

A KNOT WITHOUT A NONORIENTABLE ESSENTIAL SPANNING SURFACE

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ABSTRACT. This note gives the first example of a hyperbolic knot in the 3-sphere that lacks a nonorientable essential spanning surface; this disproves the Strong Neuwirth Conjecture formulated by Ozawa and Rubinstein. Moreover, this knot has no even strict boundary slopes, disproving the Even Boundary Slope Conjecture of the same authors. The proof is a rigorous calculation using Thurston’s spun-normal surfaces in the spirit of Haken’s original normal surface algorithms.

1. Introduction

Let me start with the definitions needed to precisely state the first result outlined in the abstract. Throughout, all 3-manifolds will be orientable. A properly embedded orientable surface $S \subset M^3$ is *essential* if it is incompressible, ∂ -incompressible, and not isotopic (rel boundary) into ∂M . A nonorientable S is defined to be essential when the boundary of a regular neighborhood $N(S)$ is essential. For a tame knot K in S^3 , consider its exterior $E(K) = S^3 \setminus \overset{\circ}{N}(K)$, which is a compact manifold with torus boundary. Any such K has a Seifert surface, that is, there is an embedded surface S in S^3 with $\partial S = K$. Moreover, there is always a Seifert surface whose intersection with $E(K)$ is essential, for example, any Seifert surface of minimal genus. Additionally, any K bounds a *nonorientable* spanning surface (just add a small half-twisted band to the boundary of a Seifert surface). Ichihara, Ohtouge, and Teragaito studied nonorientable spanning surfaces and asked whether there is always such a surface that is essential in $E(K)$ [IOT02]. While torus knots $T_{p,q}$ with p and q both odd lack nonorientable essential spanning surfaces [OR12, Example 3.11], Ozawa and Rubinstein [OR12] posited that these are the only such examples:

Received September 23, 2015; received in final form October 9, 2016.
2010 *Mathematics Subject Classification*. 57M.

STRONG NEUWIRTH CONJECTURE 1.1 ([OR12]). Every prime non-torus knot in S^3 has a nonorientable essential spanning surface K .

As the name suggests, this conjecture implies the Neuwirth Conjecture from 1963 [Neu63, Conjecture B], which remains open:

NEUWIRTH CONJECTURE 1.2. Every non-trivial knot K in S^3 lies on a closed surface F in S^3 where K is nonseparating in F and $F \cap E(K)$ is essential.

Part of the motivation in [OR12] for formulating Conjecture 1.1 is that in almost all cases where Conjecture 1.2 is known it is by proving this stronger statement; please see [OR12] for details and an overview of work in this direction. My main result here disproves Conjecture 1.1, cutting off this approach to proving the Conjecture 1.2 in full generality.

THEOREM 1.3. *Let K in S^3 be the braid closure of $(\sigma_1\sigma_2\sigma_3\sigma_4)^{13}\sigma_1\sigma_4\sigma_3\sigma_2$. Then K is a hyperbolic knot without a nonorientable essential spanning surface.*

The knot K was introduced in [CDW99] as $k6_{36}$ where they described it as the twisted torus knot $T(5, 17)_{4, -1}$. While its diagram in Figure 1 has some 56 crossings, its exterior is not complicated: it is the hyperbolic 3-manifold $s800$ from [CHW99] which can be triangulated with 6 ideal tetrahedra and has volume about 5.34821999.

Theorem 1.3 is an immediate corollary of a more technical result for which I need more definitions. Any essential S in $E(K)$ with nonempty boundary has a *boundary slope*, namely the common unoriented isotopy class of the components of ∂S in the torus $\partial E(K)$; as usual, boundary slopes are recorded as elements of $\mathbb{Q} \cup \{\infty\}$ using the standard homological framing on $\partial E(K)$. In our context, the boundary slope of an essential surface S is *strict* if S is not a Seifert surface corresponding to a fibration of $E(K)$ over the circle. I will show.

THEOREM 1.4. *Let $K \subset S^3$ be as in Theorem 1.3. Then $E(K)$ has strict boundary slopes exactly $\{-77, -71, -211/3, -69, -67\}$.*

Theorem 1.4 implies Theorem 1.3 because the boundary slope of a nonorientable essential spanning surface S must be an even integer: it is integral because S intersects a meridian curve exactly once, and it is even as S demonstrates that the boundary of $S \cap E(K)$ is zero in $H_1(E(K); \mathbb{F}_2)$. As promised in the abstract, Theorem 1.4 disproves.

EVEN BOUNDARY SLOPE CONJECTURE 1.5 ([OR12]). For any prime non-torus knot K , there is an essential surface $E(K)$, not a Seifert surface, whose boundary slope is a rational number with even numerator.

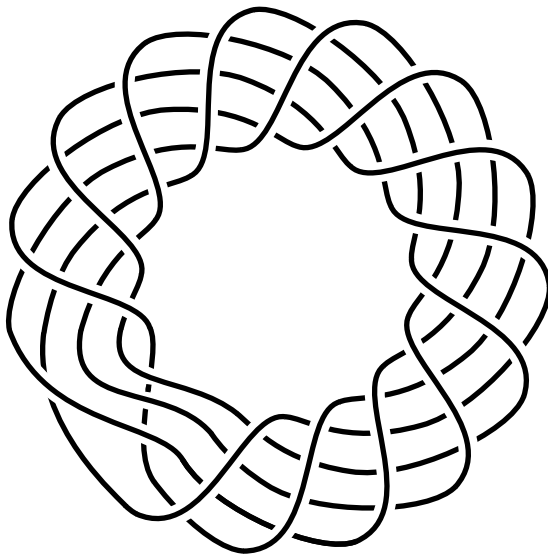


FIGURE 1. The knot $K = k6_{36}$ is the braid closure of $(\sigma_1\sigma_2\sigma_3\sigma_4)^{13}\sigma_1\sigma_4\sigma_3\sigma_2$ as well as the twisted torus knot $T(5, 17)_{4, -1}$ [CDW99].

2. Proof

The proof of Theorem 1.4 will be a straightforward rigorous computation using Thurston's theory of spun-normal surfaces, which is a version of Haken's normal surface theory tuned to the setting of ideal triangulations of cusped manifolds (see [Til08] or [DG12] for general background). Let \mathcal{T} be the standard 6-tetrahedra ideal triangulation of the exterior $E(K)$ of $K = k6_{36}$ used in [CDGW16], which is the one given in [CHW99] with its peripheral framing changed to the homologically natural one for the complement of a knot in S^3 .

LEMMA 2.1. *The set of strict boundary slopes for $E(K)$ is contained in the set of boundary slopes of spun-normal surfaces in \mathcal{T} , which is*

$$\{-77, -71, -211/3, -69, -67\}.$$

Before proving this, let me point out that Lemma 2.1 is already enough to establish Theorem 1.3. Also, you might be troubled by the absence of 0 on the above list of slopes; however, as K is the closure of a positive braid, the manifold $E(K)$ fibers over the circle and so 0 need not be a *strict* boundary slope.

Proof of Lemma 2.1. Let S be any essential surface in $E(K)$ which is not a fiber. We will assume that S is orientable, since if not we can replace it

with the boundary of its regular neighborhood, which is an orientable essential surface with the same boundary slope. By [HIK+16], there is a geometric solution to the gluing and completeness equations for \mathcal{T} , that is, one where all tetrahedra are positively oriented. Thus, the interior of $E(K)$ is hyperbolic and no edge of \mathcal{T} is homotopically peripheral. Therefore, by [Wal11, Theorem 1.6], the surface S can be isotoped into spun-normal form with respect to \mathcal{T} . (Technical note: the hypotheses in [Wal11] require that S is not a *virtual* fiber, but the only virtual fibers in the exterior of a knot in S^3 are actual fibers.) Thus, the boundary slope of S is also the boundary slope of a spun-normal surface, proving the first part of the lemma. To see that the spun-normal surfaces have only the boundary slopes listed above, one simply computes the boundary slopes of the finite collection of vertex spun-normal surfaces. This is easily done rigorously using SnapPy [CDGW16] or Regina [BBP+15]; for example, in the former one simply does: `Manifold('K6_36').normal_boundary_slopes()`. \square

As mentioned, Lemma 2.1 immediately proves Theorem 1.3. The stronger statement of Theorem 1.4 now follows by combining Lemma 2.1 with the following.

LEMMA 2.2. *The exterior $E(K)$ contains no closed essential surfaces, and the Dehn fillings of $E(K)$ along the slopes $\{-77, -71, -211/3, -69, -67\}$ each yield Haken manifolds.*

I established Lemma 2.2 using the breakthrough work of [BO], [BCT13] as implemented in [BBP+15]. The complete script I used for this can be found at [Dun] and the total running time was less than 20 seconds; since Lemma 2.2 is not needed to prove Theorem 1.3, I simply refer you to the code for details.

REMARK 2.3. Given the original motivation for Conjecture 1.1, you might wonder whether K satisfies Conjecture 1.2, namely that K lies on a closed surface F in S^3 where K is nonseparating in F and $F \cap E(K)$ is essential. In fact it does, and here is one way to see this. SnapPy finds a vertex spun-normal surface S in \mathcal{T} which has exactly two boundary components, each of slope -77 , and which is essential by [DG12, Theorem 1.1]. As the boundary slope of S is an integer, we can piece together the two boundary components of S to get a surface F in S^3 on which K lies. The knot K does not separate F because, as a vertex surface, the original surface S is connected.

FURTHER EXAMPLES 2.4. The knot K was found by a computer search through all 502 knots whose complements can be triangulated with 8 or fewer tetrahedra [CDW99], [CKP04], [CKM14]. Other examples which lack nonori-

entable essential spanning surfaces include $k5_{17}$, $k7_{29}$, $k7_{64}$ and $k8_{114}$. The complete search took less than 10 seconds to run.

Acknowledgments. This work was partially supported by US NSF grant DMS-1510204 and partially conducted at the Institute for Advanced Study with the support of the Giorgio and Elena Petronio Fellowship Fund and the Bell Companies Fellowship Fund. I thank the lunch table crowd at IAS for comments and encouragement, as well as the referee for their comments.

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