FIXED-POINT INDEX, THE INCOMPATIBILITY THEOREM, AND TORUS PARAMETRIZATION

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ABSTRACT. The fixed-point index of a homeomorphism of Jordan curves measures the number of fixed-points, with multiplicity, of the extension of the homeomorphism to the full Jordan domains in question. The now-classical Circle Index Lemma says that the fixed-point index of a positive-orientation-preserving homeomorphism of round circles is always non-negative. We begin by proving a generalization of this lemma, to accommodate Jordan curves bounding domains which do not disconnect each other. We then apply this generalization to give a new proof of Schramm's Incompatibility Theorem, which was used by Schramm to give the first proof of the rigidity of circle packings filling the complex and hyperbolic planes. As an example application, we include outlines of proofs of these circle packing theorems.

We then introduce a new tool, the so-called torus parametrization, for working with fixed-point index, which allows some problems concerning this quantity to be approached combinatorially. We apply torus parametrization to give the first purely topological proof of the following lemma: given two positively oriented Jordan curves, one may essentially prescribe the images of three points of one of the curves in the other, and obtain an orientation-preserving homeomorphism between the curves, having non-negative fixed-point index, which respects this prescription. This lemma is essential to our proof of the Incompatibility Theorem.

1. Introduction

This article is concerned with a topological quantity, the so-called *fixed-point index* of a homeomorphism of Jordan curves, which has proven useful in the study of various areas of complex analysis. We begin with its definition:

DEFINITION 1.1. A *Jordan curve* is a homeomorphic image of a topological circle $\mathbb{S}^1$ in the complex plane $\mathbb{C}$. A *Jordan domain* is a bounded open set in $\mathbb{C}$ with Jordan curve boundary. We use the term *closed Jordan domain* or *compact Jordan domain* to refer to the closure of a Jordan domain. We define the *positive orientation* on a Jordan curve as usual. That is, if K is a closed Jordan domain, then as we traverse ∂K in what we call the *positive* direction, the interior of K stays to the left.

Let K and $\tilde{K}$ be closed Jordan domains. Let $\phi \colon \partial K \to \partial \tilde{K}$ be a homeomorphism of Jordan curves which is fixed-point-free and orientation-preserving. We call such a homeomorphism *indexable*. Then $\{\phi(z) - z\}_{z \in \partial K}$ is a closed curve in the plane which misses the origin. It has a natural orientation induced by traversing ∂K positively. Then we define the *fixed-point index* of ϕ, denoted $\eta(\phi)$, to be the winding number of $\{\phi(z) - z\}_{z \in \partial K}$ around the origin.

Two examples are shown in Figures 1 and 2. We remark that the fixed-point index depends crucially on the choice of homeomorphism, and also on the way that the sets K and $\tilde{K}$ are juxtaposed. It is a worthwhile exercise to construct an indexable homeomorphism $\partial K \to \partial \tilde{K}$, for K and $\tilde{K}$ as in Figure 2, having fixed-point index unequal to -1.

Fixed-point index has found applications for example in the theories of circle packing [HS93], Koebe uniformization [HS93], and Sierpinski carpets
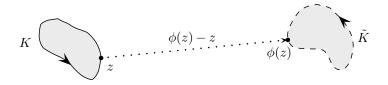

FIGURE 1. Two closed Jordan domains K and $\tilde{K}$ so that any indexable homeomorphism $\phi \colon \partial K \to \partial \tilde{K}$ satisfies $\eta(\phi) = 0$. The arrows on ∂K and $\partial \tilde{K}$ indicate the positive orientations on these Jordan curves. In this case ϕ is indexable so long as it is orientation-preserving; the fixed-point-free condition is automatic because ∂K and $\partial \tilde{K}$ do not meet. The dashed arrow represents a vector of the form $\phi(z) - z$. The vector $\phi(z) - z$ must always point "to the right," so the curve $\{\phi(z) - z\}_{z \in \partial K}$ has winding number 0 around the origin, thus $\eta(\phi) = 0$.
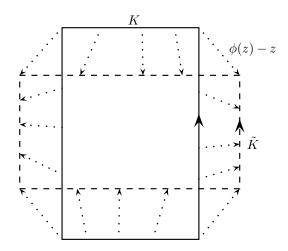
FIGURE 2. An indexable homeomorphism $\phi : \partial K \to \partial \tilde{K}$ so that $\eta(\phi) = -1$. Suppose we insist that ϕ identifies the respective corners as shown. Then tracing the path of the dashed vectors $\phi(z) - z$ as z traverses ∂K positively, we see that $\phi(z) - z$ must wind once clockwise around the origin, thus $\eta(\phi) = -1$.

[Mer12], Section 12. In all of these settings, it has been applied to prove powerful existence, rigidity, and uniformization statements. Most recently, the current author has used fixed-point index, including torus parametrization, to prove rigidity statements for collections of possibly-overlapping round disks, see [Mis12], [Mis13].

The fixed-point index measures the following topological quantity: suppose that K and $\tilde{K}$ are closed Jordan domains, and $\Phi : K \to \tilde{K}$ is a homeomorphism, having finitely many fixed points, which restricts to an indexable homeomorphism $\partial \Phi : \partial K \to \partial \tilde{K}$. There is a well-understood notion of the *multiplicity* of a fixed point of Φ. Then the fixed-point index $\eta(\partial \Phi)$ counts the number of fixed-points of Φ, with multiplicity. For more discussion along these lines, see [HS93, Section 2].

In this article, we describe a new technique for working with fixed-point index, which we call the *torus parametrization* of a pair of Jordan curves, defined in Section 5. We apply torus parametrization to give a new, elementary proof of the following fundamental lemma.

THREE POINT PRESCRIPTION LEMMA 1.2. *Let K and $\tilde{K}$ be compact Jordan domains in transverse position, with boundaries oriented positively. Let $z_1, z_2, z_3 \in \partial K \setminus \partial \tilde{K}$ appear in counterclockwise order, similarly $\tilde{z}_1, \tilde{z}_2, \tilde{z}_3 \in$*

$\partial \tilde{K} \setminus \partial K$. *Then there is an indexable homeomorphism* $\phi : \partial K \to \partial \tilde{K}$ *sending* $z_i \mapsto \tilde{z}_i$ *for* $i = 1, 2, 3$, *so that* $\eta(\phi) \geq 0$.

Two Jordan domains are in *transverse position* if their boundary Jordan curves cross wherever they meet, cf. Definition 2.3 in Section 2. The example given in Figure 2 shows that if we prescribe the images of four points, then a negative fixed-point index may be forced.

A version of the Three Point Prescription Lemma 1.2 is stated in [Ste05, Lemma 8.14], but we have not been able to fill in the details of the argument. The idea of the approach is as follows: first, any Riemann mapping $\Phi : \Omega \to \tilde{\Omega}$ between open Jordan domains having non-self-intersecting boundaries extends to a homeomorphism $\partial \Phi : \partial \Omega \to \partial \tilde{\Omega}$ of their boundaries, and we may prescribe the images of three points of $\partial \Omega$ in $\partial \tilde{\Omega}$ by post-composing with self-biholomorphisms of $\tilde{\Omega}$. Next, it is known that any isolated fixed point of a holomorphic map has non-negative multiplicity, see [HS93, Section 2]. Thus the map $\partial \Phi$, if it is indexable, has non-negative fixed-point index, because the fixed-point index of $\partial \Phi$ counts the fixed points of Φ with multiplicity, completing the argument. However, it is not clear how to deal with possible fixed points in the induced boundary map $\partial \Phi$. Our proof of Lemma 1.2 uses only induction and plane topology arguments and is given in Section 7.

For a discussion on the strength of the hypotheses of Lemma 1.2, refer to Remark 7.1 at the end of the article.

We also state and prove a new fundamental lemma on fixed-point index, generalizing the well-known Circle Index Lemma 2.1 which states that the fixed-point index of an indexable homeomorphism between circles is always non-negative. The Circle Index Lemma was a crucial ingredient in all of the applications of fixed-point index described above. In our generalization, round disks are replaced by closed Jordan domains which do not disconnect each other. In particular, the closed Jordan domains K and $\tilde{K}$ are said to *cut each other* if $K \setminus \tilde{K}$ or $\tilde{K} \setminus K$ is disconnected. Then there is the following lemma.

LEMMA 1.3. *Let* K *and* $\tilde{K}$ *be closed Jordan domains in transverse position, which do not cut each other, having boundaries oriented positively. Let* $\phi : \partial K \to \partial \tilde{K}$ *be an indexable homeomorphism. Then* $\eta(\phi) \geq 0$.

The proof appears at the end of Section 3.

As an example of the power of fixed-point index, we apply the Three Point Prescription Lemma 1.2 and Lemma 1.3 to prove a version of the Incompatibility Theorem of Schramm [SCH91, Theorem 3.1], as described in Section 3. The Incompatibility Theorem is then easily applied in Section 4 to prove some well-known rigidity theorems for circle packings. The ideas for these proofs are borrowed from [HS93], [Ste05, Chapter 8].

2. Background lemmas and definitions

In the upcoming discussion, it will be useful to have access to two well-known lemmas on fixed-point index. This section is devoted to introducing these two lemmas.

Our first background lemma says essentially that "the fixed-point index between two circles is always non-negative":

CIRCLE INDEX LEMMA 2.1. *Let K and $\tilde{K}$ be closed Jordan domains in $\mathbb{C}$, with boundaries oriented positively, and let $\phi : \partial K \to \partial \tilde{K}$ be an indexable homeomorphism. Then the following hold.*

(1) *We have $\eta(\phi) = \eta(\phi^{-1})$.*
(2) *If $K \subseteq \tilde{K}$ or $\tilde{K} \subseteq K$, then $\eta(\phi) = 1$.*
(3) *If K and $\tilde{K}$ have disjoint interiors, then $\eta(\phi) = 0$.*
(4) *If ∂K and $\partial \tilde{K}$ intersect in exactly two points, then $\eta(\phi) \geq 0$.*

As a consequence of the above, if K and $\tilde{K}$ are closed disks in the plane, then $\eta(\phi) \geq 0$.

Lemma 2.1 can be found in [HS93, Lemma 2.2], with a clear and complete proof. There it is indicated that a version of the lemma appeared earlier in [Str51].

The moral of our second background lemma is that fixed-point indices "add nicely":

INDEX ADDITIVITY LEMMA 2.2. *Suppose that K and L are interiorwise disjoint closed Jordan domains which meet along a single positive-length Jordan arc $\partial K \cap \partial L$, similarly for $\tilde{K}$ and $\tilde{L}$. Then $K \cup L$ and $\tilde{K} \cup \tilde{L}$ are closed Jordan domains.*

Let $\phi : \partial K \to \partial \tilde{K}$ and $\psi : \partial L \to \partial \tilde{L}$ be indexable homeomorphisms. Suppose that ϕ and ψ agree on $\partial K \cap \partial L$. Let $\theta : \partial(K \cup L) \to \partial(\tilde{K} \cup \tilde{L})$ be induced via restriction to ϕ or ψ as necessary. Then θ is an indexable homeomorphism and $\eta(\theta) = \eta(\phi) + \eta(\psi)$.

Proof. The situation is as depicted in Figure 3. We may consider $\eta(\phi)$ to be $1/2\pi$ times the change in argument of the vector $\phi(z) - z$, as z traverses ∂K once in the positive direction. Then as z varies positively in ∂K and in ∂L the contributions to the sum $\eta(\phi) + \eta(\psi)$ along $\partial K \cap \partial L$ cancel. $\square$

Next, we wish to make precise our notion of *transverse position*:

DEFINITION 2.3. Two Jordan curves γ and $\tilde{\gamma}$ are in *transverse position* if for any point $z \in \gamma \cap \tilde{\gamma}$ where they meet, they *cross transversely*, which here means that there is an open neighborhood $U \subset \mathbb{C}$ of z and a homeomorphism $\phi : U \to \mathbb{D}$ between U and the open unit disk $\mathbb{D}$, so that $\phi(\gamma \cap U) = \mathbb{R} \cap \mathbb{D}$ and $\phi(\tilde{\gamma} \cap U) = i\mathbb{R} \cap \mathbb{D}$. Two (open or closed) Jordan domains are said to be in *transverse position* if their boundary Jordan curves are in transverse position.
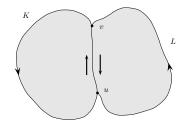
FIGURE 3. An illustration of fixed-point index additivity.

Note that if two Jordan curves are in transverse position, then they meet finitely many times, by a compactness argument.

Finally, it will be helpful to have available to us the terminology of the following definition.

DEFINITION 2.4. Suppose that $X_1, \ldots, X_n$ and $X_1', \ldots, X_n'$ are all subsets of $\mathbb{C}$. Then we say that the collections $\{X_1, \ldots, X_n\}$ and $\{X_1', \ldots, X_n'\}$ are in the same *topological configuration* if there is an orientation-preserving homeomorphism $\varphi : \mathbb{C} \to \mathbb{C}$ so that $\varphi(X_i) = X_i'$ for all $1 \leq i \leq n$. In practice, the collections of objects under consideration will not be labeled X_i and X_i', but there will be some natural bijection between them. Then our requirement is that φ respects this natural bijection. We say that certain conditions on some objects *uniquely determine* their topological configuration if any two collections of objects satisfying the given conditions are in the same topological configuration.

For example, considering a point $z \in \mathbb{C}$ and an open Jordan domain Ω, the condition that $z \in \Omega$ uniquely determines the topological configuration of $\{z, \Omega\}$, but the condition that $z \notin \Omega$ does not uniquely determine the topological configuration of $\{z, \Omega\}$. (We may have $z \in \partial\Omega$, or $z \in \mathbb{C} \setminus (\Omega \cup \partial\Omega)$, and these situations are topologically distinct.)

The following lemma says that when working with fixed-point index, we need to consider our Jordan domains only "up to topological configuration."

LEMMA 2.5. *Suppose K and $\tilde{K}$ are closed Jordan domains. Let $f : \partial K \to \partial\tilde{K}$ be an indexable homeomorphism. Suppose that K' and $\tilde{K}'$ are also closed Jordan domains, so that $\{K, \tilde{K}\}$ and $\{K', \tilde{K}'\}$ are in the same topological configuration, via the homeomorphism $\varphi : \mathbb{C} \to \mathbb{C}$. Let $f' : \partial K' \to \partial\tilde{K}'$ be induced in the natural way, explicitly as $f' = \varphi|_{\partial\tilde{K}} \circ f \circ \varphi^{-1}|_{\partial K'}$. Then f' is indexable with respect to the usual orientation on $\partial K'$ and $\partial\tilde{K}'$, and $\eta(f) = \eta(f')$.*

Proof. The following is well known. For a reference, see Chapters 1 and 2 of [FM12].

FACT 2.6. *Every orientation-preserving homeomorphism $\mathbb{C} \to \mathbb{C}$ is homotopic to the identity map via homeomorphisms.*

Thus let $H_t : \mathbb{C} \times [0, 1] \to \mathbb{C}$ be such a homotopy from the identity to φ. Explicitly, for fixed t we have that H_t is an orientation-preserving homeomorphism $\mathbb{C} \to \mathbb{C}$, with H_0 equal to the identity on $\mathbb{C}$ and $H_1 = \varphi$.

Let $K_t = H_t(K)$ and $\tilde{K}_t = H_t(\tilde{K})$. Then K_t and $\tilde{K}_t$ are closed Jordan domains, because H_t is a homeomorphism. Let $f_t : \partial K_t \to \partial \tilde{K}_t$ be induced in the natural way, explicitly as $H_t|_{\partial \tilde{K}} \circ f \circ H_t^{-1}|_{\partial K_t}$. Let $\gamma_t = \{f_t(z) - z\}_{z \in \partial K_t}$. Then tautologically $\eta(f)$ is the winding number of γ_0 around the origin, and $\eta(f')$ is the winding number of γ_1 around the origin.

Every γ_t is a closed curve because ∂K_t is a closed curve and f_t is continuous. Once we establish that no γ_t passes through the origin, Lemma 2.5 will be proved because we have an induced homotopy from γ_0 to γ_1, and two curves homotopic in $\mathbb{C} \setminus \{0\}$ have the same winding number around the origin. Suppose for contradiction that $0 \in \gamma_t$. Then there is a $z \in \partial K_t$ so that $f_t(z) = z$. Thus $H_t \circ f \circ H_t^{-1}(z) = z$, and so $f(H_t^{-1}(z)) = H_t^{-1}(z)$, contradicting the fixed-point-free condition on f. □

3. The Incompatibility Theorem

In this section, we state and prove the Incompatibility Theorem of Oded Schramm, appearing in [SCH91, Theorem 3.1]. Before doing so, we need some preliminary definitions:

First, a *topological rectangle* is a closed Jordan domain R with four marked points on its boundary ∂R, which we naturally call its *corners*. A *side* of R is a closed sub-arc of ∂R having two corners of R as its endpoints, and containing no other corner of R. We abuse notation slightly and use the same symbol, in this case R, to refer both to a topological rectangle and to its constituent closed Jordan domain. We define *topological triangles*, their *corners*, and their *sides* analogously, and employ the same abuse of notation.

A *packing* of a topological rectangle R consists of a finite collection $\{K_1, \ldots, K_n\}$ of closed Jordan domains so that the following hold:

- Every K_i is contained in R.
- The K_i are pairwise interiorwise disjoint, and any two of them meet at at most one point.
- Each of the K_i meets ∂R at at most one point, and no K_i meets a corner of R.
- For every connected component U of $R \setminus \bigcup_{i=1}^{n} K_i$, we have that the closure of U is a topological triangle T each of whose sides is contained in one of the ∂K_i, or in a side of R. Then every corner of T is either a corner of R, or an intersection point of some ∂K_i either with some other ∂K_j or with ∂R.

See Figure 4 for two examples of packings of topological rectangles. Let S_a, S_b, S_c, S_d denote the sides of R. The *contact graph* of the packing of R by $K_1, \ldots, K_n$ is the graph having vertices $v_1, \ldots, v_n$ corresponding to the
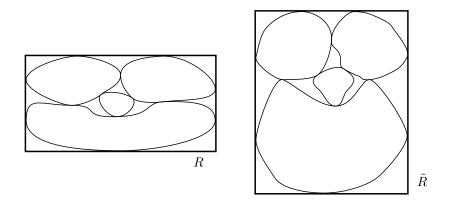
FIGURE 4. Two topological rectangles packed with shapes.

$K_1, \ldots, K_n$ and v_a, v_b, v_c, v_d corresponding to the sides of R, both in the natural way, so that two distinct vertices share an edge if and only if the corresponding sets meet. Note that for example, the contact graph of a packing of a rectangle is always a triangulation of a square, that is, a triangulation of a topological closed disk having four boundary edges.

Next, suppose that we have packings of topological rectangles R and $\tilde{R}$ by collections of closed Jordan domains $K_1, \ldots, K_n$ and $\tilde{K}_1, \ldots, \tilde{K}_n$, respectively. The two packings are said to be in *transverse position* if:

- For every pair of integers i and $\tilde{i}$, both between 1 and n, with possibly $i = \tilde{i}$, we have that K_i and $\tilde{K}_{\tilde{i}}$ are in transverse position as closed Jordan domains.
- For every $1 \leq i \leq n$, we have that K_i and $\tilde{R}$ are in transverse position as closed Jordan domains, as are $\tilde{K}_i$ and R.
- No intersection point of a pair of the sets $\partial K_1, \ldots, \partial K_n, \partial R$ lies on any $\partial \tilde{K}_i$ nor on $\partial \tilde{R}$. Similarly, no intersection point of a pair of the sets $\partial \tilde{K}_1, \ldots, \partial \tilde{K}_n, \partial \tilde{R}$ lies on any ∂K_i nor on ∂R.

For two examples of packings in transverse position, see Figure 5.

Finally, two closed Jordan domains K and $\tilde{K}$ which are in transverse position are said to *cut* one another if at least one of the sets $K \setminus \tilde{K}$ and $\tilde{K} \setminus K$ is disconnected. Though it is not important for us, it holds for such K and $\tilde{K}$ that $K \setminus \tilde{K}$ is connected if and only if $\tilde{K} \setminus K$ is.

We are now ready to state our version of the Incompatibility Theorem originally due to Schramm. The original appears in [SCH91, Theorem 3.1], where it is used to give the first proof of the rigidity of circle packings filling the complex plane.

When processing our statement of the Incompatibility Theorem 3.1, it may be helpful to keep the following in mind: in Schramm's original formulation,
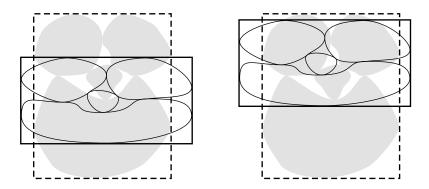
FIGURE 5. Shapes cutting each other. We have drawn R and $\tilde{R}$ on top of each other in two different "incompatible" ways, and in both cases some pair of corresponding shapes are "incompatible."
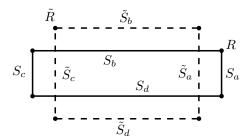

FIGURE 6. Two rectangles cutting one another.

our notion of shapes *cutting one another* is referred to as *incompatibility*. Then the Incompatibility Theorem 3.1 may be remembered as, "if two incompatible rectangles are packed in the same combinatorial way, then at least one pair of corresponding shapes of the two packings will be incompatible." The precise statement given in [SCH91] is somewhat different from the one we give here. Our statement communicates the main idea of the theorem, and suffices for our applications. Our proof uses fixed-point index, with the main new tool being Lemma 1.3. The original proof by Schramm is via different methods, but is also elementary.

INCOMPATIBILITY THEOREM 3.1. *Let R and $\tilde{R}$ be topological rectangles having sides S_a, S_b, S_c, S_d and $\tilde{S}_a, \tilde{S}_b, \tilde{S}_c, \tilde{S}_d$ respectively, in the topological configuration depicted in Figure 6. Suppose that we are given packings of R and $\tilde{R}$ by collections of closed Jordan domains $K_1, \ldots, K_n$ and $\tilde{K}_1, \ldots, \tilde{K}_n$*

respectively, in transverse position, so that the packings are combinatorially equivalent in the following precise sense: denoting the contact graphs of the packings by G and $\tilde{G}$ respectively, we insist that the following holds: letting $v_a, v_b, v_c, v_d, v_1, \ldots, v_n$ and $\tilde{v}_a, \tilde{v}_b, \tilde{v}_c, \tilde{v}_d, \tilde{v}_1, \ldots, \tilde{v}_n$ denote the vertex sets of G and $\tilde{G}$ in the natural way, we have that G and $\tilde{G}$ are isomorphic via the identification $v_i \mapsto \tilde{v}_i$ for $i = a, b, c, d, 1, \ldots, n$. Then, there is an $1 \leq i \leq n$ so that K_i and $\tilde{K}_i$ cut each other.

For example, the packings shown in Figure 4 share a contact graph. In Figure 5, we have overlaid them in two different ways, in both cases ensuring that the hypotheses of Theorem 3.1 are satisfied. We see that in each case, a pair of corresponding closed Jordan domains cut one another.

The rest of this section consists of a proof of the Incompatibility Theorem 3.1. Our proof relies on the following simple lemma, which appeared in the introduction but is restated here for the convenience of the reader.

LEMMA 1.3. *Let K and $\tilde{K}$ be closed Jordan domains in transverse position, which do not cut each other, having boundaries oriented positively. Let $\phi : \partial K \to \partial \tilde{K}$ be an indexable homeomorphism. Then $\eta(\phi) \geq 0$.*

We now give the proof of Theorem 3.1 assuming Lemma 1.3:

First, note that there is a natural bijection between the U_f and the $\tilde{U}_f$, where we write $\{U_f\}_{f \in F}$ to denote the connected components of $R \setminus \partial R \cup \bigcup_{i=1}^{n} K_i$, similarly $\{\tilde{U}_f\}_{f \in F}$. Moreover, for fixed f, we have that U_f and $\tilde{U}_f$ are topological triangles, and that the corners of U_f correspond in a natural way to those of its partner $\tilde{U}_f$. (To see this, one may consider the graphs G and $\tilde{G}$. Each is the 1-skeleton of a triangulation of a topological closed disk, and because the graphs are isomorphic, we get that the triangulations are combinatorially equivalent. Then the U_f are in natural bijection with the faces F of this combinatorial triangulation, as are the $\tilde{U}_f$.)

Given such a pair U_f and $\tilde{U}_f$, we have that U_f and $\tilde{U}_f$ are in transverse position as closed Jordan domains, because the packings we began with are in transverse position. For every such pair U_f and $\tilde{U}_f$, orient ∂U_f and $\tilde{\partial} U_f$ positively, and let $\phi_f : \partial U_f \to \partial \tilde{U}_f$ be an indexable homeomorphism identifying corresponding corners, satisfying $\eta(\phi_f) \geq 0$. We may do so by the Three Point Prescription Lemma 1.2.

Next, for every $1 \leq i \leq n$, orienting ∂K_i and $\partial \tilde{K}_i$ positively, we obtain an indexable homeomorphism $\phi_i : \partial K_i \to \partial \tilde{K}_i$ by restriction to the ϕ_f as necessary. Also, via the same procedure, orienting ∂R and $\partial \tilde{R}$ positively, we obtain an indexable homeomorphism $\phi_R : \partial R \to \partial \tilde{R}$. Then, by the Index Additivity Lemma 2.2, we have:

$$\eta(\phi_R) = \sum_{i=1}^{n} \eta(\phi_i) + \sum_{f \in F} \eta(\phi_f).$$

Now, as we saw in Figure 2, we have that $\eta(\phi_R) = -1$. On the other hand, every $\eta(\phi_f)$ is non-negative by construction, and the $\eta(\phi_i)$ are non-negative by Lemma 1.3, which gives us a contradiction.

To complete the proof of the Incompatibility Theorem 3.1, we require the proofs of several lemmas. The proof of the Three Point Prescription Lemma 1.2 is given in Section 7 using torus parametrization. Our proof of Lemma 1.3 is inspired[1] by an argument given by Schramm in [HS93, Proof of Lemma 2.2], in support of the Circle Index Lemma 2.1. We first need to prove a topological lemma on Jordan domains which do not cut each other.
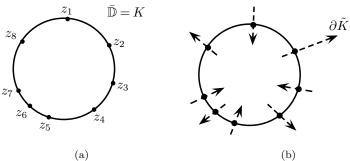
LEMMA 3.2. *Suppose K and $\tilde{K}$ are closed Jordan domains in transverse position which do not cut each other, whose boundaries meet at least twice. Then the topological configuration of $\{K, \tilde{K}\}$ is determined by how many times ∂K and $\partial \tilde{K}$ meet.*
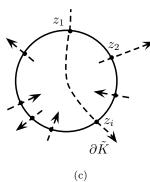
Proof. First, suppose without loss of generality that K is the closed unit disk $\bar{\mathbb{D}}$. Let $2m \geq 2$ be the number of intersection points of ∂K with $\partial \tilde{K}$. Note also that we may without loss of generality pick these arbitrarily along $\partial K = \partial \mathbb{D}$. Label these $z_1, \ldots, z_{2m}$ in clockwise order around $\partial K = \partial \mathbb{D}$. This brings us to the situation of Figure 7(a).
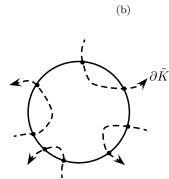
Orient ∂K and $\partial \tilde{K}$ as usual. We now follow what happens as we traverse $\partial \tilde{K}$. By relabeling, we may suppose that $\partial \tilde{K}$ enters K at z_1. Because the interior of $\tilde{K}$ stays to the left of $\partial \tilde{K}$, and because $\partial \tilde{K}$ crosses ∂K at every point where the two curves meet, it follows that $\partial \tilde{K}$ exits K at z_2. The same reasoning allows us to conclude that $\partial \tilde{K}$ enters K at z_3, etc., so we get that $\partial \tilde{K}$ enters K at z_i for all odd $1 \leq i \leq 2m$, and exits K at z_i for all even $1 \leq i \leq 2m$. This brings us to the situation of Figure 7(b).

We now consider where $\partial \tilde{K}$ goes after it crosses z_1. Denote by z_i the point of $\partial K \cap \partial \tilde{K}$ at which it arrives immediately after crossing z_1, noting that then i is even. We wish to establish that then $i = 2$, so suppose for contradiction that $i \neq 2$. This brings us to the situation of Figure 7(c). Then $[z_1 \to z_i]_{\partial \tilde{K}}$ disconnects $K = \bar{\mathbb{D}}$ into two components, call them A_1 and A_2. We have that every connected component of $K \setminus \tilde{K}$ must then be completely contained in one of A_1 and A_2. But $K \setminus \tilde{K}$ is connected by hypothesis, so one of A_1 and A_2 must be disjoint from $K \setminus \tilde{K}$. We then get a contradiction, because, keeping careful track of the orientation of $\partial \tilde{K}$, we see that there are points of $K \setminus \tilde{K}$ immediately counterclockwise from z_1 along ∂K, and points of $K \setminus \tilde{K}$ immediately clockwise from z_2 along ∂K, and these lie in different components of $K \setminus [z_1 \to z_i]_{\partial \tilde{K}}$ unless $i = 2$. The same reasoning allows us to conclude that, for any odd $1 \leq j \leq 2m$, after entering K at z_j, the curve $\partial \tilde{K}$ exits K at z_{j+1}, bringing us to the situation of Figure 7(d).

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[1] Thanks to Mario Bonk for suggesting this line of proof, greatly simplifying the required arguments.
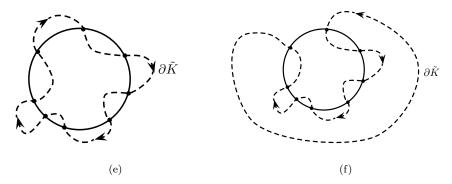
FIGURE 7. The construction of the topologically unique pair K and $\tilde{K}$ of transversely positioned closed Jordan domains, not cutting each other, having boundaries meeting at 8 points. The orientation on $\partial K = \partial \mathbb{D}$ is the usual, positive one, that is, the counterclockwise one. In all cases, dashed arcs are sub-arcs of $\partial \tilde{K}$.

By the same reasoning as in the previous paragraph, we get that, for any even $1 \leq j \leq 2m$, after exiting K at z_j, the curve $\partial \tilde{K}$ enters K at z_{j+1}, adopting the convention that $z_{2m+1} = z_1$. However, in this case, there are two ways to connect z_j to z_{j+1}: roughly speaking, we may either travel clockwise around the complement of $K = \mathbb{D}$ from z_j to z_{j+1}, or counterclockwise. If we connect every such pair z_j and z_{j+1} with "clockwise" arcs, then the resulting orientation on $\partial \tilde{K}$ is not positive with respect to the Jordan domain it bounds, see Figure 7(e). Thus suppose without loss of generality, by relabeling if necessary, that z_{2m} and z_1 are connected with a "counterclockwise" arc. Now for the remaining pairs z_j, z_{j+1}, with j even, there are no choices (up to topological equivalence) about how to draw the connecting sub-arcs of $\partial \tilde{K}$ between them, and we arrive at the situation of Figure 7(f). $\qquad \square$

REMARK 3.3. Which *a priori* topological configurations can occur for two Jordan curves in transverse position is a poorly understood question, and is known as the study of *meanders*. We are fortunate that our setting is nice enough that a statement like that of Lemma 3.2 is possible. Thanks to Thomas Lam for informing us of the topic of meander theory.

Proof of Lemma 1.3. In light of Lemmas 2.5 and 3.2, we may suppose that K and $\tilde{K}$ are as in Figure 8, of course drawn with the correct number of meeting points between ∂K and $\partial \tilde{K}$. Recalling that $\eta(f)$ counts the winding number of the vector $f(z) - z$ around the origin, we consider when it is possible for $f(z) - z$ to be a positive real number. If z does not lie on the intersection of the left-hand side of the rectangle ∂K with the interior of $\tilde{K}$, then $f(z) - z$ certainly has either a negative real component, or a non-zero imaginary component. Thus, pick a z lying in this intersection. Then the only way for $f(z) - z$ to be real and positive is for $f(z)$ to lie on the semicircular sub-arc of $\partial \tilde{K}$ to the right of z, as in the figure. But then, considering the orientations on ∂K and $\partial \tilde{K}$, we get that the vector $f(z) - z$ must always be turning counterclockwise at such a z, as we traverse ∂K in the positive direction. We conclude that whenever the curve $\{f(z) - z\}_{\partial K}$ crosses the positive real axis, it does so in the positive direction, thus the winding number of this curve around the origin is non-negative. $\qquad \square$

REMARK 3.4. The same Figure 8 can be used to show that under the hypotheses of Lemma 1.3, we get that $\eta(f) \leq 2$. Thus the only fixed-point indices which can be achieved in this setting are $0, 1, 2$, and all three of these occur. We do not use these facts, so working out the details is left as an exercise for the interested reader.

4. Proof of rigidity and uniformization of circle packings

As an example of the power of fixed-point index, in this section we prove three rigidity and uniformization theorems on circle packings. Theorem 4.1
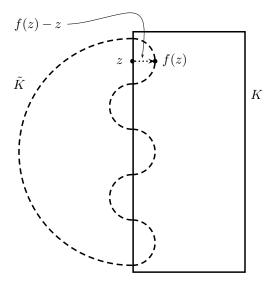
FIGURE 8. We see that whenever $f(z) - z$ is positive and real, as we traverse ∂K and $\partial \tilde{K}$ positively, the point $f(z)$ is moving upward, and the point z is moving downward, so the vector $f(z) - z$ is winding counterclockwise, thus in the positive direction.

is usually credited to Koebe [Koe36], Andreev [And70], and Thurston.[2] All of their proofs were via methods different from ours. Theorems 4.2 and 4.3 are originally due to Schramm [SCH91, Rigidity Theorems 1.1, 5.1], who proved them essentially via the Incompatibility Theorem 3.1, although his proof of the Incompatibility Theorem is not via fixed-point index techniques. The first ones to study circle packings by fixed-point index techniques were He and Schramm in [HS93], although they did not proceed via the Incompatibility Theorem, instead applying other normalizations to the packings in question, to similarly argue by contradiction. For further references on circle packing, see, for example, the articles [Sac94], [Roh11] and their bibliographies.

We begin with some basic definitions. A *circle packing* is a collection $\mathcal{P} = \{D_i\}$ of pairwise interiorwise disjoint round closed disks in the Riemann sphere $\hat{\mathbb{C}}$. The *contact graph* of a circle packing is the graph having a vertex for every disk of the packing, so that two vertices share an edge if and only if the corresponding disks meet. Then the following hold.

[2] Originally at his talk at the International Congress of Mathematicians, Helsinki, 1978, according to [Sac94, p. 135]. See also [Thu, Chapter 13].

THEOREM 4.1. *Suppose that $\mathcal{P}$ and $\tilde{\mathcal{P}}$ are circle packings in $\hat{\mathbb{C}}$, sharing a contact graph that triangulates the 2-sphere $\mathbb{S}^2$. Then $\mathcal{P}$ and $\tilde{\mathcal{P}}$ differ by a Möbius or anti-Möbius transformation.*

THEOREM 4.2. *Suppose that $\mathcal{P}$ and $\tilde{\mathcal{P}}$ are circle packings which are locally finite in $\mathbb{C}$, sharing a contact graph that triangulates a topological open disk. Then $\mathcal{P}$ and $\tilde{\mathcal{P}}$ differ by a Euclidean similarity.*

THEOREM 4.3. *There cannot be two circle packings $\mathcal{P}$ and $\tilde{\mathcal{P}}$ sharing a contact graph triangulating a topological open disk, so that one is locally finite in $\mathbb{C}$ and the other is locally finite in the hyperbolic plane $\mathbb{H}^2$, equivalently the open unit disk $\mathbb{D}$.*

The rest of this section consists of the proofs of these three theorems. Before moving on, we make one note.

REMARK 4.4. The statement for locally finite packings in the hyperbolic plane $\mathbb{H}^2 \cong \mathbb{D}$ analogous to Theorem 4.2 also holds. However, to apply our techniques to prove it, one would need to show that two combinatorially equivalent packings, both locally finite in $\mathbb{H}^2$, having contact graphs triangulating $\mathbb{H}^2$, induce a homeomorphism, or at least some appropriately behaved identification, on the boundary $\partial_\infty \mathbb{H}^2 \cong \partial \mathbb{D}$. This turns out to be true (in fact, it follows from the rigidity theorem we discuss in this remark), but nontrivial.

Proof of Theorem 4.1. The argument is by contradiction. The main idea is to superimpose the packings $\mathcal{P}$ and $\tilde{\mathcal{P}}$ on the Riemann sphere in a convenient way. In particular, we isolate two topological quadrilaterals Q and $\tilde{Q}$ so that they are packed in combinatorially equivalent ways by disks of $\mathcal{P}$ and $\tilde{\mathcal{P}}$, but so that Q and $\tilde{Q}$ cut each other, as in Figure 5. Then the simple observation that two round disks cannot cut each other gives us our desired contradiction, via the Incompatibility Theorem 3.1.

Let $X = (V, E, F)$ be the common triangulation of $\mathbb{S}^2$ which $\mathcal{P}$ and $\tilde{\mathcal{P}}$ realize. Suppose for contradiction that $\mathcal{P}$ and $\tilde{\mathcal{P}}$ are not equivalent under any Möbius or anti-Möbius transformation. For the first part of the proof, we apply a sequence of normalizations to $\mathcal{P}$ and to $\tilde{\mathcal{P}}$. Let $f_0 = \langle v_1, v_2, v_3 \rangle \in F$ be a face of X. We first normalize via a Möbius transformation so that $D_i = \tilde{D}_i$ for $i = 1, 2, 3$. Here $D_i \in \mathcal{P}$ is the disk corresponding to the vertex $v_i \in V$ of X, similarly $\tilde{D}_i \in \tilde{\mathcal{P}}$. In particular, the correct Möbius transformation is the one that sends the intersection points of the ∂D_i to the corresponding ones of the $\partial \tilde{D}_i$.

Our next normalization is in our initial choice of f_0 and our labeling of the v_i, as per the following observation.

OBSERVATION 4.5. *Let v_4 denote the vertex of X other than v_1 so that $\langle v_2, v_3, v_4 \rangle$ is a face of X. Then there is some choice of $f_0 = \langle v_1, v_2, v_3 \rangle$ so*

that the disks D_4 and $\tilde{D}_4$ are not equal after our normalization identifying $D_i = \tilde{D}_i$ for $i = 1, 2, 3$.

If there were no such choice of f_0, then in fact every pair of corresponding disks D_i and $\tilde{D}_i$ would coincide after our first normalization, and so $\mathcal{P}$ and $\tilde{\mathcal{P}}$ coincide.

An *interstice* of the packing $\mathcal{P}$ is a connected component of $\hat{\mathbb{C}} \setminus \mathcal{P}$. Every interstice is necessarily a curvilinear triangle, because the packing's contact graph triangulates $\hat{\mathbb{C}}$. For our next normalization, we insist that ∞ lies in the interstice formed by $D_1 = \tilde{D}_1, D_2 = \tilde{D}_2, D_3 = \tilde{D}_3$ which contains no other disks of either packing. Finally, we insist that $D_1 = \tilde{D}_1, D_2 = \tilde{D}_2, D_3 = \tilde{D}_3$ all have Euclidean radius 1, and that D_2 and D_3 are tangent at a point lying on the horizontal axis, so that D_1 lies to their left. The situation for $\mathcal{P}$ is depicted in Figure 9.

From now on, we work in the plane $\mathbb{C}$, in the sense that $\infty \in \hat{\mathbb{C}}$ will not move again for the remainder of the proof. Note that every face f of F corresponds to some interstices $U_f \subset \mathbb{C}$ and $\tilde{U}_f \subset \mathbb{C}$ of $\mathcal{P}$ and $\tilde{\mathcal{P}}$ respectively, except for f_0, for which the interstices $U_{f_0} = \tilde{U}_{f_0}$ contain ∞. Let Q be the topological quadrilateral shown in Figure 9(b). More precisely, let $V_Q = V \setminus$
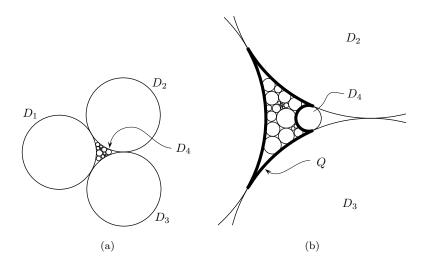

FIGURE 9. The packing $\mathcal{P}$ after some normalizations. The disks of $\mathcal{P}$ all lie between D_1, D_2, D_3. Note that the interstice formed by D_1, D_2, D_3 on $\hat{\mathbb{C}}$ is the outside region in these figures. The disk D_4 is "the first disk of $\mathcal{P} \setminus \{D_1, D_2, D_3\}$ we get to if we start scanning from the right." In (b) the topological quadrilateral Q is outlined in bold.

$\{v_1, v_2, v_3\}$, and let $F_Q = F \setminus \{f_0 = \langle v_1, v_2, v_3 \rangle, \langle v_2, v_3, v_4 \rangle\}$. Then we define $Q = \bigcup_{v \in V_Q} D_v \cup \bigcup_{f \in F_Q} U_f$. Define the analogous objects for $\tilde{\mathcal{P}}$ in the obvious way.

We now apply one final transformation to $\mathcal{P}$. First, suppose without loss of generality that the Euclidean radius of D_4 is larger than that of $\tilde{D}_4$. Then translate every disk of $\mathcal{P}$ to the right by a small amount $\varepsilon > 0$, leaving the disks of $\tilde{\mathcal{P}}$ unchanged. Denote this transformation by T_ε. We will discuss more precise requirements on ε later. The situation is depicted in Figure 10. The essential point is that there is an open interval of values that $\varepsilon > 0$ may take so that after all of our transformations, the topological quadrilaterals Q and $\tilde{Q}$ are arranged qualitatively as in Figure 10(b). In particular, we may choose ε so that the packings of Q and $\tilde{Q}$ by the remaining disks of $\mathcal{P}$ and $\tilde{\mathcal{P}}$ are in transverse position, because there are only finitely many values of $\varepsilon > 0$ for which this fails. Then our desired contradiction follows immediately from the Incompatibility Theorem 3.1, because round closed disks cannot cut one another. $\square$
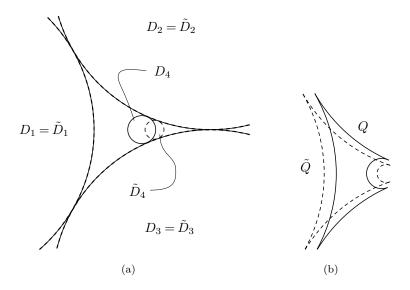

FIGURE 10. The interaction between $\mathcal{P}$ and $\tilde{\mathcal{P}}$ before and after applying T_ε. In (a), we see the superimposition of the D_i with the $\tilde{D}_i$ before applying T_ε to $\mathcal{P}$. The disks D_i are drawn solid, and the disks $\tilde{D}_i$ are drawn dashed. In (b), we see the relative positions of Q and $\tilde{Q}$ after applying T_ε to $\mathcal{P}$.

Proof of Theorem 4.2. The proof of Theorem 4.2 proceeds along the same lines, except that after our first round of normalizations identifying D_i and $\tilde{D}_i$ for $i = 1, 2, 3$, and sending a point of their common interstice to ∞, the remaining disks of $\mathcal{P}$ accumulate around a point $z_\infty \in \mathbb{C}$, as do those of $\tilde{\mathcal{P}}$ around a point $\tilde{z}_\infty \in \mathbb{C}$. The points z_∞ and $\tilde{z}_\infty$ may coincide or may be different. We define and apply T_ε as before, this time making sure that z_∞ and $\tilde{z}_\infty$ differ after applying T_ε.

Next, pick small disjoint neighborhoods W and $\tilde{W}$ of z_∞ and $\tilde{z}_\infty$ respectively, and contained in Q and $\tilde{Q}$ respectively. Then, let V_L be the set of vertices $v \in V$ so that both $D_v \subset W$ and $\tilde{D}_v \subset \tilde{W}$. Remove vertices from V_L until the sub-triangulation of X having vertices $V \setminus V_L$ is a triangulation of a topological closed disk. Let F_L be the set of faces of X corresponding to interstices formed by disks whose vertices are in V_L. Let L (which stands for leftovers) be the union $\bigcup_{v \in V_L} D_v \cup \bigcup_{f \in F_L} U_f \cup z_\infty$, and define $\tilde{L}$ similarly. Then $\{D_v\}_{v \in V \setminus V_L}$ together with L form a packing of the topological quadrilateral Q by closed Jordan domains, as do $\{\tilde{D}_v\}_{v \in V \setminus V_L}, \tilde{L}$ in $\tilde{Q}$. Furthermore, because L and $\tilde{L}$ are disjoint by construction, these two domains do not cut each other. Then we get our desired contradiction by the Incompatibility Theorem 3.1 as before.   □

Proof of Theorem 4.3. The adaptation here is along similar lines as the adaptation to prove Theorem 4.2. Suppose for contradiction that $\mathcal{P}$ is locally finite in $\mathbb{C}$, and $\tilde{\mathcal{P}}$ is locally finite in $\mathbb{H}^2 \cong \mathbb{D}$. This time, after our normalizations, the disks of $\mathcal{P}$ accumulate around a single point z_∞, and the disks of $\tilde{\mathcal{P}}$ accumulate around some round circle C contained in the bounded region in the plane formed between $D_1 = \tilde{D}_1, D_2 = \tilde{D}_1, D_3 = \tilde{D}_3$. This time, ensure that we chose ε so that z_∞ does not lie on the circle C. We define L and $\tilde{L}$ by throwing away disks of our circle packings, as before, but this time, either L and $\tilde{L}$ are disjoint, or one contains the other. In either case, the two do not cut each other, and the conclusion of the proof proceeds as before.   □

5. Torus parametrization

Before defining torus parametrization, it will be helpful to have access to the following simple lemma.

LEMMA 5.1. *Suppose K and $\tilde{K}$ are closed Jordan domains in transverse position. Suppose that $z \in \partial K \cap \partial \tilde{K}$. Orient ∂K and $\partial \tilde{K}$ positively as usual. Then one of the following two mutually exclusive possibilities holds at the point z.*

(1) *The curve $\partial \tilde{K}$ is entering K, and the curve ∂K is exiting $\tilde{K}$.*
(2) *The curve ∂K is entering $\tilde{K}$, and the curve $\partial \tilde{K}$ is exiting K.*

Thus as we traverse ∂K, we alternate arriving at points of $\partial K \cap \partial \tilde{K}$ where (1) occurs and those where (2) occurs, and the same holds as we traverse $\partial \tilde{K}$.
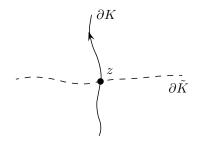
FIGURE 11. A meeting point between two Jordan curves in transverse position. The orientation shown on ∂K implies that K lies to the left. Depending on the orientation chosen for $\partial \tilde{K}$ we will get that $\tilde{K}$ lies above $\partial \tilde{K}$ or below it.

Proof. Let $z \in \partial K \cap \partial \tilde{K}$. We may assume, by applying a homeomorphism, that locally near z the picture looks like Figure 11, with ∂K oriented down-to-up as shown. Then K lies to the left of ∂K. Now, certainly $\partial \tilde{K}$ is either entering or exiting K at z. Suppose $\partial \tilde{K}$ is entering K at z. Then $\partial \tilde{K}$ is oriented right-to-left, and so $\tilde{K}$ is below $\partial \tilde{K}$. Thus ∂K is exiting $\tilde{K}$, and case (1) occurs. Similarly, if $\partial \tilde{K}$ is exiting K at z then ∂K is entering $\tilde{K}$ at z, so case (2) occurs. $\square$

We are now ready to define *torus parametrization*. Throughout the definition, refer to Figure 12 for an example.

DEFINITION 5.2. Let K and $\tilde{K}$ be closed Jordan domains in transverse position, so that ∂K and $\partial \tilde{K}$ meet at $2M \geq 0$ points, with boundaries oriented as usual. Let $\partial K \cap \partial \tilde{K} = \{P_1, \ldots, P_M, \tilde{P}_1, \ldots, \tilde{P}_M\}$, where P_i and $\tilde{P}_i$ are labeled so that at every P_i we have that ∂K is entering $\tilde{K}$, and at every $\tilde{P}_i$ we have that $\partial \tilde{K}$ is entering K. Imbue $\mathbb{S}^1$ with an orientation and let $\kappa : \partial K \to \mathbb{S}^1$ and $\tilde{\kappa} : \partial \tilde{K} \to \mathbb{S}^1$ be orientation-preserving homeomorphisms. We refer to this as fixing a *torus parametrization* for K and $\tilde{K}$.

We consider a point $(x, \tilde{x})$ on the 2-torus $\mathbb{T} = \mathbb{S}^1 \times \mathbb{S}^1$ to be parametrizing simultaneously a point $\kappa^{-1}(x) \in \partial K$ and a point $\tilde{\kappa}^{-1}(\tilde{x}) \in \partial \tilde{K}$. We denote by $p_i \in \mathbb{T}$ the unique point $(x, \tilde{x}) \in \mathbb{T}$ satisfying $\kappa^{-1}(x) = \tilde{\kappa}^{-1}(\tilde{x}) = P_i$, similarly $\tilde{p}_i \in \mathbb{T}$. Note that by the transverse position hypothesis no pair of points in $\{p_1, \ldots, p_M, \tilde{p}_1, \ldots, \tilde{p}_M\}$ share a first coordinate, nor a second coordinate.

Suppose we pick $(x_0, \tilde{x}_0) \in \mathbb{S}^1 \times \mathbb{S}^1$. Then we may draw an image of $\mathbb{T} = \mathbb{S}^1 \times \mathbb{S}^1$ by letting $\{x_0\} \times \mathbb{S}^1$ be the vertical axis and letting $\mathbb{S}^1 \times \{\tilde{x}_0\}$ be the horizontal axis. Then we call $(x_0, \tilde{x}_0)$ a *base point* for the drawing.

Suppose that $\phi : \partial K \to \partial \tilde{K}$ is an orientation-preserving homeomorphism. Then ϕ determines an oriented curve γ in $\mathbb{T}$ for us, namely its graph $\gamma = \{(\kappa(z), \tilde{\kappa}(\phi(z)))\}_{z \in \partial K}$, with orientation obtained by traversing ∂K positively.
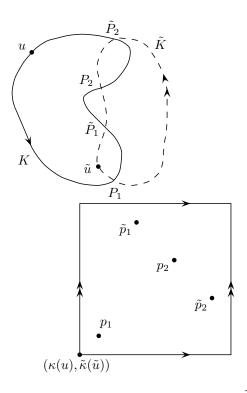
FIGURE 12. A pair of closed Jordan domains K and $\tilde{K}$ and a torus parametrization for them, drawn with base point $(\kappa(u), \tilde{\kappa}(\tilde{u}))$. The key points to check are that as we vary the first coordinate of $\mathbb{T}$ positively starting at u, we arrive at $\kappa(P_1)$, $\kappa(\tilde{P}_1)$, $\kappa(P_2)$, and $\kappa(\tilde{P}_2)$ in that order, and as we vary the second coordinate of $\mathbb{T}$ positively starting at $\tilde{\kappa}(\tilde{u})$, we arrive at $\tilde{\kappa}(P_1)$, $\tilde{\kappa}(\tilde{P}_2)$, $\tilde{\kappa}(P_2)$, and $\tilde{\kappa}(\tilde{P}_1)$ in that order.

Note that ϕ is fixed-point-free if and only if its associated curve γ misses all of the p_i and $\tilde{p}_i$. Pick $u \in \partial K$ and denote $\tilde{u} = \phi(u)$. Then if we draw the torus parametrization for K and $\tilde{K}$ using the base point $(\kappa(u), \tilde{\kappa}(\tilde{u}))$, the curve γ associated to ϕ essentially looks like the graph of a strictly increasing function from a closed interval to itself. The converse is also true: given any such γ, it determines for us an orientation-preserving homeomorphism $\partial K \to \partial \tilde{K}$ sending u to $\tilde{u}$, which is fixed-point-free if and only if γ misses all of the p_i and $\tilde{p}_i$.

   Suppose that $\phi(u) = \tilde{u}$, equivalently that $(\kappa(u), \tilde{\kappa}(\tilde{u})) \in \gamma$. The curve γ and the horizontal and vertical axes $\{\tilde{\kappa}(\tilde{u})\} \times \mathbb{S}^1$ and $\mathbb{S}^1 \times \{\kappa(u)\}$ divide $\mathbb{T}$ into two

simply connected open sets $\Delta_\uparrow(u, \gamma)$ and $\Delta_\downarrow(u, \gamma)$ as shown in Figure 13. We suppress the dependence on $\tilde{u}$ in the notation because $\tilde{u} = \phi(u)$. If neither $u \in \partial \tilde{K}$ nor $\tilde{u} \in \partial K$, then every p_i and every $\tilde{p}_i$ lies in either $\Delta_\downarrow(u, \gamma)$ or $\Delta_\uparrow(u, \gamma)$. In this case, we write $\#p_\downarrow(u, \gamma)$ to denote $|\{p_1, \ldots, p_M\} \cap \Delta_\downarrow(u, \gamma)|$ the number of points p_i which lie in $\Delta_\downarrow(u, \gamma)$, and we define $\#p_\uparrow(u, \gamma)$, $\#\tilde{p}_\downarrow(u, \gamma)$, and $\#\tilde{p}_\uparrow(u, \gamma)$ in the analogous way. Denote by $\omega(\alpha, z)$ the winding number of the closed curve $\alpha \subset \mathbb{C}$ around the point $z \notin \alpha$.

Suppose γ_0 is any oriented closed curve in $\mathbb{T} \setminus \{p_1, \ldots, p_M, \tilde{p}_1, \ldots, \tilde{p}_M\}$. Then the closed curve $\{\tilde{\kappa}^{-1}(\tilde{x}) - \kappa^{-1}(x)\}_{(x, \tilde{x}) \in \gamma_0}$ misses the origin, and has a natural orientation obtained from that of γ_0. We denote by $w(\gamma_0)$ the winding number around the origin of $\{\tilde{\kappa}^{-1}(\tilde{x}) - \kappa^{-1}(x)\}_{(x, \tilde{x}) \in \gamma_0}$.

The following central lemma says that given an indexable homeomorphism $\phi : \partial K \to \partial \tilde{K}$, we may read off its fixed-point index $\eta(\phi)$ simply by examining the curve γ associated to ϕ in the way we just described:

LEMMA 5.3. *Let K and $\tilde{K}$ be closed Jordan domains. Fix a torus parametrization of K and $\tilde{K}$ via κ and $\tilde{\kappa}$. Let $\phi : \partial K \to \partial \tilde{K}$ be an indexable homeomorphism, with graph γ in $\mathbb{T}$. Suppose that $\phi(u) = \tilde{u}$, where $u \notin \partial \tilde{K}$ and $\tilde{u} \notin \partial K$. Then:*

$$(1) \qquad \eta(\phi) = w(\gamma) = \omega(\partial K, \tilde{u}) + \omega(\partial \tilde{K}, u) - \#p_\downarrow(u, \gamma) + \#\tilde{p}_\downarrow(u, \gamma)$$

$$(2) \qquad = \omega(\partial K, \tilde{u}) + \omega(\partial \tilde{K}, u) + \#p_\uparrow(u, \gamma) - \#\tilde{p}_\uparrow(u, \gamma).$$

The remainder of the section is spent proving Lemma 5.3.
We begin with an observation.

OBSERVATION 5.4. *If γ_1 and γ_2 are homotopic in $\mathbb{T} \setminus \{p_1, \ldots, p_M, \tilde{p}_1, \ldots, \tilde{p}_M\}$, then $w(\gamma_1) = w(\gamma_2)$.*

This is because the homotopy between γ_1 and γ_2 in $\mathbb{T} \setminus \{p_1, \ldots, p_M, \tilde{p}_1, \ldots, \tilde{p}_M\}$ induces a homotopy between the closed curves $\{\tilde{\kappa}^{-1}(\tilde{x}) - \kappa^{-1}(x)\}_{(x, \tilde{x}) \in \gamma_1}$ and $\{\tilde{\kappa}^{-1}(\tilde{x}) - \kappa^{-1}(x)\}_{(x, \tilde{x}) \in \gamma_2}$ in the punctured plane $\mathbb{C} \setminus \{0\}$.

Suppose that $\phi : \partial K \to \partial \tilde{K}$ is a fixed-point-free orientation-preserving homeomorphism. Let γ be its graph in $\mathbb{T}$. If γ has orientation induced by traversing ∂K and $\partial \tilde{K}$ positively, then the following is a tautology.

OBSERVATION 5.5. $\eta(\phi) = w(\gamma)$.

Orient $\partial \Delta_\downarrow(u, \gamma)$ as shown in Figure 13. Then $\partial \Delta_\downarrow(u, \gamma)$ is the concatenation of the curve γ traversed backwards with $\mathbb{S}^1 \times \{\tilde{\kappa}(\tilde{u})\}$ and $\{\kappa(u)\} \times \mathbb{S}^1$, where the two latter curves are oriented according to the positive orientation on $\mathbb{S}^1$.

OBSERVATION 5.6. *If $\mathbb{S}^1 \times \{\tilde{\kappa}(\tilde{u})\}$ and $\{\kappa(u)\} \times \mathbb{S}^1$ are oriented according to the positive orientation on $\mathbb{S}^1$, then $w(\mathbb{S}^1 \times \{\tilde{\kappa}(\tilde{u})\}) = \omega(\partial K, \tilde{u})$ and $w(\{\kappa(u)\} \times \mathbb{S}^1) = \omega(\partial \tilde{K}, u)$.*

It is also easy to see that if we concatenate two closed curves γ_1 and γ_2 that meet at a point, we get $w(\gamma_1 \circ \gamma_2) = w(\gamma_1) + w(\gamma_2)$. Thus, in light of the orientations on $\partial\Delta_\downarrow(u,\gamma)$ and all other curves concerned we get:

$$w\big(\partial\Delta_\downarrow(u,\gamma)\big) = w\big(\mathbb{S}^1 \times \{\tilde{\kappa}(\tilde{u})\}\big) + w\big(\{\kappa(u)\} \times \mathbb{S}^1\big) - w(\gamma)$$
$$= \omega(\partial K, \tilde{u}) + \omega(\partial\tilde{K}, u) - \eta(\phi).$$

For every i let $\zeta(p_i)$ and $\zeta(\tilde{p}_i)$ be small squares around p_i and $\tilde{p}_i$ respectively in $\mathbb{T}$, oriented as shown in Figure 13. By *square* we mean a simple closed curve which decomposes into four "sides," so that on a given side one of the two coordinates of $\mathbb{S}^1 \times \mathbb{S}^1 = \mathbb{T}$ is constant. Pick the squares small enough so that the closed boxes they bound are pairwise disjoint and do not meet $\partial\Delta_\downarrow(u,\gamma)$.

Let Γ be the closed curve in $\Delta_\downarrow(u,\gamma)$ obtained in the following way. First, start with every loop $\zeta(p_i)$ and $\zeta(\tilde{p}_j)$ for those p_i and $\tilde{p}_j$ lying in $\Delta_\downarrow(u,\gamma)$. Let δ_0 be an arc contained in the interior of $\Delta_\downarrow(u,\gamma)$ which meets each $\zeta(p_i)$ and $\zeta(\tilde{p}_j)$ contained in $\Delta_\downarrow(u,\gamma)$ at exactly one point. It is easy to prove inductively that such an arc exists. Let δ be the closed curve obtained by traversing δ_0 first in one direction, then in the other. Then let Γ be obtained by concatenating δ with every $\zeta(p_i)$ and $\zeta(\tilde{p}_j)$ contained in $\Delta_\downarrow(u,\gamma)$.
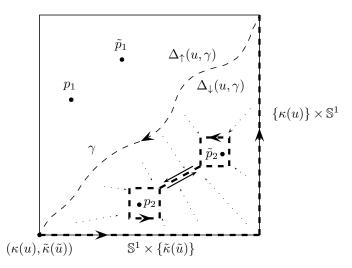

FIGURE 13. A homotopy from $\partial\Delta_\downarrow(u,\gamma)$ to Γ. Here the orientation shown on γ is the opposite of the orientation induced by traversing ∂K positively.

OBSERVATION 5.7. *The curves* Γ *and* $\partial\Delta_\downarrow(u,\gamma)$ *are homotopic in* $\mathbb{T} \setminus \{p_1,\ldots,p_M,\tilde{p}_1,\ldots,\tilde{p}_M\}$. *Also* $w(\delta) = 0$. *It follows that:*

$$w\big(\partial\Delta_\downarrow(u,\gamma)\big) = w(\Gamma) = \sum_{p_i \in \Delta_\downarrow(u,\gamma)} w\big(\zeta(p_i)\big) + \sum_{\tilde{p}_j \in \Delta_\downarrow(u,\gamma)} w\big(\zeta(\tilde{p}_j)\big).$$

See Figure 13 for an example. On the other hand, the following holds.

OBSERVATION 5.8. $w(\zeta(p_i)) = 1$, $w(\zeta(\tilde{p}_i)) = -1$.

To see why, suppose that $\zeta(p_i) = \partial([x_0 \to x_1]_{\mathbb{S}^1} \times [\tilde{x}_0 \to \tilde{x}_1]_{\mathbb{S}^1})$. (If α is an oriented topological circle with $a_0, a_1 \in \alpha$, then we denote by $[a_0 \to a_1]_\alpha$ the closed oriented sub-arc of α beginning at a_0 and ending at a_1.) Then up to orientation-preserving homeomorphism the picture near P_i is as in Figure 14. We let $(x,\tilde{x})$ traverse $\zeta(p_i)$ positively starting at $(x_0,\tilde{x}_0)$, keeping track of the vector $\tilde{\kappa}^{-1}(\tilde{x}) - \tilde{\kappa}^{-1}(x)$ as we do so. The vector $\tilde{\kappa}^{-1}(\tilde{x}_0) - \tilde{\kappa}^{-1}(x_0)$ points to the right. As x varies from x_0 to x_1, the vector $\tilde{\kappa}^{-1}(\tilde{x}) - \tilde{\kappa}^{-1}(x)$ rotates in the positive direction, that is, counter-clockwise, until it arrives at $\tilde{\kappa}^{-1}(\tilde{x}_0) - \kappa^{-1}(x_1)$, which points upward. Continuing in this fashion, we see that $\tilde{\kappa}^{-1}(\tilde{x}) - \tilde{\kappa}^{-1}(x)$ makes one full counter-clockwise rotation as we traverse $\zeta(p_i)$. The proof that $w(\zeta(\tilde{p}_i)) = -1$ is similar. Combining all of our observations establishes equation (1). The proof that equation (2) holds is similar.
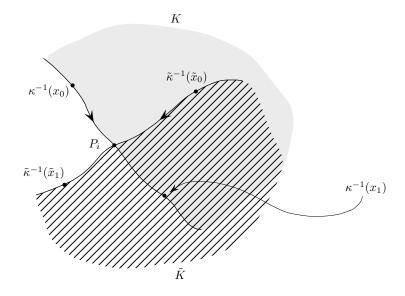

FIGURE 14. The local picture near P_i. This allows us to compute the "local fixed-point index" $w(\zeta(p_i))$ of f near P_i.

6. Preparatory plane-topological lemmas

As usual, let K and $\tilde{K}$ be compact Jordan domains in transverse position, whose boundaries meet at least twice. Note that the connected components of $\partial K \cap \partial \tilde{K}$ are points, that those of $(\partial K \setminus \partial \tilde{K}) \cup (\partial \tilde{K} \setminus \partial K)$ are topological open intervals, and that those of $\hat{\mathbb{C}} \setminus (\partial K \cup \partial \tilde{K})$ are topological open disks. Denote by $X = (V, E, F)$ the cellular decomposition of the Riemann sphere obtained by taking the connected components of $\partial K \cap \partial \tilde{K}$ to be the vertices, those of $(\partial K \setminus \partial \tilde{K}) \cup (\partial \tilde{K} \setminus \partial K)$ to be the edges, and those of $\hat{\mathbb{C}} \setminus (\partial K \cup \partial \tilde{K})$ to be the faces.

For a face $f \in F$, define $\deg(f)$ to be the number of sides of f. Note that $\deg(f)$ is even for all $f \in F$. Let $F_n = \{f \in F : \deg(f) = n\}$, and $F_{>n} = \{f \in F : \deg(f) > n\}$.

LEMMA 6.1. $|F_2| \geq 4$. Furthermore, if there is an $f \in F$ having $\deg(f) > 4$, then $|F_2| > 4$.

Proof. Recall that for a cellular decomposition of a topological sphere, we have $|V| - |E| + |F| = 2$. In our setup, it is easy to see that $|E| = 2|V|$. It follows that $|F| = |E|/2 + 2$.

Next, note that

$$2|E| = \sum_{f \in F} \deg(f) = 2|F_2| + 4|F_4| + 6|F_6| + \cdots$$

so we get that $|E| = |F_2| + 2|F_4| + 3|F_6| + \cdots$. Finally, note that $|F| = |F_2| + |F_{>2}|$.

Putting these together gives

$$
\begin{aligned}
|E| &= |F_2| + 2|F_4| + 3|F_6| + 4|F_8| + \cdots \\
&\geq |F_2| + 2|F_4| + 2|F_6| + 2|F_8| + \cdots \\
&= |F_2| + 2|F_{>2}| \\
&= 2|F| - |F_2| = |E| + 4 - |F_2|
\end{aligned}
$$

noting that the inequality on the second line is strict if there is an $f \in F$ having $\deg(f) > 4$. The lemma follows. $\square$

Let $F_K = \{f \in F : f \subset K \setminus \tilde{K}\}$, and define $F_{\tilde{K}}$ analogously. Let $F_\varnothing = \{f \in F : f \subset \hat{\mathbb{C}} \setminus (K \cup \tilde{K})\}$, and let $F_\cap = \{f \in F : f \subset K \cap \tilde{K}\}$. Then, for example, we have that $F = F_K \sqcup F_{\tilde{K}} \sqcup F_\varnothing \sqcup F_\cap$. We call $f \in F$ a *bigon* of X if $\deg(f) = 2$, and we say that f is *finite* if the point $\infty \in \hat{\mathbb{C}}$ is not in f. For example, by Lemma 6.1, we always have at least 3 finite bigons. Then F_2 denotes the set of all of the bigons of X.

LEMMA 6.2. $|F_K| = |F_{\tilde{K}}|$.

Proof. This is apparent if ∂K and $\partial \tilde{K}$ meet twice, because in that case K and $\tilde{K}$ can be in only one topological configuration (for example, by Lemma 3.2). We proceed by induction on the size of $|V| = |\partial K \cap \partial \tilde{K}|$, using $|V| = 2$ for our base case.

Suppose that $|V| \geq 4$ and let f be a finite bigon of F. We have depicted the picture near f in Figure 15. In this figure, we have that a, b, c, d, f are faces of $X = (V, E, F)$, the cellular decomposition induced by K and $\tilde{K}$. Note that $c \neq d$ because $|V| \geq 4$.

Our plan is to modify K and $\tilde{K}$ as shown in the figure, to create two new compact Jordan domains K' and $\tilde{K}'$. Using the prime symbol in the natural way, we have that a', b', c' in the figure are faces of $X' = (V', E', F')$, the cellular decomposition induced by K' and $\tilde{K}'$. The modification results in c and d coalescing to form c', and causes f to effectively disappear. It also reduces the number of intersection points of our Jordan domains, allowing us to apply induction.

There are four possibilities to check. We work out the first in detail. Suppose that $f \in F_K$. In this case it follows that $a \in F_\cap$, $b \in F_\varnothing$, and $c, d \in F_{\tilde{K}}$. Our modification to $K, \tilde{K}$ in this case has the effect of decreasing each of $|F_K|$ and $|F_{\tilde{K}}|$ by 1. We apply the induction hypothesis and we are done.

The remaining three cases to check are $f \in F_{\tilde{K}}, F_\cap, F_\varnothing$. These are left as straightforward exercises for the reader. $\qquad\square$

LEMMA 6.3. $|F_\cap| = |F_\varnothing|$.

Proof. This follows from Lemma 6.2. Let z be a point in the interior of $\tilde{K} \setminus K$, let $\phi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a homeomorphism interchanging z and ∞, let $K' = \phi(K)$, let $\tilde{K}'$ be the closure of $\hat{\mathbb{C}} \setminus \phi(\tilde{K})$, and apply Lemma 6.2 to K' and $\tilde{K}'$. The details are left to the reader. $\qquad\square$
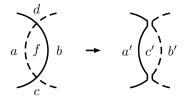

FIGURE 15. Applying the induction step in the proof of Lemma 6.2. The solid arcs represent segments of ∂K and of $\partial K'$ on the left-hand and right-hand sides respectively, similarly the dashed arcs for $\partial \tilde{K}$ and $\partial \tilde{K}'$.

LEMMA 6.4. $|V| = \sum_{f \in F_K} \deg(f) = \sum_{f \in F_{\tilde{K}}} \deg(f) = \sum_{f \in F_\varnothing} \deg(f) = \sum_{f \in F_\cap} \deg(f)$.

Proof. Every vertex $v \in V$ lies on the boundaries of exactly four faces of X, one from each of $F_K, F_{\tilde{K}}, F_\varnothing, F_\cap$, and a face's edge degree and its vertex degree are the same. Thus every vertex can be thought of as being counted exactly once in each of the sums in the statement of the lemma.   □

LEMMA 6.5. *At least one of the following holds*:
(1) *There are strictly more than three finite bigons in* X.
(2) *There is a* (*necessarily finite*) *bigon of* X *in* $F_\cap$.

Proof. Suppose that Condition 1 fails. Then, by Lemma 6.1, there are exactly three finite bigons in X, and one infinite bigon, necessarily belonging to $F_\varnothing$. Consider the sums $S_\varnothing = \sum_{f \in F_\varnothing} \deg(f)$ and $S_\cap = \sum_{f \in F_\cap} \deg(f)$. The two sums are equal by Lemma 6.4, and they have the same number of terms, all of which are positive even integers, by Lemma 6.3. The sum $S_\varnothing$ has at least one term which is equal to 2 (coming from the infinite bigon). By the strictness portion of Lemma 6.1, no term of the sum $S_\cap$ is strictly greater than 4, so this sum must have at least one term which is equal to 2, which is equivalent to Condition 2.   □

Suppose that $z_1, z_2, z_3 \in \partial K \setminus \partial \tilde{K}$ and $\tilde{z}_1, \tilde{z}_2, \tilde{z}_3 \in \partial \tilde{K} \setminus \partial K$ appear in counterclockwise order as in the statement of the Three Point Prescription Lemma 1.2. Let f be a bigon of X. We define the notion of *constraint point count* for f, denoted CPC, as follows:

$$\mathrm{CPC}_K(f) = \big| \partial f \cap \{z_1, z_2, z_3\} \big|,$$
$$\mathrm{CPC}_{\tilde{K}}(f) = \big| \partial f \cap \{\tilde{z}_1, \tilde{z}_2, \tilde{z}_3\} \big|,$$
$$\mathrm{CPC}(f) = \big(\mathrm{CPC}_K(f), \mathrm{CPC}_{\tilde{K}}(f) \big).$$

Then, for example, we get that:

$$\sum_{f \in F_2} \mathrm{CPC}_K(f) \leq 3,$$
$$\sum_{f \in F_2} \mathrm{CPC}_{\tilde{K}}(f) \leq 3.$$

Of course the constraint point count of f depends on the choices of $z_i, \tilde{z}_i$. This dependence is suppressed in the notation as it is normally clear which $z_i, \tilde{z}_i$ are meant.

LEMMA 6.6. *At least one of the following holds*:
(1) *There is a finite bigon* f *in* X *for which* $\mathrm{CPC}(f)$ *is equal to one of* $(0,0), (1,0), (0,1)$.

(2) *There is a finite bigon f in X satisfying $f \subset K \cap \tilde{K}$, for which $\mathrm{CPC}(f)$ is equal to one of $(2,0)$, $(1,1)$, $(0,2)$.*

Proof. Recall that by Lemma 6.1, there are always at least three finite bigons in X.

First, suppose that we have strictly more than three finite bigons. In this case, we will see that Condition 1 holds. Let f_1, f_2, f_3, f_4 be four of our finite bigons, and suppose for contradiction that Condition 1 fails. Then, for each f_i we have that $\mathrm{CPC}_K(f_i) + \mathrm{CPC}_{\tilde{K}}(f_i) \geq 2$. It follows that $\sum_{i=1}^{4} \mathrm{CPC}_K(f_i) + \mathrm{CPC}_{\tilde{K}}(f_i) \geq 8$. On the other hand, we know that $\sum_{i=1}^{4} \mathrm{CPC}_K(f_i) + \mathrm{CPC}_{\tilde{K}}(f_i) \leq 6$, which is a contradiction.

Next, suppose that we have exactly three finite bigons, implying by Lemma 6.5 that at least one of them, call it f_0, lies in $K \cap \tilde{K}$. Let f_1, f_2 denote the other two. Suppose that Condition 1 fails. Then, we have that $\mathrm{CPC}_K(f_i) + \mathrm{CPC}_{\tilde{K}}(f_i) \geq 2$ for $i = 0, 1, 2$. On the other hand, we know as before that $\sum_{i=0}^{2} \mathrm{CPC}_K(f_i) + \mathrm{CPC}_{\tilde{K}}(f_i) \leq 6$. Putting these inequalities together gives that $\mathrm{CPC}_K(f_i) + \mathrm{CPC}_{\tilde{K}}(f_i) = 2$, in particular for $i = 0$, implying Condition 2. $\square$

7. Proof of the Three Point Prescription Lemma 1.2

Let K and $\tilde{K}$ be compact Jordan domains in transverse position. Let $z_1, z_2, z_3 \in \partial K \setminus \partial \tilde{K}$ appear in counterclockwise order, similarly $\tilde{z}_1, \tilde{z}_2, \tilde{z}_3 \in \partial \tilde{K} \setminus \partial K$. We wish to find an indexable homeomorphism $\phi : \partial K \to \partial \tilde{K}$ sending $z_i \mapsto \tilde{z}_i$ for $i = 1, 2, 3$.

(We remind the interested reader to refer to Remark 7.1 at the end of the section for a discussion on the strength of the hypotheses on $K, \tilde{K}, z_i, \tilde{z}_i$ in the statement of the lemma.)

We proceed by induction on the number of intersection points $\partial K \cap \partial \tilde{K}$, recalling that this number is always even. The Circle Index Lemma 2.1 takes care of the cases where ∂K and $\partial \tilde{K}$ meet 0 or 2 times. Thus, suppose for the remainder of the argument that ∂K and $\partial \tilde{K}$ meet at least 4 times.

Our plan is to apply basically the same modification to K and $\tilde{K}$ as we did in Lemma 6.2 and Figure 15, except this time, we will be particular in our choice of which finite bigon to focus on. For the remainder of the proof, we use K' and $\tilde{K}'$ to denote the domains obtained via this modification. Figure 16 gives an alternative, topologically equivalent, example visualization.

We continue with the notation of Section 6. Suppose that we have chosen a finite bigon f of X. Beginning with a torus parametrization for K and $\tilde{K}$, we pick a torus parametrization for K' and $\tilde{K}'$, and show how we can move between these two parametrizations.

Let $U \subset \mathbb{C}$ be a simply connected open neighborhood of $f \subset \mathbb{C}$, with closure $\bar{U}$, so that $\bar{U} \cap (\partial K \cap \partial \tilde{K}) = \partial f \cap (\partial K \cap \partial \tilde{K})$, and so that every z_i and every $\tilde{z}_i$
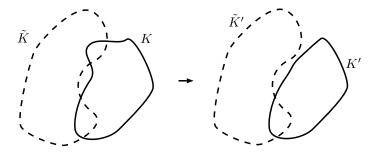
FIGURE 16. A before-and-after view of the modification we apply to K and $\tilde{K}$ in the induction step of the proof of the Three Point Prescription Lemma 1.2. We essentially get rid of one of our finite bigons. In this case, we had three choices of which finite bigon to focus on.

in $\bar{U}$ also lies in ∂f. Conceptually, we want U to be a very small neighborhood of f. As before, we obtain K' and $\tilde{K}'$ by modifying K and $\tilde{K}$ within U, so that $\partial K'$ and $\partial \tilde{K}'$ do not meet in U. Let $\psi_K : \partial K \cap U \to \partial K' \cap U$ denote an orientation-preserving homeomorphism which agrees with the identity map on the endpoints of its domain, and define $\psi_{\tilde{K}}$ similarly. Extend ψ_K and $\psi_{\tilde{K}}$ to all of ∂K and $\partial \tilde{K}$ via the identity map. Define $z_i' = \psi_K(z_i)$ and $\tilde{z}_i' = \psi_{\tilde{K}}(\tilde{z}_i')$ for $i = 1, 2, 3$.

Suppose that we have fixed a torus parametrization $\kappa : \partial K \to \mathbb{S}^1, \tilde{\kappa} : \partial \tilde{K} \to \mathbb{S}^1$ for K and $\tilde{K}$. Let $\kappa' = \psi_K^{-1} \circ \kappa$ and $\tilde{\kappa}' = \psi_{\tilde{K}}^{-1} \circ \tilde{\kappa}$. Then $\kappa' : \partial K' \to \mathbb{S}^1, \tilde{\kappa}' : \partial \tilde{K}' \to \mathbb{S}^1$ is a torus parametrization for K' and $\tilde{K}'$ so that $\kappa \equiv \kappa'$ and $\tilde{\kappa} \equiv \tilde{\kappa}'$ outside of U, and so that $\kappa(z_i) = \kappa'(z_i')$ and $\tilde{\kappa}(\tilde{z}_i) = \tilde{\kappa}'(\tilde{z}_i')$ for $i = 1, 2, 3$.

By the induction hypothesis, there exists an indexable homeomorphism $\phi' : \partial K' \to \partial \tilde{K}'$ sending $z_i' \mapsto \tilde{z}_i'$ for $i = 1, 2, 3$, satisfying $\eta(\phi') \geq 0$. Let γ' be the graph of ϕ' in $\mathbb{T} = \mathbb{S}^1 \times \mathbb{S}^1$ according to the parametrization $\kappa', \tilde{\kappa}'$. Our goal is to modify γ' to obtain a new simple closed curve $\gamma \subset \mathbb{T}$ which will be the graph, according to the parametrization $\kappa, \tilde{\kappa}$, of an indexable homeomorphism $\phi : \partial K \to \partial \tilde{K}$, in such a way that ϕ has the properties required to complete the proof of the lemma.

We break the conclusion of the proof into two cases according to Lemma 6.6.

Case 1 of Lemma 6.6. Without loss of generality, by interchanging the roles of K and $\tilde{K}$ if necessary, let f be a finite bigon in X satisfying $\mathrm{CPC}_K(f) = 0$, with $\mathrm{CPC}_{\tilde{K}}(f)$ equal to either 0 or 1. Consider the strips $S_{\tilde{K}} = \mathbb{S}^1 \times \tilde{\kappa}(U \cap \partial \tilde{K}) = \mathbb{S}^1 \times \tilde{\kappa}'(U \cap \partial \tilde{K}') \subset \mathbb{T}$ and $S_K = \kappa(U \cap \partial K) \times \mathbb{S}^1 = \kappa'(U \cap \partial K') \times \mathbb{S}^1 \subset \mathbb{T}$, let $S_\cup = S_K \cup S_{\tilde{K}}$ denote their union, and let $S_\cap = S_K \cap S_{\tilde{K}}$ denote their intersection. Note that there is no point in $\partial K' \cap \partial \tilde{K}'$ whose parametrization under $\kappa', \tilde{\kappa}'$ lies in $S_\cup$, more formally that $\{(\kappa'(z), \tilde{\kappa}'(z)) : z \in \partial K' \cap \partial \tilde{K}'\} \cap$

$S_\cup = \varnothing$. It follows that if we modify γ' however we like within $S_\cup$, leaving the points $\gamma' \cap \partial S_\cup$ fixed, to obtain a new simple closed curve $\gamma \subset \mathbb{T}$ which

- is strictly increasing in the sense of Section 5,
- contains (as is the case with γ') the points $(\kappa'(z_i'), \tilde{\kappa}'(\tilde{z}_i'))$, equivalently $(\kappa(z_i), \tilde{\kappa}(\tilde{z}_i))$, for $i = 1, 2, 3$, and
- does not contain (as is the case with γ') any of the points $(\kappa(z), \tilde{\kappa}(z))$ for $z \in \partial K \cap \partial \tilde{K}$, noting that $\partial K' \cap \partial \tilde{K}' \subset \partial K \cap \partial \tilde{K}$,

then γ will be the graph of both of two indexable homeomorphisms $\phi'' : \partial K' \to \partial \tilde{K}'$ and $\phi : \partial K \to \partial \tilde{K}$, where furthermore $\eta(\phi'') = \eta(\phi') \geq 0$.

Next, note that the only points of $\partial K \cap \partial \tilde{K}$ whose parametrizations under $\kappa, \tilde{\kappa}$ lie in $S_\cup$ are the corners of the bigon f, call them P and $\tilde{P}$. Let $p = (\kappa(P), \tilde{\kappa}(P)) \in \mathbb{T}$, and define $\tilde{p}$ similarly. Our goal is to obtain γ as above so that the two points p and $\tilde{p}$ lie on the same side of γ. More formally, let $1 \leq j \leq 3$ be chosen so that $z_j, \tilde{z}_j \notin U$. Then, in the language of Section 5, we will arrange so that either both p and $\tilde{p}$ lie in $\Delta_\downarrow(z_j, \gamma)$, or both lie in $\Delta_\uparrow(z_j, \gamma)$. For example, if γ' does not meet $S_\cap \ni p, \tilde{p}$, then this condition is satisfied automatically by taking $\gamma = \gamma'$. Having found such a γ, it will follow from Lemma 5.3 that $\eta(\phi) = \eta(\phi'')$, completing the proof in this case.

Without loss of generality, by enlarging U slightly if necessary, we may assume that $\partial S_K \cap \partial S_{\tilde{K}} \cap \gamma' = \varnothing$. Then, because γ' is strictly increasing, we get that γ' meets $\partial S_\cap$ either 0 or 2 times. In the former case, we are done by our earlier observation, so suppose γ' meets $\partial S_\cap$ twice. It follows by the pigeonhole principle that γ' meets $\partial S_\cup$ twice, since γ' meets each of S_K and $S_{\tilde{K}}$ twice.

Let $\{v_\nearrow, v_\nwarrow, v_\swarrow, v_\searrow\} = \partial S_K \cap \partial S_{\tilde{K}}$ denote the corners of $S_\cap$ in the natural way. Let $g_1, g_2 \in \gamma' \cap \partial S_\cup$ denote the points where γ' enters and exits $S_\cup$, respectively. If there is an i so that $(\kappa(z_i), \tilde{\kappa}(\tilde{z}_i)) \in S_\cup$, then let g_z denote $(\kappa(z_i), \tilde{\kappa}(\tilde{z}_i))$, otherwise let g_z remain undefined. Note that if g_z is defined then $g_z \in S_{\tilde{K}}$ by our assumption that $\mathrm{CPC}_K(f) = 0$. We are now ready to describe the construction of γ. We refer the reader to Figure 17 for an example.

First, suppose that g_z is undefined (equivalently that $\mathrm{CPC}_{\tilde{K}}(f) = 0$). Then, let γ be obtained by replacing $[g_1 \to g_2]_{\gamma'}$ by a strictly increasing arc starting at g_1, progressing first to a point located infinitesimally northwest of $v_\searrow$, then to g_2. This γ satisfies our requirements and we are done.

Next suppose that g_z is defined. One of two cases occurs: starting at g_1, either γ' passes through g_z and then through $S_\cap$, or it passes through $S_\cap$ and then through g_z. In the first case, construct γ from γ' by replacing $[g_z \to g_2]_{\gamma'}$ by a strictly increasing arc starting at g_z, progressing first to a point located infinitesimally southeast of $v_\nwarrow$, then to g_2. In the second case, construct γ from γ' by replacing $[g_1 \to g_z]_{\gamma'}$ by a strictly increasing arc starting at g_1, progressing first to a point located infinitesimally northwest of $v_\searrow$, then to g_z. In both cases γ satisfies our requirements and we are done.
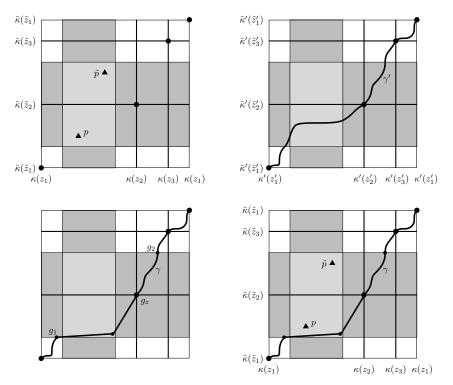
FIGURE 17. An example construction of γ in the proof of the
Three Point Prescription Lemma 1.2 in Case 1 of Lemma 6.6.
Here the lightly shaded region is $S_\cap$, and the darker regions
comprise $S_\cup \setminus S_\cap$. Here $\mathrm{CPC}_{\tilde{K}}(f) = 1$, and in the notation of
the proof, we have chosen $j = 1$, noting that $z_1, \tilde{z}_1 \notin U$.

Case 2 of Lemma 6.6. Without loss of generality, by interchanging the roles
of K and $\tilde{K}$ if necessary, let f be a finite bigon of X satisfying $f \subset K \cap \tilde{K}$
with $\mathrm{CPC}(f)$ equal to either $(1, 1)$ or $(0, 2)$.

Note that if $\mathrm{CPC}(f) = (0, 2)$, then

- $U \cap \partial K$ is a strict subset of exactly one of $[z_1 \to z_2]_{\partial K}$, $[z_2 \to z_3]_{\partial K}$, or
  $[z_3 \to z_1]_{\partial K}$, and
- $U \cap \partial \tilde{K}$ is a strict superset of exactly one of $[\tilde{z}_1 \to \tilde{z}_2]_{\partial \tilde{K}}$, $[\tilde{z}_2 \to \tilde{z}_3]_{\partial \tilde{K}}$, or
  $[\tilde{z}_3 \to \tilde{z}_1]_{\partial \tilde{K}}$.

By cyclically and simultaneously permuting the indices of the z_i and $\tilde{z}_i$ if
necessary, we assume from now on without loss of generality that if $\mathrm{CPC}(f) =$
$(0, 2)$, then $U \cap \partial \tilde{K} \supset [\tilde{z}_2 \to \tilde{z}_3]_{\partial \tilde{K}}$.

First, suppose that either

- CPC$(f) = (1,1)$, and there does not exist an i so that $(\kappa(z_i), \tilde{\kappa}(\tilde{z}_i)) \in S_\cap$, or
- CPC$(f) = (0,2)$, and $U \cap \partial K \not\subset [z_2 \to z_3]_{\partial K}$.

In both of these cases, it is possible to obtain γ from γ' via a construction very similar to that given in Case 1. This is left to the reader.

Finally, we have that either

- CPC$(f) = (1,1)$, and there exists an i so that $(\kappa(z_i), \tilde{\kappa}(\tilde{z}_i)) \in S_\cap$, or
- CPC$(f) = (0,2)$, and $U \cap \partial K \subset [z_2 \to z_3]_{\partial K}$.

In both of these cases, we set $\gamma = \gamma'$. We refer the reader to Figure 18 for an example. Let $P, \tilde{P}$ denote the points of $\partial K \cap \partial \tilde{K} \cap f$ where ∂K enters $\tilde{K}$, respectively where $\partial \tilde{K}$ enters K, using notation analogous to that of Section 5. Because of the orientations on $\partial f \cap \partial K$ and $\partial f \cap \partial \tilde{K}$ forced by the condition
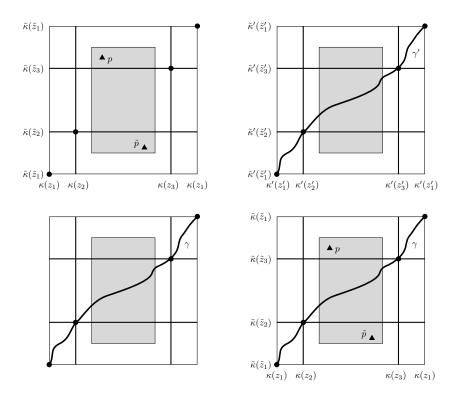

FIGURE 18. An example construction of γ in the proof of the Three Point Prescription Lemma 1.2 in Case 2 of Lemma 6.6. Here the shaded region is $S_\cap$. In this example CPC$(f) = (0,2)$ with $U \cap \partial K \subset [z_2 \to z_3]_{\partial K}$.

that $f \subset K \cap \tilde{K}$, we have that $p = (\kappa(P), \tilde{\kappa}(P))$ is in the northwest corner of $S_\cap$, and that $\tilde{p} = (\kappa(\tilde{P}), \tilde{\kappa}(\tilde{P}))$ is in the southeast corner. Furthermore, in both cases (whether $\mathrm{CPC}(f)$ is $(1,1)$ or is $(0,2)$), the curve γ separates $S_\cap$ so that $p \in \Delta_\uparrow(z_1, \gamma)$ and $\tilde{p} \in \Delta_\downarrow(z_1, \gamma)$. It follows from Lemma 5.3 that $\eta(\phi) = \eta(\phi') + 1$, and we are done.

REMARK 7.1. The conditions of Lemma 1.2 that $\tilde{z}_1, \tilde{z}_2, \tilde{z}_3 \notin \partial K$, that $z_1, z_2, z_3 \notin \partial \tilde{K}$, and that K and $\tilde{K}$ are in transverse position may appear to the reader to be too strong. There are several reasons that the lemma is stated as it is.

First, it may seem that even if those conditions are violated, it might be be possible to modify K and $\tilde{K}$ "very slightly" so that the stated hypotheses of Lemma 1.2 *do* hold, and then, for example, apply the lemma to the modified Jordan domains, hoping to recover the desired conclusion for our original K and $\tilde{K}$ by reversing our modifications to them. A major issue with this approach is that Lemma 1.2 as stated does not give any lower bound on $|\phi(z) - z|$; the author is not aware of a simple amendment to the proof that would provide one. Because of this, it is *a priori* not trivial to arrange the modifications to K and $\tilde{K}$ mentioned earlier so that they are guaranteed to be reversible without altering $\eta(\phi)$.

Next, it is not clear what a good set of hypotheses is for a strengthened version of Lemma 1.2. For the conclusion of the lemma to hold, it is necessary that there be *some* indexable $\phi : \partial K \to \partial \tilde{K}$ identifying $z_i \mapsto \tilde{z}_i$, and without the transverse position hypothesis, it is not trivial to give necessary and sufficient conditions for the existence of such a ϕ. For example, no such ϕ exists if $[z_1 \to z_2]_{\partial K} \subset [\tilde{z}_1 \to \tilde{z}_2]_{\partial \tilde{K}}$. For a more involved example, treating real numbers as points in the complex plane, let $z_1 = 2, z_2 = 0, \tilde{z}_1 = 1, \tilde{z}_2 = 3$, so that $[z_1 \to z_2]_{\partial K} \subset \mathbb{R}$ and $[\tilde{z}_1 \to \tilde{z}_2]_{\partial \tilde{K}} \subset \mathbb{R}$. Then no indexable $\phi : \partial K \to \partial \tilde{K}$ exists which identifies $z_1 \mapsto \tilde{z}_1$ and $z_2 \mapsto \tilde{z}_2$.

It is possible that Lemma 1.2 holds under the hypothesis that there exists *any* indexable $\phi : \partial K \to \partial \tilde{K}$ identifying $z_i \mapsto \tilde{z}_i$. However, this formulation would be somewhat unsatisfying absent an understanding of when such a ϕ exists, and the author does not know of a modification of the proof given in this article which could prove this stronger statement.

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