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A CHARACTERIZATION OF THE MACAULAY DUAL GENERATORS FOR QUADRATIC COMPLETE INTERSECTIONS

TADAHITO HARIMA, AKIHITO WACHI AND JUNZO WATANABE

Abstract. Let F be a homogeneous polynomial in n variables of degree d over a field K. Let $A(F)$ be the associated Artinian graded K-algebra. If $B \subset A(F)$ is a subalgebra of $A(F)$ which is Gorenstein with the same socle degree as $A(F)$, we describe the Macaulay dual generator for B in terms of F. Furthermore when $n = d$, we give necessary and sufficient conditions on the polynomial F for $A(F)$ to be a complete intersection.

1. Introduction

Let $R = K[x_1, \dots, x_n]$ be the polynomial ring in n variables over a field of characteristic zero and R_d the homogeneous space of degree d. For $F \in R_d$, let $A = A(F)$ be the graded Artinian Gorenstein algebra associated with F. So A has socle degree d and embedding dimension at most n. It is a long standing problem to characterize forms $F \in R_d$ for which the associated Artinian Gorenstein algebras $A(F)$ are complete intersections. If F is a monomial, then $A(F)$ is a monomial complete intersection. The only other known cases are a few sporadic examples (cf. [12], [5, Examples 2.82–2.85]) that occur as the algebra of co-invariants by pseudo reflection groups. It seems that there is a tendency among the experts to think there are no easily verifiable conditions which enable us to tell, for a given F, whether or not the algebra $A(F)$ is a complete intersection.

However, it is easy to see that if the degree of F is less than n, then $A(F)$ cannot be a complete intersection, since the socle degree of $A(F)$ is equals to the degree of the Jacobian of the generators. When $A(F) = R/I$ is a

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graded complete intersection with quadratic generators for the defining ideal $I = \mathrm{Ann}_R(F)$, then the degree of F is n. In Theorem 3.1 of this paper, we give necessary and sufficient conditions on a form F for $A(F) = R/I$ to be a complete intersection.

There is yet another result of this paper. We discuss the relation of Macaulay dual generators for two Artinian Gorenstein algebras A and B, with $A \supset B$, when the two algebras have the same socle degree. This was one of the topics discussed in the workshop at BIRS in March 2016, under the title "The Lefschetz Properties and Artinian algebras." There is a good reason to think that many complete intersections can be obtained as subrings of quadratic complete intersections (cf. [6], [10]). We will show that a Macaulay dual generator for B can be obtained from that of A by substituting the linear forms for the variables with duplications allowed. In Theorem 4.4, we show that the Gorenstein Artinian algebra A has a sub-quotient B of the same socle degree if and only if a Macaulay dual generator for B can be obtained from that of A by substituting the linear forms for the variables. This is independent of Theorem 3.1 and in this theorem the socle degree and embedding dimension of A are arbitrary.

When we speak about the Macaulay dual generator of a Gorenstein algebra, it is important to specify the structure of the inverse system or in the modern term, the injective hull of the residue field. The injective hull does not have a structure of a ring; it is possible, however, to regard it as the divided power algebra, induced by the natural structure of the Hopf algebra associated with the polynomial ring. In characteristic zero, the divided power algebra is the same as the polynomial ring. Throughout Sections 2–3, we assume, for simplicity, that the characteristic of the ground field is zero, and assume that the injective hull is the same as the polynomial ring itself but the action of the algebra is defined through differentiation. Nonetheless, all arguments are valid for a positive characteristic p provided that p is greater than the degree of F. Verification is left to the reader. For Macaulay's double annihilator theorem and his inverse system, we refer the reader to Meyer–Smith [11], Iarrobino–Kanev [8] and Geramita [2].

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2. Some necessary conditions for a homogeneous form to define a quadratic complete intersection

Throughout this section, K denotes a field of characteristic 0, and $R = K[x_1, x_2, \ldots, x_n]$ denotes the polynomial ring over K. We assume each variable has degree 1. We denote by R_d the homogeneous space of R of degree d. Thus we may write

$$R = \bigoplus_{d=0}^{\infty} R_d.$$

We regard R as an R-module via the operation "$\circ$" defined by

$$f(x_1, x_2, \dots, x_n) \circ F = f\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n}\right) F,$$

for $(f, F) \in R \times R$. With this operation R is the injective hull of the residue field in the category of finitely generated modules (see [5], Theorem 6.20). Thus if $f \in R_i$ and $F \in R_d$, then $f \circ F$ is an element of R_{d-i}. For $F \in R$, $\operatorname{Ann}_R(F)$ denotes

$$\operatorname{Ann}_R(F) = \{f \in R | f \circ F = 0\}.$$

It is the *annihilator* of F. $A(F)$ denotes the algebra $A(F) = R/\operatorname{Ann}_R(F)$. We will say that $A(F)$ is the *Gorenstein algebra defined by* F or simply $A(F)$ *is defined by* F. We will call the vector space $\operatorname{Ann}(F)_2 \subset R_2$ the *quadratic space defined by* F and denote it by $Q(F)$. Namely, the quadratic space $Q(F) = \operatorname{Ann}(F)_2 \subset R_2$ is the kernel of the homomorphism

$$f \in R_2 \mapsto f \circ F \in R_{d-2}.$$

Note that we have the exact sequence

$$0 \to Q(F) \to R_2 \to R_2/\operatorname{Ann}_R(F)_2 \to 0.$$

For a graded vector space $V = \bigoplus_{i=0}^{\infty} V_i$, we write

$$H_V(T) = \sum_{i=0}^{\infty} (\dim_K V_i) T^i$$

for the Hilbert series of V.

Definition 2.1. An Artinian algebra $A = R/I$ will be called a *quadratic complete intersection* if it is a complete intersection and the Hilbert series is $(1+T)^d$ for some d. (The ideal I may contain linear forms.)

Proposition 2.2. *Let $F \in R = K[x_1, x_2, \dots, x_n]$ be a homogeneous polynomial. Suppose that $H_{A(F)}(T) = (1+T)^n$. Then we have*

(1) $\deg F = n$.
(2) *No linear forms are contained in* $\operatorname{Ann}_R(F)$.
(3) *The partial derivatives* $\frac{\partial F}{\partial x_1}, \frac{\partial F}{\partial x_2}, \dots, \frac{\partial F}{\partial x_n}$ *are linearly independent.*
(4) *The quadratic space $Q(F)$ defined by F has dimension n.*
(5) $\dim_K(R_2 \circ F) = \dim_K(R_{n-2} \circ F) = \binom{n}{2}$.

Proof. (1) Recall that the homomorphism of R-modules

$$R \to R$$

defined by $f \mapsto f \circ F$ induces the degree reversing isomorphism of vector spaces

$$R/\operatorname{Ann}_R(F) = A(F) \to R \circ F \subset R,$$
$$\overline{f} \mapsto f \circ F,$$

where

$$\overline{f} = f \bmod \operatorname{Ann}_R(F).$$

This shows that if the algebra $A(F)$ has the Hilbert series $(1+T)^n$, then F has degree n.

(2) Note that $A(F)_1 = R_1/\operatorname{Ann}_R(F)_1$. Since $\dim_K A(F)_1 = n$, this shows that $\operatorname{Ann}_R(F)_1 = 0$. Hence $\operatorname{Ann}_R(F)$ contains no linear forms.

(3) Note that $A(F)_1 \cong R_1 \circ F$. Since $\dim_K R_1 \circ F = n$, this shows that the first partials of F are linearly independent.

(4) (5) Consider the exact sequence

$$0 \to Q(F) \to R_2 \to R_2/\operatorname{Ann}_R(F)_2 \to 0.$$

Note that $\dim R_2 = \binom{n+1}{2}$ and $\dim_K A(F)_2 = \binom{n}{2}$. The assertions follow from the isomorphisms $A(F)_2 = R_2/\operatorname{Ann}_R(F)_2 \cong R_2 \circ F \cong R_{n-2} \circ F$. □

3. A characterization of the Macaulay dual generator for quadratic complete intersections

Following is a characterization of $F \in R$ which defines a quadratic complete intersection.

THEOREM 3.1. *As before R denotes the polynomial ring in n variables over a field K of characteristic zero. Let $F \in R$ be a polynomial of degree n. Suppose that the partial derivatives $\frac{\partial F}{\partial x_1}, \dots, \frac{\partial F}{\partial x_n}$ are linearly independent. Then the Artinian Gorenstein algebra $A(F) = R/\operatorname{Ann}_R(F)$ is a quadratic complete intersection if and only if one of the following conditions is satisfied.*

(1) *The quadratic space $Q(F)$ is n-dimensional and generates $\operatorname{Ann}_R(F)$ as an ideal of R.*
(2) *The quadratic space $Q(F)$ contains a regular sequence of length n in R.*

Proof. Assume that $A(F)$ is a quadratic complete intersection. Then the ideal $\operatorname{Ann}_R(F)$ is generated by a regular sequence consisting of n homogeneous polynomials of degree two. Hence, we have both (1) and (2).

Conversely assume (2). Let I be the ideal generated by a regular sequence in $Q(F)$. Then we have a surjective map

$$R/I \to R/\operatorname{Ann}_R(F) = A(F).$$

Since R/I and $R/\operatorname{Ann}_R(F)$ are Gorenstein with the same socle degree, we have $A(F) = R/I$. (To see this, recall that an Artinian Gorenstein local ring has the smallest nonzero ideal.) Assume (1). Then a basis for $Q(F)$ is a regular sequence in R. Hence, $\operatorname{Ann}_R(F)$ is generated by a regular sequence consisting of quadrics. □

REMARK 3.2. Theorem 3.1 gives us an algorithm which determines whether or not the algebra $A(F) = R/\operatorname{Ann}_R(F)$ is a quadratic complete intersection for a given $F \in K[x_1, \ldots, x_n]$. The algorithm proceeds as follows.

(1) Let $F \in K[x_1, \ldots, x_n]_n$ be a homogeneous form of degree n.
(2) Check if the partials $\frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n}$ are linearly independent. If they are linearly dependent, F reduces to a polynomial of a smaller number of variables, but it has degree n. So in this case $A(F)$ is not a quadratic complete intersection. (It could be a complete intersection with a smaller embedding dimension than n.) If they are linearly independent, compute the second partials of F. If $\dim_K(R_2 \circ F) \neq \binom{n}{2}$ or equivalently $\dim_K Q(F) \neq n$, then $A(F)$ cannot be a quadratic complete intersection.
(3) If $\dim_K Q(F) = n$, let $\langle f_1, \ldots, f_n \rangle$ be a K-basis of $Q(F)$. Compute the rank of the vector space $V = R_{n-1}f_1 + R_{n-1}f_2 + \cdots + R_{n-1}f_n$. If $\dim_K V = \binom{2n}{n+1}$, then $\operatorname{Ann}_R(F)$ is a quadratic complete intersection; otherwise it is not. (Note that $V \subset R_{n+1}$ and $\dim_K R_{n+1} = \binom{2n}{n+1}$, and $V = R_{n+1}$ if and only if $f_1, f_2, \ldots, f_n$ is a complete intersection.)

REMARK 3.3. Suppose that $\dim Q(F) = n$ and $Q(F) = \langle f_1, f_2, \ldots, f_n \rangle$. It is easy to see that the following conditions are equivalent.

(1) $f_1, f_2, \ldots, f_n$ is a regular sequence.
(2) $R_{n-1}f_1 + R_{n-1}f_2 + \cdots + R_{n-1}f_n = R_{n+1}$.
(3) The initial ideal of $(f_1, f_2, \ldots, f_n)$ contains all high powers of the variables.
(4) The resultant of $f_1, f_2, \ldots, f_n$ does not vanish. (For the theory of resultants see [1]. There is a related result in [7].)

REMARK 3.4. Tony Iarrobino pointed that Theorem 3.1 can be generalized to any homogeneous polynomial $F \in R_{n(d-1)}$ to define a complete intersection with generators of any uniform degree d. We confined ourselves to the quadratic case ($d = 2$), since the generalization is straightforward, and since we had in mind the results of [6] and [7].

EXAMPLE 3.5. Let $R = K[v, w, x, y, z]$.

(1) $F = vwxyz + wxyz^2$. It is easy to see that $\operatorname{Ann}_R(F)$ contains 5 quadratic relations. In fact, $(v^2, w^2, x^2, y^2, z^2 - 2vz) \subset \operatorname{Ann}_R(F)$. So $\operatorname{Ann}_R(F)$ is a complete intersection.
(2) $F = vwxyz + xyz^3$. $\operatorname{Ann}_R(F) = (v^2, w^2, x^2, y^2, z^2 - 6vw)$. Similarly to the previous example, this is a complete intersection.
(3) $F = vwxyz + yz^4$. It is easy to see that we have the relations
$$\left(\frac{\partial}{\partial v}\right)^2 F = \left(\frac{\partial}{\partial w}\right)^2 F = \left(\frac{\partial}{\partial x}\right)^2 F = \left(\frac{\partial}{\partial y}\right)^2 F = 0.$$
With a little contemplation we see that no more quadratic relations are possible. So this is not a complete intersection. In fact, we can compute $\operatorname{Ann}_R(F) = (v^2, w^2, x^2, y^2, wz^2, vz^2, xz^2, z^3 - 24vwx)$.

PROBLEM 3.6. For what binomial $F \in K[x_1, x_2, \dots, x_n]_n$ is $\mathrm{Ann}_R(F)$ a complete intersection? (Define F to be a binomial by $F = \alpha M + \beta N$, where M, N are power products of variables and $\alpha, \beta \in K$.)

REMARK 3.7. The vector space R_n may be regarded as the parameter variety for the Gorenstein algebras of socle degree n with embedding dimension at most n. Thus, the projective space $\mathbb{P}^N$ where $N = \binom{2n-1}{n} - 1$ is the paremeter space for such Gorenstein algebras. By the Double Annihilator Theorem of Macaulay, each orbit of the general linear group $GL(n)$ contains precisely one isomorphism type of Gorenstein algebras. (See [11].) Thus, the dimension of the parameter space for the isomorphism types of Gorenstein algebras with socle degree n and embedding dimension at most n is

$$\left(\binom{2n-1}{n} - 1\right) - \left(n^2 - 1\right).$$

(We roughly estimated that each orbit is $(n^2 - 1)$-dimensional.)

On the other hand, the set of quadratic complete intersections may be parametrized by the n-dimensional subspaces in R_2. Thus, the dimension of the parameter space is $n \times \binom{n}{2}$. The linear transformation of the variables gives us an isomorphism of such complete intersections. Hence, the dimension of the parameter space for the isomorphism types is

$$n \times \binom{n}{2} - \left(n^2 - 1\right).$$

Here is a list of these dimensions for small values of n.

n	2	3	4	5	6
$\binom{2n-1}{n} - n^2$	–	1	19	101	426
$n\binom{n}{2} - n^2 + 1$	–	1	9	26	55

REMARK 3.8. Suppose $F \in R_n$ defines a quadratic binomial complete intersection. We may assume that the square free monomials are linearly independent in $A(F)$. (For this fact see [7].) Let $B_k \subset R_k$ be the set of square free monomials of degree k and define the $\binom{n}{k} \times \binom{n}{k}$ matrix $H_k(F)$ as follows:

$$H_k(F) = \big((\alpha\beta) \circ F\big)_{(\alpha,\beta) \in B_k \times B_k}.$$

The rows and columns of $H_k(F)$ are indexed by B_k. In [9], the authors call the determinant of $H_k(F)$ the higher Hessian of order k of F (with respect to the basis B_k). By [13], Theorem 4, the algebra $A(F)$ has the strong Lefschetz property if and only if

$$\det H_k(F) \neq 0, \quad \text{for all } k = 1, 2, \dots, [n/2].$$

We conjecture that $\det H_k(F)$ does not vanish for $F \in R_n$ for all k, if F defines a quadratic complete intersection. This is a part of a larger conjecture which claims that all complete intersections over a field of characteristic zero have the strong Lefschetz property. For more detail, see [5], Conjecture 3.46 and Theorem 3.76.

In the next example, we show that there exists a Gorenstein algebra with the same Hilbert series as a quadratic complete intersection, but fails the SLP.

EXAMPLE 3.9 (*R. Gondim*). Consider the polynomial

$$F = v^3wx + vw^3y + y^2z^3$$

of degree 5 in 5 variables. With an aid of a computer algebra system one sees that $A(F)$ has the Hilbert series $(1+T)^5$, but $A(F)$ is not a complete intersection. It is not difficult to see that the 2nd Hessian of F with respect to certain bases for A_2 and A_3 is identically zero, so the algebra $A(F)$ fails the SLP. The set of square free monomials of degree 2 is linearly dependent in $A(F)_2$, but this is not essential to the failure of the SLP. In fact if F is expressed in generic variables, the sets of square-free monomials can be bases for A_2 and A_3. This example was constructed by Gondim [3]. (See [3], Theorem 2.3 and the paragraph preceding it.)

4. The Macaulay dual generator for a subring of a Gorenstein algebra

In this section, we consider Artinian algebras A over a field K with the assumption characteristic K is zero or greater than the socle degree of A. The socle degree and the embedding dimension of A are arbitrary.

THEOREM 4.1. *Suppose that $A = \bigoplus_{i=0}^{d} A_i$ is a standard graded Artinian Gorenstein algebra with $A_d \neq 0$. Assume that $A_0 = K$ is a field of characteristic $p = 0$ or $p > d$. Let $\overline{x_1}, \overline{x_2}, \ldots, \overline{x_n} \in A_1$ be the images of the variables of the polynomial ring and let $\xi_1, \xi_2, \ldots, \xi_n \in K$. Fix a nonzero socle element $s \in A_d$. Define the map*

$$\Phi : K^n \to K$$

which sends $(\xi_1, \xi_2, \ldots, \xi_n)$ to c, where c is defined by

$$(\xi_1\overline{x_1} + \xi_2\overline{x_2} + \cdots + \xi_n\overline{x_n})^d = cs.$$

Since c is a function of $\xi_1, \xi_2, \ldots, \xi_n$, we may write $c = \Phi(\xi_1, \xi_2, \ldots, \xi_n)$. The map Φ is a polynomial map and $\Phi(\xi_1, \xi_2, \ldots, \xi_n)$, as a polynomial, is a Macaulay dual generator for the Gorenstein algebra A.

An extremely elegant proof is given in [5], Lemma 3.74. However, the reader has to be very alert to understand the proof, since it is written in a style which most commutative algebraists are unfamiliar with. Here we give another proof.

Proof. Let $R=K[x_1,x_2,\dots,x_n]$ be the polynomial ring and $F\in R_d$ a Macaulay dual generator for A. Then we have the isomorphism $R/\operatorname{Ann}_R(F)\cong A$ defined by $x_i\mapsto\overline{x_i}$. Let $S=S(x_1,\dots,x_n)\in R_d$ be a pre-image of a nonzero socle element $s\in A_d$. Put $\alpha=S\circ F\in K$. Since $S\notin\operatorname{Ann}_R(F)$, we have $\alpha\neq 0$. Given $(\xi_1,\xi_2,\dots,\xi_n)\in K^n$, we want to find $c=c(\xi_1,\dots,\xi_n)\in K$ which satisfies

$$cs=(\xi_1\overline{x_1}+\xi_2\overline{x_2}+\cdots+\xi_n\overline{x_n})^d.$$

Such c should satisfy $(\xi_1x_1+\xi_2x_2+\cdots+\xi_nx_n)^d-cS\in\operatorname{Ann}_R(F)$. Thus, we should have

$$\big((\xi_1x_1+\xi_2x_2+\cdots+\xi_nx_n)^d-cS\big)\circ F=0.$$

We compute the left-hand side as follows:

$$\begin{aligned}&\big((\xi_1x_1+\xi_2x_2+\cdots+\xi_nx_n)^d-cS\big)\circ F\\ &\quad=(\xi_1x_1+\xi_2x_2+\cdots+\xi_nx_n)^d\circ F-cS\circ F\\ &\quad=d!F(\xi_1,\dots,\xi_n)-c\alpha.\end{aligned}$$

We used Lemma 4.2 which we prove below for the last equality. It turned out that

$$c=\frac{d!}{\alpha}F(\xi_1,\dots,\xi_n).$$

Hence, we have

$$\Phi(x_1,x_2,\dots,x_n)=\frac{d!}{\alpha}F(x_1,x_2,\dots,x_n).\qquad\square$$

LEMMA 4.2. *Let $R=K[x_1,x_2,\dots,x_n]$ be the polynomial ring and let*

$$\xi_1,\xi_2,\dots,\xi_n\in K$$

be any elements in K. Put

$$D=\xi_1\frac{\partial}{\partial x_1}+\xi_2\frac{\partial}{\partial x_2}+\cdots+\xi_n\frac{\partial}{\partial x_n}.$$

Then for any homogeneous polynomial $F\in R_d$ of degree d, we have

$$D^dF=d!F(\xi_1,\xi_2,\dots,\xi_n).$$

Proof. Let $F=\sum_{d_1+\cdots+d_n=d}a_{(d_1,\dots,d_n)}x_1^{d_1}\cdots x_n^{d_n}$. Then:

$$\begin{aligned}D^dF&=\left(\xi_1\frac{\partial}{\partial x_1}+\cdots+\xi_n\frac{\partial}{\partial x_n}\right)^dF\\ &=\left(\sum_{d_1+\cdots+d_n=d}\frac{d!}{d_1!\cdots d_n!}\xi_1^{d_1}\left(\frac{\partial}{\partial x_1}\right)^{d_1}\cdots\xi_n^{d_n}\left(\frac{\partial}{\partial x_n}\right)^{d_n}\right)F\\ &=\sum_{d_1+\cdots+d_n=d}\frac{d!}{d_1!\cdots d_n!}a_{(d_1,\dots,d_n)}\xi_1^{d_1}\cdots\xi_n^{d_n}d_1!\cdots d_n!\\ &=d!F(\xi_1,\dots,\xi_n).\end{aligned}\qquad\square$$

DEFINITION 4.3. Suppose that $F = F(x_1, \ldots, x_n)$ is a polynomial in $x_1, \ldots, x_n$ and $G = G(y_1, \ldots, y_m)$ is a polynomial in $y_1, \ldots, y_m$. (Assume that the sets $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_m\}$ are independent sets of variables.) We will say that G is obtained from F *by substitution by linear forms*, if there exists a full rank $m \times n$ matrix $M = (m_{ij})$ such that

$$F(x_1, x_2, \ldots, x_n) = G(y_1, y_2, \ldots, y_m),$$

if we make a substitution:

$$\begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix} = \begin{pmatrix} y_1 & y_2 & \cdots & y_m \end{pmatrix} M.$$

THEOREM 4.4. *Suppose that $A = \bigoplus_{i=0}^{d} A_i$ is a standard graded Artinian Gorenstein algebra over $A_0 = K$, a field, with socle degree d. Assume that the characteristic of K is zero or greater than d. Suppose that $B = \bigoplus_{i=0}^{d} B_i$ is a Gorenstein subalgebra of $A = \bigoplus_{i=0}^{d} A_i$ with the same socle degree. (We assume that $B = K[B_1]$, $A = K[A_1]$, and $B_1 \subset A_1$.) Then a Macaulay dual generator for B is obtained from that of A by substitution by linear forms.*

Proof. We have shown that $\Phi = \Phi(\xi_1, \ldots, \xi_n)$ defined in Theorem 11 is a Macaulay dual generator for A. Likewise we let $\Phi' = \Phi'(\eta_1, \ldots, \eta_m)$ be a Macaulay dual generator for B. Let s be a nonzero socle element of A. We may assume that $A_d = B_d = \langle s \rangle$. Let $M = (m_{ij})$ be the $m \times n$ matrix which satisfies

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix} = (m_{ij}) \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix},$$

where $\langle x_1, x_2, \ldots, x_n \rangle$ is a basis for A_1 and $\langle y_1, y_2, \ldots, y_m \rangle$ for B_1. Then, since

$$\begin{aligned} (\eta_1 y_1 + \eta_2 y_2 + \cdots + \eta_m y_m)^d &= \left(\sum_{i=1}^{m} \eta_i \left(\sum_{j=1}^{n} m_{ij} x_j \right) \right)^d \\ &= \left(\sum_{j=1}^{n} \left(\sum_{i=1}^{m} \eta_i m_{ij} \right) x_j \right)^d, \end{aligned}$$

we have

$$\Phi'(\eta_1, \ldots, \eta_m) = \Phi\left(\sum_{i=1}^{m} \eta_i m_{i1}, \ldots, \sum_{i=1}^{m} \eta_i m_{in} \right).$$

This shows that Φ' is obtained from Φ by linear substitution of the variables with the matrix (m_{ij}):

$$\begin{pmatrix} \xi_1 & \xi_2 & \cdots & \xi_n \end{pmatrix} = \begin{pmatrix} \eta_1 & \eta_2 & \cdots & \eta_m \end{pmatrix} (m_{ij}). \qquad \square$$

EXAMPLE 4.5. Let R be the polynomial ring in n variables. Put S_n be the symmetric group acting on R by permutation of the variables. Let $I = (f_1, f_2, \dots, f_n)$ be a quadratic complete intersection such that

$$\mathrm{Ann}_R(F) = (f_1, \dots, f_n).$$

So F is a Macaulay dual generator of $R/(f_1, \dots, f_n)$. Suppose that $F^\sigma = F$ for all $\sigma \in S_n$. Let $\mathfrak{G}$ be a Young subgroup of S_n such that

$$\mathfrak{G} = S_{n_1} \times S_{n_2} \times \cdots \times S_{n_r},$$
$$n_1 + n_2 + \cdots + n_r = n.$$

Then the ring of invariants $B = A^{\mathfrak{G}} \subset A$ is a complete intersection and in many cases $A^{\mathfrak{G}}$ is generated by linear forms (see [6]). When this is the case, the generators can be chosen as follows:

$$\begin{aligned}
y_1 &= x_1 + x_2 + \cdots + x_{n_1}, \\
y_2 &= \underbrace{x_{n_1+1} + x_{n_1+2} + \cdots + x_{n_1+n_2}}_{n_2}, \\
&\ \vdots \\
y_r &= \underbrace{x_{n_1+\cdots+n_{r-1}+1} + x_{n_1+\cdots+n_{r-1}+2} + \cdots + x_n}_{n_r}.
\end{aligned}$$

Then a Macaulay dual generator G for B is obtained as follows:

$$G = F(\underbrace{y_1, \dots, y_1}_{n_1}, \underbrace{y_2, \dots, y_2}_{n_2}, \dots, \underbrace{y_r, \dots, y_r}_{n_r}).$$

EXAMPLE 4.6. Let $R = K[u, v, w, x, y, z]$ be the polynomial ring in 6 variables. Put

$$\begin{aligned}
f_1 &= u^2 - 2u(v + w + x + y + z), \\
f_2 &= v^2 - 2v(u + w + x + y + z), \\
f_3 &= w^2 - 2w(u + v + x + y + z), \\
f_4 &= x^2 - 2x(u + v + w + y + z), \\
f_5 &= y^2 - 2y(u + v + w + x + z), \\
f_6 &= z^2 - 2z(u + v + w + x + y).
\end{aligned}$$

Let $A = R/(f_1, \dots, f_6)$. Then A is an Artinian complete intersection. A Macaulay dual generator is given as follows:

$$\begin{aligned}
F = 80m_6 &+ 48m_{51} + 120m_{42} - 30m_{411} + 160m_{33} - 60m_{321} \\
&+ 60m_{3111} - 90m_{222} + 90m_{2211} - 225m_{21111} + 1575m_{111111},
\end{aligned}$$

where m_{ijk} etc. denotes the monomial symmetric polynomial. For example,

$$\begin{aligned}
m_6 &= u^6 + v^6 + w^6 + x^6 + y^6 + z^6,\\
m_{51} &= u^5 v + uv^5 + \cdots + wz^5 + xz^5 + yz^5,\\
m_{411} &= u^4 vw + \cdots + xyz^4,\\
&\vdots\\
m_{111111} &= uvwxuz.
\end{aligned}$$

The polynomial F was obtained by solving a system of linear equations in 462 variables by Mathematica. (462 is the dimension of $K[u,\dots,z]_6$.) The polynomial F looks complicated, but it has the striking property that any substitution of variables by another set of variables defines a complete intersection. For example,

$$F(p,p,q,q,r,s) \in K[p,q,r,s]_6$$

is a complete intersection in 4 variables. This corresponds to the ring $A^{\mathfrak{G}}$ of invariants by the Young subgroup

$$\mathfrak{G} := S_2 \times S_2 \times S_1 \times S_1.$$

5. Subalgebras of Gorenstein algebras generated by linear forms

THEOREM 5.1. *Let $A = \bigoplus_{i=0}^{d} A_i$ be an Artinian Gorenstein algebra. Let B be a subring of A generated by a subspace of A_1 such that $B_d = A_d \neq 0$. Then we have the following*:

(1) *There exists an irreducible ideal $\mathfrak{b}$ of B such that $B/\mathfrak{b}$ is a Gorenstein algebra with the socle degree d.*

(2) *A Macaulay dual generator of $B' = B/\mathfrak{b}$ is obtained from that of A by a substitution by linear forms.*

Proof. (1) Let

$$\mathfrak{b}_1 \cap \mathfrak{b}_2 \cap \cdots \cap \mathfrak{b}_r = 0$$

be an irredundant decomposition of 0 in B by irreducible ideals. Then we have the injection:

$$0 \to B \to \bigoplus_{i=1}^{r} B/\mathfrak{b}_i.$$

Note that there exists an i, say $i = 1$, such that $B' = B/\mathfrak{b}_1$ has the same socle as B.

(2) Let $\eta_1, \eta_2, \dots, \eta_m$ be elements of K. The evaluation of the map $(\eta_1 y_1 + \eta_2 y_2 + \cdots + \eta_m y_m)^d$ at B_d or B'_d are the same. Hence, the assertion follows in the same way as Theorem 4.4. □

EXAMPLE 5.2. Let K be a field of characteristic zero or greater than three. We consider $A = K[x,y,z]/(x^2 - 6yz, y^2 - 6xz, z^2 - 6xy)$. Let $S = K[r,s]$, define $\psi : S \to A$ by $r \mapsto x$, $s \mapsto y + az$. Then we have

$$\ker\psi = \bigl(3(a^3+1)rs^2 - as^3, r^2s, (a^3+1)r^3 - s^3\bigr),$$

provided that $a \neq \pm 1$.

On the other hand a Macaulay dual generator for A is $F = F(x,y,z) = x^3 + y^3 + z^3 + xyz$. Let $G = F(r,s,as)$. That is, the polynomial G is obtained from F by the substitution by the linear forms:

$$(x,y,z) = (r,s)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & a \end{pmatrix}.$$

A primary decomposition of the ideal I is given by $I = \mathrm{Ann}_S(G) \cap (r+s, s^3)$, where

$$\mathrm{Ann}_S(G) = \bigl(a^2r^2 + 9(a^3+1)rs - 3as^2, 3(a^3+1)rs^2 - as^3\bigr).$$

If $a = 1$, then $\ker\psi$ is a complete intersection:

$$\ker\psi = \bigl(r^2 + 18rs - 3s^2, 6rs^2 - s^3\bigr).$$

The case $a = 0$ works as well as other general cases. The computation was done with the computer algebra system Macaulay2 [4].

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Tadahito Harima, Department of Mathematics Education, Niigata University, Niigata, 950-2181, Japan

E-mail address: harima@ed.niigata-u.ac.jp

Akihito Wachi, Department of Mathematics, Hokkaido University of Education, Kushiro, 085-8580, Japan

E-mail address: wachi.akihito@k.hokkyodai.ac.jp

Junzo Watanabe, Department of Mathematics, Tokai University, Hiratsuka, 259-1201, Japan

E-mail address: watanabe.junzo@tokai-u.jp