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WEIGHTED LOCAL HARDY SPACES ASSOCIATED TO SCHRÖDINGER OPERATORS

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ABSTRACT. In this paper, we characterize the weighted local Hardy spaces $h^p_\rho(\omega)$ related to the critical radius function ρ and weights $\omega \in A^{\rho,\infty}_\infty(\mathbb{R}^n)$ which locally behave as Muckenhoupt's weights and actually include them, by the local vertical maximal function, the local nontangential maximal function and the atomic decomposition. Then, we establish the equivalence of the weighted local Hardy space $h^1_\rho(\omega)$ and the weighted Hardy space $H^1_{\mathcal{L}}(\omega)$ associated to Schrödinger operators $\mathcal{L}$ with $\omega \in A^{\rho,\infty}_1(\mathbb{R}^n)$. By the atomic characterization, we also prove the existence of finite atomic decompositions associated with $h^p_\rho(\omega)$. Furthermore, we establish boundedness in $h^p_\rho(\omega)$ of quasi-Banach-valued sublinear operators.

1. Introduction

The theory of classical local Hardy spaces, originally introduced by Goldberg [14], plays an important role in various fields of analysis and partial differential equations; see [6], [20], [23], [28], [29], [30] and their references. In particular, pseudo-differential operators are bounded on local Hardy spaces $h^p(\mathbb{R}^n)$ for $p \in (0,1]$, but they are not bounded on Hardy spaces $H^p(\mathbb{R}^n)$ for $p \in (0,1]$; see [14] (also [29], [30]). In [6], Bui studied the weighted local Hardy space $h^p_\omega(\mathbb{R}^n)$ with $\omega \in A_\infty(\mathbb{R}^n)$, where and in what follows, $A_p(\mathbb{R}^n)$ for $p \in [1,\infty]$ denotes the class of Muckenhoupt's weights; see [7], [12], [15], [23] for their definition and properties.

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In [19], Rychkov introduced and studied some properties of the weighted Besov–Lipschitz spaces and Triebel–Lizorkin spaces with weights that are locally in $A_p(\mathbb{R}^n)$ but may grow or decrease exponentially, which contain Hardy spaces. In particular, Rychkov [19] generalized some of theories of weighted local Hardy spaces developed by Bui [6] to $A_\infty^{\mathrm{loc}}(\mathbb{R}^n)$ weights, where $A_\infty^{\mathrm{loc}}(\mathbb{R}^n)$ weights denote local $A_\infty(\mathbb{R}^n)$ weights which are non-doubling weights, and $A_\infty^{\mathrm{loc}}(\mathbb{R}^n)$ weights include $A_\infty(\mathbb{R}^n)$ weights. Recently, Tang [24] established the weighted atomic decomposition characterization of the weighted local Hardy space $h_\omega^p(\mathbb{R}^n)$ with $\omega \in A_\infty^{\mathrm{loc}}(\mathbb{R}^n)$ via the local grand maximal function, and gave some criteria about boundedness of $\mathcal{B}_\beta$-sublinear operators on $h_\omega^p(\mathbb{R}^n)$ which was first introduced in [33]; meanwhile, Tang [24] also proved that pseudo-differential operators are bounded on local Hardy spaces $h_\omega^p(\mathbb{R}^n)$ for $p \in (0,1]$ by using the above criteria and main results in [25]. Furthermore, Yang–Yang [32] extended the main results in [24] to the weighted local Orlicz–Hardy space $h_\omega^\Phi(\mathbb{R}^n)$ case.

On the other hand, the study of Schrödinger operator $\mathcal{L} = -\Delta + V$ recently attracted much attention; see [1], [2], [3], [9], [10], [21], [27], [26], [31], [33], [34], [35], [36]. In particular, J. Dziubański and J. Zienkiewicz [9], [10] studied Hardy space $H_{\mathcal{L}}^1$ associated to Schrödinger operators $\mathcal{L}$ with potential satisfying reverse Hölder inequality. Recently, Bongioanni et al. [2] introduced new classes of weights, related to Schrödinger operators $\mathcal{L}$, that is, $A_p^{\rho,\infty}(\mathbb{R}^n)$ weight which are in general larger than Muckenhoupt's (see Section 2 for notions of $A_p^{\rho,\infty}(\mathbb{R}^n)$ weight). Naturally, it is a very interesting problem whether we can give an atomic characterization for weighted Hardy space $H_{\mathcal{L}}^1(\omega)$ with $\omega \in A_1^{\rho,\infty}(\mathbb{R}^n)$.

The purpose of this paper is to give a positive answer. More precisely, we first introduce the weighted local Hardy spaces $h_\rho^p(\omega)$ with $A_q^{\rho,\infty}(\mathbb{R}^n)$ weights, and establish the atomic characterization of the weighted local Hardy spaces $h_\rho^p(\omega)$ with $\omega \in A_q^{\rho,\infty}(\mathbb{R}^n)$ weights. Then, we establish the equivalence between the weighted local Hardy spaces $h_\rho^1(\omega)$ and the weighted Hardy space $H_{\mathcal{L}}^1(\omega)$ associated to Schrödinger operator $\mathcal{L}$ with $\omega \in A_1^{\rho,\infty}(\mathbb{R}^n)$. In particular, it should be pointed out that we cannot directly obtain the atomic characterization of $H_{\mathcal{L}}^1(\omega)$ with $A_1^{\rho,\infty}(\mathbb{R}^n)$ weights by using the methods in [9], [10], [11], which forces us to use the above weighted local Hardy spaces $h_\rho^1(\omega)$ theory to overcome the difficulty.

The paper is organized as follows. In Section 2, we review some notions and notations concerning the weight classes $A_p^{\rho,\theta}(\mathbb{R}^n)$ introduced in [2], [27], [26]. In Section 3, we first introduce the weighted local Hardy space $h_{\rho,N}^p(\omega)$ via the local grand maximal function, and then the weighted atomic local Hardy space $h_\rho^{p,q,s}(\omega)$ for any admissible triplet $(p,q,s)_\omega$ (see Definition 3.3 below), furthermore, we establish the local vertical and the local nontangential maximal function characterizations of $h_{\rho,N}^p(\omega)$ via a local Calderón reproducing formula and some useful estimates established by Rychkov [19]. In Section 4, we establish the Calderón–Zygmund decomposition associated with the grand maximal function. In Section 5, we prove that for any given admissible triplet $(p,q,s)_\omega$, $h^p_{\rho,N}(\omega)=h^{p,q,s}_\rho(\omega)$ with equivalent norms. It is worth pointing out that we obtain Theorem 5.5 by a way different from the methods in [14], [6], but close to those in [4], [24], [32]. For simplicity, in the rest of this introduction, we denote by $h^p_\rho(\omega)$ the weighted local Hardy space $h^p_{\rho,N}(\omega)$. In Section 6, we apply the atomic characterization of the weighted local Hardy spaces $h^1_\rho(\omega)$ to establish atomic characterization of weighted Hardy space $H^1_{\mathcal{L}}(\omega)$ associated to Schrödinger operator $\mathcal{L}$ with $A^{\rho,\infty}_1(\mathbb{R}^n)$ weights. In Section 7, we prove that $\|\cdot\|_{h^{p,q,s}_{\rho,\mathrm{fin}}(\omega)}$ and $\|\cdot\|_{h^p_\rho(\omega)}$ are equivalent quasi-norms on $h^{p,q,s}_{\rho,\mathrm{fin}}(\omega)$ with $q<\infty$, and we obtain criteria for boundedness of $\mathcal{B}_\beta$-sublinear operators in $h^p_\rho(\omega)$. We remark that this extends both the results of Meda–Sjögren–Vallarino [17] and Yang–Zhou [33] to the setting of weighted local Hardy spaces.

Throughout this paper, we let C denote constants that are independent of the main parameters involved but whose value may differ from line to line. By $A\sim B$, we mean that there exists a constant $C>1$ such that $1/C\le A/B\le C$. The symbol $A\lesssim B$ means that $A\le CB$. The symbol $[s]$ for $s\in\mathbb{R}$ denotes the maximal integer not more than s. We also set $\mathbb{N}\equiv\{1,2,\ldots\}$ and $\mathbb{Z}_+\equiv\mathbb{N}\cup\{0\}$. The multi-index notation is usual: for $\alpha=(\alpha_1,\ldots,\alpha_n)$ and $\partial^\alpha=(\partial/\partial_{x_1})^{\alpha_1}\cdots(\partial/\partial_{x_n})^{\alpha_n}$. Given a function g on $\mathbb{R}^n$, we let $L_g\in\mathbb{Z}_+$ denote the maximal number such that g has vanishing moments up to the order L_g, i.e., $\int x^\alpha g(x)\,dx=0$ for all multi-indices α with $|\alpha|\le L_g$. If no vanishing moments of g, then we put $L_g=-1$.

2. Preliminaries

In this section, we review some notions and notations concerning the weight classes $A^{\rho,\theta}_p(\mathbb{R}^n)$ introduced in [2], [27], [26]. Given $B=B(x,r)$ and $\lambda>0$, we will write λB for the λ-dilate ball, which is the ball with the same center x and with radius λr. Similarly, $Q(x,r)$ denotes the cube centered at x with side length r (here and below only cubes with sides parallel to the axes are considered), and $\lambda Q(x,r)=Q(x,\lambda r)$. Especially, we will denote $2B$ by B^*, and $2Q$ by Q^*.

Let $\mathcal{L}=-\Delta+V$ be a Schrödinger operator on $\mathbb{R}^n$, $n\ge 3$, where $V\not\equiv 0$ is a fixed non-negative potential. We assume that V belongs to the reverse Hölder class $RH_s(\mathbb{R}^n)$ for some $s\ge n/2$; that is, there exists $C=C(s,V)>0$ such that

$$\left(\frac{1}{|B|}\int_B V(x)^s\,dx\right)^{\frac{1}{s}}\le C\left(\frac{1}{|B|}\int_B V(x)\,dx\right),$$

for every ball $B\subset\mathbb{R}^n$. Trivially, $RH_q(\mathbb{R}^n)\subset RH_p(\mathbb{R}^n)$ provided $1<p\le q<\infty$. It is well known that, if $V\in RH_q(\mathbb{R}^n)$ for some $q>1$, then there exists $\varepsilon>0$, which depends only on d and the constant C in the above inequality, such that $V\in RH_{q+\varepsilon}(\mathbb{R}^n)$ (see [13]). Moreover, the measure $V(x)\,dx$ satisfies the doubling condition:

$$\int_{B(y,2r)} V(x)\,dx\le C\int_{B(y,r)} V(x)\,dx.$$

With regard to the Schrödinger operator $\mathcal{L}$, we know that the operators derived from $\mathcal{L}$ behave "locally" quite similar to those corresponding to the Laplacian (see [8], [21]). The notion of locality is given by the critical radius function

$$\rho(x)=\frac{1}{m_V(x)}=\sup_{r>0}\left\{r:\frac{1}{r^{n-2}}\int_{B(x,r)} V(y)\,dy\le 1\right\}. \tag{2.1}$$

Throughout the paper, we assume that $V\not\equiv 0$, so that $0<\rho(x)<\infty$ (see [21]). In particular, $m_V(x)=1$ with $V=1$ and $m_V(x)\sim(1+|x|)$ with $V=|x|^2$.

Lemma 2.1 (See [21]). *There exist $C_0\ge 1$ and $k_0\ge 1$ so that for all $x,y\in\mathbb{R}^n$,*

$$C_0^{-1}\rho(x)\left(1+\frac{|x-y|}{\rho(x)}\right)^{-k_0}\le\rho(y)\le C_0\rho(x)\left(1+\frac{|x-y|}{\rho(x)}\right)^{\frac{k_0}{k_0+1}}. \tag{2.2}$$

In particular, $\rho(x)\sim\rho(y)$ when $y\in B(x,r)$ and $r\le C\rho(x)$, where C is a positive constant.

A ball of the form $B(x,\rho(x))$ is called critical, and in what follows we will call critical radius function to any positive continuous function ρ that satisfies (2.2), not necessarily coming from a potential V. Clearly, if ρ is such a function, so is $\beta\rho$ for any $\beta>0$. As the consequence of the above lemma, we acquire the following result.

Lemma 2.2 (See [9]). *There exists a sequence of points $x_j\in\mathbb{R}^n$, $j\ge 1$, such that the family $B_j=B(x_j,\rho(x_j))$, $j\ge 1$ satisfies*:

(a) $\bigcup_j B_j=\mathbb{R}^n$.
(b) *For every $\sigma\ge 1$ there exist constants C and N_1 such that $\sum_j\chi_{\sigma B_j}\le C\sigma^{N_1}$.*

In this paper, we write $\Psi_\theta(B)=(1+r/\rho(x_0))^\theta$, where $\theta\ge 0$, x_0 and r denote the center and radius of B, respectively.

A weight always refers to a positive function which is locally integrable. As in [2], we say that a weight ω belongs to the class $A_p^{\rho,\theta}(\mathbb{R}^n)$ for $1<p<\infty$, if there is a constant C such that, for all balls B

$$\left(\frac{1}{\Psi_\theta(B)|B|}\int_B\omega(y)\,dy\right)\left(\frac{1}{\Psi_\theta(B)|B|}\int_B\omega^{-\frac{1}{p-1}}(y)\,dy\right)^{p-1}\le C.$$

We also say that a nonnegative function ω satisfies the $A_1^{\rho,\theta}(\mathbb{R}^n)$ condition if there exists a constant C such that

$$M_{V,\theta}(\omega)(x) \le C\omega(x), \quad \text{a.e. } x \in \mathbb{R}^n,$$

where

$$M_{V,\theta} f(x) \equiv \sup_{x\in B} \frac{1}{\Psi_\theta(B)|B|} \int_B \left|f(y)\right| dy.$$

When $V = 0$, we denote $M_0 f(x)$ by $Mf(x)$ (the standard Hardy–Littlewood maximal function). It is easy to see that $|f(x)| \le M_{V,\theta} f(x) \le Mf(x)$ for a.e. $x \in \mathbb{R}^n$ and any $\theta \ge 0$.

Clearly, the classes $A_p^{\rho,\theta}$ are increasing with θ, and we denote $A_p^{\rho,\infty} = \bigcup_{\theta\ge 0} A_p^{\rho,\theta}$. By Hölder's inequality, we see that $A_{p_1}^{\rho,\theta} \subset A_{p_2}^{\rho,\theta}$, if $1 \le p_1 < p_2 < \infty$, and we also denote $A_\infty^{\rho,\infty} = \bigcup_{p\ge 1} A_p^{\rho,\infty}$. In addition, for $1 \le p \le \infty$, we denote by p' the adjoint number of p, i.e., $1/p + 1/p' = 1$.

Since $\Psi_\theta(B) \ge 1$ with $\theta \ge 0$, then $A_p \subset A_p^{\rho,\theta}$ for $1 \le p < \infty$, where A_p denotes the classical Muckenhoupt weights; see [12] and [18]. Moreover, the inclusions are proper. In fact, as the example given in [27], let $\theta > 0$ and $0 \le \gamma \le \theta$, it is easy to check that $\omega(x) = (1 + |x|)^{-(n+\gamma)} \notin A_\infty = \bigcup_{p\ge 1} A_p$ and $\omega(x)\,dx$ is not a doubling measure, but $\omega(x) = (1 + |x|)^{-(n+\gamma)} \in A_1^{\rho,\theta}$ provided that $V = 1$ and $\Psi_\theta(B(x_0, r)) = (1 + r)^\theta$.

In what follows, given a Lebesgue measurable set E and a weight ω, $|E|$ will denote the Lebesgue measure of E and $\omega(E) := \int_E \omega(x)\,dx$. For any $\omega \in A_\infty^{\rho,\infty}$, the space $L^p_\omega(\mathbb{R}^n)$ with $p \in (0, \infty)$ denotes the set of all measurable functions f such that

$$\|f\|_{L^p_\omega(\mathbb{R}^n)} \equiv \left(\int_{\mathbb{R}^n} \left|f(x)\right|^p \omega(x)\,dx \right)^{1/p} < \infty,$$

and $L^\infty_\omega(\mathbb{R}^n) \equiv L^\infty(\mathbb{R}^n)$. The symbol $L^{1,\infty}_\omega(\mathbb{R}^n)$ denotes the set of all measurable functions f such that

$$\|f\|_{L^{1,\infty}_\omega(\mathbb{R}^n)} \equiv \sup_{\lambda>0} \left\{ \lambda\omega\left(\left\{ x \in \mathbb{R}^n : \left|f(x)\right| > \lambda \right\} \right) \right\} < \infty.$$

We define the local Hardy–Littlewood maximal operator by

$$M^{\text{loc}} f(x) \equiv \sup_{\substack{x\in B(x_0,r)\\ r\le\rho(x_0)}} \frac{1}{|B|} \int_B \left|f(y)\right| dy. \tag{2.3}$$

We remark that balls can be replaced by cubes in definition of $A_p^{\rho,\theta}$ and $M_{V,\theta}$, since $\Psi(B) \le \Psi(2B) \le 2^\theta \Psi(B)$. In fact, for the cube $Q = Q(x_0, r)$, we can also define $\Psi_\theta(Q) = (1 + r/\rho(x_0))^\theta$. Then we give the weighted boundedness of $M_{V,\theta}$.

LEMMA 2.3 (See [27]). *Let* $1<p<\infty$, $p'=p/(p-1)$ *and assume that* $\omega\in A_p^{\rho,\theta}$. *There exists a constant* $C>0$ *such that*

$$\|M_{V,p'\theta}f\|_{L^p_\omega(\mathbb{R}^n)}\le C\|f\|_{L^p_\omega(\mathbb{R}^n)}.$$

Next, we give some properties of weights class $A_p^{\rho,\theta}$ for $p\ge 1$.

LEMMA 2.4. *Let* $\omega\in A_p^{\rho,\infty}=\bigcup_{\theta\ge 0}A_p^{\rho,\theta}$ *for* $p\ge 1$. *Then*

(i) *If* $1\le p_1<p_2<\infty$, *then* $A_{p_1}^{\rho,\theta}\subset A_{p_2}^{\rho,\theta}$.

(ii) $\omega\in A_p^{\rho,\theta}$ *if and only if* $\omega^{-\frac{1}{p-1}}\in A_{p'}^{\rho,\theta}$, *where* $1/p+1/p'=1$.

(iii) *If* $\omega\in A_p^{\rho,\infty}$, $1<p<\infty$, *then there exists* $\epsilon>0$ *such that* $\omega\in A_{p-\epsilon}^{\rho,\infty}$.

(iv) *Let* $f\in L_{\mathrm{loc}}(\mathbb{R}^n)$, $0<\delta<1$, *then* $(M_{V,\theta}f)^\delta\in A_1^{\rho,\theta}$.

(v) *Let* $1<p<\infty$, *then* $\omega\in A_p^{\rho,\infty}$ *if and only if* $\omega=\omega_1\omega_2^{1-p}$, *where* $\omega_1,\omega_2\in A_1^{\rho,\infty}$.

(vi) *For* $\omega\in A_p^{\rho,\theta}$, $Q=Q(x,r)$ *and* $\lambda>1$, *there exists a positive constant* C *such that*

$$\omega(\lambda Q)\le C\big(\Psi_\theta(\lambda Q)\big)^p\lambda^{np}\omega(Q).$$

(vii) *If* $p\in(1,\infty)$ *and* $\omega\in A_p^{\rho,\theta}(\mathbb{R}^n)$, *then the local Hardy–Littlewood maximal operator* M^{loc} *is bounded on* $L^p_\omega(\mathbb{R}^n)$.

(viii) *If* $\omega\in A_1^{\rho,\theta}(\mathbb{R}^n)$, *then* M^{loc} *is bounded from* $L^1_\omega(\mathbb{R}^n)$ *to* $L^{1,\infty}_\omega(\mathbb{R}^n)$.

Proof. (i)–(viii) have been proved in [2], [26]. □

For any $\omega\in A_\infty^{\rho,\infty}(\mathbb{R}^n)$, define the critical index of ω by

$$q_\omega\equiv\inf\big\{p\in[1,\infty):\omega\in A_p^{\rho,\infty}\big(\mathbb{R}^n\big)\big\}. \tag{2.4}$$

Obviously, $q_\omega\in[1,\infty)$. If $q_\omega\in(1,\infty)$, then $\omega\notin A_{q_\omega}^{\rho,\infty}$.

The symbols $\mathcal{D}(\mathbb{R}^n)=C_0^\infty(\mathbb{R}^n),\mathcal{D}'(\mathbb{R}^n)$ is the dual space of $\mathcal{D}(\mathbb{R}^n)$, and for $\mathcal{D}(\mathbb{R}^n),\mathcal{D}'(\mathbb{R}^n)$ and $L^p_\omega(\mathbb{R}^n)$, we have the following conclusions.

LEMMA 2.5. *Let* $\omega\in A_\infty^{\rho,\infty}(\mathbb{R}^n)$, q_ω *be as in* (2.4) *and* $p\in(q_\omega,\infty]$.

(i) *If* $\frac{1}{p}+\frac{1}{p'}=1$, *then* $\mathcal{D}(\mathbb{R}^n)\subset L^{p'}_{\omega^{-1/(p-1)}}(\mathbb{R}^n)$.

(ii) $L^p_\omega(\mathbb{R}^n)\subset\mathcal{D}'(\mathbb{R}^n)$ *and the inclusion is continuous.*

By the same method as the proof of Lemma 2.2 in [24], we can get Lemma 2.5, and we omit the details here.

For any $\varphi,\psi\in\mathcal{D}(\mathbb{R}^n)$, let $\varphi_t(x)=t^{-n}\varphi(x/t)$ for $t>0$ and $\psi_j(x)=2^{jn}\psi(2^jx)$ for $j\in\mathbb{Z}$. It is easy to see that we have the following results.

LEMMA 2.6 (See [24]). *Let* $\varphi\in\mathcal{D}(\mathbb{R}^n)$ *and* $\int_{\mathbb{R}^n}\varphi(x)\,dx=1$.

(i) *For any* $\Phi\in\mathcal{D}(\mathbb{R}^n)$ *and* $f\in\mathcal{D}'(\mathbb{R}^n)$, $\Phi*\varphi_t\to\Phi$ *in* $\mathcal{D}(\mathbb{R}^n)$ *as* $t\to 0$, *and* $f*\varphi_t\to f$ *in* $\mathcal{D}'(\mathbb{R}^n)$ *as* $t\to 0$.

(ii) *Let* $\omega\in A_\infty^{\rho,\infty}$ *and* q_ω *be as in* (2.3). *If* $q\in(q_\omega,\infty)$, *then for any* $f\in L^q_\omega(\mathbb{R}^n)$, $f*\varphi_t\to f$ *in* $L^q_\omega(\mathbb{R}^n)$ *as* $t\to 0$.

Now, let us introduce some local maximal functions. For $N \in \mathbb{Z}_+$ and $R \in (0,\infty)$, let

$$\mathcal{D}_{N,R}(\mathbb{R}^n) \equiv \Big\{ \varphi \in \mathcal{D}(\mathbb{R}^n) : \mathrm{supp}(\varphi) \subset B(0,R), \\ \|\varphi\|_{\mathcal{D}_N(\mathbb{R}^n)} \equiv \sup_{\substack{x\in\mathbb{R}^n \\ \alpha\in\mathbb{Z}_+^n, |\alpha|\le N}} \big|\partial^\alpha \varphi(x)\big| \le 1 \Big\}.$$

Definition 2.7. Let $N \in \mathbb{Z}_+$, $R \in (0,\infty)$ and $f \in \mathcal{D}'(\mathbb{R}^n)$, the local non-tangential grand maximal function of f is defined as:

$$\widetilde{\mathcal{M}}_{N,R}(f)(x) \equiv \sup\big\{ \big|\varphi_l * f(z)\big| : |x-z| < 2^{-l} < \rho(x), \varphi \in \mathcal{D}_{N,R}(\mathbb{R}^n) \big\}, \tag{2.5}$$

and the local vertical grand maximal function of f is defined as:

$$\mathcal{M}_{N,R}(f)(x) \equiv \sup\big\{ \big|\varphi_l * f(x)\big| : 0 < 2^{-l} < \rho(x), \varphi \in \mathcal{D}_{N,R}(\mathbb{R}^n) \big\}. \tag{2.6}$$

For simplicity, we denote $\mathcal{D}_{N,R}(\mathbb{R}^n)$, $\widetilde{\mathcal{M}}_{N,R}(f)$ and $\mathcal{M}_{N,R}(f)$ as $\mathcal{D}_N^0(\mathbb{R}^n)$, $\widetilde{\mathcal{M}}_N^0(f)$ and $\mathcal{M}_N^0(f)$ when $R=1$, and as $\mathcal{D}_N(\mathbb{R}^n)$, $\widetilde{\mathcal{M}}_N(f)$ and $\mathcal{M}_N(f)$ when $R = \max\{R_1,R_2,R_3\} > 1$ (in which R_1, R_2 and R_3 are defined as in the proof of Lemma 4.2, 4.5 and 4.8). Obviously, for any $N \in \mathbb{Z}_+$ and $x \in \mathbb{R}^n$,

$$\mathcal{M}_N^0(f)(x) \le \mathcal{M}_N(f)(x) \le \widetilde{\mathcal{M}}_N(f)(x).$$

Here and in what follows, the space $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ denotes the set of all locally integrable functions on $\mathbb{R}^n$. We have the following Proposition 2.8, which can be proved by the same method as in [24, Proposition 2.2].

Proposition 2.8. *Let $N \ge 2$. Then*

(i) *For all $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) \cap \mathcal{D}'(\mathbb{R}^n)$ and almost every $x \in \mathbb{R}^n$,*

$$\big|f(x)\big| \le \mathcal{M}_N^0(f)(x) \lesssim M^{\mathrm{loc}}(f)(x).$$

(ii) *If $\omega \in A_p^{\rho,\theta}(\mathbb{R}^n)$ with $p \in (1,\infty)$, then $f \in L^p_\omega(\mathbb{R}^n)$ if and only if $f \in \mathcal{D}'(\mathbb{R}^n)$ and $\mathcal{M}_N^0(f) \in L^p_\omega(\mathbb{R}^n)$; moreover,*

$$\|f\|_{L^p_\omega(\mathbb{R}^n)} \sim \big\|\mathcal{M}_N^0(f)\big\|_{L^p_\omega(\mathbb{R}^n)}.$$

(iii) *If $\omega \in A_1^{\rho,\theta}(\mathbb{R}^n)$, then $\mathcal{M}_N^0$ is bounded from $L^1_\omega(\mathbb{R}^n)$ to $L^{1,\infty}_\omega(\mathbb{R}^n)$.*

3. Weighted local Hardy spaces

In this section, we introduce the weighted local Hardy spaces $h^p_{\rho,N}(\omega)$ and weighted atomic local Hardy space $h^{p,q,s}_\rho(\omega)$. Furthermore, we give several equivalent characterizations of the weighted local Hardy spaces by a local Calderón reproducing formula and some properties of the weighted atomic local Hardy space.

The weighted local Hardy space is defined as follows.

DEFINITION 3.1. Let $\omega \in A_{\infty}^{\rho,\infty}(\mathbb{R}^n)$, q_ω be as in (2.4), $p \in (0,1]$ and $\widetilde{N}_{p,\omega} \equiv [n(\frac{q_\omega}{p}-1)]+2$. For each $N \in \mathbb{N}$ with $N \geq \widetilde{N}_{p,\omega}$, the weighted local Hardy space is defined by

$$h_{\rho,N}^p(\omega) \equiv \{f \in \mathcal{D}'(\mathbb{R}^n) : \mathcal{M}_N(f) \in L_\omega^p(\mathbb{R}^n)\}.$$

Moreover, we define $\|f\|_{h_{\rho,N}^p(\omega)} \equiv \|\mathcal{M}_N(f)\|_{L_\omega^p(\mathbb{R}^n)}$.

For any integers N_1 and N_2 with $N_1 \geq N_2 \geq \widetilde{N}_{p,\omega}$, we have

$$h_{\rho,\widetilde{N}_{p,\omega}}^p(\omega) \subset h_{\rho,N_2}^p(\omega) \subset h_{\rho,N_1}^p(\omega),$$

and the inclusions are continuous.

Now, we introduce the weighted local atoms and weighted atomic local Hardy space.

DEFINITION 3.2. Let $\omega \in A_{\infty}^{\rho,\infty}(\mathbb{R}^n)$, q_ω be as in (2.4). A triplet $(p,q,s)_\omega$ is called to be admissible, if $p \in (0,1]$, $q \in (q_\omega,\infty]$ and $s \in \mathbb{N}$ with $s \geq [n(q_\omega/p-1)]$. A function a on $\mathbb{R}^n$ is said to be a $(p,q,s)_\omega$-atom if

(i) $\operatorname{supp} a \subset Q(x,r)$ and $r \leq L_1\rho(x)$,
(ii) $\|a\|_{L_\omega^q(\mathbb{R}^n)} \leq [\omega(Q)]^{1/q-1/p}$,
(iii) $\int_{\mathbb{R}^n} a(x)x^\alpha\,dx = 0$ for all $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| \leq s$, when $Q = Q(x,r)$, $r < L_2\rho(x)$,

where $L_1 \equiv 4C_0(3\sqrt{n})^{k_0}$, $L_2 \equiv 1/C_0^2(3\sqrt{n})^{k_0+1}$, and C_0, k_0 are constant given in Lemma 2.1. Moreover, for $q \in (q_\omega,\infty]$, a function $a(x)$ is called a $(p,q)_\omega$-single-atom if

$$\|a\|_{L_\omega^q(\mathbb{R}^n)} \leq [\omega(\mathbb{R}^n)]^{1/q-1/p}.$$

DEFINITION 3.3. Let $\omega \in A_{\infty}^{\rho,\infty}(\mathbb{R}^n)$, q_ω be as in (2.4), and $(p,q,s)_\omega$ be admissible, we define the weighted atomic local Hardy space $h_\rho^{p,q,s}(\omega)$ by the set of all $f \in \mathcal{D}'(\mathbb{R}^n)$ satisfying that

$$f = \sum_{i=0}^{\infty} \lambda_i a_i$$

in $\mathcal{D}'(\mathbb{R}^n)$, where $\{\lambda_i\}_{i\in\mathbb{Z}_+} \subset \mathbb{C}$, $\sum_{i=0}^{\infty} |\lambda_i|^p < \infty$ and $\{a_i\}_{i\in\mathbb{N}}$ are $(p,q,s)_\omega$-atoms and a_0 is a $(p,q)_\omega$-single-atom. Moreover, the quasi-norm of $f \in h_\rho^{p,q,s}(\omega)$ is defined by

$$\|f\|_{h_\rho^{p,q,s}(\omega)} \equiv \inf\left\{\left[\sum_{i=0}^{\infty} |\lambda_i|^p\right]^{1/p}\right\}.$$

It is easy to see that if triplets $(p,q,s)_\omega$ and $(p,\bar{q},\bar{s})_\omega$ are admissible and satisfy $\bar{q} \leq q$ and $\bar{s} \leq s$, then $(p,q,s)_\omega$-atoms are $(p,\bar{q},\bar{s})_\omega$-atoms, which implies that $h_\rho^{p,q,s}(\omega) \subset h_\rho^{p,\bar{q},\bar{s}}(\omega)$ and the inclusion is continuous.

Next, we will give several equivalent characterizations of the weighted local Hardy spaces $h_{\rho,N}^p(\omega)$ by the following local maximal functions.

DEFINITION 3.4. Let

$$\psi_0 \in \mathcal{D}(\mathbb{R}^n) \quad \text{with} \int_{\mathbb{R}^n} \psi_0(x)\,dx \neq 0. \tag{3.1}$$

For every $x \in \mathbb{R}^n$, there exists an integer $j_x \in \mathbb{Z}$ satisfying $2^{-j_x} < \rho(x) \leq 2^{-j_x+1}$, and then for $j \geq j_x$, $A, B \in [0, \infty)$ and $y \in \mathbb{R}^n$, we define

$$m_{j,A,B,x}(y) \equiv \big(1 + 2^j|y|\big)^A 2^{B|y|/\rho(x)}.$$

We define the local vertical maximal function of f associated to ψ_0 as

$$\psi_0^+(f)(x) \equiv \sup_{j \geq j_x} \big|(\psi_0)_j * f(x)\big|, \tag{3.2}$$

the local tangential Peetre-type maximal function of f associated to ψ_0 as

$$\psi_{0,A,B}^{**}(f)(x) \equiv \sup_{j \geq j_x, y \in \mathbb{R}^n} \frac{|(\psi_0)_j * f(x-y)|}{m_{j,A,B,x}(y)}, \tag{3.3}$$

and the local nontangential maximal function of f associated to ψ_0 as

$$(\psi_0)_\triangledown^*(f)(x) \equiv \sup_{|x-y|<2^{-l}<\rho(x)} \big|(\psi_0)_l * f(y)\big|, \tag{3.4}$$

where $l \in \mathbb{Z}$.

Obviously, for any $x \in \mathbb{R}^n$, we have

$$\psi_0^+(f)(x) \leq (\psi_0)_\triangledown^*(f)(x) \lesssim \psi_{0,A,B}^{**}(f)(x).$$

It should be pointed out that these local maximal functions were introduced by Rychkov in [19] and Yang in [32].

We introduce a lemma on the local reproducing formula, which can be deduced from Lemma 1.6 in [19], and we omit the details of its proof here.

LEMMA 3.5. *Let ψ_0 be as in (3.1), $f \in \mathcal{D}'(\mathbb{R}^n)$ and $\psi(x) \equiv \psi_0(x) - (1/2^n)\psi_0(x/2)$ for all $x \in \mathbb{R}^n$. Then there exist $\varphi_0, \varphi \in \mathcal{D}(\mathbb{R}^n)$ such that for any given integers $j \in \mathbb{Z}$ and $L \in \mathbb{Z}_+$, we have $L_\varphi \geq L$ and*

$$f = (\varphi_0)_j * (\psi_0)_j * f + \sum_{k=j+1}^{\infty} \varphi_k * \psi_k * f \tag{3.5}$$

in $\mathcal{D}'(\mathbb{R}^n)$.

LEMMA 3.6. *Let $0 < r < \infty$, ψ_0 be as in (3.3) and $\psi(x) \equiv \psi_0(x) - (1/2^n)\psi_0(x/2)$. Then there exists a positive constant A_0 depending only on the support of ψ_0 such that for any $A \in (A_0, \infty)$ and $B \in [0, \infty)$, there exists a positive constant C depending only on n, r, ψ_0, A and B, such that for all $f \in \mathcal{D}'(\mathbb{R}^n)$, $x, x_0 \in \mathbb{R}^n$ and $j \geq j_{x_0}$ (where $2^{-j_{x_0}} < \rho(x_0) \leq 2^{-j_{x_0}+1}$), we have*

$$\big|\psi_j * f(x)\big|^r \leq C \sum_{k=j}^{\infty} 2^{(j-k)Ar} 2^{kn} \int \frac{|\psi_k * f(x-y)|^r}{m_{j,Ar,Br,x_0}(y)}\,dy. \tag{3.6}$$

Proof. By Lemma 3.5, we can find $\varphi_0, \varphi \in \mathcal{D}(\mathbb{R}^n)$ so that $L_\varphi \geq A$ and (3.5) is true. Hence, we have

$$\psi_j * f = (\varphi_0)_j * (\psi_0)_j * \psi_j * f + \sum_{k=j+1}^{\infty} \psi_j * \varphi_k * \psi_k * f. \tag{3.7}$$

The function $\psi_j * \varphi_k$ $(k \geq j+1)$ have support size $\leq C2^{-j}$ and enjoy the uniform estimate

$$\|\psi_j * \varphi_k\|_{L^\infty(\mathbb{R}^n)} \leq C2^{(j-k)A}2^{jn}, \tag{3.8}$$

which can be easily deduced by the moment condition on φ (see [19, (2.13)]). Therefore, we may write

$$\left|\psi_j * \varphi_k(y)\right| \leq C\frac{2^{(j-k)A}2^{kn}}{m_{j,A,B,x_0}(y)} \quad \left(y \in \mathbb{R}^n\right). \tag{3.9}$$

Putting (3.9) together with the similar estimate for $(\varphi_0)_j * (\psi_0)_j$ into (3.7) gives (3.6) for $r = 1$, and the case $r > 1$ follows by Hölder's inequality. To obtain the case $r < 1$, we introduce the maximal functions

$$M_{A,B,x_0}(x,j) = \sup_{k \geq j, y \in \mathbb{R}^n} 2^{(j-k)A}\frac{|\psi_k * f(x-y)|}{m_{j,A,B,x_0}(y)}.$$

The (3.6) with $r = 1$ gives

$$2^{(j-k)A}\left|\psi_k * f(x-y)\right| \leq C\sum_{l=k}^{\infty} 2^{(j-l)A}2^{ln}\int \frac{|\psi_l * f(x-z)|}{m_{k,A,B,x_0}(z-y)}\,dz, \tag{3.10}$$

and the right-hand side of (3.10) decreases as k increases. Hence, to get the estimate for $M_{A,B,x_0}(x,j)$, we may only consider (3.10) with $k = j$. Combining with the elementary inequality

$$m_{j,A,B,x_0}(z) \leq m_{j,A,B,x_0}(y)m_{k,A,B,x_0}(z-y), \tag{3.11}$$

we can get

$$\begin{aligned} M_{A,B,x_0}(x,j) &\leq C\sum_{k=j}^{\infty} 2^{(j-k)A}2^{kn}\int \frac{|\psi_l * f(x-z)|}{m_{j,A,B,x_0}(z)}\,dz \\ &\leq CM_{A,B,x_0}(x,j)^{1-r}\sum_{k=j}^{\infty} 2^{(j-k)Ar}2^{kn}\int \frac{|\psi_l * f(x-z)|^r}{m_{j,Ar,Br,x_0}(z)}\,dz. \end{aligned} \tag{3.12}$$

Considering $|\psi_j * f(x)| \leq M_{A,B,x_0}(x,j)$, (3.12) implies (3.6), if $M_{A,B,x_0}(x,j) < \infty$. By [15, Proposition 2.3.4(a)], for any $f \in \mathcal{D}'(\mathbb{R}^n)$, we have $M_{A,B,x_0}(x,j) < \infty$ for all $x \in \mathbb{R}^n$ and $j \geq j_{x_0}$, provided $A > A_0$, where A_0 is a positive constant depending only on the support of ψ_0. This finishes the proof. □

For $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, $B \in [0,\infty)$ and $x \in \mathbb{R}^n$, define

$$K_B f(x) \equiv \frac{1}{(\rho(x))^n} \int_{\mathbb{R}^n} |f(y)| 2^{-B\frac{|x-y|}{\rho(x)}}\, dy, \tag{3.13}$$

and for the operator K_B, we have the following lemma:

LEMMA 3.7. *Let $p \in (1,\infty)$ and $\omega \in A_p^{\rho,\theta}(\mathbb{R}^n)$, then there exist constants $C > 0$ and $B_0 \equiv B_0(\omega, n) > 0$ such that for all $B > B_0/p$,*

$$\|K_B f\|_{L^p_\omega(\mathbb{R}^n)} \le C\|f\|_{L^p_\omega(\mathbb{R}^n)},$$

for all $f \in L^p_\omega(\mathbb{R}^n)$.

Proof. It is suffice to show that there exists a constant $C > 0$ such that for all $B > B_0$,

$$K_B f(x) \le C M_{V,p'\theta} f(x),$$

then combining with Lemma 2.3, we get the boundedness of the operator K_B.

To control $K_B f(x)$, we argue as follows:

$$\begin{aligned} K_B f(x) &= \frac{1}{(\rho(x))^n} \int_{\mathbb{R}^n} |f(y)| 2^{-B\frac{|x-y|}{\rho(x)}}\, dy \\ &= \frac{1}{(\rho(x))^n} \int_{|y-x|<\rho(x)} |f(y)| 2^{-B\frac{|x-y|}{\rho(x)}}\, dy \\ &\quad + \frac{1}{(\rho(x))^n} \int_{|y-x|\ge\rho(x)} |f(y)| 2^{-B\frac{|x-y|}{\rho(x)}}\, dy \\ &= \frac{1}{(\rho(x))^n} \int_{|y-x|<\rho(x)} |f(y)| 2^{-B\frac{|x-y|}{\rho(x)}}\, dy \\ &\quad + \sum_{k=0}^{\infty} \frac{1}{(\rho(x))^n} \int_{|y-x|\sim 2^k\rho(x)} |f(y)| 2^{-B\frac{|x-y|}{\rho(x)}}\, dy \\ &\equiv I_1 + I_2. \end{aligned}$$

For I_1, it is easy to get

$$I_1 \le \frac{C}{\Psi_{p'\theta}(B_1)|B_1|} \int_{B_1} |f(y)|\, dy \le C M_{V,p'\theta} f(x),$$

in which $B_1 = B(x,\rho(x))$ is a critical ball.

For I_2, we have

$$\begin{aligned} I_2 &\le C \sum_{k=0}^{\infty} \frac{(1+2^{k+1})^{p'\theta} 2^{kn}}{2^{B2^k}} \frac{1}{\Psi_{p'\theta}(2^{k+1}B_1)|2^{k+1}B_1|} \int_{2^{k+1}B_1} |f(y)|\, dy \\ &\le C \left(\sum_{k=0}^{\infty} \frac{(1+2^{k+1})^{p'\theta} 2^{kn}}{2^{B2^k}} \right) M_{V,p'\theta} f(x) \\ &\le C M_{V,p'\theta} f(x), \end{aligned}$$

where the sum converges when $B > B_0/p$. $\square$

LEMMA 3.8. *Let ψ_0 be as in (3.3) and $r \in (0,\infty)$. Then for any $A \in (\max\{A_0, n/r\},\infty)$ (where A_0 is as in Lemma 3.6) and $B \in [0,\infty)$, there exists a positive constant C, depending only on n, r, ψ_0, A and B, such that for all $f \in \mathcal{D}'(\mathbb{R}^n)$, $x \in \mathbb{R}^n$ and $j \geq j_x$ (where $2^{-j_x} < \rho(x) \leq 2^{-j_x+1}$),*

$$\big[(\psi_0)^*_{j,A,B}(f)(x)\big]^r \leq C\sum_{k=j}^{\infty} 2^{(j-k)(Ar-n)}\big\{M^{\mathrm{loc}}\big(\big|(\psi_0)_k * f\big|^r\big)(x) + K_{Br}\big(\big|(\psi_0)_k * f\big|^r\big)(x)\big\},$$

where

$$(\psi_0)^*_{j,A,B}(f)(x) \equiv \sup_{y\in\mathbb{R}^n} \frac{|(\psi_0)_j * f(x-y)|}{m_{j,A,B,x}(y)}$$

for all $x \in \mathbb{R}^n$.

Proof. First, we can get the stronger version of (3.8) by virtue of (3.11), that is:

$$\begin{aligned}
&\big[(\psi_0)^*_{j,A,B}(f)(x)\big]^r \\
&\leq C\sum_{k=j}^{\infty} 2^{(j-k)Ar}2^{kn}\int_{\mathbb{R}^n} \frac{|(\psi_0)_k * f(y)|^r}{m_{j,Ar,Br,x}(x-y)}\,dy \\
&\leq C\sum_{k=j}^{\infty} 2^{(j-k)(Ar-n)}\bigg\{2^{jn}\int_{|y-x|<2^{-j_x}} \frac{|(\psi_0)_k * f(y)|^r}{(1+2^j|x-y|)^{Ar}}\,dy \\
&\quad + 2^{jn}\int_{|y-x|\geq 2^{-j_x}} \frac{|(\psi_0)_k * f(y)|^r}{(2^j|x-y|)^{Ar}2^{Br|x-y|/\rho(x)}}\,dy\bigg\} \\
&\equiv C\sum_{k=j}^{\infty} 2^{(j-k)(Ar-n)}\{I + II\}.
\end{aligned}$$

Since $2^{-j_x} < \rho(x) \leq 2^{-j_x+1}$ and $j \geq j_x$, for I we have

$$\begin{aligned}
I &= 2^{jn}\int_{2^{-j}\leq|y-x|<2^{-j_x}} \frac{|(\psi_0)_k * f(y)|^r}{(1+2^j|x-y|)^{Ar}}\,dy \\
&\quad + 2^{jn}\int_{|y-x|\leq 2^{-j}} \frac{|(\psi_0)_k * f(y)|^r}{(1+2^j|x-y|)^{Ar}}\,dy \\
&\equiv I_1 + I_2.
\end{aligned}$$

According to the definition of $M^{\mathrm{loc}}f(x)$ (see (2.3)), for I_2 we have

$$I_2 \leq 2^{jn}\int_{|y-x|\leq 2^{-j}} \big|(\psi_0)_k * f(y)\big|^r\,dy \leq CM^{\mathrm{loc}}\big(\big|(\psi_0)_k * f\big|^r\big)(x),$$

and for I_1 we have

$$\begin{aligned}
I_1 &\le 2^{jn} \sum_{l=j_x+1}^{j} \int_{2^{-l}\le|y-x|<2^{-l+1}} \frac{|(\psi_0)_k * f(y)|^r}{(2^j|x-y|)^{Ar}}\,dy \\
&\le \sum_{l=j_x+1}^{j} \frac{2^{jn}(2^{-l+1})^n}{(2^{j-l})^{Ar}} \frac{1}{(2^{-l+1})^n} \int_{|y-x|\le 2^{-l+1}} \left|(\psi_0)_k * f(y)\right|^r dy \\
&\le \sum_{l=j_x+1}^{j} \frac{2^n}{2^{(Ar-n)(j-l)}} M^{\mathrm{loc}}\left(\left|(\psi_0)_k * f\right|^r\right)(x) \\
&\le C M^{\mathrm{loc}}\left(\left|(\psi_0)_k * f\right|^r\right)(x),
\end{aligned}$$

where $Ar > n$. In addition, with regard to II, we have the following estimate,

$$\begin{aligned}
II &\le \frac{2^{jn}(\rho(x))^n}{(2^{j-j_x})^{Ar}} \frac{1}{(\rho(x))^n} \int_{\mathbb{R}^n} \left|(\psi_0)_k * f(y)\right|^r 2^{-Br\frac{|x-y|}{\rho(x)}}\,dy \\
&\le C \frac{2^{jn}(2^{-j_x})^n}{(2^{j-j_x})^{Ar}} K_{Br}\left(\left|(\psi_0)_k * f\right|\right)(x) \\
&\le C K_{Br}\left(\left|(\psi_0)_k * f\right|\right)(x),
\end{aligned}$$

where the last inequality is a consequence of the fact that $j \ge j_x$ and $Ar > n$. This finishes the proof. □

Now we can establish weighted norm inequalities of $\psi_0^+(f)$, $\psi_{0,A,B}^{**}(f)$ and $\widetilde{\mathcal{M}}_{N,R}(f)$.

THEOREM 3.9. *Assume $\omega \in A_\infty^{\rho,\infty}(\mathbb{R}^n)$, $R \in (0,\infty)$, $p \in (0,1]$, ψ_0 and q_ω be as in (3.1) and (2.4). Let $A_1 \equiv \max\{A_0, nq_\omega/p\}$, $B_1 \equiv B_0/p$ and $N_0 \equiv [2A_1]+1$, where A_0 and B_0 are defined as in Lemmas 3.6 and 3.7. Then for any $f \in \mathcal{D}'(\mathbb{R}^n)$, $A \in (A_1,\infty)$, $B \in (B_1,\infty)$ and integer $N \ge N_0$, there exists a positive constant C, depending only on $A, B, N, R, \psi_0, \omega$ and n, such that*

$$\left\|\psi_{0,A,B}^{**}(f)\right\|_{L_\omega^p(\mathbb{R}^n)} \sim \left\|\psi_0^+(f)\right\|_{L_\omega^p(\mathbb{R}^n)}, \tag{3.14}$$

and

$$\left\|\widetilde{\mathcal{M}}_{N,R}(f)\right\|_{L_\omega^p(\mathbb{R}^n)} \le C \left\|\psi_0^+(f)\right\|_{L_\omega^p(\mathbb{R}^n)}. \tag{3.15}$$

Proof. We first prove (3.14). For $A \in (A_1,\infty)$ and $B \in (B_1,\infty)$, since $A_1 \equiv \max\{A_0, nq_\omega/p\}$ and $B_1 \equiv B_0/p$, there exists $r_0 \in (0, p/q_\omega)$ such that $A > n/r_0$ and $Br_0 > B_0/q_\omega$. Then, for all $x \in \mathbb{R}^n$ and $j \ge j_x$, by Lemma 3.8, we get

$$\begin{aligned}
\left[(\psi_0)_{j,A,B}^*(f)(x)\right]^{r_0} \lesssim \sum_{k=j}^{\infty} 2^{(j-k)(Ar_0-n)} \Big\{ & M^{\mathrm{loc}}\left(\left|(\psi_0)_k * f\right|^{r_0}\right)(x) \\
& + K_{Br_0}\left(\left|(\psi_0)_k * f\right|^{r_0}\right)(x) \Big\}.
\end{aligned} \tag{3.16}$$

Thus, by (3.16) and

$$\left|(\psi_0)_k * f(x)\right| \leq \psi_0^+(f)(x)$$

for any $x \in \mathbb{R}^n$ and $k \geq j_x$, we have

$$\left[\psi_{0,A,B}^{**}(f)(x)\right]^{r_0} \lesssim M^{\mathrm{loc}}\left(\left[\psi_0^+(f)\right]^{r_0}\right)(x) + K_{Br_0}\left(\left[\psi_0^+(f)\right]^{r_0}\right)(x). \tag{3.17}$$

Then by (3.17) we have

$$\begin{aligned} \int_{\mathbb{R}^n} \left|\psi_{0,A,B}^{**}(f)(x)\right|^p \omega(x)\,dx & \\ \lesssim \int_{\mathbb{R}^n} \left|\left\{M^{\mathrm{loc}}\left(\left[\psi_0^+(f)\right]^{r_0}\right)(x)\right\}\right|^{p/r_0} \omega(x)\,dx & \\ + \int_{\mathbb{R}^n} \left|\left\{K_{Br_0}\left(\left[\psi_0^+(f)\right]^{r_0}\right)(x)\right\}\right|^{p/r_0} \omega(x)\,dx & \\ \equiv I_1 + I_2. & \end{aligned} \tag{3.18}$$

For I_1, as $r_0 < p/q_\omega$, we have $q \equiv p/r_0 > q_\omega$ and $\omega \in A_q^{\rho,\infty}(\mathbb{R}^n)$, therefore by Lemma 2.4(vii) we get

$$\int_{\mathbb{R}^n} \left|M^{\mathrm{loc}}\left(\left[\psi_0^+(f)\right]^{r_0}\right)(x)\right|^{p/r_0} \omega(x)\,dx \lesssim \int_{\mathbb{R}^n} \left|\psi_0^+(f)\right|^p \omega(x)\,dx \tag{3.19}$$

and for I_2 by Lemma 3.7 we get

$$\int_{\mathbb{R}^n} \left|K_{Br_0}\left(\left[\psi_0^+(f)\right]^{r_0}\right)(x)\right|^{p/r_0} \omega(x)\,dx \lesssim \int_{\mathbb{R}^n} \left|\psi_0^+(f)\right|^p \omega(x)\,dx, \tag{3.20}$$

which together with (3.19) and

$$\psi_0^+(f)(x) \leq (\psi_0)_\triangledown^*(f)(x) \lesssim \psi_{0,A,B}^{**}(f)(x)$$

implies (3.14).

Next, we prove (3.15). For any $\gamma \in \mathcal{D}_{N,R}(\mathbb{R}^n)$, $x \in \mathbb{R}^n$, $l \in \mathbb{Z}$ (where l satisfies $2^{-l} \in (0, \rho(x)))$ and $j \geq j_x$ (where $2^{-j_x} < \rho(x) \leq 2^{-j_x+1}$), by Lemma 3.5, we have

$$\gamma_l * f = \gamma_l * (\varphi_0)_j * (\psi_0)_j * f + \sum_{k=j+1}^{\infty} \gamma_l * \varphi_k * \psi_k * f. \tag{3.21}$$

For any given $l_0 \in \mathbb{Z}$ which satisfies $2^{-l_0} \in (0, \rho(x))$, and $z \in \mathbb{R}^n$ which satisfies $|z - x| < 2^{-l_0}$, by (3.21) we have

$$\begin{aligned} \left|\gamma_{l_0} * f(z)\right| &\leq \left|\gamma_{l_0} * (\varphi_0)_{l_0} * (\psi_0)_{l_0} * f(z)\right| + \sum_{k=l_0+1}^{\infty} \left|\gamma_{l_0} * \varphi_k * \psi_k * f(z)\right| \\ &\leq \int_{\mathbb{R}^n} \left|\gamma_{l_0} * (\varphi_0)_{l_0}(y)\right| \left|(\psi_0)_{l_0} * f(z-y)\right| dy \\ &\quad + \sum_{k=l_0+1}^{\infty} \int_{\mathbb{R}^n} \left|\gamma_{l_0} * \varphi_k(y)\right| \left|\psi_k * f(z-y)\right| dy \\ &\equiv I_3 + I_4. \end{aligned} \tag{3.22}$$

For I_3, by

$$\psi^{**}_{0,A,B}(f)(x) = \sup_{j\ge j_x, y\in\mathbb{R}^n} \frac{|(\psi_0)_j * f(x-y)|}{m_{j,A,B,x}(y)} = \sup_{j\ge j_x, y\in\mathbb{R}^n} \frac{|(\psi_0)_j * f(x-(y+x-z))|}{m_{j,A,B,x}(y+x-z)} = \sup_{j\ge j_x, y\in\mathbb{R}^n} \frac{|(\psi_0)_j * f(z-y)|}{m_{j,A,B,x}(y+x-z)},$$

we have

$$\big|(\psi_0)_{l_0} * f(z-y)\big| \le \psi^{**}_{0,A,B}(f)(x) m_{l_0,A,B,x}(y+x-z),$$

which together with

$$m_{l_0,A,B,x}(y+x-z) \le m_{l_0,A,B,x}(x-z) m_{l_0,A,B,x}(y),$$

and

$$m_{l_0,A,B,x}(x-z) = \big(1+2^{l_0}|x-z|\big)^A 2^{B\frac{|x-z|}{\rho(x)}} \lesssim 2^A,$$

deduces that

$$\big|(\psi_0)_{l_0} * f(z-y)\big| \lesssim 2^A \psi^{**}_{0,A,B}(f)(x) m_{l_0,A,B,x}(y).$$

Then, we get

$$I_3 \lesssim 2^A \bigg\{\int_{\mathbb{R}^n} \big|\gamma_{l_0} * (\varphi_0)_{l_0}(y)\big| m_{l_0,A,B,x}(y)\, dy\bigg\} \psi^{**}_{0,A,B}(f)(x).$$

For I_4, when $k\in\mathbb{Z}$, we have

$$\big|\psi_k * f(z-y)\big| \le \big|(\psi_0)_k * f(z-y)\big| + \big|(\psi_0)_{k-1} * f(z-y)\big|$$

and

$$m_{k,A,B,x}(y+x-z) \le m_{k,A,B,x}(x-z) m_{k,A,B,x}(y),$$

since $m_{k,A,B,x}(x-z) \lesssim 2^{(k-l_0)A}$, we can get

$$\begin{aligned}\big|(\psi_0)_k * f(z-y)\big| &\le \psi^{**}_{0,A,B}(f)(x) m_{k,A,B,x}(y+x-z)\\ &\le \psi^{**}_{0,A,B}(f)(x) m_{k,A,B,x}(x-z) m_{k,A,B,x}(y)\\ &\lesssim 2^{(k-l_0)A} m_{k,A,B,x}(y) \psi^{**}_{0,A,B}(f)(x).\end{aligned}$$

We also have

$$\big|(\psi_0)_{k-1} * f(z-y)\big| \lesssim 2^{(k-l_0)A} m_{k,A,B,x}(y) \psi^{**}_{0,A,B}(f)(x).$$

Thus,

$$I_4 \lesssim \sum_{k=l_0+1}^{\infty} 2^{(k-l_0)A} \bigg\{\int_{\mathbb{R}^n} \big|\gamma_t * \varphi_k(y)\big| m_{k,A,B,x}(y)\, dy\bigg\} \psi^{**}_{0,A,B}(f)(x).$$

Therefore, we have

$$
\begin{aligned}
(3.23)\quad & \left|\gamma_{l_0} * f(z)\right| \\
& \lesssim \left\{ \int_{\mathbb{R}^n} \left|\gamma_{l_0} * (\varphi_0)_{l_0}(y)\right| m_{l_0,A,B,x}(y)\, dy \right. \\
& \quad \left. + \sum_{k=l_0+1}^{\infty} 2^{(k-l_0)A} \int_{\mathbb{R}^n} \left|\gamma_{l_0} * \varphi_k(y)\right| m_{k,A,B,x}(y)\, dy \right\} \psi^{**}_{0,A,B}(f)(x).
\end{aligned}
$$

Let $\mathrm{supp}(\varphi_0) \subset B(0,R_0)$, then $\mathrm{supp}((\varphi_0)_j) \subset B(0,2^{-j}R_0)$ for all $j \geq j_x$. Moreover, since $\mathrm{supp}(\gamma) \subset B(0,R)$, we have $\mathrm{supp}(\gamma_{l_0}) \subset B(0,2^{-l_0}R)$. Then, we get $\mathrm{supp}(\gamma_{l_0} * (\varphi_0)_{l_0}) \subset B(0,2^{-l_0}(R_0+R))$ and

$$
\left|\gamma_{l_0} * (\varphi_0)_{l_0}(y)\right| \lesssim \int_{\mathbb{R}^n} \left|\gamma_{l_0}(s)\right| \left|(\varphi_0)_{l_0}(y-s)\right| ds \lesssim 2^{l_0 n} \int_{\mathbb{R}^n} \left|\gamma_{l_0}(s)\right| ds \sim 2^{l_0 n},
$$

which implies that

$$
\begin{aligned}
(3.24)\quad & \int_{\mathbb{R}^n} \left|\gamma_{l_0} * (\varphi_0)_{l_0}(y)\right| m_{l_0,A,B,x}(y)\, dy \\
& \lesssim 2^{l_0 n} \int_{B(0,2^{-l_0}(R_0+R))} \left(1+2^{l_0}|y|\right)^A 2^{\frac{B|y|}{\rho(x)}}\, dy \lesssim 1.
\end{aligned}
$$

Furthermore, for φ with vanishing moments up to order N, by [19, (2.13)] we have

$$
\|\gamma_{l_0} * \varphi_k\|_{L^\infty(\mathbb{R}^n)} \lesssim 2^{(l_0-k)N} 2^{l_0 n}
$$

for all $k \in \mathbb{Z}$ with $k \geq l_0+1$. Then, for $l_0 \geq j_x$, $N > 2A$ and $\mathrm{supp}(\gamma_{l_0} * \varphi_k) \subset B(0,2^{-l_0}R_0 + 2^{-k}R)$, we get

$$
\begin{aligned}
(3.25)\quad & \sum_{k=l_0+1}^{\infty} 2^{(k-l_0)A} \int_{\mathbb{R}^n} \left|\gamma_{l_0} * \varphi_k(y)\right| m_{k,A,B,x}(y)\, dy \\
& \lesssim \sum_{k=l_0+1}^{\infty} 2^{(k-l_0)A} 2^{(l_0-k)N} 2^{l_0 n} \left(2^{-l_0}R_0 + 2^{-k}R\right)^n \\
& \quad \times \left[1 + 2^k\left(2^{-l_0}R_0 + 2^{-k}R\right)\right]^A 2^{B(2^{-l_0}R_0+2^{-k}R)/\rho(x)} \\
& \lesssim \sum_{k=l_0+1}^{\infty} 2^{(l_0-k)(N-2A)} \lesssim 1.
\end{aligned}
$$

Thus, by (3.23), (3.24) and (3.25), we have $|\gamma_{l_0} * f(z)| \lesssim \psi^{**}_{0,A,B}(f)(x)$, and by the arbitrariness of $l_0 \geq j_x$ and $z \in B(x,2^{-l_0})$, we can further obtain

$$
(3.26)\qquad \widetilde{\mathcal{M}}_{N,R}(f)(x) \lesssim \psi^{**}_{0,A,B}(f)(x),
$$

which deduces (3.15) and finishes the proof of this theorem. □

Here and in what follows, we define

$$N_{p,\omega} \equiv \max\{\widetilde{N}_{p,\omega}, N_0\}, \tag{3.27}$$

where $\widetilde{N}_{p,\omega}$ and N_0 are respectively as in Definition 3.1 and Theorem 3.9. Then we have the following equivalent characterizations of the weighted local Hardy spaces.

THEOREM 3.10. *Let $\omega \in A_\infty^{\rho,\infty}(\mathbb{R}^n)$, ψ_0 and $N_{p,\omega}$ be respectively as in (3.3) and (3.27). Then for any $f \in \mathcal{D}'(\mathbb{R}^n)$ and integer $N \ge N_{p,\omega}$, the following are equivalent*:

$$\|f\|_{h^p_{\rho,N}(\omega)} \sim \|\widetilde{\mathcal{M}}_N(f)\|_{L^p_\omega(\mathbb{R}^n)} \sim \|\widetilde{\mathcal{M}}^0_N(f)\|_{L^p_\omega(\mathbb{R}^n)} \sim \|\mathcal{M}^0_N(f)\|_{L^p_\omega(\mathbb{R}^n)}. \tag{3.28}$$

Proof. For any $N \ge N_{p,\omega}$, $f \in h^p_{\rho,N}(\omega)$, $\widetilde{\psi}_0$ satisfy (3.3) and $\widetilde{\psi}_0 \in \mathcal{D}_N(\mathbb{R}^n)$. by the definition of $\mathcal{M}_N(f)$, we get $\widetilde{\psi}_0^+(f) \le \mathcal{M}_N(f)$ and hence $\widetilde{\psi}_0^+(f) \in L^p_\omega(\mathbb{R}^n)$. Suppose $\operatorname{supp}(\psi_0) \subset B(0,R)$, then by (3.15), we have

$$\|\widetilde{\mathcal{M}}_{N,R}(f)\|_{L^p_\omega(\mathbb{R}^n)} \lesssim \|\widetilde{\psi}_0^+(f)\|_{L^p_\omega(\mathbb{R}^n)} \lesssim \|f\|_{h^p_{\rho,N}(\omega)}, \tag{3.29}$$

which combined with $\psi_0^+(f) \lesssim \widetilde{\mathcal{M}}_{N,R}(f)$ infers that $\psi_0^+(f) \in L^p_\omega(\mathbb{R}^n)$ and

$$\|\psi_0^+(f)\|_{L^p_\omega(\mathbb{R}^n)} \lesssim \|f\|_{h^p_{\rho,N}(\omega)}.$$

Then by the estimate

$$\psi_0^+(f) \le (\psi_0)^*_\triangledown(f) \lesssim \psi^{**}_{0,A,B}(f),$$

(3.14) and (3.15), we have $(\psi_0)^*_\triangledown(f) \in L^p_\omega(\mathbb{R}^n)$, $\widetilde{\mathcal{M}}_N(f) \in L^p_\omega(\mathbb{R}^n)$ and

$$\|\widetilde{\mathcal{M}}_N(f)\|_{L^p_\omega(\mathbb{R}^n)} \lesssim \|(\psi_0)^*_\triangledown(f)\|_{L^p_\omega(\mathbb{R}^n)} \lesssim \|\psi_0^+(f)\|_{L^p_\omega(\mathbb{R}^n)}.$$

Let ψ_1 satisfy (3.3) and $\psi_1 \in \mathcal{D}^0_N(\mathbb{R}^n)$. Then by (3.15), we have

$$\|\widetilde{\mathcal{M}}_N(f)\|_{L^p_\omega(\mathbb{R}^n)} \lesssim \|\psi_1^+(f)\|_{L^p_\omega(\mathbb{R}^n)},$$

which combined with $\psi_1^+(f) \le \mathcal{M}^0_N(f)$ and $\mathcal{M}_N(f) \le \widetilde{\mathcal{M}}_N(f)$ infers

$$\|\mathcal{M}_N(f)\|_{L^p_\omega(\mathbb{R}^n)} \lesssim \|\mathcal{M}^0_N(f)\|_{L^p_\omega(\mathbb{R}^n)}.$$

Then, by the definition of $h^p_{\rho,N}(\omega)$, we have $f \in h^p_{\rho,N}(\omega)$ and

$$\|f\|_{h^p_{\rho,N}(\omega)} \lesssim \|\mathcal{M}^0_N(f)\|_{L^p_\omega(\mathbb{R}^n)}.$$

On the other hand, by the facts that $\mathcal{M}^0_N(f) \le \widetilde{\mathcal{M}}^0_N(f) \le \widetilde{\mathcal{M}}_N(f)$ for any $f \in \mathcal{D}'(\mathbb{R}^n)$, we have

$$\|f\|_{h^p_{\rho,N}(\omega)} \lesssim \|\mathcal{M}^0_N(f)\|_{L^p_\omega(\mathbb{R}^n)} \le \|\widetilde{\mathcal{M}}^0_N(f)\|_{L^p_\omega(\mathbb{R}^n)} \le \|\widetilde{\mathcal{M}}_N(f)\|_{L^p_\omega(\mathbb{R}^n)},$$

which combined with (3.29) finishes the proof. □

By Theorems 3.9 and 3.10, we have the following corollary, and we omit the details here.

COROLLARY 3.11. *Let* $\omega \in A_\infty^{\rho,\infty}(\mathbb{R}^n)$, ψ_0 *be as in* (3.3), $N_{p,\omega}$ *be as in* (3.27), A *and* B *be as in Theorem* 3.9. *Then for any integer* $N \geq N_{p,\omega}$, $f \in h^p_{\rho,N}(\omega)$ *if and only if* $f \in \mathcal{D}'(\mathbb{R}^n)$ *and* $\psi^{**}_{0,A,B}(f) \in L^p_\omega(\mathbb{R}^n)$; *moreover,*

$$\|f\|_{h^p_{\rho,N}(\omega)} \sim \left\|\psi^{**}_{0,A,B}(f)\right\|_{L^p_\omega(\mathbb{R}^n)}.$$

Next, we give some basic properties of $h^p_{\rho,N}(\omega)$ and $h^{p,q,s}_\rho(\omega)$.

PROPOSITION 3.12. *Let* $\omega \in A_\infty^{\rho,\infty}(\mathbb{R}^n)$, $p \in (0,1]$ *and* $N_{p,\omega}$ *be as in* (3.29). *For any integer* $N \geq N_{p,\omega}$, *the inclusion* $h^p_{\rho,N}(\omega) \hookrightarrow \mathcal{D}'(\mathbb{R}^n)$ *is continuous.*

Proof. First, for any $x \in B(0,\rho(0))$, by Lemma 2.1, there exist $C_0 \geq 1$ and $k_0 \geq 1$, such that

$$\rho(0) \leq C_0\left(1+\frac{|x|}{\rho(0)}\right)^{k_0}\rho(x) \leq C_0 2^{k_0}\rho(x).$$

We take $r_1 \equiv \rho(0)/C_0 2^{k_0+1} < \min\{\rho(x),\rho(0)\}$, then we have $B(0,r_1) \subset B(0,\rho(0))$. In addition, for any $x \in B(0,r_1)$, we also have $|x| < r_1 < \rho(x)$.

Next, let $f \in h^p_{\rho,N}(\omega)$. For any given $\varphi \in \mathcal{D}(\mathbb{R}^n)$, suppose that $\mathrm{supp}(\varphi) \subset B(0,R)$ with $R \in (0,\infty)$. Then by Theorem 3.9 and 3.10, we have

$$\begin{aligned}\left|\langle f,\varphi\rangle\right| &= \left|f*\widetilde{\varphi}(0)\right| \leq \|\widetilde{\varphi}\|_{\mathcal{D}_{N,R}(\mathbb{R}^n)} \inf_{x\in B(0,r_1)} \widetilde{\mathcal{M}}_{N,R}(f)(x)\\ &\leq \|\widetilde{\varphi}\|_{\mathcal{D}_{N,R}(\mathbb{R}^n)}\left[\omega\left(B(0,r_1)\right)\right]^{-1/p}\left\|\widetilde{\mathcal{M}}_{N,R}(f)\right\|_{L^p_\omega(\mathbb{R}^n)}\\ &\lesssim \|\widetilde{\varphi}\|_{\mathcal{D}_{N,R}(\mathbb{R}^n)}\left[\omega\left(B(0,r_1)\right)\right]^{-1/p}\|f\|_{h^p_{\rho,N}(\omega)},\end{aligned}$$

where $\widetilde{\mathcal{M}}_{N,R}(f)$ is as in (2.5) and $\widetilde{\varphi}(x) \equiv \varphi(-x)$ for all $x \in \mathbb{R}^n$. This implies $f \in \mathcal{D}'(\mathbb{R}^n)$ and the inclusion is continuous. The proof is finished. □

PROPOSITION 3.13. *Let* $\omega \in A_\infty^{\rho,\infty}(\mathbb{R}^n)$, $p \in (0,1]$ *and* $N_{p,\omega}$ *be as in* (3.29). *For any integer* $N \geq N_{p,\omega}$, *the space* $h^p_{\rho,N}(\omega)$ *is complete.*

Proof. For any $\psi \in \mathcal{D}_N(\mathbb{R}^n)$ and $\{f_i\}_{i\in\mathbb{N}} \subset \mathcal{D}'(\mathbb{R}^n)$ such that $\{\sum_{i=1}^j f_i\}_{j\in\mathbb{N}}$ converges in $\mathcal{D}'(\mathbb{R}^n)$ to a distribution f, the series $\sum_i f_i*\psi(x)$ converges pointwise to $f*\psi(x)$ for each $x \in \mathbb{R}^n$. Therefore,

$$\left(\mathcal{M}_N(f)(x)\right)^p \leq \left(\sum_{i=1}^\infty \mathcal{M}_N(f_i)(x)\right)^p \leq \sum_{i=1}^\infty\left(\mathcal{M}_N(f_i)(x)\right)^p \quad \text{for all } x \in \mathbb{R}^n,$$

and hence $\|f\|_{h^p_{\rho,N}(\omega)} \leq \sum_i^\infty \|f_i\|_{h^p_{\rho,N}(\omega)}$.

In order to prove $h^p_{\rho,N}(\omega)$ is complete, it suffices to prove that for every sequence $\{f_j\}_{j\in\mathbb{N}}$ with $\|f_j\|_{h^p_{\rho,N}(\omega)} < 2^{-j}$ and $j \in \mathbb{N}$, the series $\sum_{j\in\mathbb{N}} f_j$ converges in $h^p_{\rho,N}(\omega)$. In fact, since $\{\sum_{i=1}^j f_i\}_{j\in\mathbb{N}}$ is a Cauchy sequence in $h^p_{\rho,N}(\omega)$, by

Proposition 3.12 and the completeness of $\mathcal{D}'(\mathbb{R}^n)$, $\{\sum_{i=1}^j f_i\}_{j\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{D}'(\mathbb{R}^n)$ as well and thus converges to some $f\in\mathcal{D}'(\mathbb{R}^n)$. Thus,

$$\left\|f-\sum_{i=1}^j f_i\right\|_{h^p_{\rho,N}(\omega)}^p=\left\|\sum_{i=j+1}^\infty f_i\right\|_{h^p_{\rho,N}(\omega)}^p\le\sum_{i=j+1}^\infty 2^{-ip}\to 0$$

as $j\to\infty$. This finishes the proof. □

THEOREM 3.14. *Let $\omega\in A_\infty^{\rho,\infty}(\mathbb{R}^n)$ and $N_{p,\omega}$ be as in (3.29). If $(p,q,s)_\omega$ is an admissible triplet and integer $N\ge N_{p,\omega}$, then*

$$h_\rho^{p,q,s}(\omega)\subset h^p_{\rho,N_{p,\omega}}(\omega)\subset h^p_{\rho,N}(\omega),$$

and moreover, there exists a positive constant C such that for all $f\in h_\rho^{p,q,s}(\omega)$,

$$\|f\|_{h^p_{\rho,N}(\omega)}\le\|f\|_{h^p_{\rho,N_{p,\omega}}(\omega)}\le C\|f\|_{h_\rho^{p,q,s}(\omega)}.$$

Proof. It is suffices to prove $h_\rho^{p,q,s}(\omega)\subset h^p_{\rho,N_{p,\omega}}(\omega)$, and for any $f\in h_\rho^{p,q,s}(\omega)$,

$$\|f\|_{h^p_{\rho,N_{p,\omega}}(\omega)}\lesssim\|f\|_{h_\rho^{p,q,s}(\omega)}.$$

By Definition 3.3 and Theorem 3.10, we just need to prove that there exists a positive constant C such that

$$\left\|\mathcal{M}^0_{N_{p,\omega}}(a)\right\|_{L^p_\omega(\mathbb{R}^n)}\le C,\quad\text{for all }(p,q)_\omega\text{-single-atoms }a,\tag{3.30}$$

and

$$\left\|\mathcal{M}^0_{N_{p,\omega}}(a)\right\|_{L^p_\omega(\mathbb{R}^n)}\le C,\quad\text{for all }(p,q,s)_\omega\text{-atoms }a.\tag{3.31}$$

For (3.30), since $q\in(q_\omega,\infty]$, we get $\omega\in A_q^{\rho,\infty}(\mathbb{R}^n)$. Let a be a $(p,q)_\omega$-single-atom. When $\omega(\mathbb{R}^n)=\infty$, by the definition of the single atom, we know that $a=0$ for almost every $x\in\mathbb{R}^n$. In this case, it is easy to obtain (3.30). When $\omega(\mathbb{R}^n)<\infty$, by Hölder's inequality, $\omega\in A_q^{\rho,\infty}(\mathbb{R}^n)$ and Proposition 2.8(i), we get

$$\begin{aligned}\left\|\mathcal{M}^0_{N_{p,\omega}}(a)\right\|^p_{L^p_\omega(\mathbb{R}^n)}&=\int_{\mathbb{R}^n}\left|\mathcal{M}^0_{N_{p,\omega}}(a)(x)\right|^p\omega(x)\,dx\\&\le\left(\int_{\mathbb{R}^n}\left|\mathcal{M}^0_{N_{p,\omega}}(a)(x)\right|^q\omega(x)\,dx\right)^{p/q}\left(\int_{\mathbb{R}^n}\omega(x)\,dx\right)^{1-p/q}\\&\le C\|a\|^p_{L^q_\omega(\mathbb{R}^n)}\left[\omega\left(\mathbb{R}^n\right)\right]^{1-p/q}\\&\le C.\end{aligned}$$

For (3.31), let a be a $(p,q,s)_\omega$-atom supported in the cube $Q\equiv Q(x_0,r)$. We consider two cases of Q.

The first case is when $r < L_2\rho(x_0)$. Let $\widetilde{Q} \equiv 2\sqrt{n}Q$, then we have

$$\begin{aligned}
(3.32)\qquad &\int_{\mathbb{R}^n} \left|\mathcal{M}^0_{N_{p,\omega}}(a)(x)\right|^p \omega(x)\,dx \\
&= \int_{\widetilde{Q}} \left|\mathcal{M}^0_{N_{p,\omega}}(a)(x)\right|^p \omega(x)\,dx + \int_{\widetilde{Q}^{\complement}} \left|\mathcal{M}^0_{N_{p,\omega}}(a)(x)\right|^p \omega(x)\,dx \\
&\equiv I_1 + I_2.
\end{aligned}$$

For I_1, by Hölder's inequality and the properties of $A^{\rho,\theta}_q(\mathbb{R}^n)$ (see Lemma 2.4(vii)), we have

$$I_1 \le C\|a\|^p_{L^q_\omega(\mathbb{R}^n)} \left[\omega(\widetilde{Q})\right]^{1-p/q} \le C. \tag{3.33}$$

For I_2, we claim that for $x \in \widetilde{Q}^{\complement}$

$$\begin{aligned}
(3.34)\qquad \mathcal{M}^0_{N_{p,\omega}}(a)(x) \le{}& C|Q|^{(s_0+n+1)/n}\left[\omega(Q)\right]^{-1/p} \\
&\times |x-x_0|^{-(s_0+n+1)}\chi_{B(x_0,c_1\rho(x_0))}(x),
\end{aligned}$$

where $s_0 \equiv [n(q_\omega/p-1)]$ and $c_1 > 2\sqrt{n}$ is an constant independent of the atom a. Indeed, for any $\psi \in \mathcal{D}^0_N(\mathbb{R}^n)$ and $2^{-l} \in (0,\rho(x))$, let P be the Taylor expansion of ψ about $(x-x_0)/2^{-l}$ with degree s_0. By Taylor's remainder theorem, for any $y \in \mathbb{R}^n$, we have

$$\begin{aligned}
&\left|\psi\left(\frac{x-y}{2^{-l}}\right) - P\left(\frac{x-x_0}{2^{-l}}\right)\right| \\
&\quad \le C \sum_{\substack{\alpha\in\mathbb{Z}^n_+\\ |\alpha|=s_0+1}} \left|(\partial^\alpha\psi)\left(\frac{\theta(x-y)+(1-\theta)(x-x_0)}{2^{-l}}\right)\right| \left|\frac{x_0-y}{2^{-l}}\right|^{s_0+1},
\end{aligned}$$

where $\theta \in (0,1)$. Since $2^{-l} \in (0,\rho(x))$ and $x \in \widetilde{Q}^{\complement}$, we have $\operatorname{supp}(a*\psi_l) \subset B(x_0,c_1\rho(x_0))$, and by $a*\psi_l(x) \ne 0$ we have $2^{-l} > |x-x_0|/2$. Then, for any $x \in \widetilde{Q}^{\complement}$, by the above estimates and Definition 3.2, we get

$$\begin{aligned}
&\left|a*\psi_l(x)\right| \\
&\quad \le \frac{1}{2^{-ln}}\left\{\int_Q |a(y)|\left|\psi\left(\frac{x-y}{2^{-l}}\right) - P\left(\frac{x-x_0}{2^{-l}}\right)\right| dy\right\}\chi_{B(x_0,c_1\rho(x_0))}(x) \\
&\quad \le C|x-x_0|^{-(s_0+n+1)}\left\{\int_Q |a(y)||x_0-y|^{s_0+1}\,dy\right\}\chi_{B(x_0,c_1\rho(x_0))}(x) \\
&\quad \le C|Q|^{(s_0+1)/n}\|a\|_{L^q_\omega(\mathbb{R}^n)}\left(\int_Q \left[\omega(y)\right]^{-q'/q} dy\right)^{1/q'} \\
&\qquad \times |x-x_0|^{-(s_0+n+1)}\chi_{B(x_0,c_1\rho(x_0))}(x) \\
&\quad \le C|Q|^{(s_0+n+1)/n}\left[\omega(Q)\right]^{-1/p}|x-x_0|^{-(s_0+n+1)}\chi_{B(x_0,c_1\rho(x_0))}(x),
\end{aligned}$$

which combined with the arbitrariness of $\psi \in \mathcal{D}^0_N(\mathbb{R}^n)$ infers (3.34).

Let $Q_i \equiv 2^i\sqrt{n}Q$ with $i \in \mathbb{N}$ and $i_0 \in \mathbb{N}$ satisfying $2^{i_0}r \le c_1\rho(x_0) < 2^{i_0+1}r$. Since $s_0 = [n(q_\omega/p-1)]$, there exists $q_0 \in (q_\omega, \infty)$ such that $p(s_0+n+1) > nq_0$. Then by Lemma 2.4, we have

$$\begin{aligned} I_2 &\le \int_{\sqrt{n}r\le|x-x_0|<c_1\rho(x_0)} \left|\mathcal{M}^0_{N_{p,\omega}}(a)(x)\right|^p \omega(x)\,dx \\ &\le C|Q|^{p(s_0+n+1)/n}\left[\omega(Q)\right]^{-1}\int_{\sqrt{n}r\le|x-x_0|<c_1\rho(x_0)} |x-x_0|^{-p(s_0+n+1)}\omega(x)\,dx \\ &\le Cr^{p(s_0+n+1)}\left[\omega(Q)\right]^{-1}\sum_{i=0}^{i_0}\int_{Q_{i+1}\setminus Q_i} |x-x_0|^{-p(s_0+n+1)}\omega(x)\,dx \\ &\le C\left[\omega(Q)\right]^{-1}\sum_{i=0}^{i_0} 2^{-ip(s_0+n+1)}\omega(Q_{i+1}) \le C, \end{aligned}$$

which combine with (3.32) and (3.33) implies (3.31) in the first case.

For the case $L_2\rho(x_0) \le r \le L_1\rho(x_0)$, let $Q^* \equiv Q(x_0, c_2r)$, in which $c_2 > 1$ is a constant independent of atom a. Thus, by $\mathrm{supp}(\mathcal{M}^0_{N_{p,\omega}}(a)) \subset Q^*$, Hölder's inequality and Lemma 2.4, we get

$$\begin{aligned} \int_{\mathbb{R}^n} \left|\mathcal{M}^0_{N_{p,\omega}}(a)(x)\right|^p \omega(x)\,dx &= \int_{Q^*} \left|\mathcal{M}^0_{N_{p,\omega}}(a)(x)\right|^p \omega(x)\,dx \\ &\le C\|a\|^p_{L^q_\omega(\mathbb{R}^n)}\left[\omega\big(Q^*\big)\right]^{1-p/q} \\ &\le C. \end{aligned}$$

The proof of Theorem 3.14 is complete. □

4. Calderón–Zygmund decompositions

In this section, we establish the Calderón–Zygmund decompositions associated with local grand maximal functions on weighted Euclidean space $\mathbb{R}^n$. We follow the constructions in [23], [4] and [5].

In this section, we consider a distribution f satisfying that for all $\lambda > 0$,

$$\omega\big(\{x \in \mathbb{R}^n : \mathcal{M}_N(f)(x) > \lambda\}\big) < \infty.$$

For any given $\lambda > \inf_{x\in\mathbb{R}^n} \mathcal{M}_N(f)(x)$, set

$$\Omega_\lambda \equiv \{x \in \mathbb{R}^n : \mathcal{M}_N(f)(x) > \lambda\},$$

which is a proper open subset of $\mathbb{R}^n$. As in [23], we give the usual Whitney decomposition of Ω_λ. Thus there will be closed cubes Q_i, and their interiors distance from $\Omega_\lambda^\complement$, with $\Omega_\lambda = \bigcup_i Q_i$ and

$$\mathrm{diam}(Q_i) \le 2^{-(6+n)}\,\mathrm{dist}\big(Q_i, \Omega_\lambda^\complement\big) \le 4\,\mathrm{diam}(Q_i).$$

In what follows, fix $a \equiv 1 + 2^{-(11+n)}$ and $b \equiv 1 + 2^{-(10+n)}$, and if we denote $\bar{Q}_i = aQ_i, Q_i^* = bQ_i$, we have $Q_i \subset \bar{Q}_i \subset Q_i^*$. Moveover, $\Omega_\lambda = \bigcup_i Q_i^*$, and $\{Q_i^*\}_i$ have the bounded interior property, that is, each point in Ω_λ is contained in at most a fixed number of $\{Q_i^*\}_i$.

Take a function $\xi \in \mathcal{D}(\mathbb{R}^n)$ such that $0 \le \xi \le 1$, $\mathrm{supp}(\xi) \subset aQ(0,1)$ and $\xi \equiv 1$ on $Q(0,1)$. For $x \in \mathbb{R}^n$, set $\xi_i(x) \equiv \xi((x-x_i)/l_i)$, where x_i is the center of the cube Q_i and l_i is its sidelength. Then for any $x \in \mathbb{R}^n$, we have $1 \le \sum_i \xi_i(x) \le M$, where M is a fixed positive integer independent of x. Let $\eta_i \equiv \xi_i/(\sum_j \xi_j)$, then $\{\eta_i\}_i$ can form a smooth partition of unity for Ω_λ subordinate to the locally finite covering $\{Q_i^*\}_i$ of Ω_λ, that is, $\chi_{\Omega_\lambda} = \sum_i \eta_i$ with each $\eta_i \in \mathcal{D}(\mathbb{R}^n)$ supported in $\bar{Q}_i$.

Let $s \in \mathbb{Z}_+$ be some fixed integer and $\mathcal{P}_s(\mathbb{R}^n)$ denote the linear space of polynomials in n variables of degrees no more than s. For each $i \in \mathbb{N}$ and $P \in \mathcal{P}_s(\mathbb{R}^n)$, set

$$\|P\|_i \equiv \left[\frac{1}{\int_{\mathbb{R}^n} \eta_i(y)\,dy} \int_{\mathbb{R}^n} \left|P(x)\right|^2 \eta_i(x)\,dx\right]^{1/2}. \tag{4.1}$$

Then $(\mathcal{P}_s(\mathbb{R}^n), \|\cdot\|_i)$ is a finite dimensional Hilbert space. Let $f \in \mathcal{D}'(\mathbb{R}^n)$, then f can induce a linear functional on $\mathcal{P}_s(\mathbb{R}^n)$ by

$$P \mapsto \frac{1}{\int_{\mathbb{R}^n} \eta_i(y)\,dy} \langle f, P\eta_i\rangle.$$

By the Riesz represent theorem, there exists a unique polynomial $P_i \in \mathcal{P}_s(\mathbb{R}^n)$ for each i such that for any $Q \in \mathcal{P}_s(\mathbb{R}^n)$,

$$\langle f, Q\eta_i\rangle = \langle P_i, Q\eta_i\rangle = \int_{\mathbb{R}^n} P_i(x) Q(x) \eta_i(x)\,dx.$$

For each i, define the distribution $b_i \equiv (f - P_i)\eta_i$ when $l_i \in (0, L_3\rho(x_i))$ (where $L_3 = 2^{k_0} C_0$, x_i is the center of the cube Q_i) and $b_i \equiv f\eta_i$ when $l_i \in [L_3\rho(x_i), \infty)$.

We will show that for suitable choices of s and N, the series $\sum_i b_i$ converge in $\mathcal{D}'(\mathbb{R}^n)$, and in this case, we define $g \equiv f - \sum_i b_i$ in $\mathcal{D}'(\mathbb{R}^n)$.

The representation $f = g + \sum_i b_i$, where g and b_i are as above, is called a Calderón–Zygmund decomposition of f of degree s and height λ associated with $\mathcal{M}_N(f)$.

In the following section, we give some lemmas. In Lemmas 4.1 and 4.2, we give some properties of the smooth partition of unity $\{\eta_i\}_i$. From Lemmas 4.3 to 4.6, we get some estimates for the bad parts $\{b_i\}_i$. Lemmas 4.7 and 4.8 give some estimates of the good part g, and Corollary 4.10 shows the density of $L^q_\omega(\mathbb{R}^n) \cap h^p_{\rho,N}(\omega)$ in $h^p_{\rho,N}(\omega)$, where $q \in (q_\omega, \infty)$.

LEMMA 4.1. *There exists a positive constant C_1 depending only on N, such that for all i and $l \le l_i$,*

$$\sup_{|\alpha| \le N} \sup_{x \in \mathbb{R}^n} \left|\partial^\alpha \eta_i(lx)\right| \le C_1.$$

Lemma 4.1 is essentially Lemma 5.2 in [4].

LEMMA 4.2. *If $l_i < L_3\rho(x_i)$, then there exists a constant $C_2 > 0$ independent of $f \in \mathcal{D}'(\mathbb{R}^n)$, l_i and $\lambda > 0$ so that*

$$\sup_{y\in\mathbb{R}^n} \left|P_i(y)\eta_i(y)\right| \le C_2\lambda.$$

Proof. As in the proof of Lemma 5.3 in [4]. Let $\pi_1,\dots,\pi_m$ ($m = \dim \mathcal{P}_s$) be an orthonormal basis of $\mathcal{P}_s$ with respect to the norm (4.1). we have

$$P_i = \sum_{k=1}^{m}\left(\frac{1}{\int \eta_i}\int f(x)\pi_k(x)\eta_i(x)\,dx\right)\bar{\pi}_k, \tag{4.2}$$

where the integral is understood as $\langle f, \pi_k\eta_i\rangle$. Therefore,

$$\begin{aligned} 1 &= \frac{1}{\int \eta_i}\int_{\bar{Q}_i}\left|\pi_k(x)\right|^2\eta_i(x)\,dx \ge \frac{2^{-n}}{|Q_i|}\int_{\bar{Q}_i}\left|\pi_k(x)\right|^2\eta_i(x)\,dx \\ &\ge \frac{2^{-n}}{|Q_i|}\int_{Q_i}\left|\pi_k(x)\right|^2dx = 2^{-n}\int_{Q^0}\left|\widetilde{\pi}_k(x)\right|^2dx, \end{aligned} \tag{4.3}$$

where $\widetilde{\pi}_k(x) = \pi_k(x_i + l_ix)$ and Q^0 denotes the cube of side length 1 centered at the origin.

Since $\mathcal{P}_s$ is finite dimensional, all norms on $\mathcal{P}_s$ are equivalent, then there exists $A_1 > 0$ such that for all $P \in \mathcal{P}_s$

$$\sup_{|\alpha|\le s}\sup_{z\in bQ^0}\left|\partial^\alpha P(z)\right| \le A_1\left(\int_{Q^0}\left|P(z)\right|^2dz\right)^{1/2}.$$

From this and (4.3), for $k = 1,\dots,m$, we have

$$\sup_{|\alpha|\le s}\sup_{z\in bQ^0}\left|\partial^\alpha\widetilde{\pi}_k(z)\right| \le A_12^{n/2}. \tag{4.4}$$

If $z \in 2^{8+n}nQ_i \cap \Omega^{\complement}$, by Lemma 2.1, we have $\rho(x_i) \le C_0(1 + 2^{8+n}n^2L_3)^{k_0}\rho(z)$, then we let $\widetilde{L} \equiv 1/C_0(1 + 2^{8+n}n^2L_3)^{k_0}L_3$. For $k = 1,\dots,m$, we define

$$\Phi_k(y) = \frac{2^{-k_in}}{\int \eta_i}\pi_k\big(z - 2^{-k_i}y\big)\eta_i\big(z - 2^{-k_i}y\big),$$

where $z \in 2^{8+n}nQ_i \cap \Omega^{\complement}$ and $2^{-k_i} \le \widetilde{L}l_i < 2^{-k_i+1}$. It is easy to see that $\operatorname{supp}\Phi_k \subset B(0,R_1)$ where $R_1 \equiv 2^{9+n}n^2/\widetilde{L}$, and $\|\Phi_k\|_{\mathcal{D}_N} \le A_2$ by Lemma 4.1.

Note that

$$\frac{1}{\int \eta_i}\int f(x)\pi_k(x)\eta_i(x)\,dx = \big(f * (\Phi_k)_{k_i}\big)(z),$$

since $2^{-k_i} \le \widetilde{L}l_i < \widetilde{L}L_3\rho(x_i) \le \rho(z)$, then we have

$$\left|\frac{1}{\int \eta_i}\int f(x)\pi_k(x)\eta_i(x)\,dx\right| \le \mathcal{M}_Nf(z)\|\Phi_k\|_{\mathcal{D}_N} \le A_2\lambda.$$

By (4.2), (4.4) and the above estimate, we have

$$\sup_{z\in Q_i^*}\big|P_i(z)\big|\le m2^{n/2}A_1A_2\lambda.$$

Thus,

$$\sup_{z\in\mathbb{R}^n}\big|P_i(z)\eta_i(z)\big|\le C_2\lambda.$$

The proof is complete. □

By the same method, we can get the following lemma as Lemma 4.3 in [24], and we omit the details here.

LEMMA 4.3. *There exists a constant* $C_3>0$ *such that*

$$\mathcal{M}_N^0 b_i(x)\le C_3\mathcal{M}_N f(x)\quad \textit{for } x\in Q_i^*. \tag{4.5}$$

LEMMA 4.4. *Suppose that* $Q\subset\mathbb{R}^n$ *is bounded, convex, and* $0\in Q$, *and* N *is a positive integer. Then there is a constant* C *depending only on* Q *and* N *such that for every* $\phi\in\mathcal{D}(\mathbb{R}^n)$ *and every integer* s, $0\le s<N$ *we have*

$$\sup_{z\in Q}\sup_{|\alpha|\le N}\big|\partial^\alpha R_y(z)\big|\le C\sup_{z\in y+Q}\ \sup_{s+1\le|\alpha|\le N}\big|\partial^\alpha\phi(z)\big|,$$

where R_y *is the remainder of the Taylor expansion of* ϕ *of order* s *at the point* $y\in\mathbb{R}^n$.

Lemma 4.4 is Lemma 5.5 in [4].

LEMMA 4.5. *Suppose that* $0\le s<N$. *Then there exist positive constants* C_4, C_5 *so that for* $i\in\mathbb{N}$,

$$\mathcal{M}_N^0(b_i)(x)\le C\frac{\lambda l_i^{n+s+1}}{(l_i+|x-x_i|)^{n+s+1}}\chi_{\{|x-x_i|<C_4\rho(x)\}}(x)\quad \textit{if } x\notin Q_i^*. \tag{4.6}$$

Moreover,

$$\mathcal{M}_N^0(b_i)(x)=0,\quad \textit{if } x\notin Q_i^* \textit{ and } l_i\ge C_5\rho(x).$$

Proof. Since η_i is supported in the cube $\bar{Q}_i$, and $\bar{Q}_i$ is strictly contained in Q_i^*, then if $x\notin Q_i^*$ and $\eta_i(y)\ne0$, there exists a positive constant C_4 such that $|x-y|\le|x-x_i|\le C_4|x-y|$. On the other hand, take $\varphi\in\mathcal{D}_N^0(\mathbb{R}^n)$, the support property of φ requires that $\rho(x)>2^{-l}\ge|x-y|\ge2^{-11-n}l_i$. Hence, $|x-x_i|\le C_4 2^{-l}$, $l_i<2^{11+n}\rho(x)\equiv C_5\rho(x)$ and $l_i<C_5 2^{-l}$. Take $w\in(2^{8+n}nQ_i)\cap\Omega^{\complement}$, and we discuss the following two cases.

Case I. If $L_3\rho(x_i)\le l_i<C_5 2^{-l}<C_5\rho(x)$, then according to Lemma 2.1 we have $l_i<C_5C_0(1+C_4)^{k_0}\rho(x_i)$ and

$$\rho(\omega)\ge C_0^{-1}\left(1+\frac{|\omega-x_i|}{\rho(x_i)}\right)^{-k_0}\rho(x_i)\ge C_0^{-1}\big(1+2^{8+n}n\sqrt{n}L_2\big)^{-k_0}\rho(x_i),$$

therefore, $l_i<a_1\rho(w)$, where $a_1>1$ is a constant.

Now we define $\bar{l}_i = l_i/a_1 < \rho(w)$ and take $k_i \in \mathbb{Z}$ such that $2^{-k_i} \le \bar{l}_i < 2^{-k_i+1}$, then for $\varphi \in \mathcal{D}_N^0(\mathbb{R}^n)$, $\phi(z) \equiv \varphi(2^{-k_i}z/2^{-l})$ and $2^{-l} < \rho(x)$ we have

$$\begin{aligned}(b_i * \varphi_l)(x) &= 2^{ln} \int b_i(z)\varphi\big(2^l(x-z)\big)\,dz\\ &= 2^{ln} \int b_i(z)\phi\big(2^{k_i}(x-z)\big)\,dz\\ &= 2^{ln} \int b_i(z)\phi_{2^{k_i}(x-w)}\big(2^{k_i}(w-z)\big)\,dz\\ &= \frac{2^{ln}}{2^{k_i n}}(f * \Phi_{k_i})(w),\end{aligned}$$

where

$$\Phi(z) \equiv \phi_{2^{k_i}(x-w)}(z)\eta_i\big(w - 2^{-k_i}z\big), \quad \phi_{2^{k_i}(x-w)}(z) \equiv \phi\big(z + 2^{k_i}(x-w)\big).$$

Obviously, $\operatorname{supp}\Phi \subset B(0, R_2)$, where $R_2 \equiv 2^{9+n}n^2 a_1$. Since $l_i < C_5 2^{-l}$ and $|x - x_i| \le C_4 2^{-l}$, we have

$$\big|(b_i * \varphi_l)(x)\big| \le C\frac{2^{ln}}{2^{k_i n}}\mathcal{M}_N f(w) \le C\lambda\frac{2^{ln}}{2^{k_i n}} \le C\lambda\frac{l_i^{n+s+1}}{(l_i + |x - x_i|)^{n+s+1}}. \tag{4.7}$$

Case II. If $l_i < L_3\rho(x_i)$ and $\varphi \in \mathcal{D}_N^0(\mathbb{R}^n)$, taking $j_i \in \mathbb{Z}$ such that $2^{-j_i} \le l_i < 2^{-j_i+1}$, then we define $\phi(z) = \varphi(2^{-j_i}z/2^{-l})$ and consider the Taylor expansion of ϕ of order s at the point $y = 2^{j_i}(x-w)$:

$$\phi(y+z) = \sum_{|\alpha| \le s} \frac{\partial^\alpha \phi(y)}{\alpha!} z^\alpha + R_y(z),$$

where R_y denotes the remainder. Thus we get

$$\begin{aligned}(b_i * \varphi_l)(x) &= 2^{ln} \int b_i(z)\varphi\big(2^{ln}(x-z)\big)\,dz\\ &= 2^{ln} \int b_i(z)\phi\big(2^{j_i n}(x-z)\big)\,dz\\ &= 2^{ln} \int b_i(z)R_{2^{j_i}(x-w)}\big(2^{j_i}(w-z)\big)\,dz\\ &= \frac{2^{ln}}{2^{j_i n}}(f * \Phi_{j_i})(w)\\ &\quad - 2^{ln} \int P_i(z)\eta_i(z)R_{2^{j_i}(x-w)}\big(2^{j_i}(w-z)\big)\,dz,\end{aligned} \tag{4.8}$$

where

$$\Phi(z) \equiv R_{2^{j_i}(x-w)}(z)\eta_i\big(\omega - 2^{-j_i}z\big).$$

Obviously, $\operatorname{supp}\Phi \subset B_n \equiv B(0,R_2)$. Applying Lemma 4.4 to $\phi(z) = \varphi(2^{-j_i}z/2^{-l})$, $y = 2^{j_i}(x-w)$ and B_n, we have

$$\begin{aligned}\sup_{z\in B_n}\sup_{|\alpha|\le N}\left|\partial^\alpha R_y(z)\right| &\le C\sup_{z\in y+B_n}\sup_{s+1\le|\alpha|\le N}\left|\partial^\alpha \phi(z)\right|\\ &\le C\sup_{z\in y+B_n}\left(\frac{2^{-j_i}}{2^{-l}}\right)^{s+1}\sup_{s+1\le|\alpha|\le N}\left|\partial^\alpha\varphi\big(2^{-j_i}z/2^{-l}\big)\right|\\ &\le C\left(\frac{2^{-j_i}}{2^{-l}}\right)^{s+1}.\end{aligned}$$

Notice that $l_i < C_5 2^{-l}$ and $|x-x_i| \le C_4 2^{-l}$, then by (4.8), we obtain

$$\begin{aligned}(4.9)\quad &(b_i * \varphi_l)(x)\\ &\le \frac{2^{ln}}{2^{j_i n}}\left|(f*\Phi_{j_i})(w)\right| + 2^{ln}\int\left|P_i(z)\eta_i(z)R_{2^{j_i}(x-w)}\big(2^{j_i}(w-z)\big)\right|dz\\ &\le C\frac{2^{ln}}{2^{j_i n}}\Big(\mathcal{M}_N f(w)\|\Phi\|_{\mathcal{D}_N} + \lambda\sup_{z\in B_n}\sup_{|\alpha|\le N}\left|\partial^\alpha R_y(z)\right|\Big)\\ &\le C\lambda\frac{l_i^{n+s+1}}{(l_i+|x-x_i|)^{n+s+1}}.\end{aligned}$$

By combining both cases, we obtain (4.6). □

LEMMA 4.6. *Let $\omega \in A_\infty^{\rho,\infty}(\mathbb{R}^n)$ and q_ω be as in (2.4). If $p\in(0,1]$, $s \ge [n(q_\omega/p-1)]$, $N > s$ and $N \ge N_{p,\omega}$, then there exists a positive constant C_6 such that for all $f \in h^p_{\rho,N}(\omega)$, $\lambda > \inf_{x\in\mathbb{R}^n}\mathcal{M}_N f(x)$ and $i\in\mathbb{N}$,*

$$(4.10)\qquad \int_{\mathbb{R}^n}\big(\mathcal{M}_N^0(b_i)(x)\big)^p\omega(x)\,dx \le C_6\int_{Q_i^*}\big(\mathcal{M}_N(f)(x)\big)^p\omega(x)\,dx.$$

Moreover the series $\sum_i b_i$ converges in $h^p_{\rho,N}(\omega)$ and

$$(4.11)\qquad \int_{\mathbb{R}^n}\left(\mathcal{M}_N^0\Big(\sum_i b_i\Big)(x)\right)^p\omega(x)\,dx \le C_6\int_\Omega\big(\mathcal{M}_N(f)(x)\big)^p\omega(x)\,dx.$$

Proof. By the proof of Lemma 4.5, we know $|x-x_i| < C_4\rho(x)$, $l_i < C_5\rho(x)$ and $\rho(x) \le C_0(1+C_4)^{k_0}\rho(x_i)$, thus $Q_i^* \subset a_2\rho(x_i)Q_i^0$, where $a_2 \equiv 2C_0(1+C_4)^{k_0}\max\{C_4,C_5\}$ and $Q_i^0 \equiv Q(x_i,1)$. Furthermore, we have

$$\begin{aligned}(4.12)\quad \int_{\mathbb{R}^n}\big(\mathcal{M}_N^0(b_i)(x)\big)^p\omega(x)\,dx &\le \int_{Q_i^*}\big(\mathcal{M}_N^0(b_i)(x)\big)^p\omega(x)\,dx\\ &\quad+\int_{a_2\rho(x_i)Q_i^0\setminus Q_i^*}\big(\mathcal{M}_N^0(b_i)(x)\big)^p\omega(x)\,dx.\end{aligned}$$

Notice that $s \ge [n(q_\omega/p-1)]$ implies $2^{-n(q_\omega+\eta)}2^{(s+n+1)p} > 1$ for sufficient small $\eta>0$. By Lemma 2.1(iii) with $\omega \in A^{\rho,\infty}_{q_\omega+\eta}(\mathbb{R}^n)$, Lemma 4.5 and

$\mathcal{M}_N(f)(x) > \lambda$ for all $x \in Q_i^*$, we have

$$\begin{aligned}(4.13)\qquad &\int_{a_2\rho(x_i)Q_i^0\setminus Q_i^*} \big(\mathcal{M}_N^0(b_i)(x)\big)^p \omega(x)\,dx \\ &\le \sum_{k=0}^{k_0} \int_{2^kQ_i^*\setminus 2^{k-1}Q_i^*} \big(\mathcal{M}_N^0(b_i)(x)\big)^p \omega(x)\,dx \\ &\le \lambda^p \omega\big(Q_i^*\big) \sum_{k=0}^{k_0} \big[2^{-n(q_\omega+\eta)+(s+n+1)p}\big]^{-k} \\ &\le C \int_{Q_i^*} \big(\mathcal{M}_N f(x)\big)^p \omega(x)\,dx,\end{aligned}$$

where $b = 1 + 2^{-(10+n)}$, $k_0 \in \mathbb{Z}$ such that $2^{k_0-1}bl_i \le a_2\rho(x_i) < 2^{k_0}bl_i$.

Combining the last two estimates we obtain (4.10); furthermore, we have

$$\begin{aligned}&\sum_i \int_{\mathbb{R}^n} \big(\mathcal{M}_N^0(b_i)(x)\big)^p \omega(x)\,dx \\ &\quad \le C \sum_i \int_{Q_i^*} \big(\mathcal{M}_N f(x)\big)^p \omega(x)\,dx \le C \int_{\Omega} \big(\mathcal{M}_N(f)(x)\big)^p \omega(x)\,dx,\end{aligned}$$

which together with the completeness of $h^p_{\rho,N}(\omega)$ (see Proposition 3.13) implies that $\sum_i b_i$ converges in $h^p_{\rho,N}(\omega)$. Therefore, the series $\sum_i b_i$ converges in $\mathcal{D}'(\mathbb{R}^n)$ and $\mathcal{M}_N^0(\sum_i b_i)(x) \le \sum_i \mathcal{M}_N^0(b_i)(x)$, which gives (4.11). This finishes the proof. $\square$

LEMMA 4.7. *Let $\omega \in A_\infty^{\rho,\infty}(\mathbb{R}^n)$ and q_ω be as in (2.4), $s \in \mathbb{Z}_+$, and integer $N \ge 2$. If $q \in (q_\omega, \infty]$ and $f \in L^q_\omega(\mathbb{R}^n)$, then the series $\sum_i b_i$ converges in $L^q_\omega(\mathbb{R}^n)$ and there exists a positive constant C_7, independent of f and λ, such that*

$$\Big\| \sum_i |b_i| \Big\|_{L^q_\omega(\mathbb{R}^n)} \le C_7 \|f\|_{L^q_\omega(\mathbb{R}^n)}.$$

Proof. The proof for $q = \infty$ is similar to that for $q \in (q_\omega, \infty)$. So we only give the proof for $q \in (q_\omega, \infty)$. Set $F_1 = \{i \in \mathbb{N} : l_i \ge L_3\rho(x_i)\}$ and $F_2 = \{i \in \mathbb{N} : l_i < L_3\rho(x_i)\}$. By Lemma 4.2, for $i \in F_2$, we have

$$\begin{aligned}\int_{\mathbb{R}^n} \big|b_i(x)\big|^q \omega(x)\,dx &\le \int_{Q_i^*} \big|f(x)\big|^q \omega(x)\,dx + \int_{Q_i^*} \big|P_i(x)\eta_i(x)\big|^q \omega(x)\,dx \\ &\le \int_{Q_i^*} \big|f(x)\big|^q \omega(x)\,dx + \lambda^q \omega\big(Q_i^*\big).\end{aligned}$$

For $i \in F_1$, we have

$$\int_{\mathbb{R}^n} \big|b_i(x)\big|^q \omega(x)\,dx \le \int_{Q_i^*} \big|f(x)\big|^q \omega(x)\,dx.$$

By these, we obtain

$$\begin{aligned}\sum_i \int_{\mathbb{R}^n} \big|b_i(x)\big|^q \omega(x)\,dx &= \sum_{i\in F_1} \int_{\mathbb{R}^n} \big|b_i(x)\big|^q \omega(x)\,dx + \sum_{i\in F_2} \int_{\mathbb{R}^n} \big|b_i(x)\big|^q \omega(x)\,dx \\ &\le \sum_i \int_{Q_i^*} \big|f(x)\big|^q \omega(x)\,dx + C \sum_{i\in F_2} \lambda^q \omega\big(Q_i^*\big) \\ &\le \sum_i \int_{Q_i^*} \big|f(x)\big|^q \omega(x)\,dx + C\lambda^q \omega(\Omega) \\ &\le C \int_{\mathbb{R}^n} \big|f(x)\big|^q \omega(x)\,dx.\end{aligned}$$

Combining the above estimates with the fact that $\{b_i\}_i$ have finite covering, we obtain

$$\Big\| \sum_i |b_i| \Big\|_{L^q_\omega(\mathbb{R}^n)} \le C_7 \|f\|_{L^q_\omega(\mathbb{R}^n)}.$$

This finishes the proof. □

LEMMA 4.8. *If $N > s \ge 0$ and $\sum_i b_i$ converges in $\mathcal{D}'(\mathbb{R}^n)$, then there exists a positive constant C_8, independent of f and λ, such that for all $x \in \mathbb{R}^n$,*

$$\begin{aligned}\mathcal{M}_N^0(g)(x) &\le \mathcal{M}_N^0(f)(x)\chi_{\Omega^\complement}(x) \\ &\quad + C_8\lambda \sum_i \frac{l_i^{n+s+1}}{(l_i + |x-x_i|)^{n+s+1}} \chi_{\{|x-x_i|<C_4\rho(x)\}}(x) + C_8\lambda\chi_\Omega(x),\end{aligned}$$

where x_i is the center of Q_i and C_4 is as in Lemma 4.5.

Proof. For $x \notin \Omega$, since

$$\mathcal{M}_N^0(g)(x) \le \mathcal{M}_N^0(f)(x) + \sum_i \mathcal{M}_N^0(b_i)(x),$$

by Lemma 4.5, we have

$$\begin{aligned}\mathcal{M}_N^0(g)(x) &\le \mathcal{M}_N^0(f)(x)\chi_{\Omega^\complement}(x) \\ &\quad + C\lambda \sum_i \frac{l_i^{n+s+1}}{(l_i + |x-x_i|)^{n+s+1}} \chi_{\{|x-x_i|<C_4\rho(x)\}}(x).\end{aligned}$$

For $x \in \Omega$, take $k \in \mathbb{N}$ such that $x \in Q_k^*$. Let $J \equiv \{i \in \mathbb{N} : Q_i^* \cap Q_k^* \ne \emptyset\}$. Then the cardinality of J is bounded by L. By Lemma 4.5, we have

$$\sum_{i\notin J} \mathcal{M}_N^0(b_i)(x) \le C\lambda \sum_{i\notin J} \frac{l_i^{n+s+1}}{(l_i + |x-x_i|)^{n+s+1}} \chi_{\{|x-x_i|<C_4\rho(x)\}}(x).$$

We need to estimate the grand maximal function of $g+\sum_{i\notin J} b_i = f-\sum_{i\in J} b_i$. Take $\varphi\in\mathcal{D}_N^0(\mathbb{R}^n)$ and $l\in\mathbb{Z}$ such that $0<2^{-l}<\rho(x)$, then we have

$$\left(f-\sum_{i\in J} b_i\right)*\varphi_i(x) = (f\xi)*\varphi_l(x) + \left(\sum_{i\in J} P_i\eta_i\right)*\varphi_l(x) = f*\widetilde{\Phi}_l(w) + \left(\sum_{i\in J} P_i\eta_i\right)*\varphi_l(x), \tag{4.14}$$

where $w\in(2^{8+n}nQ_k)\cap\Omega^{\complement}$, $\xi = 1-\sum_{i\in J}\eta_i$ and

$$\widetilde{\Phi}(z)\equiv\varphi\big(z+2^l(x-w)\big)\xi\big(w-2^{-l}z\big).$$

Since for $N\geq 2$ there is a constant $C>0$ such that $\|\varphi\|_{L^1(\mathbb{R}^n)}\leq C$ for all $\varphi\in\mathcal{D}_N^0(\mathbb{R}^n)$ and by Lemma 4.1, we have

$$\left|\left(\sum_{i\in J} P_i\eta_i\right)*\varphi_l(x)\right|\leq C\lambda.$$

Finally, we estimate $f*\Phi_l(w)$. There are two cases: If $2^{-l}\leq 2^{-(11+n)}l_k$, then $f*\Phi_l(w)=0$, because ξ vanishes in Q_k^* and φ_l is supported in $B(0,2^{-l})$. On the other hand, if $2^{-l}\geq 2^{-(11+n)}l_k$, then there exists a positive constant $a_3>1$ such that $2^{-l}<\rho(x)<a_3\rho(w)$. Take $\Phi(x)\equiv\widetilde{\Phi}(x/2^{m_1})$ and $m_1\in\mathbb{N}$ satisfying $2^{m_1-1}\leq a_3<2^{m_1}$, then $\operatorname{supp}\Phi\subset B(0,R_3)$ where $R_3\equiv 2^{3(11+n)}a_3$, and $\|\Phi\|_{\mathcal{D}_N}\leq C$. Therefore, $2^{-l-m_1}<\rho(x)/a_3<\rho(w)$ and

$$\big|(f*\widetilde{\Phi}_l)(w)\big| = 2^{-m_1 n}\big|(f*\Phi_{l+m_1})(w)\big|\leq C\mathcal{M}_N f(w)\|\Phi\|_{\mathcal{D}_N}\leq C\lambda.$$

According to the above estimates, we have

$$\left|\left(f-\sum_{i\in J} b_i\right)*\varphi_l\right|\leq C\lambda,$$

then we can get

$$\mathcal{M}_N^0\left(\left(f-\sum_{i\in J} b_i\right)\right)(x)\leq C\lambda.$$

This finishes the proof of the lemma. □

Lemma 4.9. *Let $\omega\in A_\infty^{\rho\infty}(\mathbb{R}^n)$, q_ω be as in (2.4), $q\in(q_\omega,\infty)$, $p\in(0,1]$ and $N\geq N_{p,\omega}$.*

(i) *If $N>s\geq[n(q_\omega/p-1)]$ and $f\in h^p_{\rho,N}(\omega)$, then $\mathcal{M}_N^0(g)\in L^q_\omega(\mathbb{R}^n)$ and there exists a positive constant C_9, independent of f and λ, such that*

$$\int_{\mathbb{R}^n}\big[\mathcal{M}_N^0(g)(x)\big]^q\omega(x)\,dx\leq C_9\lambda^{q-p}\int_{\mathbb{R}^n}\big[\mathcal{M}_N(f)(x)\big]^p\omega(x)\,dx.$$

(ii) *If $N\geq 2$ and $f\in L^q_\omega(\mathbb{R}^n)$, then $g\in L^\infty_\omega(\mathbb{R}^n)$ and there exists a positive constant C_{10}, independent of f and λ, such that $\|g\|_{L^\infty_\omega}\leq C_{10}\lambda$.*

Proof. Since $f\in h^p_{\rho,N}(\omega)$, by Lemma 4.6 and Proposition 3.12, $\sum_i b_i$ converges in both $h^p_{\rho,N}(\omega)$ and $\mathcal{D}'(\mathbb{R}^n)$. Notice that $s\geq[n(q_\omega/p-1)]$, by Lemma 4.8 and the proof of Lemma 4.6, we get

$$\begin{aligned}
&\int_{\mathbb{R}^n}\big(\mathcal{M}^0_N(g)(x)\big)^q\omega(x)\,dx\\
&\quad\leq C\lambda^q\sum_i\int_{\mathbb{R}^n}\left[\frac{l_i^{(n+s+1)}}{(l_i+|x-x_i|)^{(n+s+1)}}\chi_{B(x_i,a_2\rho(x_i))}(x)\right]^q\omega(x)\,dx\\
&\qquad+C\lambda^q\int_{\mathbb{R}^n}\chi_\Omega(x)\omega(x)\,dx+\int_{\Omega^\complement}\big(\mathcal{M}_N(f)(x)\big)^q\omega(x)\,dx\\
&\quad\leq C\lambda^q\sum_i\omega\big(Q_i^*\big)+C\lambda^q\omega(\Omega)+\int_{\Omega^\complement}\big(\mathcal{M}_N(f)(x)\big)^q\omega(x)\,dx\\
&\quad\leq C\lambda^q\omega(\Omega)+C\lambda^{q-p}\int_{\Omega^\complement}\big(\mathcal{M}_N(f)(x)\big)^p\omega(x)\,dx\\
&\quad\leq C_9\lambda^{q-p}\int_{\mathbb{R}^n}\big(\mathcal{M}_N(f)(x)\big)^p\omega(x)\,dx.
\end{aligned}$$

Thus, (i) holds.

Next, we prove (ii). If $f\in L^q_\omega(\mathbb{R}^n)$, then g and $\{b_i\}_i$ are functions. By Lemma 4.7, we know that $\sum_i b_i$ converges in $L^q_\omega(\mathbb{R}^n)$, and by Lemma 2.5(ii) we know $\sum_i b_i$ converges in $\mathcal{D}'(\mathbb{R}^n)$. If we denote

$$g=f-\sum_i b_i=f\left(1-\sum_i\eta_i\right)+\sum_{i\in F_2}P_i\eta_i=f\chi_{\Omega^\complement}+\sum_{i\in F_2}P_i\eta_i,$$

by Lemma 4.3, we have $|g(x)|\leq C\lambda$ for all $x\in\Omega$, and by Proposition 2.8(i), we also have $|g(x)|=|f(x)|\leq\mathcal{M}_Nf(x)\leq\lambda$ for almost everywhere $x\in\Omega^\complement$. Therefore, $\|g\|_{L^\infty_\omega(\mathbb{R}^n)}\leq C_{10}\lambda$ which yields (ii). □

COROLLARY 4.10. *Let $\omega\in A^{\rho,\infty}_\infty(\mathbb{R}^n)$ and q_ω be as in (2.4). If $q\in(q_\omega,\infty)$, $p\in(0,1]$ and $N\geq N_{p,\omega}$, then $h^p_{\rho,N}(\omega)\cap L^q_\omega(\mathbb{R}^n)$ is dense in $h^p_{\rho,N}(\omega)$.*

Proof. Let $f\in h^p_{\rho,N}(\omega)$. For any $\lambda>\inf_{x\in\mathbb{R}^n}\mathcal{M}_Nf(x)$, let $f=g^\lambda+\sum_i b_i^\lambda$ be the Calderón–Zygmund decomposition of f of degree s with $[n(q_\omega/p-1)]\leq s<N$ and height λ associated to $\mathcal{M}_Nf$. By Lemma 4.6, we have

$$\Big\|\sum_i b_i^\lambda\Big\|_{h^p_{\rho,N}(\omega)}\leq C\int_{\{x\in\mathbb{R}^n:\mathcal{M}_Nf(x)>\lambda\}}\big(\mathcal{M}_Nf(x)\big)^p\omega(x)\,dx.$$

Therefore, $g^\lambda\to f$ in $h^p_{\rho,N}(\omega)$ as $\lambda\to\infty$. Moreover, by Lemma 4.9, we have $\mathcal{M}^0_N(g^\lambda)\in L^q_\omega(\mathbb{R}^n)$, which combined with Proposition 2.8(ii) infers $g^\lambda\in L^q_\omega(\mathbb{R}^n)$. Thus, Corollary 4.10 is proved. □

5. Weighted atomic decompositions of $h^p_{\rho,N}(\omega)$

In this section, we will establish the equivalence between $h^p_{\rho,N}(\omega)$ and $h^{p,q,s}_{\rho}(\omega)$ by using the Calderón–Zygmund decomposition, and we will follow the proof of atomic decomposition as presented by Stein in [22].

Let $\omega\in A^{\rho,\infty}_{\infty}(\mathbb{R}^n), q_\omega$ be as in (2.4), $p\in(0,1]$, $N\ge N_{p,\omega}$, $s\equiv[n(q_\omega/p-1)]$ and $f\in h^p_{\rho,N}(\omega)$. Take $m_0\in\mathbb{Z}$ such that $2^{m_0-1}\le\inf_{x\in\mathbb{R}^n}\mathcal{M}_Nf(x)<2^{m_0}$, if $\inf_{x\in\mathbb{R}^n}\mathcal{M}_Nf(x)=0$, write $m_0=-\infty$. For each integer $m\ge m_0$ consider the Calderón–Zygmund decomposition of f of degree s and height $\lambda=2^m$ associated to $\mathcal{M}_Nf$, namely

$$f=g^m+\sum_{i\in\mathbb{N}}b_i^m, \tag{5.1}$$

and

$$\Omega_m\equiv\{x\in\mathbb{R}^n:\mathcal{M}_Nf(x)>2^m\},\qquad Q_i^m\equiv Q_{l_i^m}.$$

In this section, we denote $\{Q_i\}_i$, $\{\eta_i\}_i$, $\{P_i\}_i$ and $\{b_i\}_i$ as $\{Q_i^m\}_i$, $\{\eta_i^m\}_i$, $\{P_i^m\}_i$ and $\{b_i^m\}_i$. The center and the sidelength of Q_i^m are respectively denoted by x_i^m and l_i^m.

As in Section 4, for all i and m,

$$\sum_i\eta_i^m=\chi_{\Omega_m},\qquad \operatorname{supp}(b_i^m)\subset\operatorname{supp}(\eta_i^m)\subset Q_i^{m*}, \tag{5.2}$$

$\{Q_i^{m*}\}_i$ has the bounded interior property, and P_i^m satisfying that for any $P\in\mathcal{P}_s(\mathbb{R}^n)$,

$$\langle f,P\eta_i^m\rangle=\langle P_i^m,P\eta_i^m\rangle. \tag{5.3}$$

For each integer $m\ge m_0$ and $i,j\in\mathbb{N}$, we define $P_{i,j}^{m+1}$ as the orthogonal projection of $(f-P_j^{m+1})\eta_i^m$ on $\mathcal{P}_s(\mathbb{R}^n)$ with respect to the norm

$$\|P\|_j^2\equiv\frac{1}{\int_{\mathbb{R}^n}\eta_j^{m+1}(y)\,dy}\int_{\mathbb{R}^n}|P(x)|^2\eta_j^{m+1}(x)\,dx,$$

that is, $P_{i,j}^{m+1}$ is the unique element of $\mathcal{P}_s(\mathbb{R}^n)$ such that

$$\langle(f-P_j^{m+1})\eta_i^k,P\eta_j^{m+1}\rangle=\int_{\mathbb{R}^n}P_{i,j}^{m+1}(x)P(x)\eta_j^{m+1}(x)\,dx. \tag{5.4}$$

In what follows, we denote $Q_i^{m*}=(1+2^{-(10+n)})Q_i^m$,

$$E_1^m\equiv\{i\in\mathbb{N}:l_i^m\ge\rho(x_i^m)/(2^5n)\},\qquad E_2^k\equiv\{i\in\mathbb{N}:l_i^m<\rho(x_i^m)/(2^5n)\},$$
$$F_1^k\equiv\{i\in\mathbb{N}:l_i^m\ge L_3\rho(x_i^m)\},\qquad F_2^k\equiv\{i\in\mathbb{N}:l_i^m<L_3\rho(x_i^m)\},$$

where $L_3=2^{k_0}C_0$ is as in Section 4.

By the definition of $P_{i,j}^{m+1}$, we have

$$P_{i,j}^{m+1}\ne0\quad\text{if and only if}\quad Q_i^{m*}\cap Q_j^{(m+1)*}\ne\emptyset. \tag{5.5}$$

The following Lemmas 5.1–5.3 can be proved by similar methods of Lemmas 5.1–5.3 in [24].

LEMMA 5.1. *Notice that* $\Omega_{m+1}\subset\Omega_m$, *then*

(i) *If* $Q_i^{m*}\cap Q_j^{(m+1)*}\neq\emptyset$, *then* $l_j^{m+1}\leq 2^4\sqrt{n}l_i^m$ *and* $Q_j^{(m+1)*}\subset 2^6 nQ_i^{k*}\subset\Omega_m$.
(ii) *There exists a positive integer* L *such that for each* $i\in\mathbb{N}$, *the cardinality of* $\{j\in\mathbb{N}:Q_i^{m*}\cap Q_j^{(m+1)*}\neq\emptyset\}$ *is bounded by* L.

LEMMA 5.2. *There exists a positive constant* C *such that for all* $i,j\in\mathbb{N}$ *and integer* $m\geq m_0$ *with* $l_j^{m+1}<L_3\rho(x_j^{m+1})$,

$$\sup_{y\in\mathbb{R}^n}\left|P_{i,j}^{m+1}(y)\eta_j^{m+1}(y)\right|\leq C2^{m+1}. \tag{5.6}$$

LEMMA 5.3. *For any* $k\in\mathbb{Z}$ *with* $m\geq m_0$,

$$\sum_{i\in\mathbb{N}}\left(\sum_{j\in F_2^{m+1}}P_{i,j}^{m+1}\eta_j^{m+1}\right)=0,$$

where the series converges both in $\mathcal{D}'(\mathbb{R}^n)$ *and pointwise.*

Then we can give the weighted atomic decomposition for a dense subspace of $h^p_{\rho,N}(\omega)$ as follows.

LEMMA 5.4. *Let* $\omega\in A_\infty^{\rho,\infty}(\mathbb{R}^n)$, q_ω *and* $N_{p,\omega}$ *be respectively as in* (2.4) *and* (3.29). *If* $p\in(0,1]$, $s\geq[n(q_\omega/p-1)]$, $N>s$ *and* $N\geq N_{p,\omega}$, *then for any* $f\in(L^q_\omega(\mathbb{R}^n)\cap h^p_{\rho,N}(\omega))$, *there exist numbers* $\lambda_0\in\mathbb{C}$ *and* $\{\lambda_i^m\}_{m\geq k_0,i}\subset\mathbb{C}$, $(p,\infty,s)_\omega$*-atoms* $\{a_i^m\}_{m\geq m_0,i}$ *and a single atom* a_0 *such that*

$$f=\sum_{m\geq m_0}\sum_i\lambda_i^m a_i^m+\lambda_0 a_0, \tag{5.7}$$

where the series converges almost everywhere and in $\mathcal{D}'(\mathbb{R}^n)$. *Moreover, there exists a positive constant* C, *independent of* f, *such that*

$$\sum_{m\geq m_0,i}\left|\lambda_i^m\right|^p+|\lambda_0|^p\leq C\|f\|_{h^p_{\rho,N}(\omega)}. \tag{5.8}$$

Proof. For $f\in(L^q_\omega(\mathbb{R}^n)\cap h^p_{\rho,N}(\omega))$, in the case $m_0=-\infty$ and each $m\in\mathbb{Z}$, f has a Calderón–Zygmund decomposition of degree s and height $\lambda=2^m$ associated to $\mathcal{M}_N(f)$ as above, that is, $f=g^m+\sum_i b_i^m$. By Corollary 4.10 and Proposition 3.12, $g^m\to f$ in both $h^p_{\rho,N}(\omega)$ and $\mathcal{D}'(\mathbb{R}^n)$ as $m\to\infty$. By Lemma 4.9(i), $\|g^m\|_{L^q_\omega(\mathbb{R}^n)}\to 0$ as $m\to-\infty$, and moreover, by Lemma 2.5(ii), $g^m\to 0$ in $\mathcal{D}'(\mathbb{R}^n)$ as $m\to-\infty$. Hence,

$$f=\sum_{m=-\infty}^{\infty}\left(g^{m+1}-g^m\right) \tag{5.9}$$

in $\mathcal{D}'(\mathbb{R}^n)$. Since $\operatorname{supp}(\sum_i b_i^m) \subset \Omega_m$ and $\omega(\Omega_m) \to 0$ as $m \to \infty$, then $g^m \to f$ almost everywhere as $m \to \infty$, and (5.9) holds for almost everywhere. By Lemma 5.3 and $\sum_i \eta_i^m b_j^{m+1} = \chi_{\Omega_m} b_j^{m+1} = b_j^{m+1}$ for all j, then we have

$$(5.10)\quad \begin{aligned} g^{m+1} - g^m &= \Big(f - \sum_j b_j^{m+1}\Big) - \Big(f - \sum_i b_i^m\Big) \\ &= \sum_i b_i^m - \sum_j b_j^{m+1} + \sum_i \Big(\sum_{j \in F_2^{m+1}} P_{i,j}^{m+1} \eta_j^{m+1}\Big) \\ &= \sum_i \Big[b_i^m - \sum_j b_j^{m+1} \eta_i^m + \sum_{j \in F_2^{m+1}} P_{i,j}^{m+1} \eta_j^{m+1}\Big] \equiv \sum_i h_i^m. \end{aligned}$$

It is easy to see that the series converges in both $\mathcal{D}'(\mathbb{R}^n)$ and almost everywhere. Then, by the definitions of b_j^m and b_j^{m+1}, when $l_i^m < L_3\rho(x_i^m)$, we have

$$(5.11)\quad \begin{aligned} h_i^m = {} & f\chi_{\Omega_{m+1}^{\complement}} \eta_i^m - P_i^m \eta_i^m \\ & + \sum_{j \in F_2^{m+1}} P_j^{m+1} \eta_i^m \eta_j^{m+1} + \sum_{j \in F_2^{m+1}} P_{i,j}^{m+1} \eta_j^{m+1}, \end{aligned}$$

and when $l_i^m \geq L_3\rho(x_i^m)$, we have

$$(5.12)\quad h_i^m = f\chi_{\Omega_{m+1}^{\complement}} \eta_i^m + \sum_{j \in F_2^{m+1}} P_j^{m+1} \eta_i^m \eta_j^{m+1} + \sum_{j \in F_2^{m+1}} P_{i,j}^{m+1} \eta_j^{m+1}.$$

We can get that for almost every $x \in \Omega_{m+1}^{\complement}$,

$$|f(x)| \leq \mathcal{M}_N(f)(x) \leq 2^{m+1},$$

by Proposition 2.8(i). Then, by Lemma 4.2, Lemma 5.1(ii), Lemma 5.2, (5.11) and (5.12) we obtain that there exists a positive constant C_{11} such that for any $i \in \mathbb{N}$,

$$(5.13)\quad \|h_i^m\|_{L_\omega^\infty(\mathbb{R}^n)} \leq C_{11} 2^m.$$

Next, we need to prove h_i^m is either a multiple of a $(p, \infty, s)_\omega$-atom or a finite linear combination of $(p, \infty, s)_\omega$-atom in the following two cases of i.

Case I. For $i \in E_1^m$, $l_i^m \geq \rho(x_i^m)/2^5 n$. Clearly, h_i^m is supported in a cube $\widetilde{Q}_i^m$ that contains Q_i^{m*} as well as all the $Q_j^{(m+1)*}$ that intersect Q_i^{m*}. In fact, observe that if $Q_i^{m*} \cap Q_j^{(m+1)*} \neq \emptyset$, by Lemma 5.1, we have $Q_j^{(m+1)*} \subset 2^6 n Q_i^{m*} \subset \Omega_m$, thus, we set $\widetilde{Q}_i^m \equiv 2^6 n Q_i^{m*}$. Since $l(\widetilde{Q}_i^m) \geq 2\rho(x_i^m)$, by the same method of Lemma 3.1 in [27], $\widetilde{Q}_i^m$ can be decomposed into finite disjoint cubes $\{Q_{i,k}^m\}_k$ such that $\widetilde{Q}_i^m = \bigcup_{k=1}^{n_i} Q_{i,k}^m$ and $l_{i,k}^m/4 < \rho(x) \leq C_0(3\sqrt{n})^{k_0} l_{i,k}^m$ for some $x \in Q_{i,k}^m = Q(x_{i,k}^m, l_{i,k}^m)$, where C_0, k_0 are constants given in Lemma 2.1.

Moreover, by Lemma 2.1, we also have $l_{i,k}^m \le L_1\rho(x_{i,k}^m)$ and $l_{i,k}^m > L_2\rho(x_{i,k}^m)$. Therefore, let

$$\lambda_{i,k}^m \equiv C_{11}2^m\big[\omega\big(Q_{i,k}^m\big)\big]^{1/p} \quad \text{and} \quad a_{i,k}^m \equiv \big(\lambda_{i,k}^m\big)^{-1}\frac{h_i^m\chi_{Q_{i,k}^m}}{\sum_{k=1}^{n_i}\chi_{Q_{i,k}^m}},$$

then $\operatorname{supp} a_{i,k}^m \subset Q_{i,k}^m$ and $\|a_{i,k}^m\|_{L_\omega^\infty(\mathbb{R}^n)} \le [\omega(Q_{i,k}^m)]^{-1/p}$, hence each $a_{i,k}^m$ is a $(p,\infty,s)_\omega$-atom and $h_i^m = \sum_{k=1}^{n_i}\lambda_{i,k}^m a_{i,k}^m$.

Case II. For $i \in E_2^m$, if $j \in F_1^{m+1}$, we claim that $Q_i^{m*} \cap Q_j^{(m+1)*} = \emptyset$. In fact, if $Q_i^{m*} \cap Q_j^{(m+1)*} \ne \emptyset$, by Lemma 5.1(i), we know $l_j^{m+1} \le 2^4\sqrt{n}l_i^m$ then we can deduce that $l_i^m < l_i^m/2\sqrt{n}$ which is a contradiction, hence the claim is true. Thus, we have

$$\begin{aligned}(5.14)\quad h_i^m &= \big(f-P_i^m\big)\eta_i^m - \sum_{j\in F_1^{m+1}} f\eta_j^{m+1}\eta_i^m - \sum_{j\in F_2^{m+1}}\big(f-P_j^{m+1}\big)\eta_j^{m+1}\eta_i^m \\ &\quad + \sum_{j\in F_2^{m+1}} P_{i,j}^{m+1}\eta_j^{m+1} \\ &= \big(f-P_i^m\big)\eta_i^m - \sum_{j\in F_2^{m+1}}\big\{\big(f-P_j^{m+1}\big)\eta_j^{m+1}\eta_i^m - P_{i,j}^{m+1}\eta_j^{m+1}\big\}.\end{aligned}$$

Let $\widetilde{Q}_i^m \equiv 2^6 n Q_i^{m*}$, then $l(\widetilde{Q}_i^m) < L_1\rho(x_i^m)$ and $\operatorname{supp} h_i^m \subset \widetilde{Q}_i^m$. Furthermore, h_i^m satisfies the desired moment conditions, which can be deduced from the moment conditions of $(f-P_i^m)\eta_i^m$ and $(f-P_j^{m+1})\eta_j^{m+1}\eta_i^m - P_{i,j}^{m+1}\eta_j^{m+1}$. Let $\lambda_i^m \equiv C_{11}2^m[\omega(\widetilde{Q}_i^m)]^{1/p}$ and $a_i^m \equiv (\lambda_i^m)^{-1}h_i^m$, then a_i^m is a $(p,\infty,s)_\omega$-atom.

Thus, by (5.9), (5.10), Case I and Case II, we have

$$f = \sum_{m\in\mathbb{Z}}\left(\sum_{i\in E_1^m}\left(\sum_{k=1}^{n_i}\lambda_{i,k}^m a_{i,k}^m\right) + \sum_{i\in E_2^m}\lambda_i^m a_i^m\right)$$

in both $\mathcal{D}'(\mathbb{R}^n)$ and almost everywhere. Moreover, by Lemma 2.4, we get

$$\begin{aligned}&\sum_{k\in\mathbb{Z}}\left[\sum_{i\in E_1^m}\left[\sum_{k=1}^{n_i}\big|\lambda_{i,k}^m\big|^p\right] + \sum_{i\in E_2^m}\big|\lambda_i^m\big|^p\right] \\ &\quad\le C\sum_{k\in\mathbb{Z}}2^{mp}\left[\sum_{i\in E_1^m}\left[\sum_{k=1}^{n_i}\omega\big(Q_{i,k}^m\big)\right] + \sum_{i\in E_2^m}\omega\big(\widetilde{Q}_i^m\big)\right] \\ &\quad\le C\sum_{k\in\mathbb{Z}}2^{mp}\left[\sum_{i\in E_1^m}\omega\big(\widetilde{Q}_i^m\big) + \sum_{i\in E_2^m}\omega\big(\widetilde{Q}_i^m\big)\right] \\ &\quad\le C\sum_{m\in\mathbb{Z}}\sum_{i\in\mathbb{N}}2^{mp}\omega\big(\widetilde{Q}_i^m\big)\end{aligned}$$

$$\leq C \sum_{m\in\mathbb{Z}} \sum_{i\in\mathbb{N}} 2^{mp}\omega\big(Q_i^{m*}\big)$$
$$\leq C \sum_{m\in\mathbb{Z}} 2^{mp}\omega(\Omega_m)$$
$$\leq C\big\|\mathcal{M}_N(f)\big\|^p_{L^p_\omega(\mathbb{R}^n)} = C\|f\|^p_{h^p_{\rho,N}(\omega)},$$

by which we can obtain (5.8) in the case $m_0 = -\infty$.

Finally, when $m_0 > -\infty$, since $f \in h^p_{\rho,N}(\omega)$, we know $\omega(\mathbb{R}^n) < \infty$. By the similar arguments, we have

$$f = \sum_{m=m_0}^{\infty} \big(g^{m+1} - g^m\big) + g^{m_0} \equiv \widetilde{f} + g^{m_0}. \tag{5.15}$$

For the function $\widetilde{f}$, we have the same $(p,\infty,s)_\omega$ atomic decomposition:

$$\widetilde{f} = \sum_{m\geq m_0, i} \lambda_i^m a_i^m, \tag{5.16}$$

and

$$\sum_{m\geq m_0} \sum_{i\in\mathbb{N}} \big|\lambda_i^m\big|^p \leq C\|f\|^p_{h^p_{\rho,N}(\omega)}. \tag{5.17}$$

For the function g^{m_0}, by Lemma 4.9(ii), we have

$$\big\|g^{m_0}\big\|_{L^\infty_\omega(\mathbb{R}^n)} \leq C_{10}2^{m_0} \leq 2C_{10} \inf_{x\in\mathbb{R}^n} \mathcal{M}_N f(x), \tag{5.18}$$

where C_{10} is the same constant as in Lemma 4.9(ii).

Let $\lambda_0 \equiv C_{10}2^{m_0}[\omega(\mathbb{R}^n)]^{1/p}$ and $a_0 \equiv \lambda_0^{-1} g^{m_0}$, then

$$\|a_0\|_{L^\infty_\omega(\mathbb{R}^n)} \leq \big[\omega\big(\mathbb{R}^n\big)\big]^{-1/p} \quad \text{and} \quad |\lambda_0|^p \leq (2C_{10})^p\|f\|^p_{h^p_{\rho,N}(\omega)}. \tag{5.19}$$

Hence, $g^{m_0} = \lambda_0 a_0$ and a_0 is a $(p,\infty)_\omega$-single-atom, then by combining with (5.15) and (5.16) we can obtain (5.7) in the case $m_0 > -\infty$. Furthermore, by (5.17) and (5.19), we get

$$\sum_{m\geq m_0} \sum_{i\in\mathbb{N}} \big|\lambda_i^m\big|^p + |\lambda_0|^p \leq C\|f\|^p_{h^p_{\rho,N}(\omega)}.$$

The proof of the lemma is complete. □

Next, we can establish the weighted atomic decompositions of $h^p_{\rho,N}(\omega)$.

THEOREM 5.5. *Let $\omega \in A^{\rho,\infty}_\infty(\mathbb{R}^n)$, q_ω and $N_{p,\omega}$ be respectively as in (2.4) and (3.29). If $q \in (q_\omega,\infty]$, $p \in (0,1]$, and integers s and N satisfy $N \geq N_{p,\omega}$ and $N > s \geq [n(q_\omega/p - 1)]$, then $h^{p,q,s}_\rho(\omega) = h^p_{\rho,N}(\omega) = h^p_{\rho,N_{p,\omega}}(\omega)$ with equivalent norms.*

Proof. First of all, it is easy to get that

$$h_\rho^{p,\infty,\bar{s}}(\omega) \subset h_\rho^{p,q,s}(\omega) \subset h^p_{\rho,N_{p,\omega}}(\omega) \subset h^p_{\rho,N}(\omega) \subset h^p_{\rho,\bar{N}}(\omega),$$

where $\bar{s}$ is an integer no less than s and $\bar{N}$ is an integer larger than N, and the inclusions are continuous. Hence, we need to prove that for any $N > s \geq [n(q_\omega/p - 1)]$, $h^p_{\rho,N}(\omega) \subset h_\rho^{p,\infty,s}(\omega)$, and for all $f \in h^p_{\rho,N}(\omega)$, $\|f\|_{h_\rho^{p,\infty,s}(\omega)} \leq C\|f\|_{h^p_{\rho,N}(\omega)}$.

For $f \in h^p_{\rho,N}(\omega)$, by Corollary 4.10, there exists a sequence of functions $\{f_m\}_{m\in\mathbb{N}} \subset (h^p_{\rho,N}(\omega) \cap L^q_\omega(\mathbb{R}^n))$ such that for all $m \in \mathbb{N}$,

$$\|f_m\|_{h^p_{\rho,N}(\omega)} \leq 2^{-m}\|f\|_{h^p_{\rho,N}(\omega)} \tag{5.20}$$

and $f = \sum_{m\in\mathbb{N}} f_m$ in $h^p_{\rho,N}(\omega)$. By Lemma 5.4, for each $m \in \mathbb{N}$, f_m has an atomic decomposition

$$f_m = \sum_{i\in\mathbb{Z}_+} \lambda_i^m a_i^m$$

in $\mathcal{D}'(\mathbb{R}^n)$ with

$$\sum_{i\in\mathbb{Z}_+} \left|\lambda_i^m\right|^p \leq C\|f_m\|^p_{h^p_{\rho,N}(\omega)},$$

where $\{\lambda_i^m\}_{i\in\mathbb{Z}_+} \subset \mathbb{C}$, $\{a_i^m\}_{i\in\mathbb{N}}$ are $(p,\infty,s)_\omega$-atoms and a_0^m is a $(p,\infty)_\omega$-single-atom. Let

$$\widetilde{\lambda}_0 \equiv \left[\omega\left(\mathbb{R}^n\right)\right]^{1/p} \sum_{m=1}^\infty \left|\lambda_0^m\right| \left\|a_0^m\right\|_{L^\infty_\omega(\mathbb{R}^n)} \quad \text{and} \quad \widetilde{a}_0 \equiv (\widetilde{\lambda}_0)^{-1} \sum_{m=1}^\infty \lambda_0^m a_0^m,$$

then we have

$$\widetilde{\lambda}_0 \widetilde{a}_0 = \sum_{m=1}^\infty \lambda_0^m a_0^m$$

and

$$\|\widetilde{a}_0\|_{L^\infty_\omega(\mathbb{R}^n)} \leq \left[\omega\left(\mathbb{R}^n\right)\right]^{-1/p},$$

which implies that $\widetilde{a}_0$ is a $(\rho,\infty)_\omega$-single-atom.

Since $\|a_0^m\|_{L^\infty_\omega(\mathbb{R}^n)} \leq (\omega(\mathbb{R}^n))^{-1/p}$ and

$$\left|\lambda_0^m\right| \leq C\|f_m\|_{h^p_{\rho,N}(\omega)} \leq C2^{-m}\|f\|_{h^p_{\rho,N}(\omega)},$$

we obtain

$$|\widetilde{\lambda}_0| \leq C\left(\sum_{m=1}^\infty 2^{-m}\right)\|f\|_{h^p_{\rho,N}(\omega)} \leq C\|f\|_{h^p_{\rho,N}(\omega)},$$

moreover, we get

$$\sum_{m\in\mathbb{N}}\sum_{i\in\mathbb{N}} |\lambda_i^m|^p + |\widetilde{\lambda}_0|^p \leq C\left(\sum_{m\in\mathbb{N}} \|f_m\|^p_{h^p_{\rho,N}(\omega)} + \|f\|^p_{h^p_{\rho,N}(\omega)}\right) \leq C\|f\|^p_{h^p_{\rho,N}(\omega)}.$$

Finally, we can obtain

$$f = \sum_{m\in\mathbb{N}} \sum_{i\in\mathbb{N}} \lambda_i^m a_i^m + \widetilde{\lambda}_0 \widetilde{a}_0 \in h_\rho^{p,\infty,s}(\omega)$$

and

$$\|f\|_{h_\rho^{p,\infty,s}(\omega)} \le C\|f\|_{h_{\rho,N}^p(\omega)}.$$

The theorem is proved. □

For simplicity, from now on, we denote by $h_\rho^p(\omega)$ the weighted local Hardy space $h_{\rho,N}^p(\omega)$ when $N \ge N_{p,\omega}$.

6. Atomic characterization of $H_{\mathcal{L}}^1(\omega)$

In this section, we apply the atomic characterization of the weighted local Hardy spaces $h_\rho^1(\omega)$ with $A_1^{\rho,\theta}(\mathbb{R}^n)$ weights to establish atomic characterization of weighted Hardy space $H_{\mathcal{L}}^1(\omega)$ associated to Schrödinger operator with $A_1^{\rho,\theta}(\mathbb{R}^n)$ weights.

Let $\mathcal{L} = -\Delta + V$ be a Schrödinger operator on $\mathbb{R}^n$, $n \ge 3$, where $V \in RH_{n/2}$ is a fixed non-negative potential.

Let $\{T_t\}_{t>0}$ be the semigroup of linear operators generated by $\mathcal{L}$ and $T_t(x,y)$ be their kernels, that is,

$$T_t f(x) = e^{-t\mathcal{L}} f(x) = \int_{\mathbb{R}^n} T_t(x,y) f(y)\,dy, \quad \text{for } t>0 \text{ and } f \in L^2(\mathbb{R}^n). \tag{6.1}$$

Since V is non-negative the Feynman–Kac formula implies that

$$0 \le T_t(x,y) \le \widetilde{T}_t(x,y) \equiv (4\pi t)^{-\frac{n}{2}} \exp\left(-\frac{|x-y|^2}{4t}\right). \tag{6.2}$$

Obviously, by (6.2) the maximal operator

$$\mathcal{T}^* f(x) = \sup_{t>0} \left|T_t f(x)\right|$$

is of weak-type $(1,1)$. A weighted Hardy-type space related to $\mathcal{L}$ with $A_1^{\rho,\theta}(\mathbb{R}^n)$ weights is naturally defined by:

$$H_{\mathcal{L}}^1(\omega) \equiv \left\{f \in L_\omega^1(\mathbb{R}^n) : \mathcal{T}^* f(x) \in L_\omega^1(\mathbb{R}^n)\right\}, \quad \text{with} \tag{6.3}$$
$$\|f\|_{H_{\mathcal{L}}^1(\omega)} \equiv \left\|\mathcal{T}^* f\right\|_{L_\omega^1(\mathbb{R}^n)}.$$

The $H_{\mathcal{L}}^1(\omega)$ with $\omega \in A_1(\mathbb{R}^n)$ has been studied in [16], [36]

Now let us recall some basic properties of kernels $T_t(x,y)$ and the operator $\mathcal{T}^*$.

Lemma 6.1 (See [9]). *For every $l > 0$ there is a constant C_l such that*

$$T_t(x,y) \le C_l \left(1 + |x-y|/\rho(x)\right)^{-l} |x-y|^{-n}, \tag{6.4}$$

for $x,y\in\mathbb{R}^n$. *Moreover, there is an* $\varepsilon>0$ *such that for every* $C'>0$, *there exists* C *so that*

$$\big|T_t(x,y)-\widetilde{T}_t(x,y)\big|\le C\frac{(|x-y|/\rho(x))^{\varepsilon}}{|x-y|^n}, \tag{6.5}$$

for $|x-y|\le C'\rho(x)$.

Since $T_t(x,y)$ is a symmetric function, we also have

$$T_t(x,y)\le C_l\big(1+|x-y|/\rho(y)\big)^{-l}|x-y|^{-n},\quad \text{for } x,y\in\mathbb{R}^n. \tag{6.6}$$

LEMMA 6.2 (See [10]). *There exist a rapidly decaying function* $w\ge 0$ *and a* $\delta>0$ *such that*

$$\big|T_t(x,y)-\widetilde{T}_t(x,y)\big|\le\left(\frac{\sqrt{t}}{\rho(x)}\right)^{\delta}w_{\sqrt{t}}(x-y), \tag{6.7}$$

where $w_{\sqrt{t}}(x)=t^{-n/2}w(x/\sqrt{t})$.

LEMMA 6.3 (See [11]). *If* $V\in RH_s(\mathbb{R}^n)$, $s>n/2$, *then there exist* $\delta=\delta(s)>0$ *and* $c>0$ *such that for every* $N>0$, *there is a constant* C_N *so that, for all* $|h|<\sqrt{t}$

$$\begin{aligned}&\big|T_t(x+h,y)-T_t(x,y)\big|\\ &\quad\le C_N\left(\frac{|h|}{\sqrt{t}}\right)^{\delta}t^{-\frac{n}{2}}\left(1+\frac{\sqrt{t}}{\rho(x)}+\frac{\sqrt{t}}{\rho(y)}\right)^{-N}\exp\left(-\frac{c|x-y|^2}{t}\right).\end{aligned} \tag{6.8}$$

LEMMA 6.4 (See [2]). *For* $1<p<\infty$ *the operator* $\mathcal{T}^*$ *is bounded on* $L^p(\omega)$ *when* $\omega\in A_p^{\rho,\infty}(\mathbb{R}^n)$, *and of weak type* $(1,1)$ *when* $\omega\in A_1^{\rho,\infty}(\mathbb{R}^n)$.

Let $\{\widetilde{T}_t\}_{t>0}$ be the semigroup of linear operators, and $\widetilde{T}_t(x,y)$ be their kernels, that is,

$$\widetilde{T}_tf(x)=\int_{\mathbb{R}^n}\widetilde{T}_t(x,y)f(y)\,dy,\quad \text{for } t>0.$$

In order to achieve the desired conclusions, we need the following estimates.

LEMMA 6.5. *Let* $\omega\in A_1^{\rho,\infty}(\mathbb{R}^n)$, *then there exists a positive constant* C *such that for all* $f\in h^1_\rho(\omega)$,

$$\|f\|_{h^1_\rho(\omega)}\le C\big\|\widetilde{T}^+_\rho(f)\big\|_{L^1_\omega(\mathbb{R}^n)}, \tag{6.9}$$

where

$$\widetilde{T}^+_\rho(f)(x)\equiv\sup_{0<t<\rho(x)}\big|\widetilde{T}_{t^2}(f)(x)\big|.$$

Proof. Let $h(x)=(4\pi)^{-n/2}e^{-|x|^2/4}$, then it is easy to find that $h_t(x-y)=\widetilde{T}_{t^2}(x,y)$. Now we take a nonnegative function $\varphi\in\mathcal{D}(\mathbb{R}^n)$ such that $\varphi(x)=h(x)$ on $B(0,2)$, and we define $\varphi_\rho^+(f)(x)$ as follows:

$$\varphi_\rho^+(f)(x)\equiv\sup_{0<t<\rho(x)}\big|\varphi_t*f(x)\big|.$$

Clearly, for any $x\in\mathbb{R}^n$, we have

$$\varphi^+(f)(x)\le\varphi_\rho^+(f)(x), \tag{6.10}$$

see (3.4) for the definition of $\varphi^+(f)(x)$.

Let $f\in h_\rho^1(\omega)$. For every $N>0$, we have

$$\begin{aligned}
&\big\|\varphi_\rho^+(f)-\widetilde{T}_\rho^+(f)\big\|_{L^1_\omega(\mathbb{R}^n)}\\
&\quad\le\int_{\mathbb{R}^n}\sup_{0<t<\rho(x)}\big|\varphi_t*f(x)-h_t*f(x)\big|\omega(x)\,dx\\
&\quad\le\int_{\mathbb{R}^n}\left(\sup_{0<t<\rho(x)}t^{-n}\int_{\mathbb{R}^n}\big|f(y)\big|\left|\varphi\left(\frac{x-y}{t}\right)-h\left(\frac{x-y}{t}\right)\right|dy\right)\omega(x)\,dx\\
&\quad\lesssim\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}\big|f(y)\big|\sup_{0<t<\rho(x)}t^{-n}\left(1+\frac{|x-y|}{t}\right)^{-N}\chi_{\{|y-x|>t\}}(y)\,dy\right)\omega(x)\,dx\\
&\quad\lesssim\int_{\mathbb{R}^n}\big|f(y)\big|\left(\int_{\mathbb{R}^n}\big(\rho(x)\big)^{-n}\left(1+\frac{|x-y|}{\rho(x)}\right)^{-N}\omega(x)\,dx\right)dy.
\end{aligned}$$

In the last inequality, we used the following facts that

$$\sup_{0<t<\rho(x)}t^{-n}\left(1+\frac{|x-y|}{t}\right)^{-N}\le\big(\rho(x)\big)^{-n}\left(1+\frac{|x-y|}{\rho(x)}\right)^{-N},$$

provided that $|x-y|>t$ and $N>2n$.

We now estimate the inner integral in the last inequality. In fact,

$$\begin{aligned}
&\int_{\mathbb{R}^n}\big(\rho(x)\big)^{-n}\left(1+\frac{|x-y|}{\rho(x)}\right)^{-N}\omega(x)\,dx\\
&\quad=\int_{|x-y|<\rho(y)}\big(\rho(x)\big)^{-n}\left(1+\frac{|x-y|}{\rho(x)}\right)^{-N}\omega(x)\,dx\\
&\qquad+\int_{|x-y|\ge\rho(y)}\big(\rho(x)\big)^{-n}\left(1+\frac{|x-y|}{\rho(x)}\right)^{-N}\omega(x)\,dx\\
&\quad\equiv I+II.
\end{aligned}$$

For I, since N is large enough and (2.2), we have

$$I\le\frac{C}{(\rho(y))^n}\int_{|x-y|<\rho(y)}\omega(x)\,dx\le C\Psi_\theta(\widetilde{B}_0)M_{V,\theta}(\omega)(y)\le C\omega(y),$$

where $\widetilde{B}_0=B(y,\rho(y))$.

For II, by the same reason as above, we have

$$
\begin{aligned}
II &\le C\sum_{i=1}^{\infty}\int_{|x-y|\sim 2^i\rho(y)}\big(\rho(x)\big)^{N-n}|x-y|^{-N}\omega(x)\,dx\\
&\le C\sum_{i=1}^{\infty}\int_{|x-y|\sim 2^i\rho(y)}\big(\rho(y)\big)^{N-n}\left(1+\frac{|x-y|}{\rho(y)}\right)^{\frac{k_0(N-n)}{k_0+1}}|x-y|^{-N}\omega(x)\,dx\\
&\le C\sum_{i=1}^{\infty}\int_{|x-y|\sim 2^i\rho(y)}\big(\rho(y)\big)^{N-n}\big(1+2^i\big)^{\frac{k_0(N-n)}{k_0+1}}\big(2^i\rho(y)\big)^{-N}\omega(x)\,dx\\
&\le C\sum_{i=1}^{\infty}\big(2^{-i}\big)^{\frac{N+nk_0}{k_0+1}}\frac{1}{(\rho(y))^n}\int_{|x-y|<2^i\rho(y)}\omega(x)\,dx\\
&\le C\sum_{i=1}^{\infty}\big(2^{-i}\big)^{\frac{N+nk_0}{k_0+1}}\big(1+2^i\big)^{\theta}M_{V,\theta}(\omega)(y)\\
&\le C\sum_{i=1}^{\infty}\big(2^{-i}\big)^{\frac{N+nk_0}{k_0+1}-\theta}\omega(y)\le C\omega(y),
\end{aligned}
$$

and the last inequality holds because the real number N is large enough.

Combining the above two estimates, we get

$$
\big\|\varphi_\rho^+(f)-\widetilde{T}_\rho^+(f)\big\|_{L^1_\omega(\mathbb{R}^n)}\le C\int_{\mathbb{R}^n}\big|f(y)\big|\omega(y)\,dy=C\|f\|_{L^1_\omega(\mathbb{R}^n)}. \tag{6.11}
$$

In addition, it is easy to get $\|f\|_{L^1_\omega(\mathbb{R}^n)}\le\|\widetilde{T}_\rho^+f\|_{L^1_\omega(\mathbb{R}^n)}$. Therefore, we obtain

$$
\big\|\varphi_\rho^+(f)\big\|_{L^1_\omega(\mathbb{R}^n)}\le\big\|\widetilde{T}_\rho^+(f)\big\|_{L^1_\omega(\mathbb{R}^n)}+C\|f\|_{L^1_\omega(\mathbb{R}^n)}\le C\big\|\widetilde{T}_\rho^+(f)\big\|_{L^1_\omega(\mathbb{R}^n)}. \tag{6.12}
$$

Finally, from Theorem 3.10, (6.10) and (6.12), it follows that

$$
\|f\|_{h^1_\rho(\omega)}\le C\big\|\varphi^+(f)\big\|_{L^1_\omega(\mathbb{R}^n)}\le C\big\|\varphi_\rho^+(f)\big\|_{L^1_\omega(\mathbb{R}^n)}\le C\big\|\widetilde{T}_\rho^+(f)\big\|_{L^1_\omega(\mathbb{R}^n)},
$$

which finishes the proof. □

For $x,y\in\mathbb{R}^n$, set $E_t(x,y)=T_{t^2}(x,y)-\widetilde{T}_{t^2}(x,y)$,

$$
T_\rho^+(f)(x)\equiv\sup_{0<t<\rho(x)}\big|T_{t^2}(f)(x)\big|\quad\text{and}\quad E_\rho^+(f)(x)\equiv\sup_{0<t<\rho(x)}\big|E_t(f)(x)\big|.
$$

LEMMA 6.6. *Let $\omega\in A_1^{\rho,\infty}(\mathbb{R}^n)$. Then there exists a positive constant C such that for all $f\in L^1_\omega(\mathbb{R}^n)$,*

$$
\big\|E_\rho^+(f)\big\|_{L^1_\omega(\mathbb{R}^n)}\le C\|f\|_{L^1_\omega(\mathbb{R}^n)}.
$$

Proof. By Lemma 2.2, it suffices to prove that for all j,

$$
\big\|E_\rho^+(\chi_{B_j^*}f)\big\|_{L^1_\omega(\mathbb{R}^n)}\le C\|\chi_{B_j^*}f\|_{L^1_\omega(\mathbb{R}^n)}, \tag{6.13}
$$

in which $B_j = B(x_j, \rho(x_j))$. For any $x \in B_j^{**}$ and $y \in B_j^*$, since $\rho(y) \sim \rho(x_j) \sim \rho(x)$ via Lemma 2.1, by (6.5) we have

$$\left|E_t(x,y)\right| \le C\frac{(|x-y|/\rho(x))^{\varepsilon}}{|x-y|^n} \le \frac{C}{|x-y|^{n-\varepsilon}(\rho(x_j))^{\varepsilon}},$$

which implies that

$$\begin{aligned}
&\int_{B_j^{**}} \sup_{0<t<\rho(x)} \left|E_t(\chi_{B_j^*} f)\right|\omega(x)\,dx \\
&\quad\le C\int_{B_j^{**}}\left(\int_{B_j^*}\frac{|f(y)|}{|x-y|^{n-\varepsilon}(\rho(x_j))^{\varepsilon}}\,dy\right)\omega(x)\,dx \\
&\quad\le C\int_{B_j^{*}}\left(\int_{B_j^{**}}\frac{\omega(x)}{|x-y|^{n-\varepsilon}(\rho(x_j))^{\varepsilon}}\,dx\right)\left|f(y)\right|dy \\
&\quad\le C\int_{B_j^{*}}\left(\sum_{k=-2}^{\infty}\int_{|x-y|\sim 2^{-k}\rho(x_j)}\frac{\omega(x)}{|x-y|^{n-\varepsilon}(\rho(x_j))^{\varepsilon}}\,dx\right)\left|f(y)\right|dy \\
&\quad\le C\int_{B_j^{*}}\left(\sum_{k=-2}^{\infty}\frac{\omega(B(y,2^{-k}\rho(x_j)))}{(2^{-k}\rho(x_j))^{n-\varepsilon}(\rho(x_j))^{\varepsilon}}\,dx\right)\left|f(y)\right|dy \\
&\quad\le C\int_{B_j^{*}}\left(\sum_{k=-2}^{\infty}\frac{1}{2^{k\varepsilon}}\left(1+C_0 2^{k_0-k}\right)^{\theta}\omega(y)\right)\left|f(y)\right|dy \\
&\quad\le C\int_{B_j^{*}}\left|f(y)\right|\omega(y)\,dy = C\|\chi_{B_j^*}f\|_{L^1_\omega(\mathbb{R}^n)}.
\end{aligned}$$

For any $x \in (B_j^{**})^{\complement}$ and $y \in B_j^*$, it is easy to see that $\rho(x_j) \lesssim |x-x_j| \sim |x-y|$; in addition, by (2.2) and (6.7), we have $0 < t < \rho(x) \lesssim |x-x_j|^{k_0/(k_0+1)}(\rho(x_j))^{1/(k_0+1)}$ and $E_t(x,y) \lesssim t^N/|x-y|^{N+n} \sim t^N/|x-x_j|^{N+n}$ for any $N > 0$. Therefore, taking $N > (k_0+1)\theta$, we have

$$\begin{aligned}
&\int_{(B_j^{**})^{\complement}} \sup_{0<t<\rho(x)} \left|E_t(\chi_{B_j^*} f)\right|\omega(x)\,dx \\
&\quad\le C\int_{(B_j^{**})^{\complement}}\left(\int_{B_j^*}\frac{(\rho(x_j))^{\frac{N}{k_0+1}}|f(y)|}{|x-x_j|^{n+\frac{N}{k_0+1}}}\,dy\right)\omega(x)\,dx \\
&\quad\le C\int_{B_j^*}\left(\int_{(B_j^{**})^{\complement}}\frac{(\rho(x_j))^{\frac{N}{k_0+1}}\omega(x)}{|x-x_j|^{n+\frac{N}{k_0+1}}}\,dx\right)\left|f(y)\right|dy \\
&\quad\le C\int_{B_j^*}\left(\sum_{i=2}^{\infty}\int_{|x-x_j|\sim 2^i\rho(x_j)}\frac{(\rho(x_j))^{\frac{N}{k_0+1}}\omega(x)}{|x-x_j|^{n+\frac{N}{k_0+1}}}\,dx\right)\left|f(y)\right|dy
\end{aligned}$$

$$\begin{aligned}
&\leq C\int_{B_j^*}\left(\sum_{i=2}^{\infty}\frac{(\rho(x_j))^{\frac{N}{k_0+1}}\omega(B(x_j,2^i\rho(x_j)))}{(2^i\rho(x_j))^{n+\frac{N}{k_0+1}}}\,dx\right)\big|f(y)\big|\,dy\\
&\leq C\int_{B_j^*}\left(\sum_{i=2}^{\infty}\frac{(1+2^i)^{\theta}}{(2^i)^{\frac{N}{k_0+1}}}\omega(y)\right)\big|f(y)\big|\,dy\\
&\leq C\int_{B_j^*}\big|f(y)\big|\omega(y)\,dy=C\|\chi_{B_j^*}f\|_{L^1_\omega(\mathbb{R}^n)},
\end{aligned}$$

which completes the proof of (6.13) and hence the proof of this lemma. □

Next, we give several estimates about $(p,q,s)_\omega$-atoms and $(p,q)_\omega$-single-atom, which are important for our conclusion.

LEMMA 6.7. *Let a be a $(p,q,s)_\omega$-atom, and* $\operatorname{supp} a\subset Q(x_0,r)$, *then for any $x\in(4Q)^{\complement}$, we have the following estimates*:

(i) *If $L_2\rho(x_0)\leq r\leq L_1\rho(x_0)$, then for any $M>0$,*

$$\mathcal{T}^*a(x)\lesssim\|a\|_{L^1(\mathbb{R}^n)}\frac{r^M}{|x-x_0|^{n+M}},$$

(ii) *If $r<L_2\rho(x_0)$ and $|x-x_0|\leq 2\rho(x_0)$, then there exists $\delta>0$ such that*

$$\mathcal{T}^*a(x)\lesssim\|a\|_{L^1(\mathbb{R}^n)}\frac{r^\delta}{|x-x_0|^{n+\delta}},$$

(iii) *If $r<L_2\rho(x_0)$ and $|x-x_0|\geq\rho(x_0)/\sqrt{n}$, then there exists $\delta>0$ such that for any $M>0$,*

$$\mathcal{T}^*a(x)\lesssim\|a\|_{L^1(\mathbb{R}^n)}\frac{r^\delta}{|x-x_0|^{n+\delta}}\left(\frac{\rho(x_0)}{|x-x_0|}\right)^M.$$

Proof. If $L_2\rho(x_0)\leq r\leq L_1\rho(x_0)$, since $|x-y|\sim|x-x_0|$ and $\rho(y)\sim\rho(x_0)$ for $x\in(4Q)^{\complement}$ and $y\in Q$, by Lemma 6.1, for any $M>0$, we have

$$\begin{aligned}
T_ta(x)&\leq\int_{\mathbb{R}^n}\big|T_t(x,y)\big|\big|a(y)\big|\,dy\\
&\lesssim\int_Q\left(1+\frac{|x-y|}{\rho(y)}\right)^{-M}|x-y|^{-n}\big|a(y)\big|\,dy\\
&\lesssim\int_Q\left(1+\frac{|x-x_0|}{\rho(x_0)}\right)^{-M}|x-x_0|^{-n}\big|a(y)\big|\,dy\\
&\lesssim\|a\|_{L^1(\mathbb{R}^n)}\frac{\rho(x_0)^M}{|x-x_0|^{n+M}}\lesssim\|a\|_{L^1(\mathbb{R}^n)}\frac{r^M}{|x-x_0|^{n+M}},
\end{aligned}$$

and then we obtain (i).

If $r < L_2\rho(x_0)$, by the moment condition of a and Lemma 6.3, for any $M>0$ and $y' \in Q$ which satisfies $|y-y'| < \sqrt{t}$, we have

$$\begin{aligned} T_t a(x) &= \int_{\mathbb{R}^n} T_t(x,y)a(y)\,dy \\ &= \int_Q \big(T_t(x,y) - T_t(x,y')\big)a(y)\,dy \\ &\lesssim \int_Q \left(\frac{|y-y'|}{\sqrt{t}}\right)^{\delta} t^{-\frac{n}{2}} \left(1+\frac{\sqrt{t}}{\rho(y)}\right)^{-M} \exp\left(-\frac{c|x-y|^2}{t}\right)\big|a(y)\big|\,dy \\ &\lesssim \int_Q \left(\frac{r}{\sqrt{t}}\right)^{\delta} t^{-\frac{n}{2}} \left(1+\frac{\sqrt{t}}{\rho(x_0)}\right)^{-M} \left(\frac{t}{|x-x_0|^2}\right)^{K} \big|a(y)\big|\,dy, \end{aligned}$$

where $K>0$ is any real number.

For $|x-x_0| \le 2\rho(x_0)$, taking $K=(n+\delta)/2$, we obtain

$$\begin{aligned} T_t a(x) &\lesssim \int_Q \left(\frac{r}{\sqrt{t}}\right)^{\delta} t^{-\frac{n}{2}} \left(1+\frac{\sqrt{t}}{\rho(x_0)}\right)^{-M} \left(\frac{t}{|x-x_0|^2}\right)^{K} \big|a(y)\big|\,dy \\ &\lesssim \|a\|_{L^1(\mathbb{R}^n)} \left(\frac{r}{\sqrt{t}}\right)^{\delta} t^{-\frac{n}{2}} \left(\frac{t}{|x-x_0|^2}\right)^{K} \\ &= \|a\|_{L^1(\mathbb{R}^n)} \frac{r^{\delta}}{|x-x_0|^{n+\delta}}, \end{aligned}$$

which implies (ii).

For $|x-x_0| \ge \rho(x_0)/\sqrt{n}$, taking $K=(n+M+\delta)/2$, we obtain

$$\begin{aligned} T_t a(x) &\lesssim \int_Q \left(\frac{r}{\sqrt{t}}\right)^{\delta} t^{-\frac{n}{2}} \left(1+\frac{\sqrt{t}}{\rho(x_0)}\right)^{-M} \left(\frac{t}{|x-x_0|^2}\right)^{K} \big|a(y)\big|\,dy \\ &\lesssim \|a\|_{L^1(\mathbb{R}^n)} \left(\frac{r}{\sqrt{t}}\right)^{\delta} t^{-\frac{n}{2}} \left(\frac{\rho(x_0)}{\sqrt{t}}\right)^{M} \left(\frac{t}{|x-x_0|^2}\right)^{K} \\ &= \|a\|_{L^1(\mathbb{R}^n)} \frac{r^{\delta}}{|x-x_0|^{n+\delta}} \left(\frac{\rho(x_0)}{|x-x_0|}\right)^{M}, \end{aligned}$$

which finishes the proof of this lemma. □

LEMMA 6.8. *Let $\omega \in A_q^{\rho,\theta}(\mathbb{R}^n)$ and a be a $(p,q,s)_\omega$-atom, which satisfies* $\operatorname{supp} a \subset Q(x_0,r)$. *Then there exists a constant C such that*:

$$\|a\|_{L^1(\mathbb{R}^n)} \le C|Q|\omega(Q)^{-1/p}\Psi_\theta(Q).$$

Proof. If $q>1$, by Hölder inequality and the definition of $A_q^{\rho,\theta}(\mathbb{R}^n)$ weights, we have

$$\begin{aligned}\|a\|_{L^1(\mathbb{R}^n)} &= \int_Q |a(x)|\omega(x)^{1/q}\omega(x)^{-1/q}\,dx \\ &\le \|a\|_{L^q_\omega(\mathbb{R}^n)}\left(\int_Q \omega(x)^{-q'/q}\,dx\right)^{1/q'} \\ &\le \omega(Q)^{1/q-1/p}\left(\int_Q \omega(x)^{-q'/q}\,dx\right)^{1/q'}\left(\int_Q \omega(x)\,dx\right)^{1/q}\omega(Q)^{-1/q} \\ &\le C|Q|\omega(Q)^{-1/p}\Psi_\theta(Q).\end{aligned}$$

If $q=1$, we have

$$\omega(Q)\le C|Q|\Psi_\theta(Q)\inf_{x\in Q}\omega(x),$$

which implies

$$\left\|\omega^{-1}\right\|_{L^\infty(Q)}\le C|Q|\omega(Q)^{-1}\Psi_\theta(Q).$$

Therefore, we get

$$\|a\|_{L^1(\mathbb{R}^n)}\le\|a\|_{L^1_\omega(\mathbb{R}^n)}\left\|\omega^{-1}\right\|_{L^\infty(Q)}\le C|Q|\omega(Q)^{-1/p}\Psi_\theta(Q),$$

which finishes the proof. □

Combining above two lemmas with $\Psi_\theta(Q)\lesssim 1$, we can get the following corollary.

COROLLARY 6.9. *Let a be a $(p,q,s)_\omega$-atom, and* $\operatorname{supp} a\subset Q(x_0,r)$. *Then for any $x\in(4Q)^{\complement}$, we have the following estimates*:

(i) *If $L_2\rho(x_0)\le r\le L_1\rho(x_0)$, then for any $M>0$,*

$$\mathcal{T}^*a(x)\lesssim\omega(Q)^{-1/p}\left(\frac{r}{|x-x_0|}\right)^{n+M},$$

(ii) *If $r<L_2\rho(x_0)$ and $|x-x_0|\le 2\rho(x_0)$, then there exists $\delta>0$ such that*

$$\mathcal{T}^*a(x)\lesssim\omega(Q)^{-1/p}\left(\frac{r}{|x-x_0|}\right)^{n+\delta},$$

(iii) *If $r<L_2\rho(x_0)$ and $|x-x_0|\ge\rho(x_0)/\sqrt{n}$, then there exists $\delta>0$ such that for any $M>0$,*

$$\mathcal{T}^*a(x)\lesssim\omega(Q)^{-1/p}\left(\frac{r}{|x-x_0|}\right)^{n+\delta}\left(\frac{\rho(x_0)}{|x-x_0|}\right)^{M}.$$

Next, we give the main theorem of this section.

THEOREM 6.10. *Let $0\not\equiv V\in RH_{n/2}$ and $\omega\in A_1^{\rho,\infty}(\mathbb{R}^n)$. Then $h^1_\rho(\omega)=H^1_{\mathcal{L}}(\omega)$ with equivalent norms, that is*

$$\|f\|_{h^1_\rho(\omega)}\sim\|f\|_{H^1_{\mathcal{L}}(\omega)}.$$

Proof. Assume that $f \in H^1_{\mathcal{L}}(\omega)$, by (6.7), we have

$$|f(x)| = \lim_{t<\rho(x), t\to 0} |\widetilde{T}_t(f)(x)| \leq T^+_\rho(f)(x) + C \lim_{t\to 0}\left(\frac{t}{\rho(x)}\right)^\delta M(f)(x) \leq T^+_\rho(f)(x). \tag{6.14}$$

Then according to (6.14), Lemma 6.5 and 6.6, we get $f \in h^1_\rho(\omega)$ and

$$\begin{aligned}\|f\|_{h^1_\rho(\omega)} &\lesssim \|\widetilde{T}^+_\rho(f)\|_{L^1_\omega(\mathbb{R}^n)} \lesssim \|T^+_\rho(f)\|_{L^1_\omega(\mathbb{R}^n)} + \|E^+_\rho(f)\|_{L^1_\omega(\mathbb{R}^n)} \\ &\lesssim \|T^+_\rho(f)\|_{L^1_\omega(\mathbb{R}^n)} + \|f\|_{L^1_\omega(\mathbb{R}^n)} \lesssim \|T^+_\rho(f)\|_{L^1_\omega(\mathbb{R}^n)} \\ &\lesssim \|\mathcal{T}^*(f)\|_{L^1_\omega(\mathbb{R}^n)} = \|f\|_{H^1_{\mathcal{L}}(\omega)}.\end{aligned}$$

Conversely, we need to prove that $\mathcal{T}^*$ is bounded from $h^1_\rho(\omega)$ to $L^1_\omega(\mathbb{R}^n)$. By Lemma 2.4 and Theorem 5.5, it suffices to prove that for any $(1,q,s)_\omega$-atom or $(1,q)_\omega$-single-atom a,

$$\|\mathcal{T}^*(a)\|_{L^1_\omega(\mathbb{R}^n)} \lesssim 1, \tag{6.15}$$

where $1 < q \leq 1 + \delta/n$.

If a is a $(1,q)_\omega$-single-atom, by Hölder inequality and Lemma 6.4, we have

$$\|\mathcal{T}^*(a)\|_{L^1_\omega(\mathbb{R}^n)} \leq \|\mathcal{T}^*(a)\|_{L^q_\omega(\mathbb{R}^n)} \omega(\mathbb{R}^n)^{1-1/q} \leq C\|a\|_{L^q_\omega(\mathbb{R}^n)} \omega(\mathbb{R}^n)^{1-1/q} \lesssim 1.$$

If a is a $(1,q,s)_\omega$-atom and $\operatorname{supp} a \subset Q(x_0, r)$ with $r \leq L_1\rho(x_0)$, then we have

$$\|\mathcal{T}^*(a)\|_{L^1_\omega(\mathbb{R}^n)} \leq \|\mathcal{T}^*(a)\|_{L^1_\omega(4Q)} + \|\mathcal{T}^*(a)\|_{L^1_\omega((4Q)^\complement)} \equiv I + II.$$

For I, by Hölder inequality, Lemmas 2.4 and 6.4, we get

$$\begin{aligned}\|\mathcal{T}^*(a)\|_{L^1_\omega(4Q)} &\leq \|\mathcal{T}^*(a)\|_{L^q_\omega(4Q)} \omega(4Q)^{1-1/q} \leq C\|a\|_{L^q_\omega(\mathbb{R}^n)} \omega(4Q)^{1-1/q} \\ &\leq C\big(\omega(4Q)/\omega(Q)\big)^{1-1/q} \lesssim 1.\end{aligned}$$

For II, if $L_2\rho(x_0) \leq r \leq L_1\rho(x_0)$, by Lemma 2.4 and Corollary 6.9, taking $M > q(n+\theta) - n$, we have

$$\begin{aligned}\|\mathcal{T}^*(a)\|_{L^1_\omega((4Q)^\complement)} &= \sum_{j=3}^\infty \int_{2^jQ\setminus 2^{j-1}Q} \mathcal{T}^*(a)(x)\omega(x)\,dx \\ &\lesssim \frac{1}{\omega(Q)} \sum_{j=3}^\infty \int_{2^jQ\setminus 2^{j-1}Q} \left(\frac{r}{|x-x_0|}\right)^{n+M} \omega(x)\,dx \\ &\lesssim \frac{1}{\omega(Q)} \sum_{j=3}^\infty 2^{-j(n+M)} \omega(2^jQ)\end{aligned}$$

$$\lesssim \sum_{j=3}^{\infty} 2^{-j(n+M)} 2^{jnq} \left(1 + \frac{2^j r}{\rho(x_0)}\right)^{q\theta}$$
$$\lesssim \sum_{j=3}^{\infty} 2^{-j[n+M-nq-q\theta]} \lesssim 1;$$

if $r < L_2\rho(x_0)$, then there exists $N_0 \in \mathbb{Z}$ such that $2^{N_0-1}\sqrt{n}r \le \rho(x_0) < 2^{N_0}\sqrt{n}r$. Let us assume that $N_0 \ge 3$, otherwise, we just need to consider the I_2 in the following decomposition:

$$\|\mathcal{T}^*(a)\|_{L^1_\omega((4Q)^\complement)} = \left(\sum_{j=3}^{N_0} + \sum_{j=N_0+1}^{\infty}\right) \int_{2^jQ\setminus 2^{j-1}Q} \mathcal{T}^*(a)(x)\omega(x)\,dx \equiv I_1 + I_2,$$

for I_1, since $|x - x_0| < 2^j\sqrt{n}r \le 2^{N_0}\sqrt{n}r \le 2\rho(x_0)$, $\Psi_\theta(2^jQ) \le 3^\theta$ and $q < 1 + \delta/n$, by Lemma 2.4 and Corollary 6.9, we get

$$\begin{aligned}
I_1 &= \sum_{j=3}^{N_0} \int_{2^jQ\setminus 2^{j-1}Q} \mathcal{T}^*(a)(x)\omega(x)\,dx \\
&\lesssim \frac{1}{\omega(Q)} \sum_{j=3}^{N_0} \int_{2^jQ\setminus 2^{j-1}Q} \left(\frac{r}{|x-x_0|}\right)^{n+\delta} \omega(x)\,dx \\
&\lesssim \frac{1}{\omega(Q)} \sum_{j=3}^{N_0} 2^{-j(n+\delta)} \omega\big(2^jQ\big) \\
&\lesssim \sum_{j=3}^{N_0} 2^{-j[n+\delta-nq]} \lesssim 1,
\end{aligned}$$

for I_2, since $|x - x_0| \ge 2^{j-1}r \ge 2^{N_0}r \ge \rho(x_0)/\sqrt{n}$, then

$$\Psi_\theta\big(2^jQ\big) \le \big(2^{j+1}\sqrt{n}r/\rho(x_0)\big)^\theta,$$

thus, taking $M = q\theta$, by $q < 1 + \delta/n$, Lemma 2.4 and Corollary 6.9, we obtain

$$\begin{aligned}
I_2 &= \sum_{j=N_0+1}^{\infty} \int_{2^jQ\setminus 2^{j-1}Q} \mathcal{T}^*(a)(x)\omega(x)\,dx \\
&\lesssim \frac{1}{\omega(Q)} \sum_{j=N_0+1}^{\infty} \int_{2^jQ\setminus 2^{j-1}Q} \left(\frac{r}{|x-x_0|}\right)^{n+\delta} \left(\frac{\rho(x_0)}{|x-x_0|}\right)^{M} \omega(x)\,dx \\
&\lesssim \frac{1}{\omega(Q)} \sum_{j=N_0+1}^{\infty} 2^{-j(n+\delta)} \omega\big(2^jQ\big) \left(\frac{\rho(x_0)}{2^jr}\right)^{M} \\
&\lesssim \sum_{j=N_0+1}^{\infty} 2^{-j[n+\delta-nq]} \big(\Psi_\theta\big(2^jQ\big)\big)^q \left(\frac{\rho(x_0)}{2^jr}\right)^{M} \lesssim 1,
\end{aligned}$$

which finally implies (6.15) and finishes the proof. $\square$

7. Finite atomic decompositions

In this section, we prove that for any given finite linear combination of weighted atoms when $q<\infty$, its norm in $h^p_{\rho,N}(\omega)$ can be achieved by all its finite weighted atomic decompositions. This extends the main results in [17] to the setting of weighted local Hardy spaces.

DEFINITION 7.1. Let $\omega\in A^{\rho,\infty}_\infty(\mathbb{R}^n)$ and $(p,q,s)_\omega$ be admissible as in Definition 3.2. Then $h^{p,q,s}_{\rho,\mathrm{fin}}(\omega)$ is defined to be the vector space of all finite linear combinations of $(p,q,s)_\omega$-atoms and a $(p,q)_\omega$-single-atom, and the norm of f in $h^{p,q,s}_{\rho,\mathrm{fin}}(\omega)$ is defined by

$$\|f\|_{h^{p,q,s}_{\rho,\mathrm{fin}}(\omega)}\equiv\inf\Bigg\{\Bigg[\sum_{i=0}^k|\lambda_i|^p\Bigg]^{1/p}:f=\sum_{i=0}^k\lambda_ia_i,k\in\mathbb{Z}_+,\{\lambda_i\}_{i=0}^k\subset\mathbb{C},\\ \{a_i\}_{i=1}^k\text{ are }(p,q,s)_\omega\text{ atoms, and }a_0\text{ is a }(p,q)_\omega\text{-single-atom}\Bigg\}.$$

Obviously, for any admissible triplet $(p,q,s)_\omega$ atom and $(p,q)_\omega$-single-atom, $h^{p,q,s}_{\rho,\mathrm{fin}}(\omega)$ is dense in $h^{p,q,s}_\rho(\omega)$ with respect to the quasi-norm $\|\cdot\|_{h^{p,q,s}_\rho(\omega)}$.

THEOREM 7.2. *Let $\omega\in A^{\rho,\infty}_\infty(\mathbb{R}^n)$, q_ω be as in (2.4) and $(p,q,s)_\omega$ be admissible as in Definition 3.2. If $q\in(q_\omega,\infty)$, then $\|\cdot\|_{h^{p,q,s}_{\rho,\mathrm{fin}}(\omega)}$ and $\|\cdot\|_{h^p_\rho(\omega)}$ are equivalent quasi-norms on $h^{p,q,s}_{\rho,\mathrm{fin}}(\omega)$.*

Proof. Obviously, by Theorem 5.5, we have $h^{p,q,s}_{\rho,\mathrm{fin}}(\omega)\subset h^{p,q,s}_\rho(\omega)=h^p_\rho(\omega)$, and for all $f\in h^{p,q,s}_{\rho,\mathrm{fin}}(\omega)$, we have

$$\|f\|_{h^p_\rho(\omega)}\le C\|f\|_{h^{p,q,s}_{\rho,\mathrm{fin}}(\omega)}.$$

Therefore, it suffices to prove that for every $q\in(q_\omega,\infty)$ there exists a constant C such that for all $f\in h^{p,q,s}_{\rho,\mathrm{fin}}(\omega)$,

$$\|f\|_{h^{p,q,s}_{\rho,\mathrm{fin}}(\omega)}\le C\|f\|_{h^p_\rho(\omega)}. \tag{7.1}$$

Suppose that f is in $h^{p,q,s}_{\rho,\mathrm{fin}}(\omega)$ with $\|f\|_{h^p_\rho(\omega)}=1$. In this section, we take $m_0\in\mathbb{Z}$ such that $2^{m_0-1}\le\inf_{x\in\mathbb{R}^n}\mathcal{M}_Nf(x)<2^{m_0}$, and if $\inf_{x\in\mathbb{R}^n}\mathcal{M}_Nf(x)=0$, we write $m_0=-\infty$. For each integer $m\ge m_0$, set

$$\Omega_m\equiv\big\{x\in\mathbb{R}^n:\mathcal{M}_Nf(x)>2^m\big\},$$

where and in what follows $N=N_{p,\omega}$. For $f\in(h^p_{\rho,N}(\omega)\cap L^q_\omega(\mathbb{R}^n))$, by Lemma 5.4, there exist $\lambda_0\in\mathbb{C}$, $\{\lambda^m_i\}_{m\ge k_0,i}\subset\mathbb{C}$, a $(p,\infty)_\omega$-single-atom a_0 and $(p,\infty,s)_\omega$-atoms $\{a^m_i\}_{m\ge m_0,i}$, such that

$$f=\sum_{m\ge m_0}\sum_i\lambda^m_ia^m_i+\lambda_0a_0 \tag{7.2}$$

holds both in $\mathcal{D}'(\mathbb{R}^n)$ and almost everywhere.

For any $x\in\mathbb{R}^n$, since $\mathbb{R}^n=\bigcup_{m\geq m_0}(\Omega_{2^m}\setminus\Omega_{2^{k+1}})$, there exists $j\in\mathbb{Z}$ such that $x\in(\Omega_{2^j}\setminus\Omega_{2^{j+1}})$. By the proof of Lemma 5.4, for all $m>j$, $\mathrm{supp}(a_i^m)\subset\widetilde{Q}_i^m\subset\Omega_m\subset\Omega_{j+1}$. Then by (5.13) and (5.18), we have

$$\Big|\sum_{m\geq m_0}\sum_i\lambda_i^m a_i^m(x)\Big|+\big|\lambda_0 a_0(x)\big|\leq C\sum_{k_0\leq k\leq j}2^k+2^{k_0}\leq C2^j\leq C\mathcal{M}_N(f)(x).$$

By $f\in L^q_\omega(\mathbb{R}^n)$ and Proposition 2.8(ii), we have $\mathcal{M}_N(f)(x)\in L^q_\omega(\mathbb{R}^n)$, which together with the Lebesgue dominated convergence theorem confers that

$$\sum_{m\geq m_0}\sum_i\lambda_i^m a_i^m+\lambda_0 a_0$$

converges to f in $L^q_\omega(\mathbb{R}^n)$.

Next, let us prove (7.1) for two cases of ω.

Case I: For $\omega(\mathbb{R}^n)=\infty$, since $f\in L^q_\omega(\mathbb{R}^n)$, we know that $m_0=-\infty$ and $a_0(x)=0$ for almost every $x\in\mathbb{R}^n$ in (7.2). Thus, (7.2) can be written as

$$f=\sum_{m\in\mathbb{Z}}\sum_i\lambda_i^m a_i^m.$$

When $\omega(\mathbb{R}^n)=\infty$, all $(p,q)_\omega$-single-atoms are 0, which implies that f has compact support for $f\in h^{p,q,s}_{\rho,\mathrm{fin}}(\omega)$. Suppose $\mathrm{supp}(f)\subset Q_0\equiv Q(x_0,r_0)$ and $\widetilde{Q}_0\equiv Q(x_0,r_1)$, in which $r_1=\sqrt{n}r_0+C_0^2(1+R)^{k_0+1}(1+\sqrt{n}r_0/\rho(x_0))\rho(x_0)$, then for any $\psi\in\mathcal{D}_N(\mathbb{R}^n)$, $x\in\mathbb{R}^n\setminus\widetilde{Q}_0$ and $2^{-l}\in(0,\rho(x))$, we have

$$\begin{aligned}\psi_l*f(x)&=\int_{Q(x_0,r_0)}\psi_l(x-y)f(y)\,dy\\&=\int_{B(x,R\rho(x))\cap Q(x_0,r_0)}\psi_l(x-y)f(y)\,dy=0.\end{aligned}$$

Hence, for any $m\in\mathbb{Z}$, $\Omega_m\subset\widetilde{Q}_0$, we have $\mathrm{supp}(\sum_{m\in\mathbb{Z}}\sum_i\lambda_i^m a_i^m)\subset\widetilde{Q}_0$.

For each positive integer K, let

$$F_K\equiv\big\{(m,i):m\in\mathbb{Z},m\geq m_0,i\in\mathbb{N},|m|+i\leq K\big\},$$

and

$$f_K\equiv\sum_{(m,i)\in F_K}\lambda_i^m a_i^m.$$

Then, we have f_K converges to f in $L^q_\omega(\mathbb{R}^n)$, and for any given $\varepsilon\in(0,1)$, there exists a $K_0\in\mathbb{N}$ large enough such that $\mathrm{supp}(f-f_{K_0})/\varepsilon\subset\widetilde{Q}_0$ and

$$\big\|(f-f_{K_0})/\varepsilon\big\|_{L^q_\omega(\mathbb{R}^n)}\leq\big[\omega(\widetilde{Q}_0)\big]^{1/q-1/p}.$$

For $\widetilde{Q}_0$, since $l(\widetilde{Q}_0)=r_1>2\rho(x_0)$, we can decompose it into finite disjoint cubes $\{Q_j\}_j$ such that $\widetilde{Q}_0=\bigcup_{j=1}^{N_0}Q_j$ and $l_j/4<\rho(x)\leq C_0(3\sqrt{n})^{k_0}l_j$ for some

$x \in Q_j = Q(x_j, l_j)$. Moreover, each l_j satisfies $L_2\rho(x_j) < l_j < L_1\rho(x_j)$. It is clear that for $q \in (q_\omega, \infty)$ and $p \in (0,1]$ we have

$$\left\|(f - f_{K_0})\chi_{Q_i}/\varepsilon\right\|_{L^q_\omega(\mathbb{R}^n)} \le \left[\omega(\widetilde{Q}_0)\right]^{1/q-1/p} \le \left[\omega(Q_j)\right]^{1/q-1/p},$$

which together with $\mathrm{supp}((f - f_{K_0})\chi_{Q_j}/\varepsilon) \subset Q_j$ implies that $(f - f_{K_0})\chi_{Q_j}/\varepsilon$ is a $(p,q,s)_\omega$-atom for $j = 1, 2, \ldots, N_0$. Therefore,

$$f = f_{K_0} + \sum_{j=1}^{N_0} \varepsilon \frac{(f - f_{K_0})\chi_{Q_j}}{\varepsilon}$$

is a finite weighted atom linear combination of f almost everywhere. Then by taking $\varepsilon \equiv N_0^{-1/p}$, we obtain

$$\|f\|^p_{h^{p,q,s}_{\rho,\mathrm{fin}}(\omega)} \le \sum_{(m,i)\in F_K} \left|\lambda_i^m\right|^p + N_0\varepsilon^p \le C,$$

which implies the Case I.

Case II: For $\omega(\mathbb{R}^n) < \infty$, f may not have compact support. As in Case I, for any positive integer K, let

$$f_K \equiv \sum_{(m,i)\in F_K} \lambda_i^m a_i^m + \lambda_0 a_0$$

and $b_K \equiv f - f_K$. By above proof, we know that f_K converges to f in $L^q_\omega(\mathbb{R}^n)$. Thus, there exists a positive integer $K_1 \in \mathbb{N}$ large enough such that

$$\|b_{K_1}\|_{L^q_\omega(\mathbb{R}^n)} \le \left[\omega\big(\mathbb{R}^n\big)\right]^{1/q-1/p}.$$

Therefore, b_{K_1} is a $(p,q)_\omega$-single-atom and $f = f_{K_1} + b_{K_1}$ is a finite weighted atom linear combination of f. By Lemma 5.4, we have

$$\|f\|^p_{h^{p,q,s}_{\rho,\mathrm{fin}}(\omega)} \le C\left(\sum_{(m,i)\in F_K} \left|\lambda_i^m\right|^p + \lambda_0^p\right) \le C.$$

Thus, (7.1) holds, and the theorem is proved. □

As an application of finite atomic decompositions, we establish boundedness in $h^p_\rho(\omega)$ of quasi-Banach-valued sublinear operators.

As in [5], a quasi-Banach space space $\mathcal{B}$ is a vector space endowed with a quasi-norm $\|\cdot\|_{\mathcal{B}}$ which is nonnegative, non-degenerate (i.e., $\|f\|_{\mathcal{B}} = 0$ if and only if $f = 0$), homogeneous, and obeys the quasi-triangle inequality, i.e., there exists a positive constant K no less than 1 such that for all $f, g \in \mathcal{B}$, $\|f + g\|_{\mathcal{B}} \le K(\|f\|_{\mathcal{B}} + \|g\|_{\mathcal{B}})$.

Let $\beta \in (0,1]$, a quasi-Banach space $\mathcal{B}_\beta$ with the quasi-norm $\|\cdot\|_{\mathcal{B}_\beta}$ is called a β-quasi-Banach space if $\|f + g\|^\beta_{\mathcal{B}_\beta} \le \|f\|^\beta_{\mathcal{B}_\beta} + \|g\|^\beta_{\mathcal{B}_\beta}$ for all $f, g \in \mathcal{B}_\beta$.

For any given β-quasi-Banach space $\mathcal{B}_\beta$ with $\beta\in(0,1]$ and a linear space $\mathcal{Y}$, an operator T from $\mathcal{Y}$ to $\mathcal{B}_\beta$ is said to be $\mathcal{B}_\beta$-sublinear if for any $f,g\in\mathcal{Y}$ and $\lambda,\nu\in\mathbb{C}$,

$$\big\|T(\lambda f+\nu g)\big\|_{\mathcal{B}_\beta}\le\big(|\lambda|^\beta\big\|T(f)\big\|_{\mathcal{B}_\beta}^\beta+|\nu|^\beta\big\|T(g)\big\|_{\mathcal{B}_\beta}^\beta\big)^{1/\beta}$$

and $\|T(f)-T(g)\|_{\mathcal{B}_\beta}\le\|T(f-g)\|_{\mathcal{B}_\beta}$.

If T is linear, then it is $\mathcal{B}_\beta$-sublinear. Moreover, if $\mathcal{B}_\beta$ is a space of functions, and T is nonnegative and sublinear in the classical sense, then T is also $\mathcal{B}_\beta$-sublinear.

THEOREM 7.3. *Let $\omega\in A_\infty^{\rho,\infty}(\mathbb{R}^n), 0<p\le\beta\le1,$ and $\mathcal{B}_\beta$ be a β-quasi-Banach space. Suppose $q\in(q_\omega,\infty)$ and $T:h^{p,q,s}_{\rho,\mathrm{fin}}(\omega)\to\mathcal{B}_\beta$ is a $\mathcal{B}_\beta$-sublinear operator such that*

$$S\equiv\sup\big\{\big\|T(a)\big\|_{\mathcal{B}_\beta}: a \text{ is a } (p,q,s)_\omega \text{ atom or } (p,q)_\omega\text{-single-atom}\big\}<\infty.$$

Then there exists a unique bounded $\mathcal{B}_\beta$-sublinear operator $\widetilde{T}$ from $h^p_\rho(\omega)$ to $\mathcal{B}_\beta$ which extends T.

Proof. For any $f\in h^{p,q,s}_{\rho,\mathrm{fin}}(\omega)$, by Theorem 7.2, there exist a set of numbers $\{\lambda_j\}_{j=0}^l\subset\mathbb{C}$, $(p,q,s)_\omega$-atoms $\{a_j\}_{j=1}^l$ and a $(p,q)_\omega$-single-atom a_0 such that $f=\sum_{j=0}^l\lambda_ja_j$ pointwise and

$$\sum_{j=0}^l|\lambda_j|^p\le C\|f\|^p_{h^p_\rho(\omega)}.$$

Then by the assumption, we have

$$\big\|T(f)\big\|_{\mathcal{B}_\beta}\le C\Bigg[\sum_{j=0}^l|\lambda_j|^p\Bigg]^{1/p}\le C\|f\|_{h^p_\rho(\omega)}.$$

Since $h^{p,q,s}_{\rho,\mathrm{fin}}(\omega)$ is dense in $h^p_\rho(\omega)$, a density argument gives the desired results. □

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