Illinois Journal of Mathematics
Volume 61, Numbers 3–4, Fall and Winter 2017, Pages 399–413
S 0019-2082

SOME UNIQUENESS RESULTS FOR RICCI SOLITONS

J. F. SILVA FILHO

ABSTRACT. We investigate the relationship between Ricci soliton structures and homothetic vector fields, especially Killing vector fields. More precisely, we present some characterizations for Ricci solitons endowed with Killing vector fields of constant norm as well as homothetic vector fields. In particular, we relate different Ricci soliton structures on a Riemannian manifolds in order to deduce some uniqueness results.

1. Introduction

A *Ricci soliton* is a Riemannian manifold (M^n, g), $n \geq 2$, endowed with a vector field X satisfying

$$Ric + \frac{1}{2}\mathcal{L}_X g = \lambda g, \tag{1.1}$$

where λ is a constant and $\mathcal{L}$ stands for the Lie derivative. If X is the gradient vector field of a function f on M, such a manifold is called a *gradient Ricci soliton*. In this case, (1.1) becomes

$$Ric + \operatorname{Hess} f = \lambda g, \tag{1.2}$$

where $\operatorname{Hess} f$ stands for the Hessian of f. Moreover, the Ricci soliton (M^n, g, X, λ) will be called expanding, steady or shrinking if $\lambda < 0, \lambda = 0$ or $\lambda > 0$, respectively. When either the vector field X is trivial or f is constant the Ricci soliton is said to be called trivial. Otherwise, it will be called non-trivial. Notice that Ricci solitons are natural extensions of Einstein manifolds. Let us also highlight that if M^n has dimension $n \geq 3$ and X is conformal, then Schur's lemma guarantees that M^n is Einstein with constant scalar curvature.

The relation between conformal vector fields and Ricci solitons is well known, especially Killing vector fields. For instance, vector fields that generate

Received May 7, 2017; received in final form May 21, 2018.
2010 *Mathematics Subject Classification.* Primary 53C25, 53C20, 53C21. Secondary 53C65.

Ricci soliton structures on a fixed Riemannian manifold are related module homothetic vector fields. In this context, Perelman [18] proved that compact Ricci solitons are gradient type, that is, vector fields that generate Ricci soliton structures are gradients up to Killing vector fields. Aquino et al. [2] shed light on the Perelman potential of a compact Ricci soliton by showing that Perelman's potential is (up to a multiplication by a constant) the Hodge-De Rham potential.

On the other hand, Baird and Danielo [4] proved that Ricci soliton structures on 3-dimensional homogeneous Riemannian manifolds are unique up to Killing vector fields, while Ribeiro Jr. and Silva Filho [22] described explicitly the 3-dimensional homogeneous Ricci soliton with an isometry group of dimension 4. In [3], Baird proved similar results for 4-dimensional homogeneous Ricci solitons. Moreover, some interesting facts concerning gradient Ricci soliton structures and Killing vector field can be found in [20].

Motivated by the historical development on the study of Ricci solitons, in this article, we describe the relationship between different Ricci soliton structures on a fixed Riemannian manifold. To do so, we provide some conditions to guarantee the uniqueness of the module Killing vector fields of such structures; that is, vector fields that generate Ricci soliton structures on a fixed Riemannian manifold differ by Killing vector fields. In this approach, it is crucial to understand the relationship between such structures and homothetic vector fields.

Before describing our main results let us recall some basic definitions. First of all, we consider a smooth manifold M^n jointly with a Riemannian metric $g=\langle\cdot,\cdot\rangle$ and Levi–Civita connection ∇ associated to g. We remember that a smooth vector field X on a Riemannian manifold (M^n,g) is called *conformal* if

$$\mathcal{L}_X g = 2\psi g, \tag{1.3}$$

where $\mathcal{L}_X$ is the Lie derivative in the direction of X and $\psi\in C^\infty(M)$ is called *conformal factor*. This condition is equivalent to saying that X satisfies the equation

$$\langle\nabla_Y X, Z\rangle + \langle Y, \nabla_Z X\rangle = 2\psi\langle Y, Z\rangle, \tag{1.4}$$

for all vector fields $Y,Z\in\mathfrak{X}(M)$. In particular, we have

$$\psi = \frac{1}{n}\operatorname{div} X. \tag{1.5}$$

Furthermore, we say that X is *homothetic* (i.e., *Killing*) if its conformal factor ψ is constant (i.e., identically null). A particular case of a conformal vector field X is that which

$$\nabla_Y X = \psi Y, \tag{1.6}$$

for all $Y \in \mathfrak{X}(M)$; in this case, X is called *closed*. The expression *closed* acknowledges the fact that its dual 1-form $X^\flat = \omega$ is closed. Indeed, we have

$$\begin{aligned} d\omega(Y,Z) &= Y\langle X,Z\rangle - Z\langle X,Y\rangle - \big\langle X,[Y,Z]\big\rangle \\ &= \psi\langle Y,Z\rangle - \psi\langle Y,Z\rangle + \big\langle X,[Y,Z]\big\rangle - \big\langle X,[Y,Z]\big\rangle = 0. \end{aligned}$$

It is worthwhile to remark that closed vector fields are also known as concircular vector fields. More particularly, a closed conformal vector field X is said to be *parallel* if its conformal factor ψ vanishes identically.

After these preliminary remarks, we present our first result as follows.

THEOREM 1. *Let (M^n, g, X, λ) be a Ricci soliton endowed with a parallel vector field, then it is non-compact. In addition, if M^n is complete, then either*

(1) *(M^n, g, X, λ) is a steady Ricci soliton; or*
(2) *M^n is isometric to the direct product $N^{n-k} \times \mathbb{R}^k$, where N^k admits a Ricci soliton structure and does not have a non-trivial parallel vector field.*

In the sequel, we recall that Fernandez-Lopez and Garcia-Rio proved that a complete shrinking Ricci soliton with an associated vector field of a bounded norm must be compact. In particular, the Gaussian soliton (see [7] and Example 1 in the next section) guarantees that the assumption for the associated vector field is necessary. Next, we assume that a complete (steady, expanding or shrinking) Ricci soliton has an associated vector field of a bounded norm to obtain the following consequence of Theorem 1.

COROLLARY 1. *Let (M^n, g, X, λ) be a complete Ricci soliton such that X has a bounded norm. Then either*

(1) *M^n does not admit a parallel vector field; or*
(2) *M^n has non-constant scalar curvature.*

In particular, since homogeneous gradient Ricci solitons are not steady (cf. Theorem 2.4 of [11]) and they always admit non-trivial parallel vector fields (cf. Petersen and Wylie [20]), we deduce the following corollary.

COROLLARY 2. *Let (M^n, g, X, λ) be a homogeneous Ricci soliton such that X has a bounded norm. Then M^n does not carry any gradient Ricci soliton structure.*

We now assume that a gradient Ricci soliton admits a non-trivial Killing vector field with a constant norm to establish the following result.

COROLLARY 3. *Let $(M^n, g, \nabla f, \lambda)$ be a complete gradient Ricci soliton endowed with a Killing vector field of constant norm. Then, either*

(1) *M^n is isometric to the direct product $N^{n-k} \times \mathbb{R}^k$; or*
(2) *$(M^n, g, \nabla f, \lambda)$ is a non-expanding Ricci soliton.*

Proceeding, as it was previously mentioned, a classical result based on Perelman [18] asserts that compact Ricci solitons are gradient up to Killing vector fields, while Naber [14] proved that the same conclusion remains true for shrinking Ricci solitons with bounded curvature. Here, we prove that the same result holds for Ricci solitons endowed with a homothetic closed vector field. To be precise, we show the following result.

THEOREM 2. *Let (M^n, g, X, λ) be a Ricci soliton endowed with a non-parallel homothetic closed vector field. Then there exists a smooth function $\varphi : M^n \to \mathbb{R}$ such that $(M^n, g, \nabla\varphi, \lambda)$ is a gradient Ricci soliton.*

We recall that a complete Riemannian manifold endowed with a gradient homothetic vector fields has null identically scalar curvature (cf. Theorem 2 in Tashiro [26]). Here, we provide a version of this result for Ricci solitons (not necessarily complete).

COROLLARY 4. *Every Ricci soliton endowed with a non-parallel homothetic closed vector field has null identically scalar curvature.*

In order to proceed, we recall that Baird and Danielo [4] have showed that Ricci soliton structures on 3-dimensional homogeneous Riemannian manifolds are unique up to Killing vector fields. Here, inspired by this result, we get the following theorem.

THEOREM 3. *Let (M^n, g, X, λ) be a Ricci soliton with non-null constant scalar curvature, then this structure is unique up to Killing vector fields. In addition, if M^n is complete and $|X| \in \mathcal{L}^1(M)$, then M^n does not admit a gradient Ricci soliton structure.*

In [27], Wylie showed that every complete shrinking Ricci soliton has a finite fundamental group. In the same context, we have the following corollaries.

COROLLARY 5. *Let (M^n, g, X, λ) be a complete Ricci soliton. Then one of the following statements holds.*

(1) *This structure is unique up to Killing vector fields.*
(2) *M^n has non-negative scalar curvature and a finite fundamental group.*

COROLLARY 6. *Every complete Ricci soliton endowed with a non-Killing homothetic vector field has non-negative scalar curvature and a finite fundamental group.*

This article is organized as follows. In Section 2, we present some examples and recall some basic facts on conformal vector fields. We then prove some key lemmas that will be useful in proving our main results. In Section 3, we prove those results.

2. Background and key lemmas

In this section, we present lemmas that will be useful in the proof of the main results. To begin with, we present some examples of Ricci solitons that motivate the assumptions considered in the main results.

2.1. Some examples. First, we present four classical examples of gradient Ricci solitons. We start with the Gaussian solitons, which are described as follows.

Example 1 (*Gaussian Ricci soliton*). Consider the canonical metric g_0 of $\mathbb{R}^n$ and the function $f:\mathbb{R}^n \to \mathbb{R}$ given by

$$f(x) = \frac{1}{2}\lambda|x|^2,$$

where λ is an arbitrary constant. Since $Ric = 0$ and $\operatorname{Hess} f = \lambda g_0$ we get

$$Ric + \operatorname{Hess} f = \lambda g_0,$$

From this, $(M^n, g_0, \nabla f, \lambda)$ is a gradient Ricci soliton.

Example 2 (*Cigar soliton*). Consider the Euclidean space $\mathbb{R}^2$ endowed with the metric $g = \frac{dx^2+dy^2}{1+x^2+y^2}$ and potential function $f(x,y) = -\log(1+x^2+y^2)$. This is the first example of a complete non-compact gradient steady soliton on $\mathbb{R}^2$. It is asymptotic to cylinder at infinity

Proceeding, we present the *rigid soliton*. It can be found in the works of Petersen and Wylie [20], [21].

Example 3 (*Rigid soliton*). Consider the direct product $N \times \mathbb{R}^k$, where (N^n, g) is an Einstein manifold with Einstein constant $\lambda \in \mathbb{R}$. We further define the function $f: N \times \mathbb{R}^k \to \mathbb{R}$ by

$$f(p,x) = \frac{\lambda}{2}|x|^2.$$

With these settings $(M^n, g, \nabla f, \lambda)$ is a gradient Ricci soliton.

Finally, we present an example of non-gradient Ricci solitons on the Lie group $\mathbb{S}\mathrm{ol}_3$ obtained by Baird and Danielo [4].

Example 4. Consider the Lie group $\mathbb{S}\mathrm{ol}_3$; that is, $\mathbb{R}^3$ endowed with the left invariant metric $g = e^{2t}dx^2 + e^{-2t}dy^2 + dt^2$. Moreover, we consider $X \in \mathfrak{X}(\mathbb{S}\mathrm{ol}_3)$ given by

$$X = -4ye^{-2t}\partial_x - 2\partial_t.$$

Whence, $(\mathbb{S}\mathrm{ol}_3, X, \lambda = -2)$ is an expanding non-gradient Ricci soliton.

2.2. Conformal vector fields and Ricci solitons. Here, we recall some definitions and basic facts on conformal vector fields. To do so, we remember that a smooth vector field X on a Riemannian manifold (M^n, g) is called *conformal* if

$$\mathcal{L}_X g = 2\psi g, \tag{2.1}$$

where $\mathcal{L}_X$ denote the Lie derivative in the direction of X and $\psi \in C^\infty(M)$ is called *conformal factor*. Moreover, we say that X is *homothetic* (i.e., *Killing*) if its conformal factor ψ is constant (i.e., identically null). A direct computation says that a gradient conformal vector field $X = \nabla\varphi$ is always a closed vector field. In fact, it suffices to compute

$$\langle \nabla_Y X, Z\rangle = \operatorname{Hess}\varphi(Y, Z) = \langle \psi Y, Z\rangle,$$

where ψ is the conformal factor of X. So, it follows that $\nabla_Y X = \psi Y$, for all $Y \in \mathfrak{X}(M)$.

We now present a lemma obtained by Obata and Yanno [16].

Lemma 1 (Obata and Yanno, [16]). *Let (M^n, g) be a Riemannian manifold and let $X \in \mathfrak{X}(M)$ be a conformal vector field. Then*

$$\frac{1}{2}\langle X, \nabla R\rangle = -(n-1)\Delta\psi - R\psi,$$

where R stands for the scalar curvature and ψ is the conformal factor of X.

It has long been a goal of mathematicians to understand the geometry of Riemannian manifolds admitting a closed conformal vector field; for more details on this subject, refer to [1], [15], [16], [24], [25] and [26]. The next lemma summarizes some useful properties of closed conformal vector fields as well as some information on Riemannian manifolds endowed with such a vector field (cf. [8] and [23]).

Lemma 2. *Let (M^n, g) be a Riemannian manifold and $X \in \mathfrak{X}(M)$ a closed conformal vector field with conformal factor ψ. Then the following assertions hold:*

(1) *The set $\mathcal{Z}(X)$ of singularities of X is discrete.*
(2) *The gradient of $|X|^2$ is given by*

$$\nabla |X|^2 = 2\psi X.$$

(3) *The Hessian of $|X|^2$ is given by*

$$\operatorname{Hess}|X|^2 = 2\psi^2 g + 2X^\flat \otimes d\psi.$$

(4) *The Ricci tensor of (M^n, g) satisfies*

$$Ric(X, Y) = -(n-1)\langle Y, \nabla\psi\rangle,$$

for all $Y \in \mathfrak{X}(M)$.

In order to set the stage for the proofs to follow, let us show a useful lemma that is inspired by a result obtained by Petersen and Wylie [20].

LEMMA 3. *Let $(M_1 \times M_2, g, X, \lambda)$ be a Ricci soliton structure, where $g = g_1 + g_2$ denotes the product metric. Then each (M_i, g_i) admits a Ricci soliton structure.*

Proof. Since $T_{(p_1,p_2)}(M_1 \times M_2) \cong T_{p_1}M_1 \oplus T_{p_2}M_2$, we have the split

$$X = X_1 + X_2,$$

where $X_1 \in \mathfrak{X}(M_1)$ and $X_2 \in \mathfrak{X}(M_2)$ are unique. In particular, we deduce $Ric = Ric_{M_1} + Ric_{M_2}$ and $\mathcal{L}_X g = \mathcal{L}_{X_1} g_1 + \mathcal{L}_{X_2} g_2$. From this, it follows that

$$Ric_{M_i} + \frac{1}{2}\mathcal{L}_{X_i} g_i = \lambda g_i.$$

Hence, each (M_i, g_i, X_i, λ) is a Ricci soliton. This finishes the proof of the lemma. $\square$

To finish this section we present a fundamental identity on the Lie derivative (cf. O'Neill [17]), which will be used in the proof of Theorem 1.

PROPOSITION 1. *Let (M^n, g) be a Riemannian manifold and $X, Y \in \mathfrak{X}(M)$. Then we have*

$$\mathcal{L}_{[X,Y]} g = \mathcal{L}_X \mathcal{L}_Y g - \mathcal{L}_Y \mathcal{L}_X g.$$

Proof. By using the definition of the Lie derivative we immdiately obtain

$$\mathcal{L}_X \mathcal{L}_Y g(U,V) = X\big[\mathcal{L}_Y g(U,V)\big] - \mathcal{L}_Y g\big([X,U],V\big) - \mathcal{L}_Y g\big(U,[X,V]\big). \tag{2.2}$$

Notice moreover that

$$X\big[\mathcal{L}_Y g(U,V)\big] = X\big[\langle \nabla_U Y, V\rangle + \langle U, \nabla_V Y\rangle\big],$$

and also

$$\begin{aligned} X\big[\mathcal{L}_Y g(U,V)\big] = {} & \langle \nabla_X \nabla_U Y, V\rangle + \langle \nabla_U Y, \nabla_X V\rangle + \langle \nabla_X U, \nabla_V Y\rangle \\ & + \langle U, \nabla_X \nabla_V Y\rangle. \end{aligned} \tag{2.3}$$

On the other hand, we have

$$\mathcal{L}_Y g\big([X,U],V\big) = \langle \nabla_{[X,U]} Y, V\rangle + \langle \nabla_X U, \nabla_V Y\rangle - \langle \nabla_U X, \nabla_V Y\rangle, \tag{2.4}$$

as well as

$$\mathcal{L}_Y g\big(U,[X,V]\big) = \langle \nabla_U Y, \nabla_X V\rangle - \langle \nabla_U Y, \nabla_V X\rangle + \langle U, \nabla_{[X,V]} Y\rangle. \tag{2.5}$$

Therefore, substituting (2.3), (2.4) and (2.5) into (2.2) gives

$$\begin{aligned} \mathcal{L}_X \mathcal{L}_Y g(U,V) = {} & \langle \nabla_X \nabla_U Y, V\rangle + \langle U, \nabla_X \nabla_V Y\rangle - \langle \nabla_{[X,U]} Y, V\rangle \\ & + \langle \nabla_U X, \nabla_V Y\rangle + \langle \nabla_U Y, \nabla_V X\rangle - \langle U, \nabla_{[X,V]} Y\rangle. \end{aligned}$$

Proceeding, notice that using the definition of curvature we infer

$$R(X,U,Y,V) = \langle \nabla_X \nabla_U Y, V\rangle - \langle \nabla_U \nabla_X Y, V\rangle - \langle \nabla_{[X,U]} Y, V\rangle$$

and

$$R(X,V,Y,U) = \langle \nabla_X \nabla_V Y, U\rangle - \langle \nabla_V \nabla_X Y, U\rangle - \langle \nabla_{[X,V]} Y, U\rangle.$$

From here it follows that

$$\begin{aligned}\mathcal{L}_X \mathcal{L}_Y g(U,V) = R(X,U,Y,V) + \langle \nabla_U \nabla_X Y, V\rangle + R(X,V,Y,U)\\ + \langle \nabla_V \nabla_X Y, U\rangle + \langle \nabla_U X, \nabla_V Y\rangle + \langle \nabla_U Y, \nabla_V X\rangle.\end{aligned}$$

Similarly, we arrive at

$$\begin{aligned}\mathcal{L}_Y \mathcal{L}_X g(U,V) = R(Y,U,X,V) + \langle \nabla_U \nabla_Y X, V\rangle + R(Y,V,X,U)\\ + \langle \nabla_V \nabla_Y X, U\rangle + \langle \nabla_U Y, \nabla_V X\rangle + \langle \nabla_U X, \nabla_V Y\rangle.\end{aligned}$$

Now, it suffices to take the difference of the last two identities to achieve

$$\begin{aligned}&\mathcal{L}_X \mathcal{L}_Y g(U,V) - \mathcal{L}_Y \mathcal{L}_X g(U,V)\\ &\quad = R(X,U,Y,V) + \langle \nabla_U \nabla_X Y, V\rangle + R(X,V,Y,U)\\ &\qquad + \langle \nabla_V \nabla_X Y, U\rangle - R(Y,U,X,V) - \langle \nabla_U \nabla_Y X, V\rangle\\ &\qquad - R(Y,V,X,U) - \langle \nabla_V \nabla_Y X, U\rangle.\end{aligned}$$

But since $R(X,U,Y,V) = R(Y,V,X,U)$ e $R(X,V,Y,U) = R(Y,V,X,U)$, we then get

$$\begin{aligned}&\mathcal{L}_X \mathcal{L}_Y g(U,V) - \mathcal{L}_Y \mathcal{L}_X g(U,V)\\ &\quad = \langle \nabla_U \nabla_X Y, V\rangle + \langle \nabla_V \nabla_X Y, U\rangle - \langle \nabla_U \nabla_Y X, V\rangle\\ &\qquad - \langle \nabla_V \nabla_Y X, U\rangle\\ &\quad = \big\langle \nabla_U [X,Y], V\big\rangle + \big\langle \nabla_V [X,Y], U\big\rangle\\ &\quad = \mathcal{L}_{[X,Y]} g(U,V).\end{aligned}$$

So, the proof is completed. □

3. Proof of the main results

3.1. Proof of Theorem 1.

Proof. We argue by contraction by assuming that M^n is compact. In such a case, we consider a smooth function φ on M^n given by $\varphi = \langle X, K\rangle$, where K is a parallel vector field on M^n. Now, a simple computation gives

$$\begin{aligned}\operatorname{div} \varphi K &= \varphi \operatorname{div} K + \langle K, \nabla \varphi\rangle,\\ &= \langle \nabla_K X, K\rangle = \lambda |K|^2,\end{aligned}$$

where we have used the fourth item of Lemma 2. Then, we use the Stokes theorem to conclude that $\lambda = 0$. But, since compact Ricci solitons are shrinking, we arrive at a contradiction (cf. Hamilton [13]).

Proceeding, taking into account that K is a parallel vector field, it follows from Proposition 1 that

$$\begin{aligned}\mathcal{L}_{[K,X]}g &= \mathcal{L}_K\mathcal{L}_X g - \mathcal{L}_X\mathcal{L}_K g\\ &= -2\mathcal{L}_K Ric.\end{aligned}$$

At the same time, since parallel vector fields preserve the Ricci tensor (cf. [5], p. 4, first paragraph) we have $\mathcal{L}_K Ric = 0$ and then

$$\mathcal{L}_{[K,X]}g = 0.$$

From here it follows that

$$\mathcal{L}_{\nabla_K X}g = \mathcal{L}_{\nabla_X K}g = 0. \tag{3.1}$$

Next, for an arbitrary vector field $Z \in \mathfrak{X}(M)$, we deduce

$$\begin{aligned}Z(\varphi) &= \langle \nabla_Z X, K\rangle\\ &= 2\lambda\langle Z, K\rangle - \langle Z, \nabla_K X\rangle,\end{aligned}$$

where we have used the fourth item of Lemma 2. This immediately provides

$$\nabla\varphi = 2\lambda K - \nabla_K X$$

as well as

$$\nabla_K X = 2\lambda K - \nabla\varphi. \tag{3.2}$$

Proceeding, we combine (3.1) with (3.2) to arrive at

$$\operatorname{Hess}\varphi = 0.$$

Therefore, if φ is constant, then

$$\begin{aligned}\lambda|K|^2 &= Ric(K,K) + \langle\nabla_K X, K\rangle\\ &= K\langle X, K\rangle = 0,\end{aligned}$$

which implies that such a structure must be steady. Otherwise, we may invoke De Rham's theorem [10] and Lemma 3 to conclude that (M^n, g) isometric to a direct product $M_1 \times \mathbb{R}$, where (M_1, g_1) admits a non-steady Ricci soliton structure.

In the sequel, notice that $T_{(p_1,p_2)}(M_1 \times \mathbb{R}) \cong T_{p_1}M_1 \oplus T_{p_2}\mathbb{R}$, then we have

$$K = K_1 + K_0,$$

with $K_1 \in \mathfrak{X}(M_1)$ and $K_0 \in \mathfrak{X}(\mathbb{R})$. Whence, it follows that, for all $X \in \mathfrak{X}(M)$,

$$\begin{aligned}0 &= \nabla_X K\\ &= \nabla^1_{X_1}K_1 + \nabla^0_{X_0}K_0,\end{aligned}$$

which shows that K_1 and K_0 are parallel.

Now, suppose that K_1 is non-null identically, then it follows from Lemma 3 jointly with the first part of the proof that (M_1, g_1) is isometric to a direct product $M_2 \times \mathbb{R}$, where (M_2, g_2) admits a non-steady Ricci soliton structure.

Finally, it suffices to repeat the same arguments to conclude that M^n is isometric to either Euclidean space or $N^{n-k} \times \mathbb{R}^k$, where N^{n-k} non-admits non-trivial parallel vector fields. So, the proof is completed. □

3.2. Proof of Corollary 1.

Proof. To begin with, suppose that (M^n, g) admits a parallel vector field K. So, from the proof of Theorem 1 we already know that M^n is non-compact and $\operatorname{Hess}\varphi = 0$, where $\varphi = \langle X, K\rangle$. In particular, since X has a bounded norm, we conclude that φ is constant and therefore

$$\langle \nabla_K X, K\rangle = K\langle X, K\rangle = 0.$$

On the other hand, we have $\langle \nabla_K X, K\rangle = \lambda |K|^2$. This implies $\lambda = 0$. However, from Theorem 2.4 of [11], there are no steady non-compact Ricci solitons with constant scalar curvature. This finishes the proof of the corollary. □

3.3. Proof of Corollary 2.

Proof. We argue by contraction by supposing that (M^n, g) admits a gradient Ricci soliton structure. In this case, from Theorem 1.1 in [20] M^n must be rigid. Whence, it follows that M^n has a non-trivial parallel vector field. Therefore, we may use Corollary 1 to conclude that (M^n, g) does not have constant scalar curvature. This contradiction argument finishes the proof of the corollary. □

3.4. Proof of Corollary 3.

Proof. Let K be a Killing vector field on M^n. Therefore, we immediately deduce

$$\operatorname{Hess} K(f) = \mathcal{L}_K \operatorname{Hess} f.$$

Since parallel vector fields preserve the Ricci tensor ($\mathcal{L}_K Ric = 0$) we may use the fundamental equation to infer

$$\operatorname{Hess} K(f) = \mathcal{L}_K(\lambda g - Ric) = 0. \tag{3.3}$$

Next, suppose that $K(f)$ is constant, then

$$\operatorname{Hess} f(K, K) = \langle \nabla_K \nabla f, K\rangle = -\langle \nabla f, \nabla_K K\rangle.$$

But, since K has a constant norm we get

$$\operatorname{Hess} f(K, K) = 0. \tag{3.4}$$

Proceeding, from Proposition 29 in [19] we obtain

$$\frac{1}{2}\Delta |K|^2 = |\nabla K|^2 - Ric(K, K),$$

and using once more the fact that K has a constant norm jointly with (1.2) and (3.4) we achieve

$$\lambda|K|^2 = |\nabla K|^2,$$

which immediately ensures that such a structure cannot be expanding.

On the other hand, if $K(f)$ is not constant we use (3.3) to deduce that $\nabla K(f)$ is parallel and then the proof follows from Theorem 1. $\square$

3.5. Proof of Theorem 2.

Proof. We start by assuming that H is a non-parallel homothetic closed vector field on M^n. In particular, H is not Killing. Next, we define a smooth function $\varphi : M^n \to \mathbb{R}$ given by

$$\varphi = \frac{1}{2\psi^2}\langle \psi X - \lambda H, H\rangle,$$

where ψ is the conformal factor of H. From this, it follows that

$$\nabla\varphi = \frac{1}{2\psi}\big[\nabla\langle X, H\rangle - 2\lambda H\big]$$

(cf. Lemma 2). In particular, we get

$$\operatorname{Hess}\varphi = \frac{1}{2\psi}\big[\operatorname{Hess}\langle X, H\rangle - 2\lambda\psi g\big]. \tag{3.5}$$

On the other hand, a straightforward computation gives

$$\begin{aligned} Y\langle X, H\rangle &= \langle \nabla_Y X, H\rangle + \psi\langle X, Y\rangle \\ &= \mathcal{L}_X g(Y, H) - \langle Y, \nabla_H X\rangle + \psi\langle X, Y\rangle, \end{aligned}$$

and then by using (1.2) we obtain

$$\big\langle \nabla\langle X, H\rangle, Y\big\rangle = 2\lambda\langle Y, H\rangle - \langle Y, \nabla_H X\rangle + \psi\langle X, Y\rangle,$$

which arrives at

$$\nabla\langle X, H\rangle = 2\lambda H - \nabla_H X + \psi X. \tag{3.6}$$

Notice moreover that

$$\begin{aligned} \mathcal{L}_{\nabla_H X} g &= \mathcal{L}_{[H,X]} g + \psi\mathcal{L}_X g \\ &= \mathcal{L}_H\mathcal{L}_X g - \mathcal{L}_X\mathcal{L}_H g + \psi\mathcal{L}_X g \\ &= 2\mathcal{L}_H Ric - \psi\mathcal{L}_X g \\ &= -\psi\mathcal{L}_X g, \end{aligned}$$

which can be rewritten succinctly as

$$\operatorname{Hess}\langle X, H\rangle = 2\lambda\psi g + \psi\mathcal{L}_X g. \tag{3.7}$$

Finally, we combine (3.5), (3.6) and (3.7) to arrive at

$$\operatorname{Hess}\varphi = \frac{1}{2}\mathcal{L}_X g.$$

Then we use (1.2) once more to obtain

$$\operatorname{Hess}\varphi = \lambda g - Ric,$$

as we wanted to prove. □

3.6. Proof of Corollary 4.

Proof. Let (M^n, g, X, λ) be a Ricci soliton endowed with a homothetic closed vector field. Then, from Theorem 2 there exists a smooth function φ such that $(M^n, g, \nabla\varphi, \lambda)$ is a gradient Ricci soliton structure. Now we invoke Proposition 2.1 of [12] to infer

$$Ric(H, \nabla\varphi) = \frac{1}{2}\langle H, \nabla R\rangle.$$

Whence, it follows from Lemma 1 that

$$\langle H, \nabla R\rangle = 0$$

and then by Lemma 2 we conclude that R is null identically. So, the proof is completed. □

3.7. Proof of Theorem 3.

Proof. Let (M^n, g, X, λ) be a Ricci soliton. Now, suppose that (M^n, g, Y, ζ) is another Ricci soliton structure on (M^n, g). So, by using the fundamental equation we immediately deduce

$$\mathcal{L}_{(X-Y)}g = 2(\lambda - \zeta)g.$$

Hence, it follows that $X - Y$ is a conformal vector field on M^n. Therefore, by Lemma 1 we have

$$(\lambda - \zeta)R = 0,$$

and then $\zeta = \lambda$. Consequently,

$$\mathcal{L}_{(X-Y)}g = 0,$$

so that $X - Y$ is a Killing vector field.

In the sequel, we assume that M^n is complete and $|X| \in \mathcal{L}^1(M)$. Moreover, suppose by contradiction that there exists a gradient Ricci soliton structure $(M^n, g, \nabla\varphi, \zeta)$. Under these conditions, we have

$$X = \nabla\varphi + K, \tag{3.8}$$

where K denotes a Killing vector field. In particular, we may apply Theorem 2.4 of [11] to deduce

$$\zeta = \lambda \neq 0.$$

Hence, by Proposition 5 of [20], we arrive at

$$0 \le R \le n\lambda \quad \text{or} \quad n\lambda \le R \le 0, \tag{3.9}$$

depending on the sign of the constant λ.

Notice moreover that if the scalar curvature of (M^n, g) satisfies $R = n\lambda$, then Proposition 5 of [20] also guarantees that (M^n, g) is Einstein. In particular, by the Ricci soliton fundamental equation we get

$$\frac{R}{n} g + \frac{1}{2}\mathcal{L}_X g = \lambda g,$$

so that

$$\mathcal{L}_X g = 0.$$

Therefore, X is Killing and then such a structure must be trivial.

Proceeding, we already know that

$$0 \leq R < n\lambda \quad \text{or} \quad n\lambda < R \leq 0,$$

for shrinking or expanding cases, respectively. In particular, we infer

$$\operatorname{div} X = n\lambda - R > 0 \quad (\text{or } < 0),$$

which is a contradiction by Proposition 2.1 of [6]. This concludes the proof of the theorem. □

3.8. Proof of Corollary 5.

Proof. First of all, suppose that there exists a Ricci soliton structure (M^n, g, Y, ζ), which is not related up to a Killing vector field to the first structure (M^n, g, X, λ); that is, X and Y differ by a non-Killing vector field. Next, we define $Z_\alpha = (1-\alpha)X + \alpha Y$ to obtain

$$\begin{aligned}\frac{1}{2}\mathcal{L}_{Z_\alpha} g &= \frac{1}{2}\big[(1-\alpha)\mathcal{L}_X g + \alpha \mathcal{L}_Y g\big] \\ &= (1-\alpha)[\lambda g - Ric] + \alpha[\zeta g - Ric] \\ &= -Ric + \big[(1-\alpha)\lambda + \alpha\zeta\big] g,\end{aligned}$$

or also,

$$Ric + \frac{1}{2}\mathcal{L}_{Z_\alpha} g = \eta_\alpha g,$$

where $\eta_\alpha = (1-\alpha)\lambda + \alpha\zeta$.

We can set

$$\eta_\alpha = (1-\alpha)\lambda + \alpha\zeta = \lambda - \alpha(\lambda - \zeta),$$

However, by choosing

$$\lambda > \alpha(\lambda - \zeta),$$

we deduce $\eta_\alpha > 0$. Under these conditions, we have that $(M^n, g, Z_\alpha, \lambda)$ is shrinking and then by Theorem 1.1 of [27] we conclude that (M^n, g) has finite fundamental group.

Since λ and ζ are distinct, we consider

$$\beta = \frac{\lambda}{\lambda - \zeta},$$

then

$$\eta_\beta = \lambda - \beta(\lambda - \zeta) = 0.$$

Thus, $(M^n, g, Z_\beta, \lambda_\beta)$ is a steady Ricci soliton structure. Finally, we apply Corollary 2.5 of [9] to conclude that (M^n, g) has non-negative scalar curvature, which finishes the proof of the corollary. □

3.9. Proof of Corollary 6.

Proof. First, let (M^n, g, X, λ) be a Ricci soliton and H a non-Killing homothetic vector field with conformal factor ψ, then the vector field $Y = X + H$ satisfies

$$Ric + \frac{1}{2}\mathcal{L}_Y g = \zeta g,$$

where $\zeta = \lambda + \psi$. Under these conditions, (M^n, g, Y, ζ) is another Ricci soliton structure on (M^n, g), which is not related module Killing vector field to (M^n, g, X, λ). Therefore, it suffices to use Corollary 5 to conclude the proof of the corollary. □

Acknowledgments. The author is grateful to E. Ribeiro Jr. for helpful conversations during the preparation of this paper. Moreover, the author would like to thank the referee for his careful reading and helpful suggestions.

References

[1] K. Amur and V. S. Hedge, *Conformality of Riemannian manifolds to spheres*, J. Differential Geom. **9** (1974), 571–576. MR 0375145

[2] C. Aquino, A. Barros and E. Jr. Ribeiro, *Some applications of the Hodge–de Rham decomposition to Ricci solitons*, Results Math. **60** (2011), 235–246. MR 2836897

[3] P. Baird, *Explicit constructions of Ricci solitons*, Variational problems in differential geometry, Cambridge University Press, Cambridge, 2012, pp. 37–55. MR 2882769

[4] P. Baird and L. Danielo, *Three-dimensional Ricci solitons which project to surfaces*, J. Reine Angew. Math. **608** (2007), 65–71. MR 2339469

[5] E. Calviño-Louzao, M. Fernández-López, E. García-Rio and R. Vázquez-Lorenzo, *Homogeneous Ricci almost soliton*, Israel J. Math. **220** (2017), 531–546. MR 3666435

[6] A. Caminha, *The geometry of closed conformal vector fields on Riemann spaces*, Bull. Braz. Math. Soc. (N.S.) **42** (2011), 277–300. MR 2833803

[7] H.-D. Cao, *Recent progress on Ricci soliton*, Advanced Lectures in Mathematics (ALM), vol. 11, 2009, pp. 1–38. MR 2648937

[8] I. Castro, C. R. Montealegre and F. Urbano, *Closed conformal vector fields and Lagrangian submanifolds in complex space forms*, Pacific J. Math. **199** (2001), 269–302. MR 1847135

[9] B.-L. Chen, *Strong uniqueness of the Ricci flow*, J. Differential Geom. **82** (2009), 363–382. MR 2520796

[10] G. De Rham, *Sur la réeducibilité*, Comment. Math. Helv. **26** (1952), 328–344. MR 0052177

[11] L. Di Cerbo, *Generic properties of homogeneous Ricci solitons*, Adv. Geom. **14** (2014), 225–237. MR 3263424

[12] M. Eminenti, G. La Nave and C. Mantegazza, *Ricci solitons—The equation point of view*, Manuscripta Math. **127** (2008), 345–367. MR 2448435

[13] R. S. Hamilton, *The formation of singularities in the Ricci flow*, Surveys in differential geometry (Cambridge, MA, 1993), vol. 2, International Press, Cambridge, MA, 1995, pp. 7–136. MR 1375255

[14] A. Naber, *Noncompact shrinking four solitons with nonnegative curvature*, J. Reine Angew. Math. **645** (2010), 125–153. MR 2673425

[15] M. Obata, *Conformal transformations of Riemannian manifolds*, J. Differential Geom. **4** (1970), 311–333. MR 0267485

[16] M. Obata and K. Yanno, *Conformal changes of Riemannian metrics*, J. Differential Geom. **4** (1970), 53–72. MR 0261500

[17] B. O'Neill, *Semi-Riemannian geometry with application to relativity*, Academic Press, New York, 1983. MR 0719023

[18] G. Perelman, *The entropy formula for the Ricci flow and its geometric applications*, Preprint, 2002. Available in `arXiv.math/0211159v1` [math.DG].

[19] P. Petersen, *Riemannian geometry*, Graduate Texts in Mathematics, Springer-Verlag, New York, 1998. MR 1480173

[20] P. Petersen and W. Wylie, *On gradient Ricci solitons with symmetry*, Proc. Amer. Math. Soc. **137** (2009), 2085–2092. MR 2480290

[21] P. Petersen and W. Wylie, *Rigidity of gradient Ricci solitons*, Pacific J. Math. **241** (2009), 329–345. MR 2507581

[22] E. Jr. Ribeiro and J. F. Silva Filho, *Explicit description of three-dimensional homogeneous Ricci solitons*, Publ. Math. Debrecen **85** (2014), 31–46. MR 3231503

[23] A. Ros and F. Urbano, *Lagrangian submanifolds of $\mathbb{C}^n$ with conformal Maslov form and the Whitney sphere*, J. Math. Soc. Japan **50** (1998), 203–226. MR 1484619

[24] Y. Suyama and Y. Tsukamoto, *Riemannian manifolds admitting a certain conformal transformation group*, J. Differential Geom. **5** (1971), 415–426. MR 0298591

[25] S. Tanno and W. Webber, *Closed conformal vector fields*, J. Differential Geom. **3** (1969), 361–366. MR 0261498

[26] Y. Tashiro, *Complete Riemannian manifolds and some vector fields*, Trans. Amer. Math. Soc. **117** (1965), 251–275. MR 0174022

[27] W. Wylie, *Complete shrinking Ricci solitons have finite fundamental group*, Proc. Amer. Math. Soc. **136** (2008), 1803–1806. MR 2373611

J. F. Silva Filho, Instituto de Ciências Exatas e da Natureza, UNILAB, 62785-000, Acarape, CE, Brazil

E-mail address: `joaofilho@unilab.edu.br`