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ON THE KREIN–MILMAN–KY FAN THEOREM FOR CONVEX COMPACT METRIZABLE SETS

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ABSTRACT. We extend the extension by Ky Fan of the Krein–Milman theorem. The Φ-extreme points of a Φ-convex compact metrizable space are replaced by the Φ-exposed points in the statement of Ky Fan theorem. Our main results are based on new variational principles which are of independent interest. Several applications will be given.

1. Introduction

Let S be any nonempty set, Φ a family of real valued functions on S. A subset $X \subset S$ is said to be Φ-convex if $X = S$ or there exists a nonempty set I, such that

$$X = \bigcap_{i \in I} \{x \in S : \varphi_i(x) \leq \lambda_i\},$$

where $\varphi_i \in \Phi$ and $\lambda_i \in \mathbb{R}$ for all $i \in I$. For a nonempty set $A \subset S$, the intersection of all Φ-convex subset of S containing A is said to be the Φ-convex hull of A. By $\mathrm{conv}_\Phi(A)$, we denote the Φ-convex hull of A.

Let $a, x, y \in S$, we say that a is Φ-between x and y, if

$$\big(\varphi \in \Phi, \varphi(x) \leq \varphi(a), \varphi(y) \leq \varphi(a)\big) \quad \Longrightarrow \quad \big(\varphi(a) = \varphi(x) = \varphi(y)\big).$$

Let $\emptyset \neq A \subset B \subset S$. The set A is said to be Φ-extremal subset of B, if

$$(a \in A, a \text{ is } \Phi\text{-between the points } x, y \in B) \quad \Longrightarrow \quad (x \in A, y \in A).$$

If A is a singleton $A = \{a\}$, we say that a is Φ-extremal point of B. The set of all Φ-extremal points of a nonempty set A will be denoted by $\Phi\,\mathrm{Ext}(A)$.

When S is a Hausdorff locally convex topological vector space (in short l.c.t. space, *"Hausdorff"* will be implicit), and $\Phi = S^*$ is the topological dual of S, then the Φ-convexity and the classical convexity coincides for closed subsets

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of S and the Φ-convex hull of a set coincides with its closed convex hull. Also, the Φ-extremal points of a nonempty set, coincides with its extreme points [14, Proposition 2]. (Note that in [14], the author use a class Ψ of functions in the definitions which correspond to the class $-\Phi$ in the above definitions. The class Φ considered in this paper will be a Banach space of real valued functions.) Recall that if C is a subset of S, we say that a point $x \in C$ is an extreme point of C, and write $x \in \operatorname{Ext}(C)$, if and only if: $y, z \in C$, $0 < \alpha < 1$; $x = \alpha y + (1-\alpha)z \Longrightarrow x = y = z$.

The result in what is known as the Krein–Milman theorem [16], asserts that if K is a convex compact subset of an l.c.t. space, then K is the closed convex hull of its extreme points,

$$K = \overline{\operatorname{conv}}\big(\operatorname{Ext}(K)\big).$$

The Krein–Milman theorem has a partial converse known as Milman's theorem (see [20]) which states that if A is a subset of K and the closed convex hull of A is all of K, then every extreme point of K belongs to the closure of A,

$$\big(A \subset K; K = \overline{\operatorname{conv}}(A)\big) \quad \Longrightarrow \quad \operatorname{Ext}(K) \subset \overline{A}.$$

In [12], Ky Fan extended the Krein–Milman theorem to the more general framework of Φ-convexity. Recall that a class Φ of functions on K, is said to separate the points of K if to any two distinct points of K there exists a function $\varphi \in \Phi$ which takes distinct values at the given points.

Theorem 1 (Krein–Millman–Ky Fan). *Let S be a Hausdorff space and Φ a family of real valued functions defined on S. Let K be a nonempty compact Φ-convex subset of S and suppose that*:

1. *the restriction of each $\varphi \in \Phi$ to K, is lower semicontinuous on K*;
2. *Φ separates the points of K.*

Then, $\Phi \operatorname{Ext}(K) \neq \emptyset$ *and* $K = \operatorname{conv}_\Phi(\Phi \operatorname{Ext}(K))$.

The main results. Recall that when S is an l.c.t. space and C is a subset of S, we say that a point $x \in C$ is an exposed point of C, and write $x \in \operatorname{Ext}(C)$, if there exists some continuous linear functional $x^* \in S^*$ which attains its strict maximum over C at x. Such a functional is then said to expose C at x. Thus, an exposed point is a special sort of extreme point. If S is a dual space, a weak* exposed point x (we write $x \in w^* \operatorname{Ext}(C)$) is to simply an exposed point by a continuous functional from the predual. We introduce a general concept of Φ-exposed points that coincides with the classical exposed points when S is an l.c.t. space and $\Phi = S^*$ and coincides with the weak* exposed points where $S = E^*$ is a dual space and $\Phi = E$.

Definition 1. Let S be a Hausdorff space, C a subset of S and Φ a family of real valued functions defined on S. We say that a point x of C is Φ-exposed in C, and write $x \in \Phi \operatorname{Ext}(C)$, if there exists $\varphi \in \Phi$ such that φ has a strict maximum on C at x i.e. $\varphi(x) > \varphi(y)$ for all $y \in C \setminus \{x\}$ (when C has at least two distinct points). Such φ is then said to Φ-expose C at x.

All the subsets considered in this article are assumed having at least two distinct points. The case of sets having only one point is trivial.

It is easy to see that $\Phi\,\mathrm{Ext}(C) \subset \Phi\,\mathrm{Ext}(C)$ but the converse is not true in general (see examples in Section A.1). The first main result (Theorem 2 below) states that, for compact metrizable sets we can replace the Φ-extremal points in the Krein–Millman–Ky Fan theorem, by the Φ-exposed points.

THEOREM 2. *Let S be a Hausdorff space and $(\Phi, \|\cdot\|_{\Phi})$ be a Banach space of real valued functions defined on S. Let K be a nonempty compact metrizable Φ-convex subset of S and suppose that*:

(1) *the restriction of each $\varphi \in \Phi$ to K is continuous*, *and there exists some real number $\alpha_K > 0$ such that $\alpha_K \|\varphi\|_{\Phi} \geq \sup_{x\in K} |\varphi(x)|$ for all $\varphi \in \Phi$*;

(2) *Φ separates the points of K.*

Then, *we have that*

(i) *$\Phi\,\mathrm{Ext}(K) \neq \emptyset$ and the set of all $\varphi \in \Phi$ that Φ-expose K at some point*, *contains a dense G_δ subset of $(\Phi, \|\cdot\|_{\Phi})$*;

(ii) *$K = \mathrm{conv}_{\Phi}(\Phi\,\mathrm{Ext}(K))$.*

Note that Theorem 2 is not true in general for compact subsets that are not metrizable. For example, when $(E, \|\cdot\|) = (l^1(\Gamma), \|\cdot\|_1)$ (Γ uncountable set), $S = (E^*, \text{weak}^*)$ and $(\Phi, \|\cdot\|_{\Phi}) = (E, \|\cdot\|)$, then, all of the hypothesis of Theorem 2 are satisfied. However, for the not metrizable weak* compact subset $K = B_{E^*}$ (the dual unit ball), we have that $\Phi\,\mathrm{Ext}(K) = w^*\,\mathrm{Ext}(K) = \emptyset$ (see Remark 6). Note also that the above theorem fails in general if Φ is not a Banach space satisfying the condition (1). Indeed, in the locally convex separable metrizable space $S = \mathbb{R}^{\aleph_0}$, where $\Phi = S^*$ is its topological dual, the cube $K = [-1, 1]^{\aleph_0}$ has no exposed points (see [15]).

We give below some examples in the linear framework, where the above theorem can be applied (see the corollaries of Section 3.1).

EXAMPLE 1. Let $(E, \|\cdot\|)$ be a Banach space, $S = (E^*, \text{weak}^*)$ and $(\Phi, \|\cdot\|_{\Phi}) = (E, \|\cdot\|)$. In this case, a subset of S is Φ-convex iff it is convex weak* closed and the Φ-convex hull of a set, coincides with its weak* closed convex hull. The Φ-exposed points coincides with the weak* exposed points.

EXAMPLE 2. Let $(E, \|\cdot\|)$ be a Banach space, $S = (E, \text{weak})$ and $(\Phi, \|\cdot\|_{\Phi}) = (E^*, \|\cdot\|)$ the topological dual of $(E, \|\cdot\|)$. In this case, a subset of S is Φ-convex iff it is convex and weak closed iff it is convex and norm closed (by Mazur's lemma on the coincidence of weak and norm closure of convex sets). The Φ-convex hull of a set coincides with its weak closed convex hull which also coincides with its norm closed convex hull (by Mazur's lemma). The Φ-exposed points coincides with the classical exposed points.

The second main result of this paper is the following theorem. The space $(C(K), \|\cdot\|_\infty)$ denotes the Banach space of all real valued continuous functions on a compact space K equipped with the sup-norm. Let $(\Phi, \|\cdot\|_\infty)$ be a Banach subspace of $(C(K), \|\cdot\|_\infty)$. By B_{Φ^*}, we denote the dual unit ball of $(\Phi, \|\cdot\|_\infty)$. We also use the following notation:

$$\pm\delta\big(\Phi\,\mathrm{Ext}(K)\big) := \big\{\pm\delta_k / k \in \Phi\,\mathrm{Ext}(K)\big\},$$

where, for each $k \in \Phi\,\mathrm{Ext}(K)$, $\delta_k : \varphi \mapsto \varphi(k)$ for all $\varphi \in \Phi$.

THEOREM 3. *Let K be a compact metric space and $(\Phi, \|\cdot\|_\infty)$ be a Banach subspace of $(C(K), \|\cdot\|_\infty)$ which separates the points of K and contains the constants. Then, we have*

$$w^*\,\mathrm{Ext}(B_{\Phi^*}) = \pm\delta\big(\Phi\,\mathrm{Ext}(K)\big)$$

and

$$B_{\Phi^*} = \overline{\mathrm{conv}}^{\mathrm{w}^*}\Big(\pm\delta\big(\Phi\,\mathrm{Ext}(K)\big)\Big).$$

Note that in Theorem 3, the concept of Φ-exposed points of K appears in a natural way in the description of the weak* exposed points of the dual unit ball B_{Φ^*}. As a consequence, we deduce, under the hypothesis of Theorem 3, that the set $\Phi\,\mathrm{Ext}(K)$ of all Φ-exposed points of K, is a dense subset of the Shilov boundary $\partial\Phi$ of Φ that is, $\partial\Phi = \overline{\Phi\,\mathrm{Ext}(K)}$ (Corollary 7).

The main tool used in this paper for proving our main results, is based on the following version of variational principle in the compact metric framework (Lemma 3 in Section 2). This analogous to the Deville–Godefroy–Zizler variational principle [10], also gives a new information about the set of *"ill-posed problems"* on compact metric sets. It is shown in particular, that the σ-porosity of the set of *"ill-posed problems"* in [11], is not optimal (see comments after the following lemma). Several others consequences are obtained (the details are given in Section 2). The notion of delta-convex (we shall abbreviate d.c.) hypersurfaces will be defined later.

A KEY LEMMA. *Let K be a compact metric space and $(Y, \|\cdot\|_Y)$ be a Banach space included in $C(K)$ which separates the points of K and such that $\alpha\|\cdot\|_Y \geq \|\cdot\|_\infty$ for some real number $\alpha > 0$. Let $f : K \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous function. Then, the set*

$$N(f) = \{\varphi \in Y : f - \varphi \text{ does not have a strict minimum on } K\}$$

is of the first Baire category in Y. If moreover Y is separable then $N(f)$ can be covered by countably many d.c. hypersurfaces *in Y.*

In a separable Banach space Y, each set N which can be covered by countably many *d.c. hypersurfaces* is σ-lower porous, also σ-directionally porous; in particular it is both Aronszajn (equivalent to Gauss) null and Γ-null. For more details about "small sets", see [24] and references therein.

This paper is organized as follows. In Section 2, we prove the key lemma (Lemma 3) and give several consequences. In Section 3, we give the proofs of the main results (Theorem 2 and Theorem 3) and applications. In Section 3.2, we give some additional properties and remarks about Φ-exposed points and the Krein–Milman theorem. We introduce and start the study of a new class of Hausdorff locally convex topological vector space based on a new concept of remarkable points of a convex compact set, called "affine exposed points".

2. Variational principle and consequences

Let (M,d) be a complete metric space and $f: M \longrightarrow \mathbb{R}\cup\{+\infty\}$ be an extended real-valued function which is bounded from below and proper. By the term proper, we mean that the domain of f, $\mathrm{dom}(f) := \{x \in M / f(x) < +\infty\}$ is non-empty. We say that f has a strong minimum at x if $\inf_M f = f(x)$ and $d(x_n, x) \to 0$ whenever $f(x_n) \to f(x)$. The problem to find a strong minimum for f, is called *Tykhonov well-posed-problem.* Let $(C_b(M), \|\cdot\|_\infty)$ be the space of all real-valued bounded and continuous functions on M, equipped with the sup-norm and let $(Y, \|\cdot\|_Y)$ be a Banach space included in $C_b(M)$. Let

$$N(f) = \{\varphi \in Y : f - \varphi \text{ does not have a strong minimum on } M\}.$$

The set $N(f)$ is called the set of *"ill-posed problems"*. The problem is to find conditions on Y under which the set $N(f)$ is a *"small"* set. In [10], Deville, Godefroy and Zizler proved that the set $N(f)$ is of first Baire category in Y, and in [11], Deville and Rivalski generalize the result of Deville–Godefroy–Zizler (D–G–Z), where they showed that the set $N(f)$ is σ-porous in Y, whenever f is bounded from below, proper and lower semicontinuous and Y satisfies the following conditions:

(i) $\|g\| \geq \|g\|_\infty$, for all $g \in Y$;

(ii) for every natural number n, there exists a positive constant M_n such that for any point $x \in M$ there exists a function $h_n : M \longrightarrow [0;1]$, such that $h_n \in Y$, $\|h_n\| \leq M_n$, $h_n(x) = 1$ and $\mathrm{diam}(\mathrm{supp}(h)) < \frac{1}{n}$.

The D–G–Z variational principle has several applications in particular in optimization and in the geometry of Banach spaces and can be applied without compactness assumption. However, the assumption (ii) is crucial and so the D–G–Z variational principle cannot include the linear case, like the Stegall's variational principle. Of course, the interest in the D–G–Z variational principle, is to avoid compactness, but in our purpose (in connection with the Krein–Milman theorem), we need only to treat the compact framework. Thus, we prove in Lemma 3 that when we assume that (K,d) is compact metric space, the condition (ii) can be omitted. This allows to expand the class Y to a class of functions including the linear cases (see Examples in Section 2.3). Moreover, when we assume that $(Y, \|\cdot\|_Y)$ is a separable Banach space included in $C(K)$ and separate the points of K, then the set $N(f)$ can be more smaller than σ-porous. In fact, we prove that in this situation, the set $N(f)$ can be covered by countably many *d.c. hypersurfaces* (see the definitions below). This gives, in particular, examples showing that the σ-porosity of the set of *"ill-posed problems"* in [11], is not optimal (see Corollary 1 and Remark 2). Our version of variational principle has several consequences, in particular it allows to give the proofs of the main results of the paper. Instead of the above condition (ii), the proof of Lemma 3 is based on the use of a differentiability result of convex continuous functions on separable Banach spaces due to Zajicek [23] and a nonconvex analogue to Fenchel duality introduced in [4].

We recall from [24] the following definitions.

Definition 2. Let Y, Z be Banach spaces, $C \subset Y$ an open convex set, and $F : C \to Z$ a continuous mapping. We say that F is d.c. (that is, delta-convex) if there exists a continuous convex function $f : C \to \mathbb{R}$ such that $y^* \circ F + f$ is convex whenever $y^* \in Y^*$, $\|y^*\| \leq 1$.

Definition 3. Let Y be a Banach space and $n \in \mathbb{N}^*$, $1 \leq n < \dim Y$. We say that $A \subset X$ is a *d.c. surface* of codimension n if there exist an n-dimensional linear space $F \subset X$, its topological complement E and a d.c. mapping $\varphi : E \to F$ such that $A = \{x + \varphi(x) : x \in E\}$. A d.c. surface of codimension 1 will be called a *d.c. hypersurface.*

2.1. Preliminary results. If $(Y, \|\cdot\|_Y)$ is a Banach space included in $C_b(M)$ with $\alpha\|\cdot\| \geq \|\cdot\|_\infty$ for some real number $\alpha > 0$ and $x \in M$, we denote by δ_x the evaluation map (Dirac mass) on Y at x i.e. $\delta_x : \varphi \mapsto \varphi(x)$, for all $\varphi \in Y$. The map δ_x is a linear continuous functional on Y since $\alpha\|\cdot\| \geq \|\cdot\|_\infty$. If Z is a Banach space, by B_Z we denote the closed unit ball of Z and by S_Z we denote the unit sphere of Z. We recall the following definition from [4].

Definition 4. Let (M, d) be a complete metric space and $(Y, \|\cdot\|_Y)$ be a Banach space included in $C_b(M)$ with $\alpha\|\cdot\| \geq \|\cdot\|_\infty$ for some real number $\alpha > 0$. We say that the space Y has the property P^G if and only if, for every sequence $(x_n)_n \subset M$, the following assertions are equivalent:

(i) the sequence $(x_n)_n$ converges in (M, d),
(ii) the sequence of the Dirac masses $(\delta_{x_n})_n$ converges in (Y^*, weak^*).

The letter G in P^G is justified by the fact that the Gâteaux bornology, the Gâteaux differentiability and the weak* topology have some connection between them. We refer to [4] for more details. The space $C_b(M)$, the subspace $C_b^u(M)$ of uniformly continuous functions and the space $\mathrm{Lip}_b(M)$ of all bounded and Lipschitz continuous functions (equipped with their natural norms), satisfies the property P^G for any complete metric space (M, d) (see [4, Proposition 2.6]). The following lemma, which will be used later, shows that in the compact metric framework, the property P^G is satisfied for a large class of function spaces.

Lemma 1. *Let (K,d) be a compact metric space and $(Y,\|\cdot\|_Y)$ be a Banach space included in $C(K)$, which separates the points of K and such that $\alpha\|\cdot\| \geq \|\cdot\|_\infty$ for some real number $\alpha > 0$. Then, Y has the property P^G.*

Proof. If $(x_n)_n$ is a sequence of K that converges to some point x in (K,d), it is clear that $(\delta_{x_n})_n$ converge to δ_x for the weak* topology. Suppose now that $(\delta_{x_n})_n$ converge to some point Q in Y^* for the weak* topology. We prove that the sequence $(x_n)_n$ converge in (K,d). Indeed, suppose that l_1 and l_2 are two distinct cluster point of $(x_n)_n$. There exists two subsequences $(y_n)_n$ and $(z_n)_n$ such that $(y_n)_n$ converge to l_1 and $(z_n)_n$ converge to l_2. Since $(\delta_{x_n})_n$ converge to Q and (Y^*, weak^*) is a Hausdorff space, it follows that $\delta_{l_1} = Q = \delta_{l_2}$ which is a contradiction since Y separate the points of K. So, the sequence $(x_n)_n$ has a unique cluster point, and hence it converges to some point since K is a compact metric space. $\square$

Now, if we are interested in the property P^G for separable Banach spaces $(Y,\|\cdot\|_Y)$ included in $C_b(M)$, the following proposition shows that this situation holds only when M is compact. In fact, this characterizes the compact metric sets.

Proposition 1. *Let (K,d) be a complete metric space and $(Y,\|\cdot\|_Y)$ be a separable Banach space included in $C_b(K)$, which separate the points of K and such that $\alpha\|\cdot\| \geq \|\cdot\|_\infty$ for some real number $\alpha > 0$. Then, the following assertions are equivalent.*

(1) *K is compact.*
(2) *Y has the property P^G.*

Proof. The part (1) $\Longrightarrow$ (2) is given by Lemma 1. Let us prove the part (2) $\Longrightarrow$ (1). Indeed, since Y is separable, by the Banach–Alaoglu theorem, the dual unit ball B_{Y^*} equipped with the weak* topology, is a compact metrizable set. Let us denote $\delta(K) := \{\delta_k : k \in K\}$ and consider the map:

$$\begin{aligned} \delta : (K,d) &\to \big(\delta(K), \text{weak}^*\big), \\ x &\mapsto \delta_x. \end{aligned}$$

Since Y has the property P^G, it follows that $(\delta(K), \text{weak}^*)$ is a closed subspace of the compact metrizable set (B_{Y^*}, weak^*). Therefore, $(\delta(K), \text{weak}^*)$ is a Hausdorff compact space. Since Y separate the points of K, the map δ is one-to-one. Consequently, $\delta : (K,d) \to (\delta(X), \text{weak}^*)$ is a continuous and bijective map from (K,d) onto the compact space $(\delta(K), \text{weak}^*)$, it is then an homeomorphism (since Y has the property P^G) which implies that (K,d) is a compact space. $\square$

We also need the following lemma.

LEMMA 2. *Let K be a compact Hausdorff space and $(Y, \|\cdot\|_Y)$ be a Banach space included in $C(K)$ which separates the points of K and such that $\alpha\|\cdot\| \geq \|\cdot\|_\infty$ for some real number $\alpha > 0$. Then, the following assertions are equivalent.*

(1) *K is metrizable*
(2) *there exists a sequence $(\varphi_n)_n \subset S_Y$ which separates the points of K*
(3) *there exists a separable Banach subspace Z of $(Y, \|\cdot\|_Y)$ which separates the points of K.*

Proof. (1) $\Longrightarrow$ (2). Since K is a compact metrizable space, then $(C(K), \|\cdot\|_\infty)$ is a separable Banach space. Since Y is a subspace of $C(K)$, it is also $\|\cdot\|_\infty$-separable. Thus, there exists a sequence $(\varphi_n)_n \subset Y$ which is dense in Y for the norm $\|\cdot\|_\infty$. Since Y separate the points of K, if $x, y \in K$ are such that $x \neq y$, then there exists $\varphi \in Y$ such that $\varphi(x) \neq \varphi(y)$. Using the uniform convergence of a subsequence of $(\varphi_n)_n$ to φ, we get that the sequence $(\varphi_n)_n$ also separate the points x and y. Normalizing if necessary, we can assume that $(\varphi_n)_n \subset S_Y$.

(2) $\Longrightarrow$ (3). It suffices here, to set $Z = \overline{\text{span}}^{\|\cdot\|_Y}\{\varphi_n : n \in \mathbb{N}\}$, then Z is a separable Banach subspace of $(Y, \|\cdot\|_Y)$ that separates the points of K.

(3) $\Longrightarrow$ (1). Since Z separate the points of K and $Z \subset C(K)$, then the following Dirac map is continuous and one-to-one

$$\begin{aligned} \delta : K &\to \big(\delta(K), \text{weak}^*\big) \subset \big(B_{Z^*}, \text{weak}^*\big), \\ x &\mapsto \delta_x. \end{aligned}$$

Since K is a compact space, we get that δ is an homeomorphism and we have that $(\delta(K), \text{weak}^*)$ is a compact subset of Z^*. Now, since Z is separable then, (B_{Z^*}, weak^*) is metrizable by the Banach–Alaoglu theorem and so also $(\delta(K), \text{weak}^*)$ is metrizable. It follows that K is metrizable, since δ is an homeomorphism. $\square$

2.2. A key lemma. Recall that in [23], Zajicek proved that if F is a convex continuous function defined on a separable Banach space, the set $NG(F)$ of points where F is not Gâteaux differentiable, can be covered by countably many *d.c. hypersurfaces*. This result together with a duality result from [4], will be used in the proof of Lemma 3.

LEMMA 3. *Let (K, d) be a compact metric space and $(Y, \|\cdot\|_Y)$ be a Banach space included in $C(K)$ which separates the points of K and such that $\alpha\|\cdot\|_Y \geq \|\cdot\|_\infty$ for some real number $\alpha > 0$. Let $f : (K, d) \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous function. Then, the set*

$$N(f) = \{\varphi \in Y : f - \varphi \textit{ does not have a strong minimum on } K\}$$

is of the first Baire category in Y. Moreover, for each separable Banach subspace Z of $(Y, \|\cdot\|_Y)$ which separates the points of K, we have that $N(f) \cap$

Z can be covered by countably many d.c. hypersurfaces in Z. In particular, if Y is separable then $N(f)$ can be covered by countably many d.c. hypersurfaces in Y.

Proof. Let Z be any Banach subspace of $(Y, \|\cdot\|_Y)$. Consider the function $f^\times$ defined for all $\varphi \in Z$ by

$$f^\times(\varphi) := \sup_{x\in K}\{\varphi(x) - f(x)\}.$$

It is clear that $f^\times$ is a convex α-Lipschitz continuous function on Z.

The separable case: Suppose that Z is a separable Banach subspace of the space $(Y, \|\cdot\|_Y)$ which separates the points of K. Using [23, Theorem 2] we get that $f^\times$ is Gâteaux-differentiable outside a set N which can be covered by countably many *d.c. hypersurfaces* in Z. On the other hand, combining Lemma 1 and [4, Theorem 2.8] we get that $f^\times$ is Gâteaux-differentiable at a point $\varphi \in Z$ if and only if $f - \varphi$ has a strong minimum on K. Thus, the set N of points of non Gâteaux-differentiability of $f^\times$, coincides with the set

$$N(f) \cap Z = \{\varphi \in Z : f - \varphi \text{ does not have a strong minimum on } K\}.$$

This completes the proof in the separable case.

The general case: We proceed as in the proof of the D–G–Z variational principle, but here we don't admit the existence of a bump function in Y having a support of arbitrary small diameter. Instead, we use *"The separable case"*. Indeed, let us consider the following sequence of sets

$$O_n := \left\{\varphi \in Y; \exists x_n \in K/(f-\varphi)(x_n) < \inf\left\{(f-\varphi)(x) : d(x, x_n) \geq \frac{1}{n}\right\}\right\}.$$

The fact that O_n is open follows from the hypothesis $\alpha\|\cdot\|_Y \geq \|\cdot\|_\infty$. Let us prove that $\mathcal{G} := \bigcap_{n\geq 1} O_n$ is dense in $(Y, \|\cdot\|_Y)$. From Lemma 2, there exists a separable Banach subspace Z of $(Y, \|\cdot\|_Y)$ which separates the points of K. Let $\varphi \in Y$ and $\varepsilon > 0$. By using *"The separable case"* with the lower semicontinuous function $f - \varphi$ and the separable space Z, we get an $h \in Z$ such that $\|h\|_Y < \varepsilon$ and $(f - \varphi) - h$ has a strong minimum on K at some point x_0. It follows that $\varphi + h \in \mathcal{G}$, by taking $x_n = x_0$ for all $n \in \mathbb{N}^*$. This shows that $\mathcal{G}$ is a dense G_δ subset of Y. Now, we claim that

$$\mathcal{G} \subset \{\varphi \in Y : f - \varphi \text{ has a strong minimum on } K\}.$$

Indeed, let $\varphi \in \mathcal{G}$. For each $n \geq 1$ there exists $x_n \in K$ such that

$$(f-\varphi)(x_n) < \inf\left\{(f-\varphi)(x) : d(x, x_n) \geq \frac{1}{n}\right\}.$$

Since K is compact metric space, there exists a subsequence $(x_{n_k})_k$ that converges to some point x_∞. Using the lower semicontinuity of f, we get that

$$\begin{aligned}(f-\varphi)(x_\infty) &\le \liminf_k (f-\varphi)(x_{n_k})\\ &\le \liminf_k \inf\left\{(f-\varphi)(x) : d(x, x_{n_k}) \ge \frac{1}{n_k}\right\}\\ &\le \inf\big\{(f-\varphi)(x) : x \in K \setminus \{x_\infty\}\big\}.\end{aligned}$$

Now, to see that x_∞ is a strong minimum of $f-\varphi$, let $(y_n)_n$ be a sequence in K such that $(f-\varphi)(y_n)$ converges to $(f-\varphi)(x_\infty)$. We prove that $(y_n)_n$ converges to x_∞. Indeed, suppose that the contrary holds. Extracting, if necessary, a subsequence, we can assume that there exists $\varepsilon > 0$ such that for all $n \in N$, $d(y_n, x_\infty) \ge \varepsilon$. Thus, there exists an integer p such that $d(x_p, y_n) \ge \frac{1}{p}$ for all $n \in \mathbb{N}$. Hence,

$$(f-\varphi)(x_\infty) \le (f-\varphi)(x_p) < \inf\left\{(f-\varphi)(x) : d(x, x_p) \ge \frac{1}{p}\right\} \le (f-\varphi)(y_n)$$

for all $n \in \mathbb{N}$, which contradicts the fact that $\lim_n (f-\varphi)(y_n) = (f-\varphi)(x_\infty)$. This concludes the proof. □

Remark 1. A strong and strict minimum coincides for lower semicontinuous functions on a compact metric space.

We obtain immediately the following corollary.

Corollary 1. *Let (K, d) be a compact metric space and Y be any closed subspace of $(C(K), \|\cdot\|_\infty)$ that separates the points of K. Let $f : K \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous function. Then, the set*

$$N(f) = \{\varphi \in Y : f - \varphi \text{ does not have a strong minimum on } K\}$$

can be covered by countably many d.c. hypersurfaces in Y.

Proof. Since (K, d) is a compact metric space, $(C(K), \|\cdot\|_\infty)$ is a separable Banach space and so $(Y, \|\cdot\|_\infty)$ is a separable Banach subspace satisfying the hypothesis of Lemma 3. □

Remark 2. (1) The above corollary cannot be obtained from the D–G–Z variational principle. For example, the space $Y = \{\varphi \in C(B_{\mathbb{R}^n})/\varphi(0) = 0\}$, where $B_{\mathbb{R}^n}$ denotes the closed unit ball of $\mathbb{R}^n$ for some norm, satisfies the hypothesis of Corollary 1 but since $\varphi(0) = 0$ for all $\varphi \in Y$, it follows that Y does not satisfy the condition (ii): for every natural number n, there exists a positive constant M_n such that for any point $x \in B_{\mathbb{R}^n}$ there exists a function $h_n : B_{\mathbb{R}^n} \longrightarrow [0;1]$, such that $h_n \in Y$, $\|h_n\| \le M_n$, $h_n(x) = 1$ and $\mathrm{diam}(\mathrm{supp}(h)) < \frac{1}{n}$.

(2) The above corollary gives also a class of examples showing that the σ-porosity of the *"ill-posed problems"* in [11] is not optimal.

2.3. Applications to linear variational principles. As consequence of Lemma 3, we give in this section the following analogue of the Stegall's variational principle for compact metrizable sets. This result cannot be obtained from the D–G–Z variational principle. If E is a Banach space and $x \in E$, by $\hat{x}$ we denote the map $\hat{x} : p \mapsto p(x)$ for all $p \in E^*$.

PROPOSITION 2. *Let E be a Banach space and K be a weak* compact metrizable subset of E^*. Let $f : (K, \text{weak}^*) \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous function. Then, the set*

$$N(f) = \{x \in E : f - \hat{x} \textit{ does not have a strict minimum on } K\}$$

is of the first Baire category in E. If moreover, E is a separable Banach space, then $N(f)$ can be covered by countably many d.c. hypersurfaces in E.

Proof. Since K is a weak* compact subset of E^*, it is norm bounded. Let $\alpha_K := \sup_{x^* \in K} \|x^*\| < +\infty$. Thus, $(E, \|\cdot\|)$ is a Banach space, included in $C(K)$, which separates the points of K and satisfies $\alpha_K \|\cdot\| \geq \|\cdot\|_\infty$. So we can apply Lemma 3 with $(Y, \|\cdot\|_Y) = (E, \|\cdot\|)$. □

PROPOSITION 3. *Let E be a Banach space. Let K be a weak compact metrizable subset of E. Let $f : (K, \text{weak}) \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous function. Then, the set*

$$N(f) = \left\{x^* \in E^* : f - x^* \textit{ does not have a strict minimum on } K\right\}$$

is of the first Baire category in E^*. If moreover, E^* is a separable Banach space, then $N(f)$ can be covered by countably many d.c. hypersurfaces in E^*.

Proof. Since K is a weak compact subset of E, it is norm bounded. Let $\alpha_K := \sup_{x \in K} \|x\| < +\infty$. Thus, $(E^*, \|\cdot\|)$ is a Banach space, included in $C(K)$, which separates the points of K by the Hahn–Banach theorem and satisfies $\alpha_K \|\cdot\| \geq \|\cdot\|_\infty$. So we can apply Lemma 3 with $(Y, \|\cdot\|_Y) = (E^*, \|\cdot\|)$. □

Let K be a convex subset of a topological vector space. A function $\varphi : K \to \mathbb{R}$ is said to be affine if for all $x, y \in K$ and $0 \leq \lambda \leq 1$, $\varphi(\lambda x + (1-\lambda) y) = \lambda \varphi(x) + (1-\lambda)\varphi(y)$. The space of all continuous real-valued affine functions on K will be denoted by $\mathrm{Aff}(K)$. Note that all translates of continuous linear functionals are elements of $\mathrm{Aff}(K)$, but the converse is not true in general (see Example A in the Appendix, Proposition 5 and the reference [20], page 22).

PROPOSITION 4. *Let K be a compact metrizable convex subset of an* l.c.t. *space X and $f : K \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous function. Then the set*

$$N(f) := \left\{\varphi \in \mathrm{Aff}(K) : f - \varphi \textit{ does not have a strong minimum on } K\right\}$$

can be covered by countably many d.c. hypersurfaces in $(\mathrm{Aff}(K), \|\cdot\|_\infty)$.

Proof. We use Lemma 3 with $Y = \mathrm{Aff}(K)$. Since $(\mathrm{Aff}(K), \|\cdot\|_\infty)$ is a closed Banach subspace of the separable Banach space $(C(K), \|\cdot\|_\infty)$, it is separable. On the other hand, by the Hahn–Banach theorem, $\mathrm{Aff}(K)$ separates the points of K, since it contains the set $\{x^*_{|K} : x^* \in X^*\}$. So, from Lemma 3, the set

$$N(f) = \big\{\varphi \in \mathrm{Aff}(K) : f - \varphi \text{ does not have a strong minimum on } K\big\}$$

can be covered by countably many *d.c. hypersurfaces* in $(\mathrm{Aff}(K), \|\cdot\|_\infty)$. $\square$

2.4. Application to the Gâteaux differentiability. Recall that a weak Asplund space E is a Banach space in which every convex continuous function is Gâteaux differentiable at each point of a dense G_δ subset of E. The following corollary gives a class of convex continuous functions which are Gâteaux differentiable at each point of a dense G_δ subset of E, where E is any Banach space. Recall also that the Fenchel transform of f is defined on the dual space for all $p \in E^*$ by

$$f^*(p) := \sup_{x \in E} \big\{\langle p, x\rangle - f(x)\big\}.$$

Corollary 2. *Let E be a Banach space and $f : E \longrightarrow \mathbb{R}$ be a convex continuous function such that* $\mathrm{dom}(f^*) \subset K$, *where K is weak* compact metrizable subset of E^*. Then f is Gâteaux differentiable at each point of a dense G_δ subset of E.*

Proof. Since $f^*_{|K} : (K, \mathrm{weak}^*) \longrightarrow \mathbb{R} \cup \{+\infty\}$ is proper weak* lower semicontinuous and (K, weak^*) is metrizable, by Proposition 2 we have that the set

$$G := \big\{x \in E / f^*_{|K} - \hat{x} \text{ has a strict minimum on } K\big\}$$

contains a dense G_δ subset of E. Since $\mathrm{dom}(f^*) \subset K$, we also have that

$$G = \big\{x \in E / f^* - \hat{x} \text{ has a strict minimum on } E^*\big\}.$$

By using the classical Asplund–Rockafellar duality result [3, Corollary 1] we get that f is Gâteaux differentiable at each point of G. $\square$

Let C be a non-empty subset of E^*. We denote by σ_C the support function defined on E by

$$\sigma_C(x) = \sup_{x^* \in C} x^*(x) \quad \forall x \in E.$$

Let $f : E \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a proper lower semi-continuous convex function. The inf convolution of f and σ_C is defined for all $x \in E$ by

$$f \triangledown \sigma_C(x) := \inf_{y \in E} \big\{f(x-y) + \sigma_C(y)\big\}.$$

From the above corollary, we get that if K is a convex weak* compact metrizable subset of E^* and f is a proper lower semi-continuous convex function on E such that $\mathrm{dom}(f^*) \cap K \neq \emptyset$, then $f \triangledown \sigma_K$ is convex continuous and Gâteaux differentiable at each point of a dense G_δ subset of E. In particular, the support function σ_K is Gâteaux differentiable at each point of a dense G_δ subset of E.

3. The main results and applications

This section is devoted to the proofs of the main results of the paper. Some applications are also given.

3.1. Proof of Theorem 2. Now, we give the proof of the first main result.

Proof of Theorem 2. Let K be a compact metrizable Φ-convex subset of S. Using the conditions (1) and (2), we have that $(\Phi, \|\cdot\|_\Phi)$ is a Banach space included in $C(K)$ which separates the points of K and satisfies $\alpha_K\|\cdot\|_\Phi \geq \|\cdot\|_\infty$ for some $\alpha_K > 0$. Thus, Lemma 3 applies. Using Lemma 3, applied with the lower semi-continuous function $f = i_K$ (the indicator function which is equal to 0 on K and $+\infty$ otherwise), we get that the set of all $\varphi \in \Phi$ which Φ-exposes K at some point, contains a dense G_δ subset of Φ. In particular, $\Phi\,\mathrm{Ext}(K) \neq \emptyset$.

Since K is a Φ-convex subset of S, it is clear that $\mathrm{conv}_\Phi(\Phi\,\mathrm{Ext}(K)) \subset K$. Now, let us prove that $K = \mathrm{conv}_\Phi(\Phi\,\mathrm{Ext}(K))$. Suppose towards a contradiction that there exists $k_0 \in K \setminus \mathrm{conv}_\Phi(\Phi\,\mathrm{Ext}(K))$.

Claim. *There exist $h \in \Phi$ and $r \in \mathbb{R}$ such that*

$$\sup\big\{h(k) : k \in \mathrm{conv}_\Phi\big(\Phi\,\mathrm{Ext}(K)\big)\big\} < r < h(k_0).$$

Proof of the claim. Since $\mathrm{conv}_\Phi(\Phi\,\mathrm{Ext}(K))$ is a Φ-convex subset of S, then there exist a set I, $\varphi_i \in \Phi$ and $\lambda_i \in \mathbb{R}$ for all $i \in I$ such that

$$\mathrm{conv}_\Phi\big(\Phi\,\mathrm{Ext}(K)\big) = \bigcap_{i\in I}\big\{k \in S/\varphi_i(k) \leq \lambda_i\big\}.$$

Since $k_0 \notin \mathrm{conv}_\Phi(\Phi\,\mathrm{Ext}(K))$, there exists $i_0 \in I$ such that $\varphi_{i_0}(k_0) > \lambda_{i_0}$. On the other hand, we have $\varphi_{i_0}(k) \leq \lambda_{i_0}$ for all $k \in \mathrm{conv}_\Phi(\Phi\,\mathrm{Ext}(K))$. This finishes the proof of the claim by taking $h = \varphi_{i_0}$ and by choosing a real number $r \in \mathbb{R}$ such that $\lambda_{i_0} < r < \varphi_{i_0}(k_0)$.

Now, using Lemma 3 applied with $f = -h + i_K$, we can find $\psi \in \Phi$ close to 0 in Φ such that $h + \psi$, Φ-expose K at some point $k_1 \in \Phi\,\mathrm{Ext}(K)$. Since $\alpha\|\cdot\| \geq \|\cdot\|_\infty$, we have that $\varphi := h + \psi$ is also close to h uniformly on K. Hence, using the claim, φ satisfies also

$$\sup\big\{\varphi(k) : k \in \mathrm{conv}_\Phi\big(\Phi\,\mathrm{Ext}(K)\big)\big\} < r < \varphi(k_0). \tag{1}$$

On the other hand

$$\varphi(k_0) \leq \sup\big\{\varphi(k) : k \in K\big\} = \varphi(k_1)$$

which is a contradiction with (1), since $k_1 \in \Phi\,\mathrm{Ext}(K)$. □

REMARK 3. Under the assumptions of Theorem 2, if moreover we assume that $(\Phi, \|\cdot\|_\Phi)$ is a separable Banach space, then we have that the set of all $\varphi \in \Phi$ that Φ-exposes K at some point, has a complement in Φ which can be covered by countably many *d.c. hypersurfaces*.

For the classical convexity, we obtain the following Krein–Milman type results for convex compact metrizable subsets, where the extreme points are replaced by the exposed points:

• We know from [19, Theorem 6.2] that a Banach space E is a Gâteaux differentiability space (i.e., each convex continuous real valued function defined on E is Gâteaux differentiable at each point of a dense subset) if and only if, every convex weak* compact subset of E^* is the weak* closed convex hull of its weak* exposed points. Hence, for a non Gâteaux differentiability space (for example if $E = l^1(\Gamma)$, Γ uncountable set) there always exist a convex weak* compact subset of E^* which is not the weak* closed convex hull of its weak* exposed points. This shows in particular that Theorem 2 fails in general for not metrizable convex compact subsets. However, part (1) of Corollary 3 shows that the situation is better for a convex weak* compact metrizable subset of E^*, when E is any Banach space.

• Part (2) of Corollary 3 is an extension of a Klee result [15, Theorem 2.1] and its proof follows easily from Theorem 2. However, this result is not optimal. Indeed, there exists a more general result due to Lindenstrauss and Troyanski showing: *Let C be a convex and weakly-compact subset of a Banach space E. Then C is the closed convex hull of its strongly exposed points.* A nice geometrical proof of this result was given by Bourgain in [8].

COROLLARY 3. *Let E be a Banach space.*

(1) *Let K be a convex weak* compact metrizable subset of E^*. Then,*

$$K = \overline{\operatorname{conv}}^{w^*}\big(w^* \operatorname{Ext}(K)\big).$$

(2) *Let K be a convex weak compact metrizable subset of E. Then,*

$$K = \overline{\operatorname{conv}}^{w}\big(\operatorname{Ext}(K)\big) = \overline{\operatorname{conv}}^{\|\cdot\|}\big(\operatorname{Ext}(K)\big).$$

Proof. In part (1), we apply Theorem 2 with the convex weak* metrizable subset K of E^* and by taking $(\Phi, \|\cdot\|_\Phi) = (E, \|\cdot\|)$ (see Example 1). In part (2), we apply Theorem 2 with the convex weak compact metrizable subset K of E and by taking $(\Phi, \|\cdot\|_\Phi) = (E^*, \|\cdot\|)$, using in this case the fact that the weak and norm closure coincides for convex sets by the well-known Mazur's lemma (see Example 2). □

In [15], Klee pointed to the fact that outside the normed space, the above result is not true. He suspected that some condition rather close to normability may be needed and that the metrizability is inadequate even in the separable case, mentioning the following counterexample: in the locally convex separable metrizable space $\mathbb{R}^{\aleph_0}$, the cube $[-1, 1]^{\aleph_0}$ has no exposed points.

To answer positively this problem in the general l.c.t. spaces, we introduce an intermediate concept of remarkable points called *"affine exposed points"* which is between the concept of exposed points and extreme points.

DEFINITION 5. Let K be a convex subset of a l.c.t. space X. We say that a point $x \in K$ is an affine exposed point of K, and write $x \in \mathrm{AExp}(K)$, if there exists some affine continuous map $\tau \in \mathrm{Aff}(K)$ which attains its strict maximum over K at x.

Clearly, $\mathrm{Ext}(K) \subset \mathrm{AExp}(K) \subset \mathrm{Ext}(K)$, but these inclusions are strict in general. For example, the cube $[-1,1]^{\aleph_0}$ has affine exposed points by Proposition 4, but is without exposed points. A comparison of these three sets will be given in Section A.1.

We obtain then, the following extension of the Krein–Milman theorem in the metrizable framework.

THEOREM 4. *Let K be a convex compact metrizable subset of a* l.c.t. *space X. Then, $\mathrm{AExp}(K) \neq \emptyset$ and K is the closed convex hull of its affine exposed points*: $K = \overline{\mathrm{conv}}(\mathrm{AExp}(K))$.

Proof. The proof is given by taking $S = K$ and $(\Phi, \|\cdot\|_\Phi) = (\mathrm{Aff}(K), \|\cdot\|_\infty)$ in Theorem 2, observing that in this case, convexity and Φ-convexity coincide for closed subsets by the Hahn–Banach theorem. □

We also have the following consequences.

COROLLARY 4. *Let E be a Banach space.*

(1) *Let* (K, weak^*) *be a convex weak* compact metrizable subset of* E^*. *Then, the set* $w^* \mathrm{Ext}(K)$ *is weak* dense in the set* $\mathrm{AExp}(K)$, *which is weak* dense in the set* $\mathrm{Ext}(K)$.

(2) *Let* (K, weak) *be a convex weak compact metrizable subset of E. Then, the set* $\mathrm{Ext}(K)$ *is weak dense in the set* $\mathrm{AExp}(K)$, *which is weak dense in the set* $\mathrm{Ext}(K)$.

Proof. First, note that the spaces (E^*, weak^*) and (E, weak) are l.c.t. spaces. Combining part (1) (resp. part (2)) of Corollary 3 with Theorem 4 and the partial converse of the Krein–Milman theorem (the Milman's theorem), we get the part (1) (resp. the part (2)). □

3.2. Proof of Theorem 3. In this subsection, we give the proof of the second main result. We need the following lemma from [6].

LEMMA 4 (See [6]). *Let Z be a Banach space and $h, k : Z \to \mathbb{R}$ be two continuous and convex functions. Suppose that the function $z \to l(z) := \max(h(z), k(z))$ is Fréchet (respectively, Gâteaux) differentiable at some point $z_0 \in Z$. Then either h or k (maybe both h and k) is Fréchet (respectively, Gâteaux) differentiable at z_0 and $l'(z_0) = h'(z_0)$ or $l'(z_0) = k'(z_0)$.*

Proof. We give the proof for the Fréchet differentiability; the Gâteaux differentiability is similar. Suppose without loss of generality that $l(z_0)=h(z_0)$ and let us prove that h is Fréchet differentiable at z_0 and that $l'(z_0)=h'(z_0)$. For each $z\neq 0$ we have:

$$0\leq \frac{h(z_0+z)+h(z_0-z)-2h(z_0)}{\|z\|}\leq \frac{l(z_0+z)+l(z_0-z)-2l(z_0)}{\|z\|}.$$

Since l is Fréchet differentiable at z_0, then the right-hand side in the above inequalities tends to 0 when z tends to 0. This implies that h is Fréchet differentiable at z_0 by the convexity of h. Now, if we denote $f=h-l$, then $f(z_0)=0$, $f\leq 0$ and $f'(z_0)$ exists. Thus, for all $z\in Z$

$$f'(z_0)(z)=\lim_{t\longrightarrow 0^+}\frac{1}{t}\big(f(z_0+tz)-f(z_0)\big)\leq 0.$$

This implies that $f'(z_0)=0$. Thus $h'(z_0)=l'(z_0)$. □

We also need to establish the following lemma.

Lemma 5. *Let K be a compact metric set and $(\Phi,\|\cdot\|_\infty)$ be a closed Banach subspace of $(C(K),\|\cdot\|_\infty)$ which separates the points of K and contains the constants. Then, the following assertions are equivalent.*

(1) *A point $Q\in B_{\Phi^*}$ is a weak* exposed point,*

(2) *there exists a Φ-exposed point $k\in\Phi\,\mathrm{Ext}(K)$ such that $Q=\pm\delta_k$, where $\delta_k:\varphi\mapsto\varphi(k)$ for all $\varphi\in\Phi$.*

Proof. (1) $\Longrightarrow$ (2). Let $Q\in w^*\,\mathrm{Ext}(B_{\Phi^*})$, so there exists $\varphi\in\Phi$ which weak* expose B_{Φ^*} at Q. It follows from [19, Proposition 6.9] that the norm $\|\cdot\|_\infty$ is Gâteaux differentiable at φ with Gâteaux derivative equal to Q. On the other hand it is clear that $\|\psi\|_\infty=\max(0^\times(\psi),0^\times(-\psi))$ for all $\psi\in\Phi$, where $0^\times(\psi)=\sup_{k\in K}\varphi(k)$ for all $\varphi\in\Phi$. Thus, from Lemma 4 we have that either $\psi\mapsto 0^\times(\psi)$ or $\psi\mapsto 0^\times(-\psi)$ is Gâteaux differentiable at φ with Gâteaux derivative equal to Q. Suppose in the first case that is the function $\psi\mapsto 0^\times(\psi)$ which is Gâteaux differentiable at φ. Thus, from Lemma 1 and [4, Theorem 2.8] applied with the function $f=0$, we get that there exists $k\in K$ such that φ has a strong maximum at k and that $Q=\delta_k$. Thus, in this case k is Φ-exposed by φ and $Q=\delta_k$. For the second case, where it is the function $\psi\mapsto 0^\times(-\psi)$ which is Gâteaux differentiable at φ with Gâteaux derivative equal to Q, in a similar way, using Lemma 1, [4, Theorem 2.8] and the chain rule formula we obtain that there exists some $k\in K$ such that $-\varphi$ has a strong maximum at k (so that k is Φ-exposed point) and $Q=-\delta_k$.

(2) $\Longrightarrow$ (1). Suppose that $k\in\Phi\,\mathrm{Ext}(K)$. There exists $\varphi\in\Phi$ which Φ-exposes k. Thus $-\varphi$ has a strict minimum at k, equivalent to a strong minimum at k, since K is compact metric set. We can find a real number r such that $-(\varphi+r)$ has also a strong minimum at k and such that $\varphi+r>1$ on K. Hence, the function $0^\times$ coincides with $\|\cdot\|_\infty$ on an open neighborhood

of $\varphi + r \in \Phi$. Since $-(\varphi + r)$ has a strong minimum at k, [4, Theorem 2.8] asserts that $0^{\times}$ and so also $\|\cdot\|_\infty$ is Gâteaux differentiable at $\varphi + r$ with Gâteaux derivative equal to δ_k. It follows from [19, Proposition 6.9], that δ_k is weak* exposed by $\varphi + r$. Thus, $\delta_k \in w^* \operatorname{Ext}(B_{\Phi^*})$. By the symmetry of B_{Φ^*}, we also have that $-\delta_k \in w^* \operatorname{Ext}(B_{\Phi^*})$. □

A. The second main result. Now, we give the proof of the second main result.

Proof of Theorem 3. The fact that

$$w^* \operatorname{Ext}(B_{\Phi^*}) = \pm\delta\big(\Phi \operatorname{Ext}(K)\big)$$

is given by Lemma 5. Now, since $(\Phi, \|\cdot\|_\infty)$ is separable, the weak* compact set $(B_{\Phi^*}, \text{weak}^*)$ is metrizable. Thus, from Corollary 3 applied to the convex compact metrizable set $(B_{\Phi^*}, \text{weak}^*)$, we have that

$$B_{\Phi^*} = \overline{\text{conv}}^{w^*}\big(w^* \operatorname{Ext}(B_{\Phi^*})\big).$$

Hence,

$$B_{\Phi^*} = \overline{\text{conv}}^{w^*}\big(w^* \operatorname{Ext}(B_{\Phi^*})\big) = \overline{\text{conv}}^{w^*}\big(\pm\delta\big(\Phi \operatorname{Ext}(K)\big)\big).$$

This concludes the proof. □

We deduce immediately the following corollaries. Replacing Φ by $\operatorname{Aff}(K)$ in Theorem 3, we obtain the following corollary.

Corollary 5. *Let K be a compact metrizable convex subset of an* l.c.t. *space X. Then,*

$$w^* \operatorname{Ext}(B_{(\operatorname{Aff}(K))^*}) = \pm\delta\big(\operatorname{AExp}(K)\big)$$

and

$$B_{(\operatorname{Aff}(K))^*} = \overline{\text{conv}}^{\mathrm{w}^*}\big(\pm\delta\big(\operatorname{AExp}(K)\big)\big),$$

where $\pm\delta(\operatorname{AExp}(K)) := \{\pm\delta_k / k \in \operatorname{AExp}(K)\}$.

Replacing Φ by $C(K)$ in Theorem 3, where (K, d) is a compact metric space, and observing that $\Phi \operatorname{Ext}(K) = K$, since each point $k \in K$ is an exposed point by the continuous function $x \mapsto -d(x, k)$, we obtain:

Corollary 6. *Let (K, d) be a compact metric space. Then,*

$$w^* \operatorname{Ext}(B_{C(K)^*}) = \pm\delta(K)$$

and

$$B_{C(K)^*} = \overline{\text{conv}}^{\mathrm{w}^*}\big(\pm\delta(K)\big).$$

Remark 4. It is well known that if K is a compact Hausdorff space then $\operatorname{Ext}(B_{C(K)^*}) = \pm\delta(K)$. Thus, for compact metric space K, the extremal points and the weak* exposed points of $B_{C(K)^*}$ coincide.

B. The Shilov boundary and the Φ-exposed points. Let K be a compact space and $(\Phi, \|\cdot\|_\infty)$ be a closed Banach subspace of $(C(K), \|\cdot\|_\infty)$ which separates the points of K. A subset L of K is said to be a boundary of Φ if for every $\varphi \in \Phi$, we have

$$\|\varphi\|_\infty = \sup_{x \in L} |\varphi(x)|.$$

A closed boundary C of Φ satisfies, for every $\varphi \in \Phi$,

$$\|\varphi\|_\infty = \max_{x \in C} |\varphi(x)|.$$

The Choquet boundary of Φ, denoted $\mathrm{Ch}(\Phi)$, is defined as the set of all $x \in K$ such that δ_x is an extreme point of the unit ball of Φ^*. It is well known that $\mathrm{Ch}(\Phi)$ is a boundary for Φ (see [22], page 184). When Φ admits a unique minimal closed boundary, it is called the Shilov boundary of Φ and is denoted by $\partial\Phi$. D. P. Milman proved the existence of the Shilov boundary for every closed linear subspace of $C(K)$, separating points of K and containing the constants (see [17] and [18]). He also proved that in this case the Shilov boundary concides with the closure of the Choquet boundary: $\partial\Phi = \overline{\mathrm{Ch}(\Phi)}$. A proof of this result by H. S. Bear can be found in [7]. For other information about boundary sets we refer to [1] and [2]. As a consequence of Theorem 3, we prove below that if K is a compact metric space, then the set of Φ-exposed points $\Phi\,\mathrm{Ext}(K)$ is a boundary of Φ and its closure coincides with the Shilov boundary of Φ. Note that, $\Phi\,\mathrm{Ext}(K) \subset \mathrm{Ch}(\Phi)$, but this inclusion is strict in general.

Corollary 7. *Let K be a compact metric set and $(\Phi, \|\cdot\|_\infty)$ be a closed Banach subspace of $(C(K), \|\cdot\|_\infty)$ which separates the points of K and contains the constants. Then, the set $\Phi\,\mathrm{Ext}(K)$ is a boundary of Φ and we have that*

$$\partial\Phi = \overline{\Phi\,\mathrm{Ext}(K)} = \overline{\mathrm{Ch}(\Phi)}.$$

Proof. By the Hahn–Banach theorem, for each $\varphi \in \Phi$, we have

$$\|\varphi\|_\infty = \sup_{Q \in B_{\Phi^*}} \langle Q, \varphi \rangle.$$

By Theorem 3, we have that

$$\|\varphi\|_\infty = \sup_{Q \in \overline{\mathrm{conv}}^{w^*}(\pm\delta(\Phi\,\mathrm{Ext}(K)))} \langle Q, \varphi \rangle.$$

Since the map $\hat{\varphi} : Q \mapsto \langle Q, \varphi \rangle$ is linear and weak* continuous, we obtain that

$$\|\varphi\|_\infty = \sup_{Q \in \pm\delta(\Phi\,\mathrm{Ext}(K))} \langle Q, \varphi \rangle = \sup_{k \in \Phi\,\mathrm{Ext}(K)} |\varphi(k)|.$$

Thus, the set $\Phi\,\mathrm{Ext}(K)$ is a boundary of Φ. It follows that $\overline{\Phi\,\mathrm{Ext}(K)}$ is a closed boundary for Φ. It is clear that $\partial\Phi \subset \overline{\Phi\,\mathrm{Ext}(K)}$, since $\partial\Phi$ is the minimal closed boundary. We prove that $\Phi\,\mathrm{Ext}(K) \subset \partial\Phi$. Suppose that the contrary holds, there exists $k_0 \in \Phi\,\mathrm{Ext}(K)$ such that $k_0 \notin \partial\Phi$. Thus, on one hand there exists $\varphi \in \Phi$ that expose K at k_0 that is, $\varphi(k_0) > \varphi(k)$ for all $k \in K \setminus \{k_0\}$. On the other hand, since $\partial\Phi$ is compact and φ is continuous, there exists $k_1 \in \partial\Phi$ such that $\varphi(k_1) = \sup_{k\in\partial\Phi} \varphi(k)$. Thus, since $k_0 \neq k_1$, we have $\varphi(k_0) > \varphi(k_1) = \sup_{k\in\partial\Phi} \varphi(k)$. Since Φ contain the constant, there exists $r \in \mathbb{R}$ such that $\varphi + r \in \Phi$ and $\varphi + r \geq 0$. Hence, we have that

$$\|\varphi + r\|_\infty = \varphi(k_0) + r > \varphi(k_1) + r = \sup_{k\in\partial\Phi} \big(\varphi(k) + r\big) = \|\varphi + r\|_\infty$$

which is a contradiction. Thus, $\Phi\,\mathrm{Ext}(K) \subset \partial\Phi$ and so we have $\overline{\Phi\,\mathrm{Ext}(K)} = \partial\Phi$. □

Appendix

In this section, we give some additional properties about remarkable points and the Krein-Milman theorem.

A.1. Exposed, affine exposed and extreme points. Examples. Let K be a convex subset of a l.c.t. space X. It is easy to see that we always have

$$\mathrm{Ext}(K) \subset \mathrm{AExp}(K) \subset \mathrm{Ext}(K).$$

This section gives examples showing that these inclusions are strict in general.

Example A (*Example where* $\mathrm{Ext}(K) \subsetneq \mathrm{AExp}(K)$). The cube $[-1,1]^{\aleph_0}$ in the locally convex separable metrizable space $\mathbb{R}^{\aleph_0}$ has no exposed points; however, the set of its affine exposed points is nonempty. Indeed, for example, the point $b = (1,1,1,\dots)$ is affine exposed in $[-1,1]^{\aleph_0}$ by the affine continuous map defined on $[-1,1]^{\aleph_0}$ by $\varphi : (x_1,x_2,x_3,\dots) \mapsto \sum_{n\geq 0} 2^{-n}x_n$.

A slight change of the set $[-1,1]^{\aleph_0}$ gives also an example where $\emptyset \neq \mathrm{Ext}(K) \neq \mathrm{AExp}(K)$. For example, we can take the convex compact set $K := \{ta + (1-t)k/t \in [0,1], k \in [-1,1]^{\aleph_0}\}$, where $a = (-2,0,0,0,\dots)$. In this case the point a is exposed by the continuous functional $x^* : (x_1,x_2,x_3,\dots) \mapsto -x_1$, but the point $b = (1,1,1,\dots)$ is not an exposed point. However, b is affine exposed by the affine continuous map defined on K by $\varphi : (x_1,x_2,x_3,\dots) \mapsto \sum_{n\geq 0} 2^{-n}x_n$.

Example B (*Example where* $\mathrm{Ext}(K) = \mathrm{AExp}(K)$). Let K be a convex compact subset of an l.c.t. space. Clearly, all translates of continuous linear functionals are elements of $\mathrm{Aff}(K)$, but the converse in not true in general (see Example A; see also [20], page 22). However, we do have the following relationship.

Proposition 5 ([20], Proposition 4.5). *Assume that K is a compact convex subset of an* l.c.t. *space X, then*

$$L(K) := \big\{a \in \mathrm{Aff}(K) : a = x^*_{|K} + r \textit{ for some } x^* \in X^* \textit{ and some } r \in \mathbb{R}\big\}$$

is dense in $(\mathrm{Aff}(K), \|\cdot\|_\infty)$.

If $(E, \|\cdot\|)$ is a Banach space and E^* is its topological dual, the space $X = (E^*, \text{weak}^*)$ is an l.c.t. space. It is well known that in this case we have that $X^* = E$ (see, for instance, [13, Corollary 224]). In this case, the exposed points of a subset of X coincide, by definition, with the weak* exposed points and the closure of this subset coincides with the weak* closure.

Proposition 6. *Let E be a Banach space. Let K be a convex weak* compact subset of E^* such that the norm interior of K is nonempty. Let X be the* l.c.t. *space* (E^*, weak^*). *Then, $X^* = E$ and $w^* \operatorname{Ext}(K) = \operatorname{AExp}(K)$.*

Proof. The fact that $X^* = E$, follows from [13, Corollary 224]. Now, let a be a point in the interior of K. Replacing K by $K - a$ we can assume without loss of generality that 0 belongs to the interior of K. Thus, from [13, Corollary 224], each linear functional that is continuous on K belongs to the space E. This shows that

$$\left\{x^*_{|K} + r : x^* \in X^*, r \in \mathbb{R}\right\} = \{\hat{x}_{|K} + r : x \in E, r \in \mathbb{R}\},$$

where $\hat{x}$ denotes the map $p \mapsto p(x)$ for all $p \in E^*$. It is easy to see that the space $\{\hat{x}_{|K} + r : x \in E, r \in \mathbb{R}\}$ equipped with the sup-norm on K is isomorphic to $(E \oplus \mathbb{R}, \|\cdot\| + \|\cdot\|)$, since K is norm bounded and contains 0 in its (norm) interior. Hence, $\{x^*_{|K} + r : x^* \in X^*, r \in \mathbb{R}\}$ is a closed Banach subspace of $(\operatorname{Aff}(K), \|\cdot\|_\infty)$. This implies by Proposition 5 that

$$\operatorname{Aff}(K) = \{\hat{x}_{|K} + r : x \in E, r \in \mathbb{R}\}.$$

Note that since $X^* = E$, by definition we have that the weak* exposed points of the set K considered as a subset of the dual Banach space $(E^*, \|\cdot\|)$ coincide with the exposed points of K considered as a subset of the l.c.t. space $X = (E^*, \text{weak}^*)$. Note also that a map $\hat{x}_{|K} + r$ affine expose K if and only if $\hat{x}_{|K}$ expose K. Hence, $w^* \operatorname{Ext}(K) = \operatorname{AExp}(K)$. $\square$

Proposition 7. *In normed vector space, the exposed points and the affine exposed points coincide for a nonempty convex compact set of finite-dimension* (*i.e., compact convex set whose affine hull is finite-dimensional*).

Proof. Let V be a normed vector space and K be a convex compact set of finite-dimension. Up to a translation, we can assume without loss of generality that $0 \in K$. Let V_0 be the linear hull of K. Then K has nonempty interior in V_0, since V_0 is finite-dimentional. Hence, we have that $\operatorname{Aff}(K) = \{x^*_{|K} + r / x^* \in V_0^*; r \in \mathbb{R}\}$ by Proposition 6 (since weak and norm topology coincide in finite-dimentional). Thus, if $\varphi \in \operatorname{Aff}(K)$, affine expose $k \in K$, then there exists $x^* \in V_0^*$ and $r \in \mathbb{R}$ such that $x^*_{|K} + r$ expose K at k. This is equivalent to the fact that x^* expose K at k (since r is a constant). Now, by the Hahn–Banach theorem, there exists $\tilde{x}^* \in V^*$ such that $\tilde{x}^*$ coincides with x^* on V_0. Hence, $\tilde{x}^* \in V^*$ also expose K at k. This shows that k is an exposed point of K. $\square$

REMARK 5. We know from Proposition 5 that the set

$$L(K) := \{a \in \mathrm{Aff}(K) : a = r + x^*_{|K} \text{ for some } x^* \in X^* \text{ and some } r \in \mathbb{R}\}$$

is dense in $(\mathrm{Aff}(K), \|\cdot\|_\infty)$. As it is given in Proposition 6, there exists situations where the sets $L(K)$ and $\mathrm{Aff}(K)$ coincide, for instance, if $K = B_{E^*}$ in (E^*, weak^*), where E is a Banach space. There exist also other situations, where $L(K)$ can be a very "small" subset of $\mathrm{Aff}(K)$. Indeed, if K is a convex compact metrizable subset of a l.c.t. space X without exposed points (for example if $K = [-1, 1]^{\aleph_0}$ in $\mathbb{R}^{\aleph_0}$), then from Proposition 4 we get that $L(K) \subset N(0)$ and so $L(K)$ can be covered by countably many *d.c.* hypersurface in $(\mathrm{Aff}(K), \|\cdot\|_\infty)$.

EXAMPLE C (*Example where* $\mathrm{AExp}(K) \subsetneq \mathrm{Ext}(K)$). It is well known that even in the two dimensional space $\mathbb{R}^2$, there exists a closed unit ball B for a suitable norm, such that $\mathrm{Ext}(B) \neq \mathrm{Ext}(B)$ (see, for instance, Examples 5.9 in [19]). Thus by Proposition 6, we have also that $\mathrm{AExp}(B) \neq \mathrm{Ext}(B)$.

REMARK 6. Note that Corollary 3 and Theorem 4 fail for convex compact sets which are not metrizable. Indeed, take $K = B_{E^*}$ where $E = l^1(\Gamma)$ (Γ is uncountable set), we know that the norm $\|\cdot\|_1$ is nowhere Gâteaux differentiable (see Example 1.4(b), page 3 in [19]). So from [19, Proposition 6.9] we get that the dual unit ball $B_{(l^1(\Gamma))^*}$ in the l.c.t. space $((l^1(\Gamma))^*, \mathrm{weak}^*)$ has no (weak*) exposed points. It follows from Proposition 6 that $w^* \,\mathrm{Ext}(B_{(l^1(\Gamma))^*}) = \mathrm{AExp}(B_{(l^1(\Gamma))^*}) = \emptyset$. Note also that the assumption of local convexity cannot be omitted. Indeed, Roberts proved in [21], that there exist a Hausdorff topological vector space X which is metrizable by a complete metric, and a nonempty compact convex set $K \subset X$ such that $\mathrm{Ext}(K) = \emptyset$.

A.2. Remarks on the A.E.P.P. spaces. We introduce the following class of l.c.t. spaces.

DEFINITION 6. An l.c.t. space X is said to have the "Affine Exposed Points Property" (in short A.E.P.P.) if and only if every convex compact subset of X is the closed convex hull of its affine exposed points.

Let us define

$$\Xi := \{X \text{ l.c.t. space in which every compact subset is metrizable}\}.$$

The class Ξ was actively studied in the 1980's by several authors. This class contains of course all metrizable l.c.t. spaces, in particular Fréchet spaces but is much larger. For several examples, we refer to [9] and references therein.

We obtain immediately from Theorem 4 the following corollary.

COROLLARY 8. *Every space from the class* Ξ, *has the A.E.P.P.*

In particular, the space $\mathbb{R}^{\aleph_0}$ has the A.E.P.P. Examples of l.c.t. spaces having the A.E.P.P. who do not belong to the class Ξ are given in Remark 7.

For an example of an l.c.t. space without A.E.P.P., we mention the l.c.t. space $((l^1(\Gamma))^*, \mathrm{weak}^*)$, where Γ is uncountable (see Remark 7). Thus, spaces having A.E.P.P. encompass a broad class of spaces and it would be interesting to better know their properties.

Examples 1. Immediate examples.

(1) Every Fréchet space has the A.E.P.P.

(2) Every convex closed and bounded subset of a Fréchet–Montel space is the closed convex hull of its affine exposed points (in Fréchet–Montel space, any closed bounded set is compact metrizable). A classical example of a Fréchet–Montel space is the space $C^\infty(\Omega)$ of smooth functions on an open set Ω in $\mathbb{R}^n$.

Recall that a Banach space E is said to be a Gâteaux differentiability space (GDS) iff each convex continuous real valued function defined on E is Gâteaux differentiable at each point of a dense subset. In [19], Phelps proved the following result.

Theorem 5 ([19], Theorem 6.2, page 95). *A Banach space E is a GDS if and only if every weak* compact convex subset of E^* is the weak* closed convex hull of its weak* exposed points.*

Remark 7. (1) Since the exposed points are in particular affine exposed points, it follows from the above theorem that the space (E^*, weak^*) has the A.E.P.P. whenever E is a GDS. However, if E is a nonseparable GDS, the dual unit ball is a weak* compact not metrizable subset. Thus, the space (E^*, weak^*) has the A.E.P.P. but $(E^*, \mathrm{weak}^*) \notin \Xi$, whenever E is a nonseparable GDS (for example, the nonseparable Hilbert spaces).

(2) The l.c.t. space $((l^1(\Gamma))^*, \mathrm{weak}^*)$ does not have the A.E.P.P. (see Remark 6). More generally, the l.c.t. space (E^*, weak^*) does not have the A.E.P.P. whenever E is a Banach space equipped with a nowhere Gâteaux differentiable norm.

The next proposition provides examples of nonmetrizable spaces which belong to the class Ξ.

Proposition 8. *Let E be a separable Banach space. Then (E^*, weak^*) and (E, weak) belong to the class Ξ; in particular, they have the A.E.P.P. but are not metrizable.*

Proof. It is well known that the whole spaces (E^*, weak^*) and (E, weak) are not metrizable. It is also well known that a Banach space E is separable iff every compact subset of (E^*, weak^*) is metrizable. Thus, $(E^*, \mathrm{weak}^*) \in \Xi$. For the space (E, weak), let K be a weak compact subset of E. Since E is separable, then K is also separable. Now, consider K as a subset of E^{**} by the canonical embedding, we get that K is norm separable and weak* compact subset of E^{**}, which implies from [5, Lemma 1] that K is weak* metrizable in E^{**}. In other words, K is weak metrizable. Thus $(E, \text{weak}) \in \Xi$. □

Several other not trivial examples of spaces from Ξ can be found in [9].

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