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CONVEX SUBQUIVERS AND THE FINITISTIC DIMENSION

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ABSTRACT. Let $\mathcal{Q}$ be a quiver and K a field. We study the interrelationship of homological properties of algebras associated to convex subquivers of $\mathcal{Q}$ and quotients of the path algebra $K\mathcal{Q}$. We introduce the homological heart of $\mathcal{Q}$ which is a particularly nice convex subquiver of $\mathcal{Q}$. For any algebra of the form $K\mathcal{Q}/I$, the algebra associated to $K\mathcal{Q}/I$ and the homological heart have similar homological properties. We give an application showing that the finitistic dimension conjecture need only be proved for algebras with path connected quivers.

1. Introduction

The study of homological properties of algebras is made both interesting and challenging by the fact that such properties are usually not inherited by subalgebras or quotient algebras. In the rare cases where homological properties are inherited, one can hopefully pass to a quotient or subalgebra for which the question being studied is easier to attack. The finitistic dimension conjecture for finite dimensional algebras [5] is one such problem. The *finitistic dimension* of a finite dimensional algebra is the supremum of the projective dimensions of finitely generated modules having finite projective dimension and the Finitistic Dimension Conjecture asserts that this supremum is finite. The conjecture has been proven directly in a number of cases, for example, [10], [13], [17], [4], and attempts have been made to study the conjecture by reducing the conjecture to 'simpler' rings [11], [14], [16], [21], [15], [6], but the conjecture remains open. A survey on the state of the art until 2007 can be found in [15].

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In this paper, we study rings of the form $K\mathcal{Q}/I$ where K is a field, $\mathcal{Q}$ a quiver and I is an ideal. We find conditions on $\mathcal{Q}$, independent of I, so that there is a full subquiver $\mathcal{L}$ such that the homological properties of the algebra $K\mathcal{L}/(I\cap K\mathcal{L})$ provide information about homological properties of $K\mathcal{Q}/I$. In particular, if e' is the sum of the vertices not in $\mathcal{L}$, we investigate conditions that imply that the canonical a surjection $K\mathcal{Q}/I\to(K\mathcal{Q}/I)/\langle e'\rangle$ is a homological epimorphism; that is, the surjection induces a fully faithful functor $\mathcal{D}^b(\mathrm{mod}(K\mathcal{Q}/I)\to\mathcal{D}^b(\mathrm{mod}((K\mathcal{Q}/I)/\langle e'\rangle))$ [9], [18], [3], [22], [12].

Let $\mathcal{Q}$ be a finite quiver. We let $\mathcal{L}$ be a full convex subcategory of $\mathcal{Q}$; that is, if p is a path from a vertex in $\mathcal{L}$ to a vertex in $\mathcal{L}$, then each arrow and vertex of p is in $\mathcal{L}$. The study of convex subcategories is not new and they have appeared in the study of the representation theory of finite dimensional algebras; for example, see [23], [2], [1], [7]. Noting that if $\mathcal{L}$ is convex, then $(K\mathcal{Q}/I)/\langle e'\rangle$ is isomorphic to $K\mathcal{L}/(I\cap K\mathcal{L})$, (see Proposition 3.3). we show that if $\mathcal{L}$ is a convex subquiver of $\mathcal{Q}$, then the canonical surjection $K\mathcal{Q}/I\to K\mathcal{L}/(I\cap K\mathcal{L})$ is a homological embedding, see Theorem 4.2.

In Section 5, for any quiver Q, we define a subquiver which we will call the homological heart of $\mathcal{Q}$. We show that there is a strong homological relation between a finite dimensional algebra $K\mathcal{Q}/I$ and the algebra $K\mathcal{H}/(I\cap K\mathcal{H})$, where $\mathcal{H}$ is the homological heart of $\mathcal{Q}$. For instance, the finitistic dimension of one is finite if and only if the finitistic dimension of the other is finite. A similar concept was defined and studied in [19].

The paper ends with an application showing that the Finitistic Dimension Conjecture only needs to be proved in the case where the quiver is path connected.

If $\mathcal{Q}$ is a quiver, then $\mathcal{Q}_0$ denotes the vertex set of $\mathcal{Q}$, $\mathcal{Q}_1$ denotes the arrow set of $\mathcal{Q}$. If K is a field, $K\mathcal{Q}$ denotes the path algebra. If J denotes the ideal in a path algebra $K\mathcal{Q}$ generated by the arrows of $\mathcal{Q}$, then an ideal I in $K\mathcal{Q}$ is *admissible* if $J^n\subseteq I\subseteq J^2$, for $n\geq 2$. If Λ is an algebra and A is a subset of Λ, then $\langle A\rangle$ denotes the ideal in Λ generated by A. Unless otherwise stated, modules are right modules and if a and b are arrows in a quiver $\mathcal{Q}$, ab denotes the concatenation of first a then b.

2. Definitions and basic properties

Throughout this paper, let K be a field, $\mathcal{Q}$ a finite quiver, and let $K\mathcal{Q}$ denote the path algebra. We say a subquiver $\mathcal{L}$ of $\mathcal{Q}$ is *full* if for any two vertices v and w of $\mathcal{L}$, all the arrows in $\mathcal{Q}$ with origin v and terminus w are also arrows in $\mathcal{L}$. A full subquiver is uniquely determined by its vertex set and all subquivers considered in this paper are assumed to be full. If $X\subset\mathcal{Q}_0$, then the full subquiver of $\mathcal{Q}$ having vertex X is called the *subquiver generated by X*. We freely identify a set of vertices with the full subquiver it generates. The *empty subquiver* is defined to be the subquiver with no vertices or arrows.

If $\mathcal{L}$ and $\mathcal{M}$ are subquivers of $\mathcal{Q}$, then $\mathcal{L} \cup \mathcal{M}$ (respectively, $\mathcal{L} \cap \mathcal{M}$) is the full subquiver generated by $\mathcal{L}_0 \cup \mathcal{M}_0$ (resp., by $\mathcal{L}_0 \cap \mathcal{M}_0$).

A fundamental concept that plays a central role in this paper is convexity. We say a full subquiver $\mathcal{L}$ of $\mathcal{Q}$ is *convex* if for any two vertices v, w in $\mathcal{L}$ and for any path p from v to w, every vertex occurring in p is in $\mathcal{L}$. We introduce 3 other subquivers associated to a given subquiver. Let $\mathcal{L}$ be a full subquiver of $\mathcal{Q}$. Define $\mathcal{L}^+$ and $\mathcal{L}^-$ to be the full subquivers such that the vertex set of $\mathcal{L}^+$ is

$$\{v \in \mathcal{Q}_0 \mid v \notin \mathcal{L}_0 \text{ and there is a path in } \mathcal{Q} \text{ from a vertex in } \mathcal{L}_0 \text{ to } v\}$$

and the vertex set of $\mathcal{L}^-$ is

$$\{v \in \mathcal{Q}_0 \mid v \notin \mathcal{L}_0 \text{ and there is a path in } \mathcal{Q} \text{ from } v \text{ to a vertex in } \mathcal{L}_0\}.$$

A third construction is $\mathcal{L}^o$ whose vertex set is

$$\{v \in \mathcal{Q}_0 \mid v \notin \mathcal{L}_0 \text{ and there are no paths between } v \text{ and any vertex in } \mathcal{L}\}.$$

It is easy to see that if $\mathcal{L}$ is a full subquiver of $\mathcal{Q}$ then $\mathcal{Q} = \mathcal{L} \cup \mathcal{L}^+ \cup \mathcal{L}^- \cup \mathcal{L}^o$ and $\mathcal{L}$ is convex if and only if $\mathcal{L}_0^+ \cap \mathcal{L}_0^-$ is empty. In particular, we have

PROPOSITION 2.1. *Let $\mathcal{L}$ is a subquiver of $\mathcal{Q}$. If either $\mathcal{L}^+$ or $\mathcal{L}^-$ is empty then $\mathcal{L}$ is convex.*

3. Convexity and algebras

For the remainder of this paper, $\mathcal{L}$ will denote a full subquiver of $\mathcal{Q}$ and I is an ideal in $K\mathcal{Q}$ contained in the ideal generated by the paths of length 2. We set $\Lambda = K\mathcal{Q}/I$. Let $e_{\mathcal{L}}$ be the idempotent associated to the sum of the vertices in $\mathcal{L}$ and $e'_{\mathcal{L}}$ be the idempotent associated to the sum of vertices not in $\mathcal{L}$. Then $e_{\mathcal{L}}$ and $e'_{\mathcal{L}}$ are orthogonal idempotents in $K\mathcal{Q}$ and $1 = e_{\mathcal{L}} + e'_{\mathcal{L}}$. We abuse notation and view $e_{\mathcal{L}}$ and $e'_{\mathcal{L}}$ as idempotents in both $K\mathcal{Q}$ and Λ. The *algebra associated to the pair* $(\mathcal{L}, \Lambda)$ is defined to be $\Gamma = \Lambda/\langle e'_{\mathcal{L}} \rangle$, where $\langle e'_{\mathcal{L}} \rangle$ is the ideal generated by $e'_{\mathcal{L}}$. Another possible choice of the algebra associated to $(\mathcal{L}, \Lambda)$ could have been $e_{\mathcal{L}} \Lambda e_{\mathcal{L}}$. In general, $\Lambda/\langle e'_{\mathcal{L}} \rangle$ and $e_{\mathcal{L}} \Lambda e_{\mathcal{L}}$ are not isomorphic. For example, if $\mathcal{Q}$ is the quiver $v \to w \to x$, $\Lambda = K\mathcal{Q}$, and $\mathcal{L}$ is the subquiver generated by $\{v, x\}$, then $e_{\mathcal{L}} \Lambda e_{\mathcal{L}}$ is the hereditary algebra with quiver $v \to x$ of dimension 3 while $\Lambda/\langle e'_{\mathcal{L}} \rangle$ is the semisimple algebra of dimension 2 corresponding to the two vertices v and x with no arrows. Before proving that when $\mathcal{L}$ is convex, the two algebras are in fact isomorphic, we present the following useful result. We simplify notation by letting $e = e_{\mathcal{L}}$ and $e' = e'_{\mathcal{L}}$ when no confusion can arise.

LEMMA 3.1. *Suppose that $\mathcal{L}$ is a convex subquiver of $\mathcal{Q}$. Then if $\lambda, \gamma \in \Lambda$, $e\lambda e\gamma e = e\lambda\gamma e$. In particular, $e\lambda e'\gamma e = 0$.*

Proof. Let λ and γ be elements of Λ. If $\pi\colon K\mathcal{Q}\to\Lambda$ is the canonical surjection, then there are paths p_i and q_j and constants c_i and d_i, so that $\lambda=\pi(\sum_i c_ip_i)$ and $\gamma=\pi(\sum_j d_jq_j)$. Then, using that $1=e+e'$, we have

$$\begin{aligned} e\lambda\gamma e &= e\lambda(e+e')\gamma e\\ &= \pi\left(e\left(\sum_i c_ip_i\right)e\left(\sum_j d_jq_j\right)e\right)+\pi\left(e\left(\sum_i c_ip_i\right)e'\left(\sum_j d_jq_j\right)e\right). \end{aligned}$$

Since $\mathcal{L}$ is convex, each $ep_ie'q_je=0$ and the result follows. □

COROLLARY 3.2. *Keeping the hypothesis of Lemma 3.1, the homomorphism $\phi\colon\Lambda\to e\Lambda e$ defined by $\phi(\lambda)=e\lambda e$ is a ring surjection.*

The convexity of $\mathcal{L}$ is essential in the last part of the next result.

PROPOSITION 3.3. *Let $\mathcal{L}$ be a full subquiver of $\mathcal{Q}$ and $\Lambda=K\mathcal{Q}/I$ where $I\subseteq J^2$. The following three K-algebras are isomorphic.*

(1) *$e\Lambda e$.*
(2) *$eK\mathcal{Q}e/eIe$.*
(3) *$eK\mathcal{Q}e/I\cap eK\mathcal{Q}e$.*
(4) *If $\mathcal{L}$ is convex, then $\Lambda/\langle e'\rangle$, the algebra associated to $(\mathcal{L},\Lambda)$, is isomorphic to $e\Lambda e$ This isomorphism sends $\bar\lambda$ to $e\lambda e$, where $\lambda\in\Lambda$ and $\bar\lambda$ denotes the image of λ in $\Lambda/\langle e'\rangle$.*
(5) *If $\mathcal{L}$ is convex, then $K\mathcal{L}/(I\cap K\mathcal{L})$ is isomorphic to $\Lambda/\langle e'\rangle$.*

Proof. The proofs of parts (1)–(3) are left to the reader. By Corollary 3.2, the map λ to $e\lambda e$ is a ring surjection. We claim $\langle e'\rangle$ is the kernel of this homomorphism. Since $ee'e=0$, $\langle e'\rangle$ is contained in the kernel. Now suppose that λ is in the kernel; that is, $e\lambda e=0$. Now $\lambda=e\lambda e+e'\lambda e+e\lambda e'+e'\lambda e'$. Since $e'\lambda e+e\lambda e'+e'\lambda e'\in\langle e'\rangle$, we are done. □

Consider the map $\mu\colon e_{\mathcal{L}}\Lambda e_{\mathcal{L}}\to\Lambda$ given by inclusion and the map $\nu\colon\Lambda\to e_{\mathcal{L}}\Lambda e_{\mathcal{L}}$ given by $\nu(\lambda)=e_{\mathcal{L}}\lambda e_{\mathcal{L}}$. If $\mathcal{L}$ is convex, then $\nu\circ\mu=1_{e_{\mathcal{L}}\Lambda e_{\mathcal{L}}}$. In fact we have the following result.

PROPOSITION 3.4. *Let $\mathcal{L}$ be a convex subquiver of $\mathcal{Q}$ and let $\pi\colon\Lambda\to\Gamma=\Lambda/\langle e'\rangle$ denote the canonical surjection. Then there is a ring* (*without* 1) *splitting of ϕ mapping $\Lambda/\langle e'\rangle$ to Λ given as follows*: *if $\lambda\in\Lambda$ then $\lambda+\langle e'\rangle$ is sent to $e\lambda e$.*

Proof. Let $\delta\colon\Lambda/\langle e'\rangle\to e\Lambda e$ be the isomorphism described in Proposition 3.3. Then $incl\circ\delta\colon\Lambda/\langle e'\rangle\to\Lambda$ is easily seen to splitting to the canonical surjection $\psi\colon\Lambda\to\Lambda/\langle e'\rangle$. The reader may check that $incl\circ\delta$ is a ring homomorphism taking $1\in\Lambda/\langle e'\rangle$ to $e\in\Lambda$. □

4. Module theoretic results

Recall that a ring epimorphism $A \to B$ is called a *homological epimorphism* if the induced map $\mathcal{D}^b(\mathrm{mod}(B)) \to \mathcal{D}^b(\mathrm{mod}(A))$ is fully faithful [9], [18]. Equivalently, if $A \to B$ is a homological epimorphism and $M, N \in \mathrm{mod}(B)$, then $\mathrm{Ext}^n_B(M,N)$ is isomorphic to $\mathrm{Ext}^n_A(M,N)$ for all $n \geq 0$. Consequently, if A is a finite dimensional algebra and $A \to B$ is a homological epimorphism, then $\mathrm{gl.dim}(A) \geq \mathrm{gl.dim}(B)$ and $\mathrm{fin.dim}(A) \geq \mathrm{fin.dim}(B)$.

In this section, we show that the canonical surjection $\Lambda \to \Gamma$ is a homological epimorphism if $\mathcal{L}$ is convex subquiver of $\mathcal{Q}$. We begin by assuming $\mathcal{L}^+$ is empty. As examples, if $\mathcal{L}$ is a full subquiver of $\mathcal{Q}$, then both $(\mathcal{L}\cup\mathcal{L}^+)^+$ and $(\mathcal{L}\cup\mathcal{L}^-)^-$ are empty and hence $\mathcal{L}\cup\mathcal{L}^+$ and $\mathcal{L}\cup\mathcal{L}^-$ are convex. Another example would be $\mathcal{L}^+$ which is always convex and $(\mathcal{L}^+)^+$ is always empty.

We will apply the following result.

Proposition 4.1.

(1) *If $\mathcal{L}^+$ is empty, then the canonical surjection $\Lambda \to \Gamma$ is a homological epimorphism.*

(2) *If $\mathcal{Q} = \mathcal{L} \cup \mathcal{L}^+$, then Γ is a projective left Λ-module and the canonical surjection $\Lambda \to \Gamma$ is a homological epimorphism.*

Proof. First assume that $\mathcal{L}^+$ is empty. Then $e\Lambda e' = 0$ and hence $\Lambda = \left(\begin{smallmatrix} \Gamma & 0 \\ e'\Lambda e & e'\Lambda e' \end{smallmatrix}\right)$. Part (1) follows from [8], [20].

Next, assume that $\mathcal{Q} = \mathcal{L} \cup \mathcal{L}^+$. We claim that Γ is a left projective Λ-module. Note that $e'\Lambda e = (0)$ since there are no paths from vertices in $\mathcal{L}^+$ to $\mathcal{L}$ by convexity of $\mathcal{L}$. By Proposition 3.4, there is splitting $\psi\colon \Gamma \to \Lambda$ of the canonical surjection $\Lambda \to \Gamma$ given by $\psi(\lambda + \langle e'\rangle) = e\lambda e$. Using that $e'\Lambda e = 0$, one sees that

$$\lambda\psi(\lambda') = \lambda(e\lambda' e) = e\lambda e\lambda' e = e\lambda\lambda' e = \psi(\lambda\lambda' + \langle e'\rangle).$$

Hence, ψ is a left Λ-module homomorphism. We conclude that Γ is a left projective Λ-module. Part (2) follows from [3]. $\square$

Using the previous proposition, we prove the main result.

Theorem 4.2. *Let K be a field, $\mathcal{Q}$ a finite quiver, and $\Lambda = K\mathcal{Q}/I$, where I is an ideal in $K\mathcal{Q}$ contained in the ideal generated by the paths of length 2 in $K\mathcal{Q}$. Suppose that $\mathcal{L}$ is a convex subquiver of $\mathcal{Q}$ and let Γ be the algebra associated to $(\mathcal{L}, \Lambda)$. Then the canonical surjection $\Lambda \to \Gamma$ is a homological epimorphism.*

Proof. We begin by applying Proposition 4.1 as follows. Let $\mathcal{M} = \mathcal{L} \cup \mathcal{L}^+$. We see that $\mathcal{M}^+$ is empty. Let Δ be the algebra associated to $(\mathcal{M}, \Lambda)$. Proposition 4.1(1) yields the canonical surjection $\Lambda \to \Delta$ is a homological epimorphism.

Next, we assume that $\mathcal{L}$ is convex. View $\mathcal{L}$ as a subquiver of $\mathcal{M}$ and let Σ be the algebra associated to $(\Delta, \mathcal{L})$. We apply Proposition 4.1(2) the canonical surjection $\Delta \to \Sigma$ is a homological epimorphsim.

It is straightforward to show Σ is isomorphic to Γ and the composition of homological epimorphisms is a homological epimorphism. □

Corollary 4.3. *Suppose that Λ is a finite dimensional K-algebra. If $\mathcal{L}$ is a convex subquiver of $\mathcal{Q}$, then*

(1) $\mathrm{gl.dim}(\Lambda) \geq \mathrm{gl.dim}(\Gamma)$,
(2) $\mathrm{fin.dim}(\Lambda) \geq \mathrm{fin.dim}(\Gamma)$.

5. The homological heart

In this section, we introduce the homological heart of a quiver. Let

$$X = \{v \in \mathcal{Q}_0 \mid v \text{ is a vertex in a nontrivial cycle in } \mathcal{Q}\}$$

and let

$$Y = X \cup \{y \in \mathcal{Q}_0 \mid y \text{ is a vertex in a path with origin and terminus vertices in } X\}.$$

Let $\mathcal{H}(\mathcal{Q})$, or simply $\mathcal{H}$ when no confusion can arise, be the subquiver of $\mathcal{Q}$ with vertex set Y. We call $\mathcal{H}$ the *homological heart of* $\mathcal{Q}$.

The proof of the next result is left to the reader.

Proposition 5.1. *Let $\mathcal{H}$ be the homological heart of $\mathcal{Q}$. Then the following statements hold.*

(1) *The subquiver $\mathcal{H}$ is the empty quiver if and only if $\mathcal{Q}$ contains no nontrivial cycles*; *that is, $\mathcal{Q}$ is triangular.*
(2) *The subquiver $\mathcal{H}$ is the smallest convex subquiver of $\mathcal{Q}$ that contains all the nontrivial path connected components of $\mathcal{Q}$.*
(3) *The homological heart of $\mathcal{Q}$ is an invariant of $\mathcal{Q}$.*
(4) *The subquiver $\mathcal{H}^+ \cup \mathcal{H}^- \cup \mathcal{H}^o$ contains no oriented cycles.*

Throughout this section, we let Γ denote the algebra associated to $(\mathcal{H}, \Lambda)$. The goal of this section is to show that Γ provides information about a number homological features of Λ.

We fix the following notation. For each vertex $v \in \mathcal{Q}$, let P_v denote the projective Λ-module $v\Lambda$ and I_v the injective Λ-envelope of the simple one dimensional Λ-module associated to the vertex v.

We begin with a known result which we include for completeness.

Proposition 5.2.

(1) *Let $P^n \to P^{n-1} \to \cdots \to P^2 \to P^1 \to P^0$ be a sequence of nonzero Λ-homorphisms with each P^i a projective Λ-module which is a direct sum of projective modules of the form P_v. Assume that the image of $P^i \to P^{i-1}$ is*

contained $P^{i-1}\mathbf{r}$, where $\mathbf{r}$ is the Jacobson radical of Λ and that $\ker(P^n \to P^{n-1})$ *is contained in $P^n\mathbf{r}$, for all $1 \le i \le n$. If $v \in \mathcal{Q}_0$ is such that $v\Lambda$ is a summand of P^n, then there exist a vertex w and a path of length $\ge n$ from w to v in $\mathcal{Q}$ such that $w\Lambda$ is summand of P^0.*

(2) *Let $I_0 \to I_1 \to \cdots \to I_{n-1} \to I_n$ be a sequence of nonzero Λ-homorphisms with each I_i an injective Λ-module which is a direct sum of injective modules of the form I_v. Assume that* $\mathrm{Soc}(I_i)$ *is contained in the image of $I_{i-1} \to I_i$ and that* $\mathrm{Soc}(I_{i-1})$ *is contained in the kernel of $I_{i-1} \to I_i$, for $1 \le i \le n$. If $v \in \mathcal{Q}_0$ is such that I_v is a summand of I_n, then there exist a vertex w and a path of length $\ge n$ from v in $\mathcal{Q}$ to w such that $I_w\Lambda$ is summand of I_0.*

Proof. (Sketch) We only sketch a proof of (1) leaving (2) to the reader. The proof follows by induction and noting that if v is in the support of the Λ-module $P_w\mathbf{r}$, then there is a path from w to v of length at least 1 and if $f\colon P_v \to P_w\mathbf{r}$ is a nonzero map, then v is in the support of $P_w\mathbf{r}$. □

By Proposition 5.1(4), we set t to be the longest path in $\mathcal{Q}$ with support in $\mathcal{H}^- \cup \mathcal{H}^0 \cup \mathcal{H}^+$. The next result will be used often.

Lemma 5.3. *Let M be a Λ-module. Then, for $\ell > t$,*

(1) *the ℓth syzygy of a Λ-module has support in $\mathcal{H} \cup \mathcal{H}^+$ and*
(2) *the ℓth cosyzygy of Λ-module has support in $\mathcal{H} \cup \mathcal{H}^-$.*

Proof. We only prove (1). Let M be a Λ-module and $\cdots \to P^2 \to P^1 \to P^0 \to M \to 0$ be a minimal projective Λ-resolution of M. From the definitions of $\mathcal{H}^-$, $\mathcal{H}^+$ and $\mathcal{H}^0$, any path ending at a vertex in $\mathcal{H}^- \cup \mathcal{H}^0$ has support in $\mathcal{H}^- \cup \mathcal{H}^0$. Suppose $\ell > t$. Suppose that there is a nonzero map from P_v to P^ℓ and that v is vertex in $\mathcal{H}^- \cup \mathcal{H}^0$. Then there would be a path of length $\ge \ell$ ending at v by Proposition 5.2. This path has support in $\mathcal{H}^- \cup \mathcal{H}^0$. By our assumption on t, we must have $\ell \le t$, a contradiction. Hence if P_v is a summand of P^ℓ, v must be a vertex in $\mathcal{H} \cup \mathcal{H}^+$. It follows that P^ℓ has support in $\mathcal{H} \cup \mathcal{H}^+$ and we are done. □

Corollary 5.4. *Let M and N be Λ-modules. Then there are Λ-modules A and B such that, $\ell \ge 2t+3$,*

$$\mathrm{Ext}^{\ell}_{\Lambda}(M,N) \cong \mathrm{Ext}^{\ell-2t-2}_{\Lambda}(A,B)$$

with A having support in $\mathcal{H} \cup \mathcal{H}^+$ and B having support in $\mathcal{H} \cup \mathcal{H}^-$.

Proof. Take A to be the $(t+1)$th syzygy of M and B to be the $(t+1)$th cosyzygy of N and apply Lemma 5.3 and dimension shift. □

The next result considers modules with support contained in either $\mathcal{H} \cup \mathcal{H}^+$ or $\mathcal{H} \cup \mathcal{H}^-$.

Proposition 5.5. *Suppose that A is a Λ-module with support contained in $\mathcal{H}\cup\mathcal{H}^+$ and B is a Λ-module with support contained in $\mathcal{H}\cup\mathcal{H}^-$. If $\cdots\to P^2\to P^1\to P^0\to A\to 0$ is a minimal projective Λ-resolution of A and $0\to B\to I_0\to I_1\to I_2\to\cdots$ is a minimal injective Λ-coresolution of B, then each P^n has support contained in $\mathcal{H}\cup\mathcal{H}^+$ and each I_n has support contained in $\mathcal{H}\cup\mathcal{H}^-$.*

Proof. We use the observation that any path ending at a vertex in $\mathcal{H}^-\cup\mathcal{H}^0$ (respectively, beginning at a vertex in $\mathcal{H}^+\cup\mathcal{H}^0$) has support in $\mathcal{H}^-\cup\mathcal{H}^0$ (resp., $\mathcal{H}^+\cup\mathcal{H}^0$). The result follows from Proposition 5.2. □

Let C be a Λ-module. Define C^+ to be the largest submodule of C having support contained in $\mathcal{H}^+$ and C_- be the smallest submodule of C such that C/C_- has support contained $\mathcal{H}^-$. Let e be the sum of the vertices in $\mathcal{H}$, e^+ be the sum of the vertices in $\mathcal{H}^+$, e^- be the sum of the vertices in $\mathcal{H}^-$, and e^0 be the sum of the vertices in $\mathcal{H}^0$. Letting Γ be the algebra associated to $(\mathcal{H},\Lambda)$, we have the following result, whose proof is left to the reader.

Lemma 5.6. *Let C be a Λ-module.*

(1) *C^+ is the trace of $e^+\Lambda$ in C.*
(2) *If C has support in $\mathcal{H}\cup\mathcal{H}^-$, then C_- is the trace of $e\Lambda$ in C.*
(3) *If C has support in $\mathcal{H}\cup\mathcal{H}^+$, then C/C^+ has support contained in $\mathcal{H}$ and hence is a Γ-module.*
(4) *If C has support in $\mathcal{H}\cup\mathcal{H}^-$, then C_- has support in $\mathcal{H}$ and hence is a Γ-module.*
(5) *If C has support in $\mathcal{H}\cup\mathcal{H}^+$, then $C^+=Ce^+\Lambda$.*
(6) *If C has support in $\mathcal{H}\cup\mathcal{H}^+$, then $C/C^+\cong C\otimes_\Lambda\Gamma$.*

Before proving a result that relates projective Λ-resolutions of Λ-modules A whose support is contained in $\mathcal{H}\cup\mathcal{H}^+$ and projective Γ-resolutions of A/A^+, from Proposition 4.1(2), we know that Λe^+ is a left projective Λ-module and hence Γ is a left projective Λ-module. Noting that $\Lambda e^+\Lambda=\Lambda e^+$, we see that $\Lambda e^+\Lambda$ is a *strong idempotent ideal* as defined in [3] and that the following result is similar to Theorem 1.6 of [3].

Proposition 5.7. *Let A be a Λ-module whose support is contained in $\mathcal{H}\cup\mathcal{H}^+$. If $\cdots\to P^2\to P^1\to P^0\to A\to 0$ is a minimal projective Λ-resolution of A, then*

$$\cdots\to P^2/\big(P^2\big)^+\to P^1/\big(P^1\big)^+\to P^0/\big(P^0\big)^+\to A/A^+\to 0$$

is a minimal projective Γ-resolution of A/A^+.

Proof. By Proposition 4.1(2), we see that Γ is a left projective Λ-module hence tensoring with Γ is exact. Since $A/A^+\cong A\otimes_\Lambda\Gamma$ and $P^n/(P^n)^+\cong P^n\otimes_\Lambda\Gamma$ and minimality is clear, the result follows. □

We remark that there is a “dual” result for injective Λ-coresolutions of Λ-modules B whose support is contained in $\mathcal{H}\cup\mathcal{H}^-$ and injective Γ-coresolutions of B_-.

Proposition 5.8. *Let A and B be Λ-modules with support contained in $\mathcal{H}\cup\mathcal{H}^+$ and $\mathcal{H}\cup\mathcal{H}^-$ respectively. Then, for $n\geq 0$,*

$$\mathrm{Ext}^n_\Lambda(A,B)\cong\mathrm{Ext}^n_\Gamma\big(A/A^+,B_-\big)$$

Proof. First, we note that $\mathrm{Hom}_\Lambda(A,B)$ and $\mathrm{Hom}_\Gamma(A/A^+,B_-)$ are naturally isomorphic. To see this, if $f\colon A\to B$ is a Λ-homomorphism, then, using Lemma 5.6(6), $f\otimes 1_\Gamma\colon A/A^+\to B/B^+$. But $B^+=(0)$ since B has support in $\mathcal{H}\cup\mathcal{H}^-$ and $\mathcal{H}$ is convex. Hence, $f\otimes 1_\Gamma\colon A/A^+\to B$. Next we note that composing $f\otimes 1_\Gamma$ with the canonical surjection $B\to B/B_-$ is 0 since B/B_- has support contained in $\mathcal{H}^-$ and A/A^+ has support contained in $\mathcal{H}$. We conclude that the image of $f\otimes 1_\Gamma$ is contained in B_-. Thus, we have constructed a map from $\mathrm{Hom}_\Lambda(A,B)$ to $\mathrm{Hom}_\Gamma(A/A^+,B_-)$. To see that this map is a monomorphism we note that $\mathrm{Hom}(A^+,B)=(0)$ since A^+ has support contained in $\mathcal{H}^+$, and the convexity of $\mathcal{H}$ implies that $\mathcal{H}^+\cap\mathcal{H}^-$ is empty.

On the other hand, if $g\colon A/A^+\to B_-$ is a Γ-homomorphism, then composing with the canonical surjection $A\to A/A^+$ and the inclusion $B_-\to B$ we obtain a map from $\mathrm{Hom}_\Gamma(A/A^+,B_-)$ to $\mathrm{Hom}_\Lambda(A,B)$. The maps defined are inverse to each other and we are done.

Using the naturality of the isomorphism

$$\mathrm{Hom}_\Lambda(A,B)\to\mathrm{Hom}_\Gamma\big(A/A^+,B_-\big)$$

and Proposition 5.7, the result follows. □

The following result is the main result of this section. For a Λ-module L, we let $\Omega^n(L)$ denote the nth syzygy of L in a minimal projective Λ-resolution of L and $\Omega^{-n}(L)$ denote the nth cosyzygy of L in a minimal injective Λ-coresolution of L

Theorem 5.9. *Let K be a field, $\mathcal{Q}$ a finite quiver, and $\Lambda=K\mathcal{Q}/I$, where I is an admissible ideal I in $K\mathcal{Q}$. Let $\mathcal{H}$ be the homological heart of $\mathcal{Q}$ and Γ be the algebra associated to $(\Lambda,\mathcal{H})$. Let t be length of the longest path in $\mathcal{Q}$ having support contained in $\mathcal{H}^+\cup\mathcal{H}^-\cup\mathcal{H}^0$. Then:*

(1) $\mathrm{gl.dim}(\Lambda)$ *is finite if and only if* $\mathrm{gl.dim}(\Gamma)$ *is finite.*
(2) $\mathrm{fin.dim}(\Lambda)$ *is finite if and only if* $\mathrm{fin.dim}(\Gamma)$ *is finite.*
(3) *If M and N are Λ-modules and $\ell\geq 2t+3$, then*

$$\mathrm{Ext}^\ell_\Lambda(M,N) \textit{ is naturally isomorphic to } \mathrm{Ext}^{\ell-2t-2}_\Gamma(A_M,B_N),$$

where $A_M=\Omega^{t+1}(M)/(\Omega^{t+1}(M))^+$ and $B_N=\Omega^{-t-1}(N)_-$.

Proof. Since $\mathcal{H}$ is convex, applying Theorem 4.2, we see that $\mathrm{gl.dim}(\Lambda)\geq\mathrm{gl.dim}(\Gamma)$ and $\mathrm{fin.dim}(\Lambda)\geq\mathrm{fin.dim}(\Gamma)$. Thus, if $\mathrm{gl.dim}(\Lambda)$ is finite then gl.dim(Γ) is finite and if fin.dim(Λ) is finite then fin.dim(Γ) is finite. We note that part (3) implies that gl.dim(Λ) $\leq$ gl.dim(Γ) and hence part (1) would follow.

Thus, we must show part (3) holds and show that fin.dim(Λ) is finite if fin.dim(Γ) is finite. First, assume that fin.dim(Γ) is finite. We prove that fin.dim(Λ) $\leq$ fin.dim(Γ) $+ t$. Let M be a Λ-module having finite projective dimension. Then $\Omega^t(M)$ has finite projective dimension over Λ. By Proposition 5.7, the projective dimension of $\Omega^t(M)$ over Λ is the same as the projective dimension of $\Omega^t(M)/(\Omega^t(M))^+$ over Γ. Thus the projective dimension of $\Omega^t(M)$ over Λ $\leq$ fin.dim(Γ) $+ t$ and (2) follows.

Part (3) follows from the proof of Corollary 5.4 and Proposition 5.8. □

6. An application

We recall the definition of path connectedness. A subquiver $\mathcal{L}$ of $\mathcal{Q}$ having vertex set X, is *path connected* if for each pair of vertices v and w in X, there is both a directed path from v to w in $\mathcal{L}$ and a directed path from w to v in $\mathcal{L}$.

Viewing a vertex as an oriented cycle at v with no arrows, called the *trivial cycle at* v, we have an equivalence relation on the vertices of $\mathcal{Q}$ given by setting two vertices v and w equivalent if there is an oriented cycle in $\mathcal{Q}$ in which both vertices occur. The equivalence classes of this relation are called the *path connected components of* $\mathcal{Q}$. We call a path connected component of $\mathcal{Q}$ *trivial* if it consists of one vertex v and the only cycle in $\mathcal{Q}$ on which v lies is the trivial cycle at v. A path connected component is called *nontrivial* if it contains at least one arrow.

In this section, we show that the finitistic dimension conjecture reduces to the case where the quiver is path connected. More precisely, we have the following result.

Theorem 6.1. *If the finitistic dimension of every algebra $K\mathcal{Q}/I$, with I admissible and $\mathcal{Q}$ path connected, is finite, then the finitistic dimension of every algebra $K\mathcal{Q}/I$ is finite, for all choices of $\mathcal{Q}$ and I admissible.*

We begin with a result about the finitistic dimension of triangular matrix rings, [8], Theorem 4.20, also see [20]. The Fossum-Griffith-Reiten theorem is more general and also provides bounds which we do not need.

Theorem 6.2 ([8]). *Let A and B be artin algebras and M a finitely generated A-B-bimodule. If the finitisitic dimensions of A and B are finite, then the finitistic dimension of the upper triangular matrix ring $\left(\begin{smallmatrix} A & M \\ 0 & B \end{smallmatrix}\right)$ is finite.*

Proof of Theorem 6.1. Suppose that the finitistic dimension of an algebra $K\mathcal{Q}/I$ is finite if I is admissible and $\mathcal{Q}$ is path connected. Now let $\mathcal{Q}$ be an arbitrary finite quiver, I be an admissible ideal in $k\mathcal{Q}$, and $\Lambda = K\mathcal{Q}/I$. We wish to show that the finitistic dimension of Λ is finite.

We proceed by induction on the number of nontrivial path connected components. If there are no nontrivial path connected components, then $\mathcal{Q}$ contains no cycles and Λ has finite global dimension. Hence. we are done.

Suppose there is only one nontrivial path connected component, $\mathcal{H}$. Then $\mathcal{H}$, being convex, is the homological heart of $\mathcal{Q}$. By Theorem 5.9, we may assume that $\mathcal{Q}$ equals the homological heart $\mathcal{H}$. Since we are assuming that $\mathcal{H}$ is path connected, the hypothesis implies that the finitistic dimension of Λ is finite.

Next, suppose that $\mathcal{Q}$ has at least k nontrivial path connected components with $k \geq 2$. By induction, we assume that if a quiver $\mathcal{Q}'$ has $k-1$ nontrivial path connected components and I' is an admissible ideal in $K\mathcal{Q}'$, then the finitistic dimension of $K\mathcal{Q}'/I'$ is finite. The homological heart of $\mathcal{Q}$, $\mathcal{H}$, has k nontrivial path connected components. By Theorem 5.9 we may assume that $\mathcal{Q} = \mathcal{H}$.

Consider the set A of arrows of $\mathcal{Q}$ that do not lie on a cycle in $\mathcal{Q}$. We form a new graph Δ whose vertex set is the set of the connected path components of $\mathcal{Q}$ (including the trivial components) and the arrow set is A where if $a \in A$ with $a\colon v \to w$ in $\mathcal{Q}$ then, as an arrow in Δ, a is an arrow from the path connected component of $\mathcal{Q}$ containing v to the path connected component of $\mathcal{Q}$ containing w. Since arrows in A cannot lie on an oriented cycle in $\mathcal{Q}$, Δ contains no cycles. Thus, some vertex in Δ is a source. It follows that there is the path connected component $\mathcal{L}$ in $\mathcal{Q}$ associated a source vertex in Δ such that $\mathcal{L}$ is a path connected subquiver of $\mathcal{Q}$ and $\mathcal{L}^-$ is empty.

We claim $\mathcal{L}$ is nontrivial. If $\mathcal{L}$ is trivial, then $\mathcal{L}_0 = \{v\}$ for some $v \in \mathcal{Q}_0$. Since v is not contained in a cycle, $v \in Y$ as defined in Section 5. Hence, there is a path from v to a nontrivial path connected component to v. But then $\mathcal{L}^-$ is not empty which is a contradiction.

Let $\mathcal{L}$ be a nontrivial path connected component of $\mathcal{Q}$ such that $\mathcal{L}^-$ is empty. Let $\mathcal{M}$ be the subquiver of $\mathcal{Q}$ with vertex set $\mathcal{Q}_0 \setminus \mathcal{L}_0$, let e be the idempotent associated to the sum of the vertices of $\mathcal{L}$, and $e' = 1 - e$. Note that $\mathcal{M}$ has $k-1$ path connected components. Then $\Lambda = \left(\begin{smallmatrix} e\Lambda e & e\Lambda e' \\ e'\Lambda e & e'\Lambda e' \end{smallmatrix}\right)$. Since $\mathcal{L}^-$ is empty, we see that $e'\Lambda e = 0$. The fact that $e'\Lambda e = 0$ implies that the quiver of $e'\Lambda e'$ is $\mathcal{M}$; that is, there is no path from a vertex in $\mathcal{M}$ to a vertex in $\mathcal{M}$ such that all the 'internal' vertices of the path are in $\mathcal{L}$.

Summarizing, we have $\Lambda = \left(\begin{smallmatrix} e\Lambda e & e\Lambda e' \\ 0 & e'\Lambda e' \end{smallmatrix}\right)$ such that the quiver of $e\Lambda e$ is $\mathcal{L}$ and the quiver of $e'\Lambda e'$ is $\mathcal{M}$. We have that $\mathcal{L}$ is path connected and hence, by assumption has finite finitistic dimension. We also have that $\mathcal{M}$ has $k-1$ path connected components and, by the induction hypothesis, has finite finitistic dimension. The result now follows from Theorem 6.2. $\square$

We end with a final application. We say a cycle is *simple* if it has no repeated vertices. We say a quiver $\mathcal{Q}$ is of *simple cycle type* if every nontrivial path connected component is a simple cycle.

PROPOSITION 6.3. *If $\mathcal{Q}$ is of simple cycle type, then the finitistic dimension of $K\mathcal{Q}/I$ is finite, for all admissible ideals I.*

Proof. We note that if C is a simple cycle then KC/L has finite finitistic dimension for any admissible ideal L since KC/L is a monomial algebra [10]. Thus, if $\mathcal{Q}$ is of simple cycle type, I is an admissible ideal in $K\mathcal{Q}$, and C is a nontrivial path connected component, then $KC/(KC \cap I)$ has finite finitistic dimension. The remainder of the proof is mirrors the proof of Theorem 6.1. $\square$

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