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Some results on local cohomology modules defined by a pair of ideals

By

Lizhong Chu and Qing Wang

Abstract

Let R be a commutative Noetherian ring, and let I and J be two ideals of R. Assume that R is local with the maximal ideal $\mathfrak{m}$, we mainly prove that (i) there exists an equality

$$\inf\{i \mid H^i_{I,J}(M) \text{ is not Artinian}\} = \inf\{\operatorname{depth} M_{\mathfrak{p}} \mid \mathfrak{p} \in W(I,J)\backslash\{\mathfrak{m}\}\}$$

for any finitely generated $R-$module M, where $W(I,J) = \{\mathfrak{p} \in \operatorname{Spec}(R) \mid I^n \subseteq \mathfrak{p} + J \text{ for some positive integer } n\}$; (ii) for any finitely generated $R-$module M with $\dim M = d$, $H^d_{I,J}(M)$ is Artinian. Also, we give a characterization to the supremum of all integers r for which $H^r_{I,J}(M) \neq 0$.

1. Introduction

Local cohomology theory has been an significant tool in commutative algebra and algebraic geometry. Throughout this note, let R be a commutative Noetherian ring, and let I and J be two ideals of R. For notations and terminologies not given in this paper, the reader is referred to [1] if necessary.

As a generalization of the usual local cohomology modules, Takahashi, Yoshino and Yoshizawa [6] introduced the local cohomology modules with respect to a pair of ideals (I,J). To be more precise, let $W(I,J) = \{\mathfrak{p} \in \operatorname{Spec}(R) \mid I^n \subseteq \mathfrak{p} + J \text{ for some positive integer } n\}$. For an $R-$module M, the $(I,J)-$torsion submodule $\Gamma_{I,J}(M)$ of M, which consists of all elements x of M with $\operatorname{Supp}(Rx) \subseteq W(I,J)$, is considered. Let i ba an integer, the local cohomology functor $H^i_{I,J}$ with respect to (I,J) is defined to be the $i-$th right derived functor of $\Gamma_{I,J}$. The $i-$th local cohomology module of M with respect to (I,J) is denoted by $H^i_{I,J}(M)$. When $J = 0$, then $H^i_{I,J}$ coincides with the usual local cohomology functor H^i_I with the support in the closed subset $V(I)$.

It is interesting to discuss the vanishing and non-vanishing of the local cohomology modules. The authors of [6] investigated the local cohomology

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modules $H^i_{I,J}(M)$ extensively. In particular, they proved that, for a finitely generated $R-$module M, there is an equality

$$\inf\{i \mid H^i_{I,J}(M) \neq 0\} = \inf\{\operatorname{depth} M_{\mathfrak{p}} \mid \mathfrak{p} \in W(I,J)\},$$

which generalizes the result for the usual local chomology modules case. Huncke [3] proposed the following question: for an integer i, when is $H^i_I(M)$ Artinian? It is well-known that (i) when $\dim R/I = 0$, $H^i_I(M)$ is Artinian for all i, and (ii) let I be an ideal of R, $H^d_I(M)$ is Artinian for any finitely generated $R-$module M with $\dim M = d$. How to determine the least integer r such that $H^r_I(M)$ is not Artinian is studied by Melkersson [5], Lü and Tang [4]. In [2], Chu and Tang generalized the main result of [4] to the generalized local cohomology modules case.

This note is concerned with the Artinianness, vanishing and non-vanishing of $H^i_{I,J}(M)$. Let M be a finitely generated $R-$module. Assume that R is local with the maximal ideal $\mathfrak{m}$. We mainly prove that (i) if $\dim M = d$, then $H^d_{I,J}(M)$ is Artinian; (ii) suppose that $\sqrt{I+J} = \mathfrak{m}$, and if, for some integer t, $H^i_{I,J}(M)$ is Artinian for all $i > t$, then $H^t_{I,J}(M)/JH^t_{I,J}(M)$ is Artinian; (iii) there exists an equality

$$\inf\{i \mid H^i_{I,J}(M) \text{ is not Artinian}\} = \inf\{\operatorname{depth} M_{\mathfrak{p}} \mid \mathfrak{p} \in W(I,J)\backslash\{\mathfrak{m}\}\}.$$

On the other hand, let $\operatorname{cd}(I,J,M)$ be the supremum of all integers r for which $H^r_{I,J}(M) \neq 0$. We call this integer *cohomological dimension of $R-$module M with respect to a pair of ideals* (I,J). We give a characterization to $\operatorname{cd}(I,J,M)$ and prove that

$$\operatorname{cd}(I,J,M) = \inf\{i \mid H^i_{I,J}(R/\mathfrak{p}) = 0 \text{ for all } \mathfrak{p} \in \operatorname{Supp}_R M\} - 1.$$

2. Artinianness of $H^i_{I,J}(M)$

Theorem 2.1. *Assume $(R,\mathfrak{m})$ is local. Let M be a finitely generated $R-$module with $\dim M = d$. Then $H^d_{I,J}(M)$ is Artinian.*

Proof. Since M is a finitely generated $R-$module, there is a finite filtration

$$0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_t = M$$

of submodules of M such that $M_j/Mj-1 \cong R/\mathfrak{p}_j$ for $\mathfrak{p}_j \in \operatorname{Supp}_R M, j = 1,2,\cdots,t$. For each $j \in \{1,2,\cdots,t\}$, from the exact sequence

$$0 \longrightarrow M_{j-1} \longrightarrow M_j \longrightarrow R/\mathfrak{p}_j \longrightarrow 0,$$

we have the corresponding long exact sequence

$$\cdots \longrightarrow H^d_{I,J}(M_{j-1}) \longrightarrow H^d_{I,J}(M_j) \longrightarrow H^d_{I,J}(R/\mathfrak{p}_j) \longrightarrow \cdots.$$

From the above long exact sequence, it suffice to prove that $H^d_{I,J}(R/\mathfrak{p}_j)$ is Artinian for each $j = 1,2,\cdots,t$. Let $\mathfrak{p}_j = \mathfrak{p}$. If $J \subseteq \mathfrak{p}$, then $R/\mathfrak{p}$ is $J-$torsion and then $H^d_{I,J}(R/\mathfrak{p}) \cong H^d_I(R/\mathfrak{p})$. Since $\dim R/\mathfrak{p} \leq d$, $H^d_I(R/\mathfrak{p})$ is Artinian. Thus, $H^d_{I,J}(R/\mathfrak{p})$ is Artinian. If $J \not\subseteq \mathfrak{p}$, then $\dim (R/\mathfrak{p})/J(R/\mathfrak{p}) = \dim R/(J + \mathfrak{p}) < \dim R/\mathfrak{p} \leq d$, and so $H^d_{I,J}(R/\mathfrak{p}) = 0$ by [6, Theorem 4.3]. The proof is completed. □

Remark. Let M be a finitely generated $R-$module over a local ring $(R, \mathfrak{m})$. Suppose that $I + J$ is an $\mathfrak{m}-$primary. Let $t = \dim M/JM$. Then $t = \sup\{i \mid H^i_{I,J}(M) \neq 0\}$ (see [6, Theorem 4.5]). In general, $H^t_{I,J}(M)$ is not always Artinian. For example, let $(R, \mathfrak{m})$ be a Gorenstein local ring of dimension d and assume that J is a perfect ideal of grade $s > 1$, then $H^{d-s}_{\mathfrak{m},J}(R)$ is not Artinian (see [6, Proposition 5.6]).

Theorem 2.2. *Assume $(R, \mathfrak{m})$ is local. Let I and J be two ideals such that $\sqrt{I + J} = \mathfrak{m}$. Let M be a finitely generated $R-$module. If, for some integer t, $H^i_{I,J}(M)$ is Artinian for all $i > t$. Then $H^t_{I,J}(M)/JH^t_{I,J}(M)$ is Artinian.*

Proof. Since $\sqrt{I + J} = \mathfrak{m}$, we have $H^i_{I,J}(M) = H^i_{\mathfrak{m},J}(M)$ for any i. Thus, we may assume that $I = \mathfrak{m}$.

We prove it by induction on $n =: \dim M$.

When $n = 0$, M is $(I, J)-$torsion. Then, by [6, Corollary 4.2], $H^i_{\mathfrak{m},J}(M) = 0$ for any $i > 0$, and $H^0_{I,J}(M)$ is Artinian by Theorem 2.1. Therefore, the result is clearly true.

Next, we assume that $n > 0$ and that the claim is true for all values less than n. From the exact sequence

$$0 \longrightarrow \Gamma_J(M) \longrightarrow M \longrightarrow M/\Gamma_J(M) \longrightarrow 0,$$

we have the long exact sequence

$$\cdots \longrightarrow H^t_{\mathfrak{m},J}(M) \longrightarrow H^t_{\mathfrak{m},J}(M/\Gamma_J(M)) \longrightarrow H^{t+1}_{\mathfrak{m},J}(\Gamma_J(M) \longrightarrow \cdots .$$

Since $\Gamma_J(M)$ is $J-$torsion, we have $H^i_{\mathfrak{m},J}(\Gamma_J(M)) \cong H^i_{\mathfrak{m}}(\Gamma_J(M))$ for any i. Then $H^i_{\mathfrak{m},J}(\Gamma_J(M))$ is Artinian for any i. Therefore, from the above long exact sequence, we may assume that $\Gamma_J(M) = 0$. Take an $M-$regular element $x \in J$ and so that $\dim M/xM = n - 1$. Then we have that the exact sequence $0 \longrightarrow M \xrightarrow{x} M \longrightarrow M/xM \longrightarrow 0$, which implies the following long exact sequence

$$\cdots \longrightarrow H^t_{\mathfrak{m},J}(M) \xrightarrow{x} H^t_{\mathfrak{m},J}(M) \xrightarrow{\alpha} H^t_{\mathfrak{m},J}(M/xM) \xrightarrow{\beta} H^{t+1}_{\mathfrak{m},J}(M) \longrightarrow \cdots .$$

Since $H^i_{\mathfrak{m},J}(M)$ is Artinian for any $i > t$, the above long exact sequence implies that $H^i_{\mathfrak{m},J}(M/xM)$ is Artinian for any $i > t$. Then $H^t_{\mathfrak{m},J}(M/xM)/JH^t_{\mathfrak{m},J}(M/xM)$ is Artinian by induction hypothesis. We can get the following short exact sequences from the above long sequence

$$0 \longrightarrow \operatorname{Im}\alpha \longrightarrow H^t_{\mathfrak{m},J}(M/xM) \longrightarrow \operatorname{Im}\beta \longrightarrow 0$$

and

$$H^t_{\mathfrak{m},J}(M) \xrightarrow{x} H^t_{\mathfrak{m},J}(M) \longrightarrow \operatorname{Im}\alpha \longrightarrow 0.$$

Therefore, the following two sequences

$$\begin{aligned}\operatorname{Tor}_1^R(R/J, \operatorname{Im}\beta) \longrightarrow \operatorname{Im}\alpha/J\operatorname{Im}\alpha \longrightarrow H^t_{\mathfrak{m},J}(M/xM)/JH^t_{\mathfrak{m},J}(M/xM)\\ \longrightarrow \operatorname{Im}\beta/J\operatorname{Im}\beta \longrightarrow 0\end{aligned}$$

and

$$H^t_{\mathfrak{m},J}(M)/JH^t_{\mathfrak{m},J}(M) \xrightarrow{x} H^t_{\mathfrak{m},J}(M)/JH^t_{\mathfrak{m},J}(M) \longrightarrow \operatorname{Im}\alpha/J\operatorname{Im}\alpha \longrightarrow 0$$

are both exact. We denote the above two exact sequences by $(*)$ and $(**)$, respectively. Note that $x \in J$, it follows that $H^t_{\mathfrak{m},J}(M)/JH^t_{\mathfrak{m},J}(M) \cong \operatorname{Im}\alpha/J\operatorname{Im}\alpha$ from the exact sequence$(**)$. Since $\operatorname{Im}\beta$ is Artinian, $\operatorname{Tor}_1^R(R/J, \operatorname{Im}\beta)$ and $\operatorname{Im}\beta/J\operatorname{Im}\beta$ is Artinian. Then the result follows from the exact sequence $(*)$. □

Theorem 2.3. *Assume $(R, \mathfrak{m})$ is local. Let M be a finitely generated $R-$module such that $t = \dim M/JM$. Then $H^t_{I,J}(M)/JH^t_{I,J}(M)$ is Artinian.*

Proof. We claim that, for a finitely generated $R-$module M such that $\dim M/JM \le t$, $H^t_{I,J}(M)/JH^t_{I,J}(M)$ is Artinian. Using the similar method in the proof of Theorem 2.2, we prove this claim by induction on $n =: \dim M$. Then the result follows. □

Theorem 2.4. *Assume $(R, \mathfrak{m})$ is local. Then there is an equality*

$$\inf\{i \mid H^i_{I,J}(M) \text{ is not Artinian }\} = \inf\{\operatorname{depth} M_{\mathfrak{p}} \mid \mathfrak{p} \in W(I,J)\backslash\{\mathfrak{m}\}\}$$

for any finitely generated $R-$module M.

Proof. We set $r = \inf\{\operatorname{depth} M_{\mathfrak{p}} \mid \mathfrak{p} \in W(I,J)\backslash\{\mathfrak{m}\}\}$ and let $E^\bullet(M)$ be a minimal injective resolution of M.

Let $\mathfrak{p} \in W(I,J)\backslash\{\mathfrak{m}\}$. Then $r \le \operatorname{depth} M_{\mathfrak{p}} = \inf\{i \mid \operatorname{Ext}^i_R(R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}, M_{\mathfrak{p}}) \neq 0\} = \inf\{i \mid \mu_i(\mathfrak{p}, M) \neq 0\}$, where $\mu_i(\mathfrak{p}, M) = \dim \operatorname{Ext}^i_R(R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}, M_{\mathfrak{p}})$, which is the $i-$th Bass number of M with respect to $\mathfrak{p}$. Note that $\Gamma_{I,J}(E(R/\mathfrak{p})) = 0$ if $\mathfrak{p} \notin W(I,J)$ and that $\Gamma_{I,J}(E(R/\mathfrak{p})) = E(R/\mathfrak{p})$ if $\mathfrak{p} \in W(I,J)$ ([6, Proposition 1.11]), therefore, we have that, for any $i < r$,

$$\begin{aligned}\Gamma_{I,J}(E^i(M)) &= \bigoplus_{\mathfrak{p}\in W(I,J)\backslash\{\mathfrak{m}\}} E(R/\mathfrak{p})^{\mu_i(\mathfrak{p},M)} \bigoplus E(R/\mathfrak{m})^{\mu_i(\mathfrak{m},M)}\\ &= E(R/\mathfrak{m})^{\mu_i(\mathfrak{m},M)}\end{aligned}$$

Since $E(R/\mathfrak{m})$ is Artinian and $\mu_i(\mathfrak{m}, M)$ is finite, $\Gamma_{I,J}(E^i(M))$ is Artinian for any $i < r$, and so $H^i_{I,J}(M)$ is Artinian as a subquotient module of $\Gamma_{I,J}(E^i(M))$ for any $i < r$. On the other hand, we can see that there exists $\mathfrak{q} \in W(I,J)\backslash\{\mathfrak{m}\}$

such that $\mathfrak{q} \in \operatorname{Ass}_R(\Gamma_{I,J}(E^r(M)))$, which shows that $\Gamma_{I,J}(E^r(M))$ is not Artinian. There is a commutative graph:

$$\begin{array}{ccccccc} \cdots \xrightarrow{\Gamma_{I,J}(d^{r-1})} & \Gamma_{I,J}(E^r(M)) & \xrightarrow{\Gamma_{I,J}(d^r)} & \Gamma_{I,J}(E^{r+1}(M)) & \xrightarrow{\Gamma_{I,J}(d^{r+11})} \cdots \\ & \downarrow & & \downarrow & \\ \cdots \xrightarrow{d^{r-1}} & E^r(M) & \xrightarrow{d^r} & E^{r+1}(M) & \xrightarrow{d^{r+1}} \cdots \end{array}$$

Since $\operatorname{Ker} d^r \subseteq E^r(M)$ is an essential extension, then $\operatorname{Ker} \Gamma_{I,J}(d^r) = \operatorname{Ker} d^r \cap \Gamma_{I,J}(E^r(M)) \subseteq \Gamma_{I,J}(E^r(M))$ is an essential extension. Note that if $K \subseteq L$ is an essential extension, then K is Artinian if and only if L is Artinian. So $\operatorname{Ker} \Gamma_{I,J}(d^r)$ is not Artinian. On the other hand, since $\Gamma_{I,J}(E^{r-1}(M))$ is Artinian, we have that $\operatorname{Im} \Gamma_{I,J}(d^{r-1})$ is Artinian. Therefore, the exact sequence

$$0 \longrightarrow \operatorname{Im} \Gamma_{I,J}(d^{r-1}) \longrightarrow \operatorname{Ker} \Gamma_{I,J}(d^r) \longrightarrow H^r_{I,J}(M) \longrightarrow 0$$

implies that $H^r_{I,J}(M)$ is not Artinian. This completes the proof. □

Proposition 2.5. *Assume $(R, \mathfrak{m})$ is local. Then there is an equality*

$$\inf\{i \mid H^i_{I,J}(M) \text{ is not Artinian }\} = \inf\{i \mid H^i_{I,J}(M) \ncong H^i_{\mathfrak{m}}(M)\}$$

for any finitely generated $R-$module M.

Proof. We set $r = \inf\{i \mid H^i_{I,J}(M)$ is not Artinian $\}$. From the proof of the above theorem, we have that $\Gamma_{I,J}(E^i(M))$ is Artinian for $i < r$. Then it is clear that $\Gamma_{I,J}(E^i(M)) = \Gamma_{\mathfrak{m}}(E^i(M))$ for $i < r$. So $H^i_{I,J}(M) \cong H^i_{\mathfrak{m}}(M)$ for $i < r$. On the other hand, since $H^r_{I,J}(M)$ is not Artinian and $H^r_{\mathfrak{m}}(M)$ is Artinian, we have $H^r_{I,J}(M) \ncong H^r_{\mathfrak{m}}(M)$. □

3. Vanishing and Non-vanishing of $H^i_{I,J}(M)$

Let M be a finitely generated $R-$module over a local ring $(R, \mathfrak{m})$. Suppose that $I + J$ is an $\mathfrak{m}-$primary. Then $\dim M/JM = \sup\{i \mid H^i_{I,J}(M) \neq 0\}$ ([6, Theorem 4.5]). This result characterizes $\operatorname{cd}(I, J, M)$, cohomological dimension of $R-$module M with respect to a pair of ideals (I, J), by using the data of M/JM under some conditions. In this section, we will give a new characterization to the integer $\operatorname{cd}(I, J, M)$.

Theorem 3.1. *Let $r \geq 0$ be a given integer such that $H^r_{I,J}(R/\mathfrak{p}) = 0$ for all $\mathfrak{p} \in \operatorname{Supp}_R M$. Then*

(i) $H^r_{I,J}(N) = 0$ *for any finitely generated $R-$module N with* $\operatorname{Supp}_R N \subseteq \operatorname{Supp}_R M$. *In particular, $H^r_{I,J}(M) = 0$.*

(ii) $\operatorname{cd}(I, J, R/\mathfrak{p}) < r$ *for all* $\mathfrak{p} \in \operatorname{Supp}_R M$. *In particular,* $\operatorname{cd}(I, J, M) < r$.

Proof. (i) It suffices to show that $H^r_{I,J}(M) = 0$. Clearly, there exists a filtration of the submodules of M

$$0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_s = M$$

such that for each $1 \leq j \leq s$, $M_j/M_{j-1} \cong R/\mathfrak{p}_j$, where $\mathfrak{p}_j \in \operatorname{Supp}_R M$. From the exact sequence

$$H^r_{I,J}(M_{j-1}) \longrightarrow H^r_{I,J}(M_j) \longrightarrow H^r_{I,J}(R/\mathfrak{p}_j),$$

which is induced by the short exact sequence

$$0 \longrightarrow M_{j-1} \longrightarrow M_j \longrightarrow R/\mathfrak{p}_j \longrightarrow 0,$$

we eventually see that $H^r_{I,J}(M) = 0$.

(ii) We will show that $H^{r+1}_{I,J}(R/\mathfrak{p}) = 0$ for any $\mathfrak{p} \in \operatorname{Supp}_R M$. Suppose that $H^{r+1}_{I,J}(R/\mathfrak{p}_0) \neq 0$ for some $\mathfrak{p}_0 \in \operatorname{Supp}_R M$, so that $I \nsubseteq \mathfrak{p}_0$ and $\operatorname{Ass}_R(H^{r+1}_{I,J}(R/\mathfrak{p}_0)) \neq \emptyset$. Let $\mathfrak{q} \in \operatorname{Ass}_R(H^{r+1}_{I,J}(R/\mathfrak{p}_0))$. Clearly, $\mathfrak{p}_0 \subseteq \mathfrak{q}$.

We claim that $\mathfrak{p}_0 = \mathfrak{q}$. Otherwise, we can take $x \in \mathfrak{q} \cap I \backslash \mathfrak{p}_0$. The exact sequence $0 \longrightarrow R/\mathfrak{p}_0 \longrightarrow R/\mathfrak{p}_0 \longrightarrow R/(\mathfrak{p}_0 + xR) \longrightarrow 0$ implies that the exact sequence

$$H^r_{I,J}(R/(\mathfrak{p}_0 + xR)) \longrightarrow H^{r+1}_{I,J}(R/\mathfrak{p}_0) \xrightarrow{x} H^{r+1}_{I,J}(R/\mathfrak{p}_0).$$

By (i), $H^r_{I,J}(R/(\mathfrak{p}_0 + xR)) = 0$, and so x is $H^{r+1}_{I,J}(R/\mathfrak{p}_0)-$regular. However, the fact that $x \in \mathfrak{q}$ and $\mathfrak{q} \in \operatorname{Ass}_R(H^{r+1}_{I,J}(R/\mathfrak{p}_0))$ implies that x is a zero-divisor on $H^{r+1}_{I,J}(R/\mathfrak{p}_0)$. This is a contradiction. Hence, we have $\mathfrak{p}_0 = \mathfrak{q}$.

Since $\operatorname{Ass}_R(H^{r+1}_{I,J}(R/\mathfrak{p}_0)) \subseteq W(I,J)$, we have that $\mathfrak{p}_0 \in W(I,J)$. Then $R/\mathfrak{p}_0$ is $(I,J)-$torsion, Thus, by [6, Corollary 4.2], $H^{r+1}_{I,J}(R/\mathfrak{p}_0) = 0$, which is a contradiction. This shows that $H^{r+1}_{I,J}(R/\mathfrak{p}) = 0$ for any $\mathfrak{p} \in \operatorname{Supp}_R M$. By using (1), we see that $H^{r+1}_{I}(N) = 0$ for all finitely generated R-modules N with $\operatorname{Supp}_R N \subseteq \operatorname{Supp}_R M$. We can now apply induction to complete the proof. □

Proposition 3.2. *Let I be a proper ideal of a commutative Noetherian ring R and M, N be finitely generated R-modules such that* $\operatorname{Supp}_R N \subseteq \operatorname{Supp}_R M$. *Then* $\operatorname{cd}(I,J,N) \leq \operatorname{cd}(I,J,M)$.

Proof. By virtue of [6, Theorem 4.7], $H^i_{I,J}(M) = 0$ for $i > \dim M/JM + 1$, so it is enough to prove that $H^i_{I,J}(N) = 0$ for any i with $\operatorname{cd}(I,J,M) < i \leq \dim M/JM + 2$ and any finitely generated $R-$module with $\operatorname{Supp}_R(N) \subseteq \operatorname{Supp}_R(M)$. We prove it by using the descending induction on i, $\operatorname{cd}(I,J,M) < i \leq \dim M/JM + 2$.

If $i = \dim M/JM + 2$, note that $\dim M/JM \geq \dim N/JN$, the assertion is clear by [6, Theorem 4.7]. So we can assume that $i \leq \dim M/JM + 1$. Then, since $\operatorname{Supp}_R N \subseteq \operatorname{Supp}_R M$, by Gruson's Theorem [7, Theorem 4.1], we see that there is a chain

$$0 = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_t = N$$

such that the factors N_j/N_{j-1} are homomorphic images of a direct sum of finitely many copies of M. By using the short exact sequences $0 \longrightarrow N_{j-1} \longrightarrow$

$N_j \longrightarrow N_j/N_{j-1} \longrightarrow 0$, the situation can be reduced to the case $t = 1$. Therefore, for some positive integer m and some finitely generated R-module L, there exists an exact sequence

$$0 \longrightarrow L \longrightarrow M^m \longrightarrow N \longrightarrow 0$$

which yields the corresponding exact sequences

$$\cdots \longrightarrow H^i_{I,J}(L) \longrightarrow H^i_{I,J}(M^m) \longrightarrow H^i_{I,J}(N) \longrightarrow H^{i+1}_{I,J}(L) \longrightarrow \cdots .$$

Hence, by induction hypothesis, we see that $H^{i+1}_{I,J}(L) = 0$. Since $H^i_{I,J}(M^m) = 0$, $H^i_{I,J}(N) = 0$. □

By virtue of Theorem 3.1 and Proposition 3.2, we give a characterization to $\mathrm{cd}(I, J, M)$.

Corollary 3.3. *For a finitely generated $R-$module M, there exists an equality*

$$\mathrm{cd}(I, J, M) = \inf\{i \mid H^i_{I,J}(R/\mathfrak{p}) = 0 \textit{ for all } \mathfrak{p} \in \mathrm{Supp}_R\, M\} - 1.$$

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Department of Mathematics, Suzhou University
215006 Suzhou, China
e-mail: chulizhong@suda.edu.cn

Science and technology office, Suzhou University
215006 Suzhou, China

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