gamma = \bar{\beta} = \beta$, and relation (4.4.5) turns to $(\alpha\beta^2)^2 = 1$. Hence, the group is $\mathbb{B}_3/(\sigma_1\sigma_2)^3$. (Note that $(\sigma_1\sigma_2^2)^2 = (\sigma_1\sigma_2)^3$ in $\mathbb{B}_3$.)

### 4.8. Proof of Theorem 1.1.4

The fact that the perturbation epimorphisms $G_2', G_2'' \twoheadrightarrow \mathbb{B}_3/(\sigma_1\sigma_2)^3$ are proper is proved in Eyral, Oka [10], where it is shown that the Alexander module of a sextic with the set of singularities $(2\mathbf{E}_6 \oplus \mathbf{A}_5) \oplus \mathbf{A}_2$ has a torsion summand $\mathbb{Z}_2 \times \mathbb{Z}_2$, whereas the Alexander modules of all other groups listed in Theorem 1.1.3 can easily be shown to be $\mathbb{Z}[t]/(t^2 - t + 1)$. (In other words, the abelianization of the commutant of $G_2'$ or $G_2''$ is equal to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z} \times \mathbb{Z}$, and for all other groups it equals $\mathbb{Z} \times \mathbb{Z}$.)

The epimorphism

$$\varphi_0\colon G_0 = \mathbb{B}_4/\sigma_2\sigma_1^2\sigma_2\sigma_3^2 \twoheadrightarrow \mathbb{B}_3/(\sigma_1\sigma_2)^3$$

is considered in Oka, Pho [15]. One can observe that both braids $\sigma_2\sigma_1^2\sigma_2\sigma_3^2$ and $(\sigma_1\sigma_2)^3$ in the definition of the groups are pure, *i.e.*, belong to the kernels of the respective canonical epimorphism $\mathbb{B}_n \twoheadrightarrow \mathbb{B}_n/\sigma_1^2 = \mathbb{S}_n$. Furthermore, $\varphi_0$ takes each of the standard generators $\sigma_1$, $\sigma_2$, $\sigma_3$ of $\mathbb{B}_4$ to a conjugate of $\sigma_1$. Hence, the induced epimorphism $G_0/\varphi^{-1}(\sigma_1^2) = \mathbb{S}_4 \twoheadrightarrow \mathbb{B}_3/\sigma_1^2 = \mathbb{S}_3$ is proper, and so is $\varphi_0$.

A similar argument applies to the epimorphism $\varphi_3\colon G_3 \twoheadrightarrow G_0$, which takes each generator $\alpha$, $\beta$, $\bar{\beta}$ of $G_3$ to a conjugate of $\sigma_1 \in G_0$. The induced epimorphism

$$G_3/\varphi_3^{-1}(\sigma_1^2) = SL(2, \mathbb{F}_3) \twoheadrightarrow G_0/\sigma_1^2 = \mathbb{S}_4 = PSL(2, \mathbb{F}_3)$$

is proper; hence, so is $\varphi_3$. (Alternatively, one can compare $G_3/\varphi_3^{-1}(\sigma_1^4)$ and $G_0/\sigma_1^4$, which are finite groups of order $3 \cdot 2^9$ and $3 \cdot 2^6$, respectively. The finite quotients of $G_3$ and $G_0$ were computed using GAP [11].) $\square$

## 5. Perturbations

### 5.1. Perturbing a singular point

Consider a singular point $P$ of a plain curve $B$ and a Milnor ball $D$ around $P$. Let $B'$ be a nontrivial (*i.e.*, not equisingular) perturbation of $B$ such that, during the perturbation, the curve remains transversal to $\partial D$.

**Lemma 5.1.1.** *In the notation above, let $P$ be of type $\mathbf{E}_6$. Then $B' \cap D$ has one of the following sets of singularities*:

1. $2\mathbf{A}_2 \oplus \mathbf{A}_1$: *one has* $\pi_1(D \smallsetminus B') = \mathbb{B}_4$;
2. $\mathbf{A}_5$ *or* $2\mathbf{A}_2$: *one has* $\pi_1(D \smallsetminus B') = \mathbb{B}_3$;
3. $\mathbf{D}_5$, $\mathbf{D}_4$, $\mathbf{A}_4 \oplus \mathbf{A}_1$, $\mathbf{A}_4$, $\mathbf{A}_3 \oplus \mathbf{A}_1$, $\mathbf{A}_3$, $\mathbf{A}_2 \oplus k\mathbf{A}_1$ ($k = 0$, 1, *or* 2), *or* $k\mathbf{A}_1$ ($k = 0$, 1, 2, *or* 3): *one has* $\pi_1(D \smallsetminus B') = \mathbb{Z}$.

*Proof.* The perturbations of a simple singularity are enumerated by the subgraphs of its Dynkin graph, see E. Brieskorn [1] or G. Tjurina [17]. For the fundamental group, observe that the space $D \smallsetminus B$ is diffeomorphic to the space $\mathbb{P}^2 \smallsetminus (C \cup L)$, where $C \subset \mathbb{P}^2$ is a plane quartic with a type $\mathbf{E}_6$ singular point, and $L$ is a line with a single quadruple intersection point with $C$. Then, the perturbations of $B$ inside $D$ can be regarded as perturbations of $C$ keeping the point of quadruple intersection with $L$, see [2], and the perturbed fundamental group $\pi_1(\mathbb{P}^2 \smallsetminus (C' \cup L)) \cong \pi_1(D \smallsetminus B')$ is found in [3]. □

**Lemma 5.1.2.** *In the notation above, let $P$ be of type $\mathbf{A}_5$. Then $B' \cap D$ has one of the following sets of singularities*:

1. $2\mathbf{A}_2$: *one has* $\pi_1(D \smallsetminus B') = \mathbb{B}_3$;
2. $\mathbf{A}_3 \oplus \mathbf{A}_1$ *or* $3\mathbf{A}_1$: *one has* $\pi_1(D \smallsetminus B') = \mathbb{Z} \times \mathbb{Z}$;
3. $\mathbf{A}_4$, $\mathbf{A}_3$, $\mathbf{A}_2 \oplus \mathbf{A}_1$, $\mathbf{A}_2$, *or* $k\mathbf{A}_1$ ($k = 0$, 1, *or* 2): *one has* $\pi_1(D \smallsetminus B') = \mathbb{Z}$.

**Lemma 5.1.3.** *In the notation above, let $P$ be of type $\mathbf{A}_2$. Then $B' \cap D$ has the set of singularities $\mathbf{A}_1$ or $\varnothing$, and one has $\pi_1(D \smallsetminus B') = \mathbb{Z}$.*

*Proof of Lemmas* 5.1.2 *and* 5.1.3. Both statements are a well known property of type $\mathbf{A}$ singular points: any perturbation of a type $\mathbf{A}_p$ singular point has the set of singularities $\bigoplus \mathbf{A}_{p_i}$ with $d = (p+1) - \sum(p_i + 1) \geqslant 0$, and the group $\pi_1(D \smallsetminus B')$ is given by $\langle \alpha, \beta \mid \sigma^s \alpha = \alpha,\ \sigma^s \beta = \beta \rangle$, where $\sigma$ is the standard generator of the braid group $\mathbb{B}_2$ acting on $\langle \alpha, \beta \rangle$ and $s = 1$ if $d > 0$ or $s = \operatorname{g.c.d.}(p_i + 1)$ if $d = 0$. □

**Proposition 5.1.4.** *Let $B$ be a plane sextic of torus type with at least two type $\mathbf{E}_6$ singularities, and let $D$ be a Milnor ball about a type $\mathbf{E}_6$ singular point of $B$. Then the inclusion homomorphism $\pi_1(D \smallsetminus B) \to \pi_1(\mathbb{P}^2 \smallsetminus B)$ is onto.*

*Proof.* The proposition is an immediate consequence of Corollaries 4.2.7, 4.3.9, 4.4.12 and 4.5.6. □

**Corollary 5.1.5.** *Let $B$ be a plane sextic of torus type with at least two type $\mathbf{E}_6$ singular points, and let $B'$ be a perturbation of $B$.*

1. *If at least one of the type $\mathbf{E}_6$ singular points of $B$ is perturbed as in* 5.1.1(3)*, then $\pi_1(\mathbb{P}^2 \smallsetminus B') = \mathbb{Z}_6$.*
2. *If at least one of the type $\mathbf{E}_6$ singular points of $B$ is perturbed as in* 5.1.1(2) *and $B'$ is still of torus type, then $\pi_1(\mathbb{P}^2 \smallsetminus B') = \mathbb{B}_3/(\sigma_1\sigma_2)^3$.*

*Proof.* Let $D$ be a Milnor ball about the type $\mathbf{E}_6$ singular point in question. Due to Proposition 5.1.4, the inclusion homomorphism $\pi_1(D \smallsetminus B) \to \pi_1(\mathbb{P}^2 \smallsetminus B)$ is onto. Hence, in case (1), there is an epimorphism $\mathbb{Z} \twoheadrightarrow \pi_1(\mathbb{P}^2 \smallsetminus B')$, and in case (2), there is an epimorphism $\mathbb{B}_3 \twoheadrightarrow \pi_1(\mathbb{P}^2 \smallsetminus B')$. In the former case, the epimorphism above implies that the group is abelian, hence $\mathbb{Z}_6$. In the latter case, the central element $(\sigma_1\sigma_2)^3 \in \mathbb{B}_3$ projects to $6 \in \mathbb{Z} = \mathbb{B}_3/[\mathbb{B}_3, \mathbb{B}_3]$; since the abelianization of $\pi_1(\mathbb{P}^2 \smallsetminus B')$ is $\mathbb{Z}_6$, the epimorphism above must factor through an epimorphism $G := \mathbb{B}_3/(\sigma_1\sigma_2)^3 \twoheadrightarrow \pi_1(\mathbb{P}^2 \smallsetminus B')$. On the other hand, since $B'$ is assumed to be of torus type, there is an epimorphism $\pi_1(\mathbb{P}^2 \smallsetminus B') \twoheadrightarrow G$, and as $G \cong PSL(2, \mathbb{Z})$ is Hopfian (as it is obviously residually finite), each of the two epimorphisms is bijective. $\square$

**Corollary 5.1.6.** *Let $B$ be a plane sextic as in* 4.4*, and let $B'$ be a perturbation of $B$ such that the type $\mathbf{A}_5$ singular point is perturbed as in* 5.1.2(2) *or* (3)*. Then one has $\pi_1(\mathbb{P}^2 \smallsetminus B') = \mathbb{Z}_6$.*

*Proof.* Due to Proposition 4.6.1 and Lemma 5.1.2, the group of the perturbed sextic $B'$ is abelian. Since $B'$ is irreducible, $\pi_1(\mathbb{P}^2 \smallsetminus B') = \mathbb{Z}_6$. $\square$

**Corollary 5.1.7.** *Let $B$ be a plane sextic as in* 4.3*, and let $B'$ be a perturbation of $B$ such that an inner type $\mathbf{A}_2$ singular point of $B$ is perturbed to $\mathbf{A}_1$ or $\varnothing$. Then one has $\pi_1(\mathbb{P}^2 \smallsetminus B') = \mathbb{Z}_6$.*

*Proof.* Let $P$ be the inner type $\mathbf{A}_2$ singular point perturbed, and let $D$ be a Milnor ball about $P$. In the notation of Section 4.3, the group $\pi_1(D \smallsetminus B)$ is generated by $\alpha$ and $\beta$ (or $\bar{\alpha} = \alpha$ and $\bar{\beta}$ for the other point), and the perturbation results in an extra relation $\alpha = \beta$. Then (4.3.3) implies $\bar{\beta} = \beta$ and the group is cyclic. $\square$

### 5.2. Abelian perturbations

Theorem 5.2.1 below lists the sets of singularities obtained by perturbing at least one inner singular point from a set listed in Table 1, not covered by Nori's theorem [13], and not appearing in [8].

**Theorem 5.2.1.** *Let $\Sigma$ be a set of singularities obtained from one of those listed in Table 2 by several (possibly none) perturbations $\mathbf{A}_2 \to \mathbf{A}_1, \varnothing$ or $\mathbf{A}_1 \to \varnothing$. Then $\Sigma$ is realized by an irreducible plane sextic, not of torus type, whose fundamental group is $\mathbb{Z}_6$.*

Table 2. Sextics with abelian fundamental group

| | |
|---|---|
| $2\mathbf{E}_6 \oplus \mathbf{D}_5 \oplus \mathbf{A}_1$ | $\mathbf{D}_5 \oplus \mathbf{D}_4 \oplus 3\mathbf{A}_2$ |
| $2\mathbf{E}_6 \oplus \mathbf{A}_4 \oplus 2\mathbf{A}_1$ | $\mathbf{D}_5 \oplus \mathbf{D}_4 \oplus 2\mathbf{A}_2 \oplus 2\mathbf{A}_1$ |
| $2\mathbf{E}_6 \oplus \mathbf{A}_3 \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$ | $\mathbf{D}_5 \oplus 2\mathbf{A}_5 \oplus \mathbf{A}_1$ |
| $2\mathbf{E}_6 \oplus 2\mathbf{A}_2 \oplus \mathbf{A}_1$ | $\mathbf{D}_5 \oplus \mathbf{A}_5 \oplus \mathbf{A}_4 \oplus 2\mathbf{A}_1$ |
| $2\mathbf{E}_6 \oplus \mathbf{A}_2 \oplus 3\mathbf{A}_1$ | $\mathbf{D}_5 \oplus \mathbf{A}_5 \oplus \mathbf{A}_3 \oplus 2\mathbf{A}_2$ |
| $\mathbf{E}_6 \oplus 2\mathbf{D}_5 \oplus \mathbf{A}_1$ | $\mathbf{D}_5 \oplus 2\mathbf{A}_4 \oplus 3\mathbf{A}_1$ |
| $\mathbf{E}_6 \oplus \mathbf{D}_5 \oplus \mathbf{A}_5 \oplus \mathbf{A}_1$ | $\mathbf{D}_5 \oplus \mathbf{A}_4 \oplus \mathbf{A}_3 \oplus 2\mathbf{A}_2 \oplus \mathbf{A}_1$ |
| $\mathbf{E}_6 \oplus \mathbf{D}_5 \oplus \mathbf{A}_4 \oplus 2\mathbf{A}_1$ | $\mathbf{D}_5 \oplus \mathbf{A}_4 \oplus 2\mathbf{A}_2 \oplus 3\mathbf{A}_1$ |
| $\mathbf{E}_6 \oplus \mathbf{D}_5 \oplus \mathbf{A}_3 \oplus 2\mathbf{A}_2$ | $\mathbf{D}_5 \oplus 2\mathbf{A}_3 \oplus 2\mathbf{A}_2 \oplus \mathbf{A}_1$ |
| $\mathbf{E}_6 \oplus \mathbf{D}_5 \oplus 3\mathbf{A}_2$ | $2\mathbf{D}_4 \oplus \mathbf{A}_3 \oplus 2\mathbf{A}_2$ |
| $\mathbf{E}_6 \oplus \mathbf{D}_5 \oplus 2\mathbf{A}_2 \oplus 2\mathbf{A}_1$ | $2\mathbf{D}_4 \oplus 3\mathbf{A}_2$ |
| $\mathbf{E}_6 \oplus \mathbf{D}_4 \oplus \mathbf{A}_3 \oplus 2\mathbf{A}_2$ | $\mathbf{D}_4 \oplus \mathbf{A}_5 \oplus \mathbf{A}_3 \oplus 2\mathbf{A}_2$ |
| $\mathbf{E}_6 \oplus \mathbf{D}_4 \oplus 3\mathbf{A}_2$ | $\mathbf{D}_4 \oplus \mathbf{A}_4 \oplus \mathbf{A}_3 \oplus 2\mathbf{A}_2 \oplus \mathbf{A}_1$ |
| $\mathbf{E}_6 \oplus \mathbf{D}_4 \oplus 2\mathbf{A}_2 \oplus 2\mathbf{A}_1$ | $\mathbf{D}_4 \oplus \mathbf{A}_4 \oplus 2\mathbf{A}_2 \oplus 3\mathbf{A}_1$ |
| $\mathbf{E}_6 \oplus \mathbf{A}_5 \oplus \mathbf{A}_4 \oplus \mathbf{A}_1 \oplus \mathbf{A}_2$ | $\mathbf{D}_4 \oplus 2\mathbf{A}_3 \oplus 2\mathbf{A}_2 \oplus \mathbf{A}_1$ |
| $\mathbf{E}_6 \oplus \mathbf{A}_5 \oplus \mathbf{A}_3 \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$ | $2\mathbf{A}_5 \oplus \mathbf{A}_4 \oplus \mathbf{A}_1 \oplus \mathbf{A}_2$ |
| $\mathbf{E}_6 \oplus 2\mathbf{A}_4 \oplus 3\mathbf{A}_1$ | $2\mathbf{A}_5 \oplus \mathbf{A}_3 \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$ |
| $\mathbf{E}_6 \oplus \mathbf{A}_4 \oplus \mathbf{A}_3 \oplus 2\mathbf{A}_2 \oplus \mathbf{A}_1$ | $\mathbf{A}_5 \oplus 2\mathbf{A}_4 \oplus 2\mathbf{A}_1 \oplus \mathbf{A}_2$ |
| $\mathbf{E}_6 \oplus \mathbf{A}_4 \oplus 2\mathbf{A}_2 \oplus 3\mathbf{A}_1$ | $\mathbf{A}_5 \oplus \mathbf{A}_4 \oplus \mathbf{A}_3 \oplus 2\mathbf{A}_2 \oplus \mathbf{A}_1$ |
| $\mathbf{E}_6 \oplus 2\mathbf{A}_3 \oplus 2\mathbf{A}_2 \oplus \mathbf{A}_1$ | $\mathbf{A}_5 \oplus 2\mathbf{A}_3 \oplus 2\mathbf{A}_2 \oplus \mathbf{A}_1$ |
| $2\mathbf{D}_5 \oplus \mathbf{A}_5 \oplus \mathbf{A}_1$ | $3\mathbf{A}_4 \oplus 4\mathbf{A}_1$ |
| $2\mathbf{D}_5 \oplus \mathbf{A}_4 \oplus 2\mathbf{A}_1$ | $2\mathbf{A}_4 \oplus \mathbf{A}_3 \oplus 2\mathbf{A}_2 \oplus 2\mathbf{A}_1$ |
| $2\mathbf{D}_5 \oplus \mathbf{A}_3 \oplus 2\mathbf{A}_2$ | $2\mathbf{A}_4 \oplus 2\mathbf{A}_2 \oplus 4\mathbf{A}_1$ |
| $2\mathbf{D}_5 \oplus 3\mathbf{A}_2$ | $\mathbf{A}_4 \oplus 2\mathbf{A}_3 \oplus 2\mathbf{A}_2 \oplus 2\mathbf{A}_1$ |
| $2\mathbf{D}_5 \oplus 2\mathbf{A}_2 \oplus 2\mathbf{A}_1$ | $3\mathbf{A}_3 \oplus 2\mathbf{A}_2 \oplus 2\mathbf{A}_1$ |
| $\mathbf{D}_5 \oplus \mathbf{D}_4 \oplus \mathbf{A}_3 \oplus 2\mathbf{A}_2$ | |

Altogether, perturbations as in Theorem 5.2.1 produce 244 sets of singularities not covered by Nori's theorem; 117 of them are new as compared to [8].

*Proof.* Each set of singularities in question is obtained by a perturbation from one of the sets of singularities listed in Table 1. Furthermore, the perturbation can be chosen so that at least one type $\mathbf{E}_6$ singular point is perturbed as in 5.1.1(3), or the type $\mathbf{A}_5$ singular point is perturbed as in 5.1.2(3), or at least one inner cusp is perturbed to $\mathbf{A}_1$ or $\varnothing$. According to [8], any such (formal) perturbation is realized by a family of sextics, and due to Corollaries 5.1.5(1), 5.1.6, and 5.1.7, the perturbed sextic has abelian fundamental group. $\square$

### 5.3. Non-abelian perturbations

In this section, we treat the few perturbations of torus type that can be obtained from Table 1 and do not appear in [8].

**Theorem 5.3.1.** *Each of the eight sets of singularities listed in Table 3 is realized by an irreducible plane sextic of torus type whose fundamental group is* $\mathbb{B}_3/(\sigma_1\sigma_2)^3$.

Table 3. Sextics of torus type

| | |
|---|---|
| $(\mathbf{E}_6 \oplus 2\mathbf{A}_5) \oplus \mathbf{A}_2$ | $(3\mathbf{A}_5) \oplus \mathbf{A}_1$ |
| $(\mathbf{E}_6 \oplus 2\mathbf{A}_5) \oplus \mathbf{A}_1$ | $(3\mathbf{A}_5)$ |
| $(\mathbf{E}_6 \oplus 2\mathbf{A}_5)$ | $(\mathbf{E}_6 \oplus \mathbf{A}_5 \oplus 2\mathbf{A}_2) \oplus \mathbf{A}_3$ |
| $(3\mathbf{A}_5) \oplus \mathbf{A}_2$ | $(2\mathbf{A}_5 \oplus 2\mathbf{A}_2) \oplus \mathbf{A}_3$ |

Theorem 5.3.1 covers two tame sextics: $(\mathbf{E}_6 \oplus 2\mathbf{A}_5)$ and $(3\mathbf{A}_5)$. The fundamental groups of these curves were first found in Oka, Pho [15].

*Proof.* As in the previous section, we perturb one of the sets of singularities listed in Table 1, this time making sure that

1. each type $\mathbf{E}_6$ singular point is perturbed as in 5.1.1(1) or (2) (or is not perturbed at all),
2. each type $\mathbf{A}_5$ singular point is perturbed as in 5.1.2(1) (or is not perturbed at all),
3. none of the inner cusps is perturbed, and
4. at least one type $\mathbf{E}_6$ singular point is perturbed as in 5.1.1(2).

(Note that, in the case under consideration, inner are the cusps appearing from the cusp of $\bar{B}$.) From the arithmetic description of curves of torus type given in [5] (see also [4]) it follows that any perturbation satisfying (1)–(3) above preserves the torus structure; then, in view of (4), Corollary 5.1.5(2) implies that the resulting fundamental group is $\mathbb{B}_3/(\sigma_1\sigma_2)^3$. □

**Remark 5.3.2.** If all type $\mathbf{E}_6$ singular points are perturbed as in 5.1.1(1) (or not perturbed at all), the study of the fundamental group would require more work; in particular, one would need an explicit description of the homomorphism $\pi_1(D_{\mathbf{E}_6} \smallsetminus B) \twoheadrightarrow \pi_1(D_{\mathbf{E}_6} \smallsetminus B')$. On the other hand, it is easy to show that such perturbations do not give anything new compared to [8]. (In fact, using [4], one can even show that the deformation classes of the sextics obtained are the same; it suffices to prove the connectedness of the deformation families realizing the sets of singularities $(\mathbf{E}_6 \oplus \mathbf{A}_5 \oplus 2\mathbf{A}_2) \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$ and $(\mathbf{E}_6 \oplus 4\mathbf{A}_2) \oplus \mathbf{A}_3 \oplus \mathbf{A}_1$, which are maximal in the context.) For this reason, we do not consider these perturbations here.

Department of Mathematics
Bilkent University
06800 Ankara, Turkey
e-mail: degt@fen.bilkent.edu.tr

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