J. Math. Kyoto Univ. (JMKYAZ)
48-4 (2008), 747–755

On parabolic geometry of type $\mathbf{PGL}(d, \mathbb{C})/P$

By

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Abstract

Let P be the maximal parabolic subgroup of $\mathrm{PGL}(d, \mathbb{C})$ defined by invertible matrices $(a_{ij})_{i,j=1}^{d}$ with $a_{dj} = 0$ for all $j \in [1, d-1]$. Take a holomorphic parabolic geometry (M, E_P, ω) of type $\mathrm{PGL}(d, \mathbb{C})/P$. Assume that M is a complex projective manifold. We prove the following: If there is a nonconstant holomorphic map $f : \mathbb{CP}^1 \longrightarrow M$, then M is biholomorphic to the projective space $\mathbb{CP}^{d-1}$.

1. Introduction

Let H_0 be the hyperplane in $\mathbb{CP}^{d-1}$ defined by all $(x_1, \cdots, x_d) \in \mathbb{C}^d$ with $x_d = 0$. Let $P \subset \mathrm{PGL}(d, \mathbb{C})$ be the maximal parabolic subgroup that leaves H_0 invariant. A holomorphic parabolic geometry of type $\mathrm{PGL}(d, \mathbb{C})/P$ is a triple (M, E_P, ω), where E_P is a holomorphic principal P–bundle over a complex manifold M of complex dimension $d - 1$, and ω is a Cartan connection on E_P (the definition of a Cartan connection is recalled in Section 2).

Projective structures on complex manifolds of dimension $d-1$ are the flat holomorphic parabolic geometries of type $\mathrm{PGL}(d, \mathbb{C})/P$. See [Sh], [CS], [CSS] and references therein for parabolic geometries.

We prove the following theorem (see Theorem 3.1):

Theorem 1.1. *Let (M, E_P, ω) be a holomorphic parabolic geometry of type $\mathrm{PGL}(d, \mathbb{C})/P$. Assume that M is a complex projective manifold of dimension $d - 1$. If there is a nonconstant holomorphic map*

$$f : \mathbb{CP}^1 \longrightarrow M,$$

then M is biholomorphic to the projective space $\mathbb{CP}^{d-1}$.

As a corollary, if M in Theorem 1.1 is a Fano variety, then M is biholomorphic to the projective space $\mathbb{CP}^{d-1}$ (see Corollary 3.1).

2000 *Mathematics Subject Classification*(*s*). 53C15, 14M17.

Received September 11, 2007

2. Parabolic geometry of type $\mathbf{PGL}(d, \mathbb{C})/P$

Let V be a complex vector space of dimension d, with $d \geq 2$. Fix a hyperplane

$$H_0 \subset V. \tag{2.1}$$

Let $\mathbb{P}(V)$ be the complex projective space of dimension $d-1$ that parametrizes all the one–dimensional quotient linear spaces of V. So H_0 defines a point of $\mathbb{P}(V)$. Let

$$\mathrm{PGL}(V) := \mathrm{GL}(V)/\mathbb{C}^*$$

be the group of all projective linear automorphisms of $\mathbb{P}(V)$. Let

$$P \subset \mathrm{PGL}(V)$$

be the maximal parabolic subgroup that fixes the point of $\mathbb{P}(V)$ defined by H_0. The Lie algebra of $\mathrm{PGL}(V)$ will be denoted by $\mathfrak{g}$. Let

$$\mathfrak{p} \subset \mathfrak{g}$$

be the Lie algebra of the subgroup P.

Let

$$E_P \longrightarrow M$$

be a holomorphic principal P–bundle over a complex manifold M of dimension $d-1$. For any $g \in P$, let

$$\tau_g : E_P \longrightarrow E_P \tag{2.2}$$

be the biholomorphic map defined by $z \longmapsto zg$. Using the free action of P on E_P, we get an injective homomorphism from the Lie algebra $\mathfrak{p}$ to the Lie algebra of globally defined holomorphic vector fields on the total space of E_P. For any $v \in \mathfrak{p}$, the corresponding holomorphic vector field on E_P will be denoted by ζ_v.

A *holomorphic Cartan connection* on E_P is a holomorphic one–form

$$\omega \in H^0(E_P, \Omega^1_{E_P} \bigotimes\nolimits_{\mathbb{C}} \mathfrak{g})$$

on E_P with values in the Lie algebra $\mathfrak{g}$ that satisfies the following three conditions:

1. $\omega(z)(\zeta_v(z)) = v$ for all $v \in \mathfrak{p}$ and $z \in E_P$,
2. $\tau_g^* \omega = \mathrm{Ad}(g^{-1}) \circ \omega$ for all $g \in P$, where τ_g is defined in Eq. (2.2), and
3. for each point $z \in E_P$, the homomorphism $T_z E_P \longrightarrow \mathfrak{g}$ from the holomorphic tangent space, defined by $\alpha \longmapsto \omega(z)(\alpha)$, is an isomorphism.

See [CSS, p. 99–100], [Sh, p. 184].

Definition 2.1. A holomorphic parabolic geometry of type $\mathrm{PGL}(V)/P$ is a triple

$$(M, E_P, \omega),$$

where

• M is a connected complex manifold of complex dimension $d-1$,
• E_P is a holomorphic principal P–bundle over M, and
• ω is a Cartan connection on E_P.

See [CSS, p. 99–100], [Sh, p. 184] for holomorphic parabolic geometry (in [Sh] the terminology *Cartan geometry* is used).

Remark 1. Projective structures constitute a class of examples of holomorphic parabolic geometries of type $\mathrm{PGL}(V)/P$. We recall that a projective structure on a connected complex manifold X of complex dimension $d-1$ is given by data $\{U_i, \phi_i\}_{i\in I}$, where

• $U_i \subset X$ are open subsets with $\bigcup_{i\in I} U_i = X$, and
• $\phi_i : U_i \longrightarrow \mathbb{P}(V)$ are biholomorphic maps from U_i to $\phi_i(U_i)$ satisfying the condition that for each ordered pair $i, k \in I$, there is some $G_{i,k} \in \mathrm{PGL}(V) = \mathrm{Aut}(\mathbb{P}(V))$ such that the map

$$\phi_k \circ \phi_i^{-1} : \phi_i(U_i \cap U_k) \longrightarrow \phi_k(U_i \cap U_k)$$

coincides with the restriction of $G_{i,k}$.

Two such data $\{U_i, \phi_i\}_{i\in I}$ and $\{U_j, \phi_j\}_{j\in J}$ are called equivalent if their union $\{U_i, \phi_i\}_{i\in I\cup J}$ also satisfies the above conditions. A projective structure on X is an equivalence class of data satisfying the above conditions.

Projective structures are precisely the flat holomorphic parabolic geometries of type $\mathrm{PGL}(V)/P$; see [CS, p. 57], [Sh, p. 184] for the definition of a flat parabolic geometry. □

Let (M, E_P, ω) be a holomorphic parabolic geometry of type $\mathrm{PGL}(V)/P$ as in Definition 2.1. Let

$$E_{\mathrm{PGL}(V)} := E_P(\mathrm{PGL}(V)) = E_P \times^P \mathrm{PGL}(V) \tag{2.3}$$

be the holomorphic principal $\mathrm{PGL}(V)$–bundle over M obtained by extending the structure group of E_P using the inclusion map $P \hookrightarrow \mathrm{PGL}(V)$. Therefore, $E_{\mathrm{PGL}(V)}$ is a quotient of $E_P \times \mathrm{PGL}(V)$; two points (z_1, g_1) and (z_2, g_2) of $E_P \times \mathrm{PGL}(V)$ give the same point in the quotient space $E_{\mathrm{PGL}(V)}$ if and only if there is an element $g_0 \in P$ such that $(z_1, g_1) = (z_2 g_0, g_0^{-1} g_2)$.

The Cartan connection ω on E_P defines a holomorphic connection D^ω on the principal $\mathrm{PGL}(V)$–bundle $E_{\mathrm{PGL}(V)}$ in Eq. (2.3) (see [CS, p. 60–61], [Sh, p. 365, Proposition 3.1]). We briefly recall the construction of D^ω. Consider the $\mathfrak{g}$–valued holomorphic one–form

$$\widetilde{\omega} := p_1^*\omega + p_2^*\omega_{\mathrm{MC}} \tag{2.4}$$

on $E_P \times \mathrm{PGL}(V)$, where p_1 (respectively, p_2) is the projection of the Cartesian product $E_P \times \mathrm{PGL}(V)$ to E_P (respectively, $\mathrm{PGL}(V)$), and ω_{MC} is the Maurer–Cartan form on $\mathrm{PGL}(V)$. This form $\widetilde{\omega}$ descends to a $\mathfrak{g}$–valued holomorphic one–form on the quotient space $E_{\mathrm{PGL}(V)}$ of $E_P \times \mathrm{PGL}(V)$. It is straight–forward to check that this descended one–form defines a holomorphic connection on the principal $\mathrm{PGL}(V)$–bundle $E_{\mathrm{PGL}(V)}$. The connection D^ω on $E_{\mathrm{PGL}(V)}$ is the one defined by this descended form.

Let $L_0 := V/H_0$ be the complex line, where H_0 is the hyperplane in Eq. (2.1). Let

$$\widehat{V} := \mathrm{Hom}(V, L_0) \tag{2.5}$$

be the complex vector space of linear maps from V to L_0. Note that the quotient map

$$V \longrightarrow V/H_0 = L_0$$

gives a nonzero element

$$\theta \in \widehat{V}. \tag{2.6}$$

The parabolic subgroup $\widetilde{P} := \{g \in \mathrm{GL}(V) \mid g(H_0) = H_0\} \subset \mathrm{GL}(V)$ has a natural linear action on the vector space $\widehat{V}$ defined in Eq. (2.5). This action descends to an action of $P = \widetilde{P}/\mathbb{C}^*$ on $\widehat{V}$. In other words, $\widehat{V}$ is a P–module. The action of P on $\widehat{V}$ clearly fixes the vector θ in Eq. (2.6).

As before, let (M, E_P, ω) be a holomorphic parabolic geometry of type $\mathrm{PGL}(V)/P$. Let

$$E_{\widehat{V}} := E_P(\widehat{V}) = E_P \times^P \widehat{V} \tag{2.7}$$

be the holomorphic vector bundle over M associated to the principal P–bundle E_P for the P–module $\widehat{V}$. We noted above that the action of P on $\widehat{V}$ fixes the nonzero vector θ in Eq. (2.6). Hence θ defines a nowhere vanishing holomorphic section

$$\widetilde{\theta} \in H^0(M, E_{\widehat{V}}) \tag{2.8}$$

of the holomorphic vector bundle $E_{\widehat{V}}$ in Eq. (2.7). Let

$$L_{\widetilde{\theta}} \subset E_{\widehat{V}} \tag{2.9}$$

be the holomorphic line subbundle generated by the nowhere vanishing section $\widetilde{\theta}$ in Eq. (2.8).

Lemma 2.1. *The quotient vector bundle $E_{\widehat{V}}/L_{\widetilde{\theta}}$ (see Eq. (2.9)) is identified with the holomorphic tangent bundle TM.*

Proof. Consider the P–module $\mathfrak{g}/\mathfrak{p}$. Let

$$E_{\mathfrak{g}/\mathfrak{p}} := E_P(\mathfrak{g}/\mathfrak{p}) = E_P \times^P (\mathfrak{g}/\mathfrak{p})$$

be the holomorphic vector bundle over M associated to the principal P–bundle E_P for the P–module $\mathfrak{g}/\mathfrak{p}$. This vector bundle $E_{\mathfrak{g}/\mathfrak{p}}$ is canonically identified with the holomorphic tangent bundle TM (see [CS, p. 61, § 2.12], [Sh, p. 188]).

Therefore, to prove the lemma it suffices to show that the P–module $\mathfrak{g}/\mathfrak{p}$ is identified with $\widehat{V}/(\mathbb{C}\cdot\theta)$, where $\widehat{V}$ is defined in Eq. (2.5), and θ is constructed in Eq. (2.6). It is easy to check that both the P–modules $\mathfrak{g}/\mathfrak{p}$ and $\widehat{V}/(\mathbb{C}\cdot\theta)$ are identified with the P–module $H_0^* \bigotimes (V/H_0)$, where H_0 is the hyperplane in Eq. (2.1). This completes the proof of the lemma. □

See [At] for the definition of holomorphic connections on a holomorphic fiber bundle.

Proposition 2.1. *Let $P(E_{\widehat{V}})$ denote the projective bundle over M that parametrizes all the one–dimensional linear subspaces in the fibers of the vector bundle $E_{\widehat{V}}$ in Eq.* (2.7)*. This projective bundle $P(E_{\widehat{V}})$ has a natural holomorphic connection.*

Proof. Consider the projective bundle of relative dimension $d-1$

$$\widehat{E}_{\mathbb{P}} := E_{\mathrm{PGL}(V)}(\mathbb{P}(V)) = E_{\mathrm{PGL}(V)} \times^{\mathrm{PGL}(V)} \mathbb{P}(V) \tag{2.10}$$

over M associated to the principal $\mathrm{PGL}(V)$–bundle $E_{\mathrm{PGL}(V)}$ in Eq. (2.3) for the standard action of $\mathrm{PGL}(V)$ on $\mathbb{P}(V)$ (the projective space parametrizing hyperplanes in V). Since $E_{\mathrm{PGL}(V)}$ is an extension of structure group of the principal P–bundle E_P (see Eq. (2.3)), this projective bundle $\widehat{E}_{\mathbb{P}}$ is identified with the projective bundle

$$\widehat{E}'_{\mathbb{P}} := E_P(\mathbb{P}(V)) = E_P \times^P \mathbb{P}(V) \tag{2.11}$$

over M of relative dimension $d-1$ associated to the principal P–bundle E_P for the action of P on $\mathbb{P}(V)$ obtained by restricting the standard action of $\mathrm{PGL}(V)$.

On the other hand, the projective space $\mathbb{P}(V)$ is identified with the projective space $P(\widehat{V})$ that parametrizes all lines in the vector space $\widehat{V}$ defined in Eq. (2.5). The action of P on $\widehat{V}$ induces an action of P on $P(\widehat{V})$. The identification between $\mathbb{P}(V)$ and $P(\widehat{V})$ clearly intertwines the actions of P. Consequently, the projective bundle $\widehat{E}'_{\mathbb{P}}$ in Eq. (2.11) is identified with the projective bundle $P(E_{\widehat{V}})$ in the statement of the proposition. Finally, $P(E_{\widehat{V}})$ is identified with the projective bundle $\widehat{E}_{\mathbb{P}}$ in Eq. (2.10) using the earlier observation that $\widehat{E}_{\mathbb{P}}$ is identified with $\widehat{E}'_{\mathbb{P}}$.

We saw earlier that the principal $\mathrm{PGL}(V)$–bundle $E_{\mathrm{PGL}(V)}$ has a natural holomorphic connection D^ω which is constructed from the one–form in Eq. (2.4). The connection D^ω induces a holomorphic connection on any fiber bundle associated to the principal $\mathrm{PGL}(V)$–bundle $E_{\mathrm{PGL}(V)}$. In particular, the associated projective bundle $\widehat{E}_{\mathbb{P}}$ in Eq. (2.10) gets a holomorphic connection. Now we have a holomorphic connection on $P(E_{\widehat{V}})$ using its identification with $\widehat{E}_{\mathbb{P}}$. This completes the proof of the proposition. □

3. Rational curves and parabolic geometry of type $\mathbf{PGL}(d,\mathbb{C})/P$

We continue with the notation of the previous section.

Let (M, E_P, ω) be a holomorphic parabolic geometry of type $\mathrm{PGL}(V)/P$. Assume that M is a smooth complex projective variety of dimension $d-1$.

Theorem 3.1. *If there is a nonconstant holomorphic map*

$$f : \mathbb{CP}^1 \longrightarrow M,$$

then M is biholomorphic to the projective space $\mathbb{CP}^{d-1}$.

Proof. Let

$$f : \mathbb{CP}^1 \longrightarrow M \tag{3.1}$$

be a nonconstant holomorphic map. A theorem due to Mori says that M is biholomorphic to $\mathbb{CP}^{d-1}$ if the holomorphic vector bundle f^*TM over $\mathbb{CP}^1$ is ample for all such f [Mo, p. 594] (the above formulation of Mori's theorem can also be found in [MP, p. 41, Theorem 4.2]); we note that the condition (i) in the second paragraph of [Mo, p. 594] that the anti–canonical line bundle is ample is used in ensuring that there exists a nonconstant holomorphic map from $\mathbb{CP}^1$ to M.

Any holomorphic vector bundle over $\mathbb{CP}^1$ splits into a direct sum of holomorphic line bundle [Gr, p. 126, Théorème 2.1]. Let

$$f^*TM = \bigoplus_{i=1}^{d-1} \xi_i \tag{3.2}$$

be a holomorphic decomposition of f^*TM into a direct sum of holomorphic line bundles, where f is the map in Eq. (3.1). We note that the vector bundle f^*TM is ample if

$$\mathrm{degree}(\xi_i) > 0 \tag{3.3}$$

for all $i \in [1, d-1]$ (see [Ha, p. 66, Proposition 2.2]).

Since the section $\widetilde{\theta}$ in Eq. (2.8) does not vanish anywhere, the holomorphic line bundle $L_{\widetilde{\theta}}$ in Eq. (2.9) is trivial. Therefore, from Lemma 2.1 we have a short exact sequence of holomorphic vector bundles over M

$$0 \longrightarrow \mathcal{O}_M \longrightarrow E_{\widehat{V}} \longrightarrow TM \longrightarrow 0,$$

where $\mathcal{O}_M$ is the trivial holomorphic vector bundle $M \times \mathbb{C}$ over M. Let

$$0 \longrightarrow \mathcal{O}_{\mathbb{CP}^1} = f^*\mathcal{O}_M \longrightarrow f^*E_{\widehat{V}} \longrightarrow f^*TM \longrightarrow 0 \tag{3.4}$$

be the pull back to $\mathbb{CP}^1$ of the above short exact sequence of holomorphic vector bundles over M; here $\mathcal{O}_{\mathbb{CP}^1}$ is the trivial holomorphic line bundle over $\mathbb{CP}^1$.

Let $P(f^*E_{\widehat{V}})$ be the projective bundle over $\mathbb{CP}^1$ parametrizing all one–dimensional linear subspaces in the fibers of the vector bundle $f^*E_{\widehat{V}}$ in Eq. (3.4). The holomorphic connection on $P(E_{\widehat{V}})$ constructed in Proposition

2.1 pulls back to a holomorphic connection on $f^*P(E_{\widehat{V}}) = P(f^*E_{\widehat{V}})$. A holomorphic connection on a Riemann surface is flat because there are no nonzero $(2,0)$–forms on a Riemann surface. Since $\mathbb{CP}^1$ is simply connected, the flat projective bundle $P(f^*E_{\widehat{V}})$ over $\mathbb{CP}^1$ is holomorphically trivializable. This implies that

$$f^*E_{\widehat{V}} = \mathcal{L}^{\oplus d}, \tag{3.5}$$

where $\mathcal{L}$ is some holomorphic line bundle over $\mathbb{CP}^1$.

In Eq. (3.4), the trivial line bundle $\mathcal{O}_{\mathbb{CP}^1}$ is a subbundle of $f^*E_{\widehat{V}}$. Hence using Eq. (3.5) we have

$$H^0(\mathbb{CP}^1, \mathcal{L}) \neq 0.$$

Since any holomorphic line bundle over $\mathbb{CP}^1$ of negative degree does not admit any nonzero sections, we now have

$$\text{degree}(\mathcal{L}) \geq 0. \tag{3.6}$$

Any line bundle ξ_i in Eq. (3.2) is a quotient of f^*TM. Therefore, using the projection in Eq. (3.4), the line bundle ξ_i is a quotient of the vector bundle $f^*E_{\widehat{V}}$. In particular,

$$H^0(\mathbb{CP}^1, \mathcal{L}^* \otimes \xi_i)^{\oplus d} = H^0(\mathbb{CP}^1, (f^*E_{\widehat{V}})^* \otimes \xi_i) \neq 0.$$

Consequently,

$$\text{degree}(\mathcal{L}^* \otimes \xi_i) = \text{degree}(\xi_i) - \text{degree}(\mathcal{L}) \geq 0. \tag{3.7}$$

Hence from Eq. (3.6),

$$\text{degree}(\xi_i) \geq 0 \tag{3.8}$$

for all $i \in [1, d-1]$.

Assume that the inequality in Eq. (3.3) fails for some $i \in [1, d-1]$. Using this assumption, from Eq. (3.8) it follows that $\text{degree}(\xi_i) = 0$.

Therefore, from Eq. (3.6) and Eq. (3.7) we have $\text{degree}(\mathcal{L}) = 0$. This implies that the line bundle $\mathcal{L}$ is holomorphically trivializable. Hence using Eq. (3.5) it follows that the vector bundle $f^*E_{\widehat{V}}$ is holomorphically trivializable.

Since $f^*E_{\widehat{V}}$ is holomorphically trivializable, using the short exact sequence in Eq. (3.4) it can be shown that f^*TM is also holomorphically trivializable. To prove this, first note that all the holomorphic global sections of the trivial vector bundle $\mathbb{CP}^1 \times \mathbb{C}^d$ over $\mathbb{CP}^1$ are given by constant functions from $\mathbb{CP}^1$ to $\mathbb{C}^d$. Fix a point $x \in \mathbb{CP}^1$. Fix a hyperplane

$$S \subset (f^*E_{\widehat{V}})_x,$$

in the fiber of $f^*E_{\widehat{V}}$ over x, which is transversal to the line in $(f^*E_{\widehat{V}})_x$ given by the image of $\mathcal{O}_{\mathbb{CP}^1}$ for the homomorphism in Eq. (3.4). Let

$$\widehat{S} \subset f^*E_{\widehat{V}}$$

be the holomorphic subbundle generated by the space of all holomorphic global sections σ of $f^*E_{\widehat{V}}$ satisfying the condition

$$\sigma(x) \in S.$$

It is easy to see that $\widehat{S}$ is a direct summand of the subbundle $\mathcal{O}_{\mathbb{CP}^1} \subset f^*E_{\widehat{V}}$ in Eq. (3.4). Consequently, the subbundle $\widehat{S}$ projects isomorphically to f^*TM by the surjective homomorphism in Eq. (3.4). On the other hand, the vector bundle $\widehat{S}$ is holomorphically trivializable. Hence the pull back f^*TM is a trivializable vector bundle over $\mathbb{CP}^1$.

Since f is a nonconstant map, its differential

$$\mathrm{d}f : T\mathbb{CP}^1 \longrightarrow f^*TM$$

is a nonzero homomorphism. But there are no nonzero homomorphisms from $T\mathbb{CP}^1$ to the trivial line bundle because the degree of $T\mathbb{CP}^1$ is positive. This is in contradiction with the earlier conclusion that f^*TM is a trivializable. Therefore, we conclude that the inequality in Eq. (3.3) holds for all $i \in [1, d-1]$. This completes the proof of the theorem. □

Corollary 3.1. *Let (M, E_P, ω) be a holomorphic parabolic geometry of type* $\mathrm{PGL}(V)/P$. *If M is a Fano projective variety, then M is biholomorphic to the projective space* $\mathbb{CP}^{d-1}$.

Proof. Assume that M is a Fano projective variety, which means that the anti–canonical line bundle K_M^{-1} is ample. Then there is a nonconstant map from $\mathbb{CP}^1$ to M (it follows immediately from [KMM, p. 766, Theorem 0.1]). Therefore, the corollary is obtained from Theorem 3.1. □

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