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ONE-DOMINATION OF KNOTS

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ABSTRACT. We say that a knot k_1 in the 3-sphere 1-*dominates* another k_2 if there is a proper degree 1 map $E(k_1) \to E(k_2)$ between their exteriors, and write $k_1 \geq k_2$. When $k_1 \geq k_2$ but $k_1 \neq k_2$ we write $k_1 > k_2$. One expects in the latter eventuality that k_1 is more *complicated*. In this paper, we produce various sorts of evidence to support this philosophy.

1. Introduction

All knots are assumed tame and contained in the 3-sphere S^3 unless otherwise specified. For basic terminology in knot theory and 3-manifold theory, see [Rlf], [He] and [Ja]. If $k \subset S^3$ is a knot, then $N(k)$ denotes a closed regular neighbourhood and $E(k) = \overline{S^3 \setminus N(k)}$ the knot's exterior. Fixing an orientation of S^3 restricts to a preferred orientation of knot exteriors.

We say that a knot k_1 in the 3-sphere 1-*dominates* another knot k_2, written $k_1 \geq k_2$, if there is a degree 1 map $f : E(k_1) \to E(k_2)$ which is proper, that is, $f(\partial E(k_1)) \subset \partial E(k_2)$. If $k_1 \geq k_2$ but $k_1 \neq k_2$, we write $k_1 > k_2$, and say that the 1-domination is *non-trivial* or *strict*. Let $l_i, m_i \subset \partial E(k_i)$ be a longitude-meridian system. We can assume the f defining the 1-domination has been homotoped so that $f| : \partial E(k_1) \to \partial E(k_2)$ is a homeomorphism and $f(l_1) = l_2$. The proof of Property P for knots in $\mathbb{S}^3$ by Kronheimer and Mrowka [KM] implies that $f(m_1) = m_2$ if both knots are nontrivial. In the following, we will assume our maps realizing dominations meet these conditions.

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Standard arguments show that the relation $\geq$ provides a partial order on knots in S^3. Indeed, the transitivity and reflexivity of $\geq$ is clear. For anti-symmetry, suppose that $k_1 \geq k_2$ and $k_2 \geq k_1$. Then there are degree 1 maps $f: E(k_1) \to E(k_2)$ and $g: E(k_2) \to E(k_1)$. Since degree 1 maps are surjective on the level of fundamental groups (cf. Proposition 3 below) and knot groups are Hopfian, f induces an isomorphism $f_*: \pi_1(E(k_1)) \to \pi_1(E(k_2))$. Then since knot exteriors are Haken and f is a proper map, $E(k_1)$ and $E(k_2)$ are homeomorphic (cf. [He], Theorem 15.13). Finally, since knots are determined by their complements [GL], $k_1 = k_2$.

1.1. Algebraic consequences of 1-domination. Suppose that $k_1 \geq k_2$ and σ is a (set of) knot invariant(s). It is generally believed that $\sigma(k_1)$ "$\geq$" $\sigma(k_2)$ in some sense, and though this has been verified in various cases, the general case is unknown. See Section 1.2 and [Wan] for discussions.

Proposition 1. *Every knot 1-dominates the unknot.*

We need to show $k \geq O$ for each knot k, where O is the unknot. Note that any compact manifold M^n with spherical collared boundary, $\partial M \cong S^{n-1}$, can be degree 1 mapped onto the ball B^n, by pinching the complement of a collar of ∂M to a point. If $E(k)$ is a knot exterior, this trick can be used to map a Seifert surface in $E(k)$ to a spanning disk in $E(O)$, and another pinch to map the remainder of $E(k)$ to the remainder of $E(O)$.

A similar argument shows the following.

Proposition 2. *A connected sum $k_1 \sharp k_2$ of knots 1-dominates each summand. Moreover, if $k_1 \geq k_1'$ and $k_2 \geq k_1'$ then $k_1 \sharp k_2 \geq k_1' \sharp k_2'$.*

Next, we consider some invariants which are known to behave well under 1-domination. Proofs of the stated results will be sketched below.

Proposition 3. *If $f: E(k_1) \to E(k_2)$ is a 1-domination, then $f_*: \pi_1 E(k_1) \to \pi_1 E(k_2)$ is surjective.*

Proposition 4. *If $g(k)$ denotes the genus of k, then $k_1 \geq k_2 \implies g(k_1) \geq g(k_2)$.*

Proposition 5. *If $V(k)$ denotes the Gromov volume of $E(k)$, then $k_1 \geq k_2 \implies V(k_1) \geq V(k_2)$.*

Proposition 6. *If A_k denotes the A-polynomial of k, then $k_1 \geq k_2 \implies A_{k_2} | A_{k_1}$.*

Let Λ_k denote the Alexander module associated with the knot k. That is, consider $\widetilde{E(k)} \to E(k)$ the infinite cyclic cover associated with the (kernel of the) Hurewicz map $\pi(X) = \pi_1(E(k)) \to H_1(E(k)) \cong \mathbb{Z}$. Then Λ_k is $H_1(\widetilde{E(k)}; \mathbb{Z})$, considered as a $\mathbb{Z}[t^{\pm 1}]$-module, where t corresponds to a generator of the deck transformation group $\mathbb{Z}$.

PROPOSITION 7. $k_1 \geq k_2 \implies \Lambda_{k_1} = \Lambda_{k_2} \oplus \Lambda$, *in particular* $\Delta_{k_2} | \Delta_{k_1}$.

More generally let $\Delta(k, G) = \{\Delta_{\phi,k} | \phi : \pi_1(E(k)) \to G\}$ denote the set of all *twisted* Alexander polynomials for a given linear group G.

PROPOSITION 8. $k_1 \geq k_2 \implies \Delta(k_2, G) \subseteq \Delta(k_1, G)$.

The proof of surjectivity of f_* follows from well-known elementary facts and is left to the reader. Proposition 4 is a corollary of Gabai's result that embedded Thurston Norms and singular Thurston Norms coincide [Ga]. Proposition 5 is a basic property of Gromov volume, see [Gr] or [Th]. A sketch of proof of Proposition 6 can be found in [SWh], and also was discussed in a lecture of Boyer [Boy]. The existence of splittings provided by a degree 1 map as in Proposition 7 is a classical fact. See [Br] Theorem 1.2.5 for the $\mathbb{Z}$-coefficient case and [Wal] p. 25 for the local coefficient case. We will present a rather concrete proof based on [Mi] in Section 6. For Proposition 8 see [KMW], and also [Z].

1.2. Some open problems. The behaviour of bridge numbers $b(k)$ under dominations is largely unknown with only partial results currently available [BNW]. For crossing number $c(k)$, a positive answer to the question of whether $k_1 > k_2$ implies that $c(k_1) > c(k_2)$ would provide an alternative proof of the fact that any knot 1-dominates at most finitely many knots [BRuW]. Relatedly, Kitano asked whether $c(k_1) \geq c(k_2)$ if there is an epimorphism $\pi_1 E(k_1) \to \pi_1 E(k_2)$, which would provide an alternate proof of Simon's conjecture [AL]). It would also support the additivity of crossing number under connected sum.

Some flexibility in the interpretation of "reduces complexity" notion is necessary. For instance, for Jones polynomials it is not true that $k_1 \geq k_2$ implies that $V_{k_1} | V k_2$ (see the remark after Example 4 in Section 6), but it is possible that $k_1 \geq k_2$ implies that the degree of $V(k_1)$ is $\geq$ that of $V(k_2)$.

More problems will be raised below.

1.3. Outline of the paper. In Section 2, we will prove some rigidity results about 1-domination between knots; that is, under certain conditions, $k \geq k'$ implies that $k = k'$. Some previously known conditions include:

(1) both k and k' are hyperbolic knots and have the same Gromov volume (Gromov–Thurston's rigidity theorem [Th]);
(2) k and k' have the same genus and k is fibred [BW1, Corollary 2.3].

Theorem 1 states that $k \geq k'$ implies $k = k'$ if k is a knot with no companion of winding number zero, and if k and k' have the same genus and the same Gromov volume. We also construct a strict 1-domination $k > k'$ such that both k and k' have same genus, and same Gromov volumes (and same Alexander polynomials) to show that Theorem 1 is a best rigidity result in terms of genus and Gromov volume. Other results in a similar spirit can be found in [BNW] and [De].

Section 3 is concerned with relations between domination and double branched coverings $M_2(k)$ of S^3 over knots k. We show that if $k \geq k'$, then

(1) $M_2(k_1) \geq M_2(k_2)$ (i.e., there is a degree 1 map $M_2(k_1) \to M_2(k_2)$);
(2) If $M_2(k_1) = M_2(k_2)$, then $k_1 = k_2$.

Assertion (2) can be thought of as an extension of the fact that there is no 1-domination between distinct mutant knots with hyperbolic 2-fold branched coverings [Ru]. We use it to show that knots 1-dominated by 2-bridge knots, respectively Montesinos knots, are 2-bridge, respectively Montesinos. (See [ORS], [Li], [BB], [BBRW] for other results on 1-domination between 2-bridge knots and Montesinos knots.) We also show in Section 4 that any knot 1-dominated by a toroidally alternating knot is a connected sum of simple knots. Assertion (1) suggests some interesting questions about the relations between 1-domination among knots, the theory of left orderable groups, and Heegaard-Floer L-spaces. See [BRoW], [BGW], [OS].

In Section 5, we study upper bounds on the length n of 1-domination sequences of knots $k_0 > k_1 > k_2 > \cdots > k_n$ with given k_0, which is closely related to rigidity results. It is known that any sequence of 1-dominations $M_0 > M_1 > \cdots > M_i > \cdots$ of compact orientable 3-manifolds has a finite length [Ro1], and there is an apriori bound on this length given M_0 [So2]. Theorem 5 states that if a knot k_0 is free (see Section 5 for definitions), then the length of any 1-domination sequence of knots $k_0 > k_1 > k_2 > \cdots > k_n$ is bounded by the maximal genus $\hat{g}(k_0)$ of an incompressible Seifert surface for k_0 when $\hat{g}(k_0)$ is bounded. We point out that alternating knots, fibred knots and small knots are free with bounded $\hat{g}(k_0)$. If k_0 is either fibred or 2-bridge, then $\hat{g}(k_0)$ is equal to the genus $g(k_0)$ of k_0. One-dominations between small knots, fibred knots, and two bridge knots have also been addressed in [BW2], [ORS], [BB], [BBRW].

In Section 6, we present a proof of $\Lambda_{k_1} = \Lambda_{k_2} \oplus \Lambda$ when $k_1 \geq k_2$ along with some applications. We also point out that Gordon's approach to ribbon concordance [Go], based on Stallings' results about homology and central series of groups [St], provides some other rigidity results for 1-domination of knots in terms of Alexander polynomials. Consequently, the length of a 1-domination sequence $k_0 > k_1 > k_2 > \cdots > k_n$ of alternating knots is bounded above by the degree of Δ_{k_0} when its leading coefficient is a prime power.

It is known that any knot 1-dominates at most finitely many knots [BRuW]. (See also the stronger results of [AL], [Liu].) It is very hard to bound the number of knots 1-dominated by a given knot in general. However, the techniques of this paper provide many knots which are minimal in the sense that they only 1-dominate the trivial knot and themselves.

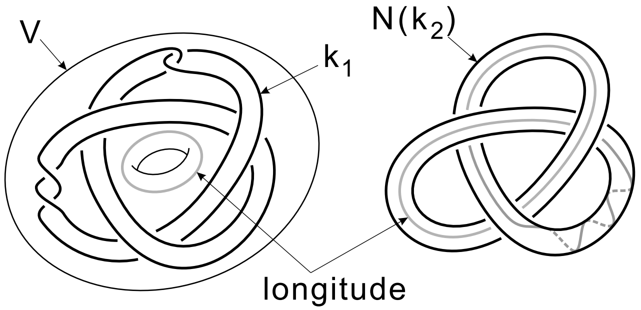

FIGURE 1. Constructing a satellite of the trefoil.

2. Rigidity via genus and Gromov volume

2.1. Satellite knots and an example. We recall the definition of satellite knots and fix some notation and terminology needed below.

Suppose that k_p is a knot contained in a solid torus V, where $V \subset S^3$ is unknotted and has longitude and meridian l, m. It is assumed that k_p does not lie in a 3-ball in V. Let k_c be another knot in S^3, with regular neighbourhood $N(k_c)$, and let $h : V \to N(k_c)$ be a homeomorphism, taking l and m respectively to the longitude and meridian of k_c. Then the knot $k_s := h(k_p)$ is called the *satellite* of k_c with *pattern* knot k_p, the latter considered in S^3. One also calls k_c a *companion* of k_s.

PROPOSITION 9. *Satellite knots* 1*-dominate their pattern knots.*

Proof. Suppose that k_s is a satellite of k_c with pattern k_p as described above. Arguing as in Proposition 1 there is a degree 1 map of the exterior of k_c to the exterior of V. Combining this with h^{-1} on the closure of $N(k_c) \setminus N(k_s)$ gives the 1-domination $k_s \geq k_p$. □

EXAMPLE 1. We construct a non-trivial 1-domination $k > k_1$ of knots with the same genus, the same Alexander polynomial, and the same Gromov volume. Moreover all those invariants are non-vanishing.

Let $k = h(k_1)$ be the satellite of the trefoil k_2 indicated by Figure 1. Here $h : V \to N(k_2)$ is a homeomorphism preserving the longitudes pictured; k itself is not drawn. Then we have a 1-domination $k \geq k_1$. Let $\mathcal{T}$ and $\mathcal{T}_1$ be the JSJ-tori of $E(k)$ and $E(k_1)$ respectively, then $E(k) \setminus \mathcal{T}$ consists of three components: two Seifert pieces and one hyperbolic piece H, which is homeomorphic to the Whitehead link complement; and $E(k_1) \setminus \mathcal{T}_1$ consists of two components: one Seifert piece and one hyperbolic piece H. Thus, $k > k_1$. On the other hand, it is clear that both k and k_1 are of genus 1, and have the same Gromov volume, which equals the hyperbolic volume of H. They also have the same Alexander polynomials, since h is longitude preserving (see [Rlf], Chapter 7) and k_1 is an untwisted double.

Remark 1. By iterating the construction in Example 1, one can provide an arbitrarily long 1-domination sequence of knots with the same genus, the same Alexander polynomial and the same Gromov volume.

Suppose k is a knot and T is an essential torus in $E(k)$. By a theorem of Alexander, T bounds a solid torus $V \cong S^1 \times D^2$ in S^3, and as T is incompressible in $E(k)$, we must have $k \subset V$. Thus, k represents some multiple of the generator of $\pi_1(V) \cong \mathbb{Z}$. We call the absolute value of this multiple the *winding number* of T relative to k. In this setting, the core curve of V is a companion of k.

The essential feature permitting the construction of satellites with the same genus, Alexander polynomial and Gromov volume is that the winding number of k in $N(k_2)$ is zero. This turns out to be necessary to the construction, as the following theorem demonstrates.

Theorem 1 (Rigidity). *Suppose that k is a non-trivial knot such that every essential torus in $E(k)$ has non-zero winding number. If k and k' have the same Gromov volume and the same genus, and $k \geq k'$, then $k = k'$.*

2.2. Proof of Theorem 1. We prove Theorem 1 by establishing a sequence of claims.

Claim 1. *Let $f : E(k) \to E(k')$ be a degree 1 map and let $(S, \partial S) \subset (E(k), \partial E(k))$ be a Seifert surface of minimal genus $g(k)$. Then the restriction $f|_* : \pi_1(S) \to \pi_1(E(k'))$ is injective.*

Proof. Otherwise there is an essential closed curve $c \subset S$ which is in the kernel of $f|_* : \pi_1(S) \to \pi_1(E(k'))$. Fix a finite covering $p : \tilde{S} \to S$ of degree d, say, so that c can be lifted to a simple closed curve $\tilde{c}$ in $\tilde{S}$ [Sc]. Since $f(S)$ carries a generator a of $H_2(E(k'), \partial E(k'); \mathbb{Z})$ and $g(k) = g(k') > 0$, the Thurston Norm of a is $|\chi(S)|$. It follows that $(f \circ p)(\tilde{S})$ carries da and realizes its Thurston norm, which is $|\chi(\tilde{S})| = d|\chi(S)|$. However since the simple essential closed curve $\tilde{c}$ lies in the kernel of $(f \circ p)_*$, we can perform surgery on $\tilde{S}$ along $\tilde{c}$ to produce a new surface $\tilde{S}^*$ and a map $g : \tilde{S}^* \to E(k')$ which also represents da. But then the singular Thurston norm of da is bounded above by $|\chi(\tilde{S}^*)|$, which is strictly less than $|\chi(\tilde{S})|$, contrary to Gabai's result that the Thurston norm and singular Thurston norm coincide. ☐

Claim 2. *If $T \subset E(k)$ is any essential torus, then the restriction $f|_* : \pi_1(T) \to \pi_1(E(k'))$ is injective.*

Proof. Let k_T be the companion of k such that $\partial E(k_T) = T$, and let (m_T, ℓ_T) be the meridian-longitude pair of k_T on $\partial E(k_T)$. If w_T denotes the winding number of T, one has $m_T = w_T m$ in $H_1(E(k); \mathbb{Z})$ and so for any integers p and q, $p\ell_T + qm_T = qw_T m$ in $H_1(E(k);\mathbb{Z})$. Since $f_* : H_1(E(k);\mathbb{Z}) \to H_1(E(k');\mathbb{Z})$ is an isomorphism given by $f_*(m) = m'$ and $\pi_1(E(k'))$ is torsion free, it follows that if the kernel of $f|_* : \pi_1(T) \to \pi_1(E(k'))$ is non-trivial, then it is generated by the longitude ℓ_T on T. As argued by Schubert, any minimal Seifert surface S for k may be assumed to intersect T in w_T longitudes. Since by hypothesis $w_T \neq 0$, we may assume $\ell_T \subset S$ and represents a nontrivial element of $\pi_1(S)$. But $f|_* : \pi_1(S) \to \pi_1(E(k'))$ is injective by Claim 1, so $f_*(\ell_T) \neq 1$. Claim 2 is proved. □

Claim 3. *If $N \subset E(k)$ is a Seifert piece of the JSJ-decomposition of $E(k)$, then the restriction $f|_* : \pi_1(N) \to \pi_1(E(k'))$ is injective.*

Proof. It follows from Seifert's classification of Seifert fibre structures on S^3 that N is either a torus knot exterior, a cable space, or a composing space (the product of a planar surface and a circle) with at least three boundary components (see Lemma VI.3.4 of [JS]). In particular, its base orbifold is orientable and therefore N admits no separating, horizontal surfaces.

Let $T \subseteq \partial N$ be either $\partial E(k)$ or the torus which separates N from $\partial E(k)$ and fix a minimal genus Seifert surface S for k. Assume that S has been isotoped to intersect ∂N minimally and recall from the proof of the previous claim that $S \cap T$ consists of $w_T > 0$ copies of the longitude ℓ_T. Fix a component S_0 of $S \cap N$ such that $S_0 \cap T \neq \emptyset$. Clearly S_0 is an essential surface in N and so can be assumed to be either vertical or horizontal with respect to a fixed Seifert structure on N. Now ℓ_T cannot be isotopic in T to a Seifert fibre of N (this can verified for each of the three types of possibilities for N), so S_0 is horizontal and therefore non-separating in N. It follows that N fibres over the circle with fibre S_0. By Claim 1, $f_*|\pi_1(S_0)$ is injective and so $f_*(\pi_1(S_0))$ is a non-Abelian free group. (It follows from the previous paragraph that $\chi(S_0) < 0$.)

Recall that the class ϕ of a regular fibre of N is central in $\pi_1(N)$ and let H be the group generated by ϕ and $\pi_1(S_0)$. Then H has finite index in $\pi_1(N)$ and since the latter is torsion free, it suffices to show that $f_*|H$ is injective. An element of H can be written $\gamma\phi^n$ for some $\gamma \in \pi_1(S_0)$ and $n \in \mathbb{Z}$. Thus if $f_*(\gamma\phi^n) = 1$, then $f_*(\phi)^n = 1$ since it is a central element of the non-abelian free group $f_*(\pi_1(S_0))$. But $f_*(\phi) \neq 1$ by Claim 2, and since $\pi_1(E(k'))$ is torsion free we see that $n = 0$. Then $f_*(\gamma) = 1$ so that $\gamma\phi^n = \gamma = 1$. Thus the claim holds. □

Let $E(k) = H_k \cup S_k$ and $E(k') = H_{k'} \cup S_{k'}$ where $H_k, H_{k'}$ and $S_k, S_{k'}$ are the unions of the hyperbolic and Seifert pieces of $E(k)$ and $E(k')$.

Claim 4. *The map f can be homotoped so that*:

(1) $f| : (H_k, \partial H_k) \to (H_{k'}, \partial H_{k'})$ *is a homeomorphism.*
(2) $f(S_k) = S_{k'}$.

Proof. Define Σ_k to be the union of S_k and regular neighbourhoods of the characteristic tori connecting two hyperbolic pieces in the JSJ decomposition of $E(k)$. Define $\Sigma_{k'}$ similarly. By Claim 3 and the enclosing property of characteristic submanifold theory [JS], we may homotope $f|\Sigma_k$ into $\Sigma_{k'}$. If $\partial E(k) \subset \Sigma_k$ we may suppose that the homotopy leaves $f|\partial E(k)$ invariant. Extend this homotopy to a homotopy of f supported in a regular neighbourhood of Σ_k. Since the Gromov norms of $E(k)$ and $E(k')$ are the same, by Soma's result [So1], one can further modify f by a homotopy fixed on S_k so that $f|H_k$ is a homeomorphism $(H_k, \partial H_k) \to (H_{k'}, \partial H_{k'})$. Then $f^{-1}(S_{k'}) \subset S_k$ and since f is surjective we have $f(S_k) = S_{k'}$. □

Claim 5. *Distinct neighbouring Seifert pieces of $E(k)$ are sent to distinct neighbouring Seifert pieces of $E(k')$ by f. Further, if $N' \subset S_{k'}$ is a Seifert piece of $E(k')$, then $f^{-1}(N')$ is a Seifert piece of $E(k)$.*

Proof. Suppose that there are distinct but non-disjoint Seifert pieces N_1, N_2 of $E(k)$ which are sent into N' by f. Since S^3 is simply-connected, $N_1 \cap N_2$ is a torus T and if $\phi_1, \phi_2 \in \pi_1(T) \cong \mathbb{Z}^2$ represent the fibre classes of N_1, N_2 respectively, they generate a $\mathbb{Z}^2$ subgroup of $\pi_1(T)$. Claim 2 shows the latter statement also holds for $f_*(\phi_1), f_*(\phi_2)$. On the other hand, Claim 3 implies that $f_*(\phi_j)$ has a non-Abelian centralizer in $\pi_1(E(k'))$ and so Addendum VI.1.8 of Theorem VI.1.6 in [JS] implies that $f_*(\phi_j)$ is a power of the fibre class of N' $(j = 1, 2)$. But then $f_*(\phi_1)$ and $f_*(\phi_2)$ lie in a $\mathbb{Z}$ subgroup of $\pi_1(E(k'))$, which we have seen is impossible. Thus distinct neighbouring Seifert pieces of $E(k)$ are sent to distinct neighbouring Seifert pieces of $E(k')$ by f.

The dual graph $\Gamma(k)$ to the JSJ-decomposition of $E(k)$ is a rooted tree where the root vertex v_0 corresponds to the vertex manifold containing $\partial E(k)$. For each vertex v of $\Gamma(k)$, we use X_v to denote the corresponding vertex manifold. Define $\Gamma(k'), v_0'$, and $X_{v'}$ similarly. Since S^3 is simply-connected, both $\Gamma(k)$ and $\Gamma(k')$ are trees. Hence, Claim 4 and the conclusion of the previous paragraph imply that f induces an isomorphism between these trees, which proves the claim. □

Claims 4 and 5 imply that for each piece X' of $E(k')$, there is a unique piece X of $E(k)$ such that $f : (X, \partial X) \to (X', \partial X')$. If v is the vertex of $\Gamma(k)$ corresponding to X, we let $f(v)$ denote the vertex of $\Gamma(k')$ corresponding to X'.

Theorem 1 is a consequence of our final claim:

Claim 6. *The map f can be homotoped to a homeomorphism.*

Proof. Given the conclusions of Claims 2, 3 and 4, classic work of Waldhausen shows that the restriction $f| : (X_{v_0}, \partial X_{v_0}) \to (X'_{v_0'}, \partial X'_{v_0'})$ is homotopic (rel $\partial E(k)$) to a covering map (see Theorem 13.6 of [He]). Since $f|\partial E(k)$ has degree 1, $f| : (X_{v_0}, \partial X_{v_0}) \to (X'_{v'_0}, \partial X'_{v'_0})$ can be homotoped to a homeomorphism. (This is automatic of course if X_{v_0} is hyperbolic.) In particular, $|\partial X_{v_0}| = |\partial X'_{v'_0}|$.

Now suppose that the vertices of $\Gamma(k)$ (respectively, $\Gamma(k')$) adjacent to v_0 (respectively, v'_0) are $v_1, \dots, v_p$ (respectively, $v'_1, \dots, v'_p$). Let T_i be the torus $X_{v_0} \cap X_{v_i}$, $1 \le i \le p$ and T'_i its image by f. By Claim 5, f cannot send X_{v_i} to $X'_{v'_0}$, $i \ne 0$, so we may assume that $f(X_i) \subset X'_{v'_i}$ for $i = 1, \dots, p$. Since $f| : T_i \to T'_i$ is a homeomorphism, the argument of the previous paragraph shows that for each i, $f : X_{v_i} \to X'_{v'_i}$ is homotopic to a homeomorphism (rel T_i). Proceeding by induction, we see that for each vertex v of $\Gamma(k)$, $f| : (X_v, \partial X_v) \to (X'_{f(v)}, \partial X'_{f(v)})$ is a homeomorphism. Since f induces an isomorphism $\Gamma(k) \to \Gamma(k')$, the proof of the claim is complete. □

3. Double branched covers

Let $M_q(k)$ denote the q-fold cyclic branched covering of S^3 over the knot k.

THEOREM 2. *Suppose $k \ge k'$. Then*

(1) *$M_2(k) \ge M_2(k')$, that is, a degree 1 map $M_2(k) \to M_2(k')$ exists; and*
(2) *$M_2(k) = M_2(k') \implies k = k'$.*

Proof. Suppose that there is a degree 1 map $f : E(k_1) \to E(k_2)$. We may assume that $f| : \partial E(k_1) \to \partial E(k_2)$ is a homeomorphism which sends m_1 to m_2.

(1) Pick a Seifert surface F' of k'. We may assume that f has been homotoped relatively to the boundary to be transverse to F' and so that $F = f^{-1}(F')$ is connected. Then F is a Seifert surface of k. Then f restricts to a proper degree 1 map between $E(k)$ cut open along F, which we denote by $E(k) \setminus F$, and $E(k') \setminus F'$. This restriction can be assumed to be a homeomorphism between the two copies of F in $\partial(E(k) \setminus F)$ to the two copies of F' in $\partial(E(k') \setminus F')$. By gluing two copies of $E(k) \setminus F$, respectively $E(k') \setminus F'$, along these copies of F, respectively of F', we obtain a degree 1 map from the 2-fold covering of $E(k)$ to the 2-fold cyclic covering of $E(k')$, which extends to a degree 1 map $\hat{f}$ from $M_2(k)$ to $M_2(k')$. In other words, the degree 1 map $f : E(k_1) \to E(k_2)$ lifts to a degree 1 map between the 2-fold coverings of the knot exteriors, which extends to a degree 1 map $\hat{f} : M_2(k) \to M_2(k')$.

(2) A degree 1 map, f induces an epimorphism $f_* : \pi_1 E(k_1) \to \pi_1 E(k_2)$ such that $f_*(m_1) = m_2$), hence it induces an epimorphism

$$\bar{f}_* : \pi_1 E(k_1)/m_1^2 \to \pi_1 E(k_2)/m_2^2$$

and we have the commutative diagram

$$\begin{tikzcd} 1 \arrow[r] & \pi_1\big(M_2(k)\big) \arrow[r] \arrow[d, "\hat{f}_*"'] & \pi_1\big(E(k)/m_1^2\big) \arrow[r] \arrow[d, "\bar{f}_*"'] & \mathbb{Z}_2 \arrow[r] \arrow[d, "\cong"'] & 1 \\ 1 \arrow[r] & \pi_1\big(M_2(k')\big) \arrow[r] & \pi_1\big(E(k')/m_2^2\big) \arrow[r] & \mathbb{Z}_2 \arrow[r] & 1 \end{tikzcd}$$

Since $\hat{f}_*$ is induced by a degree 1 map, it is surjective. Therefore, $\hat{f}_*$ is an isomorphism, because $M_2(k) = M_2(k')$ and 3-manifold groups are Hopfian. By the Five Lemma, $\bar{f}_*$ is also an isomorphism. Assertion (2) follows now from the geometrization of 3-orbifolds with singular locus a link [BP], a theorem of Boileau–Zimmermann about π-orbifolds groups [BZ] when $\pi_1(M_2(k))$ is infinite and the classification of spherical Montesinos knots when $\pi_1(M_2(k))$ is finite. □

Mutant knots have the same double branched covering, so Proposition 2 implies the following.

Corollary 1. *There is no* 1*-domination between distinct mutant knots.*

Ruberman has shown that if k, k' are mutants, then $E(k)$ is hyperbolic if and only if $E(k')$ is, and in this case, both have the same volume [Ru]. Hence, in this situation, Corollary 1 follows from the Gromov–Thurston rigidity theorem.

Assertion (1) of Theorem 2 provides a connection between 1-domination among knots, left orderable groups and L-spaces.

Definition 1. (1) A group is *left-orderable* if there is a total ordering $<$ of its elements which is left-invariant: $x < y$ if and only if $zx < zy$ for all x, y and z.

(2) An *L-space* is a closed rational homology 3-sphere whose Heegaard–Floer homology $\widehat{HF}(M)$ is a free Abelian group of rank equal to $|H_1(M, \mathbb{Z})|$.

Proposition 10 ([BRoW]). *Suppose G and G' are nontrivial fundamental groups of irreducible* 3*-manifolds and there is a surjection $G \to G'$. If G' is left orderable, then G is left orderable.*

Corollary 2. *If $\pi_1 M_2(k_1)$ is not left orderable but $\pi_1 M_2(k_2)$ is, then k_1 does not* 1*-dominate k_2.*

The left orderabilty of $\pi_1 M_2(k)$ can be determined for certain family of knots. For instance, Boyer–Gordon–Watson showed that this is never the case for non-trivial alternating knots k [BGW]. For each Montesinos knot k, $M_2(k)$ is a Seifert manifold, and work of Boyer–Rolfsen–Wiest [BRoW] combines with that of Jankins–Neumann [JN] and Naimi [Na] to determine exactly when such manifolds have left orderable fundamental groups in terms of the

Seifert invariants. As a consequence, alternating knots cannot 1-dominate certain classes of Montesinos knots.

Another result, due to Ozsvath–Szabo, states that $M_2(k)$ is an L-space for each alternating knot k [OS]. This and other evidence corroborates the following conjecture in [BGW], which is unsolved at this writing.

Conjecture 1. An irreducible 3-manifold which is a rational homology sphere is an L-space if and only if its fundamental group is not left orderable.

Ozsváth–Szabó have conjectured that an irreducible $\mathbb{Z}$-homology 3-sphere is an L-space if and only if it is the 3-sphere or the Poincaré homology sphere (cf. [Sz, Problem 11.4 and the remarks which follow it]). This combines with Conjecture 1 to yield the following conjecture: *An irreducble $\mathbb{Z}$-homology* 3-*sphere other than S^3 and the Poincaré homology sphere has a left-orderable fundamental group.*

Recall that the *determinant* of a knot k is given by $|\Delta_k(-1)|$ and coincides with $|H_1(M_2(k), \mathbb{Z})|$. Thus $M_2(k)$ is a $\mathbb{Z}$-homology 3-sphere if and only if the determinant of k is 1. The discussion above leads to the following question, whose expected answer is no.

Question 1. Suppose that k is alternating. Can k 1-dominate a nontrivial knot k' with $|\Delta_{k'}(-1)| = 1$? In particular a nontrivial knot with trivial Alexander polynomial?

Here is a related question.

Question 2. Suppose that k is alternating and $k \geq k'$. Does $|\Delta_k(-1)| = |\Delta_{k'}(-1)|$ imply that $k = k'$?

To state our next results, we need to recall some definitions: a *2-string tangle* is the 3-ball B^3 with two disjoint properly embedded arcs $a_1 \cup a_2$. A *trivial tangle* is a 2-string tangle where the arcs a_1 and a_2 bound disjoint disks together with arcs on the boundary of B^3. A *rational tangle* is the image of a trivial tangle by a homeomorphism of the ball fixing the end points of the arcs a_1 and a_2: a tangle is rational if and only if the 2-fold covering of B^3 branched along the arcs $a_1 \cup a_2$ is a solid torus $S^1 \times D^2$. A well-known fact, using this double branched covering, is that rational tangles correspond to rational numbers, called the *slopes* of the rational tangles: a rational tangle $T(r)$ corresponding to the rational number r is obtained by first drawing two strings of slope r on the boundary $S^2(2, 2, 2, 2)$ of the pillow-case B, then pushing into its interior. A *Montesinos tangle* is a tangle sum of rational tangles $T(r_1), \ldots, T(r_n)$: by adding two arcs on the boundary of B, we get the so-called 2-bridge knots and Montesinos knots. The double branched cover of those knots are, respectively, lens spaces and Seifert manifolds. Further, the action of the covering involution τ preserves the Seifert fibre of the 2-fold covering and reverses its orientation. The converse is true by the orbifold theorem [BP], see also [BS]: If $M_2(k)$ is, respectively, a lens space or a Seifert fibered manifold and τ reverses the orientation of the Seifert fibre, then k is, respectively, a 2-bridge knot or a Montesinos knot.

PROPOSITION 11. *Suppose $k \geq k'$.*

(1) *If k is a 2-bridge knot, so is k'.*
(2) *If k is a Montesinos knot, so is k'.*

Proof. Let $f : E(k) \to E(k')$ be a 1-domination and τ, τ' are the covering involutions of the 2-fold branched coverings $M_2(k)$ and $M_2(k')$. Then we have a $\mathbb{Z}_2$ equivalent degree 1 map $\tilde{f} : M_2(k) \to M_2(k')$, i.e. $\tau \circ \tilde{f} = \tilde{f} \circ \tau'$ and a surjection $\tilde{f}_* : \pi_1 M_2(k) \to \pi_1 M_2(k')$.

If k is a 2-bridge knot, $M_2(k)$ is a lens space and therefore $\pi_1(M_2(k))$ is a finite cyclic group. Hence, $\pi_1(M_2(k'))$ is finite cyclic so by the orbifol theorem [BP], $M_2(k')$ is a lens space and k' is 2-bridge. Next, suppose that k is a Montesinos knot, so that $M_2(k)$ is an irreducible Seifert manifold. Again it is known that k' is Montesinos if $\pi_1 M_2(k')$ is finite by the orbifold theorem [BP], so suppose otherwise. Then $\pi_1 M_2(k)$ is infinite and non-cyclic, so $M_2(k)$ is a $K(\pi_1 M_2(k), 1)$ space, finitely covered by a circle bundle W over a closed orientable surface F where the circle fibres of W are the inverse image of the Seifert fibres of $M_2(k)$. The class h of a regular fibre of $M_2(k)$ cannot be contained in the kernel of $\tilde{f}_*$ as otherwise the composition $W \to M_2(k) \to M_2(k')$ would factor through F, which is impossible for a non-zero degree map.

Suppose that $M_2(k')$ is reducible and let S' be an essential 2-sphere it contains. After a homotopy of $\tilde{f}$ we can suppose that the preimage S of S' in $M_2(k)$ is an essential surface. Now S cannot be vertical as this would imply that the h would be contained in the kernel of $\tilde{f}_*$. On the other hand it cannot be horizontal in $M_2(k)$ since this would imply that the odd-order Abelian group $H_1(M_2(k); \mathbb{Z})$ has a $\mathbb{Z}_2$ quotient. Thus, $M_2(k')$ is irreducible. It follows that $\pi_1 M_2(k')$ is torsion-free and therefore $\tilde{f}_*(h)$ has infinite order. But then $\pi_1 M_2(k')$ has a non-trivial centre containing $\tilde{f}_*(h)$, so $M_2(k')$ is a Seifert manifold by [CJ], [Ga2] with $\tilde{f}_*(h)$ a non-trivial power of the fibre class. It follows that k' is either a Montesinos knot or a torus knot. The former case happens if τ' reverses orientation of each fibre, and the latter case otherwise. Since $\tau'_*(\tilde{f}_*(h)) = \tilde{f}_*(\tau_*(h)) = \tilde{f}_*(h^{-1}) = \tilde{f}_*(h)^{-1}$, k' is a Montesinos knot. □

REMARK 2. Many Seifert manifolds are known to be minimal (see [HWZ] for example) from which we can deduce that many Montesinos knots are minimal.

4. AP-property and 1-domination

A knot k has the *AP-property* if every closed incompressible surface embedded in its complement carries an essential closed curve homotopic to a peripheral element. (Here *AP* refers to *accidental parabolic*.) Small knots (i.e., knots whose exteriors contain no closed essential surfaces) are AP knots, but so are *toroidally alternating knots* [Ad], a large class which contains, for instance, all hyperbolic knots which are alternating, almost alternating, or Montesinos.

A knot k is *simple* if it is either a hyperbolic or a torus knot. This condition is equivalent to the requirement that $E(k)$ contain no essential tori.

PROPOSITION 12. *Let k be a knot with the AP-property. Then k can dominate only a connected sum of simple knots or a cable of a simple knot.*

Proof. Suppose that k dominates a satellite knot k' and let T' be a JSJ torus in $E(k')$ which bounds a simple knot exterior (i.e. is innermost). Fix a degree 1 map $f : E(k) \to E(k')$ which is transverse to T'. Since $E(k')$ is irreducible, f can be homotoped so that each component F of $f^{-1}(T')$ is incompressible in $E(k)$. The AP-property implies that some essential closed curve $\gamma \subset F$ is freely homotopic to an essential closed curve $\alpha \subset \partial E(k)$. Up to replacing F by another component of $f^{-1}(T')$, we can assume that the homotopy takes place in the outermost component of $\overline{E(k) \setminus f^{-1}(T')}$ (i.e., the component which contains $\partial E(k)$). Applying f we obtain a homotopy in W, the outermost component of $\overline{E(k') \setminus T'}$. Since the restriction $f| : \partial E(k) \to \partial E(k')$ is a homeomorphism, $f(\alpha)$ is an essential closed curve on $\partial E(k')$, and therefore the annulus theorem [JS] provides an essential annulus A properly embedded in W and cobounded by essential simple closed curves on T' and $\partial E(k')$. We can assume that A intersects the JSJ tori of $E(k')$ transversely and minimally. Then it intersects each of the JSJ pieces it passes through in essential, properly embedded annuli. It follows that these pieces are Seifert fibred. Further, since S^3 contains no Klein bottles, the annuli are vertical in their respective pieces, so their Seifert structures match up. Thus, W is Seifert fibred and hence a piece of $E(k')$. Since T' was an arbitrary innermost JSJ torus of $E(k')$, the result follows. $\square$

COROLLARY 3. *A toroidally alternating knot can dominate only a connected sum of simple knots. In particular this holds for alternating knots.*

Proof. Let k be a toroidally alternating knot which 1-dominates a knot k', say $f : E(k) \to E(k')$ is a degree 1 map. We have assumed that f restricts to a homeomorphism $\partial E(k) \to \partial E(k')$ and hence sends a meridian of k to an essential simple closed curve on $\partial E(k')$ which normally generates $\pi_1 E(k')$. Thus, the Property P conjecture [KM] shows that f sends the meridian of k to the meridian of k'. By Proposition 12, k' is either a sum of simple knots or a cable of a simple knot. Let W denote the outermost JSJ piece of $E(k')$.

Each closed incompressible surface embedded in $E(k)$ carries an essential simple closed curve homotopic to a meridian of k by [Ad, Corollary 3.3], so the proof of Proposition 12 shows that there is a homotopy in W between a meridian of k' and an essential loop on an innermost JSJ torus of $E(k')$. But this never occurs when W is a cable space, so k' is a product of simple knots. □

The referee points out that the number of factors in such an image connected sum of simple knots is clearly bounded by the number of disjoint non-parallel meridianal incompressible surfaces for the domain toroidally alternating knot. This follows by surgering the pullback of the annuli in a connected sum of knots.

For Montesinos knots Corollary 3 follows from Proposition 11, since a Montesinos knot is simple by [Oe].

5. Length of 1-domination sequences via genus

DEFINITION 2. (1) We recall that a knot is *small* if each incompressible closed surface in its exterior is boundary parallel.

(2) A Seifert surface S of a knot k is *free* if $\overline{E(k)\setminus T(S)}$ is a handlebody where $T(S)$ is a tubular neighbourhood of S. A knot k is *free*, if all its incompressible Seifert surfaces are free. For example, a small knot is free.

(3) Define $\hat{g}(k) = \sup\{g(S)|S$ is an incompressible Seifert surfaces for $k\}$. Here $g(S)$ denotes the genus of the surface S.

Classic pretzel knots with three branches are small [Oe] as are 2-bridge knots [HT]. Fibered knots are free, which follows directly from the classical result that each incompressible Seifert surface of a fibred knot is isotopic to its fibre surface. In particular, $\hat{g}(k) = g(k)$ for fibred knots. If k has a companion of winding number zero, then k is not free since there is an incompressible Seifert surface for k which is contained in the complement of the companion torus.

Clearly $g(k) \leq \hat{g}(k) \leq \infty$ and it is possible that $\hat{g}(k) = \infty$ (see [Ly]). Small knots satisfy $\hat{g}(k) < \infty$ by [Wi]. Non-fibred examples of knots for which $\hat{g}(k) = g(k)$ include non-fibred 2-bridge knots, ([HT]).

QUESTION 3. Which knots k in S^3 have bounded $\hat{g}(k)$ and which are free?

Here is a construction which produces several interesting classes of free knots.

PROPOSITION 13. *A knot k with the property that each closed, essential surface in $E(k)$ contains a loop which links k homologically a non-zero number of times is free.*

Proof. Suppose that k is not free and choose an incompressible Seifert surface for k whose complement is not a handlebody. Set $H = \overline{E(k)\setminus T(S)}$ and consider a maximal compression body P for ∂H in H. There is a decomposition

$$H = P \cup V = (\partial H \times I) \cup 2\text{-handles} \cup V,$$

where V is a not necessarily connected, compact 3-manifold. Since P is maximal, ∂V either is a finite union of 2-spheres or has a component which is incompressible in V. In the former case, the incompressibility of S and irreducibility of $E(k)$ implies that V is a finite union of 3-balls. But then H is a handlebody, contrary to our assumptions. Thus there is a component F of ∂V which is incompressible in H and therefore in $H = V \cup (\text{1-handles})$ and $E(k) = H \cup (S \times I)$. Since S is contained in $\overline{E(k) \setminus V}$ and is not ∂-parallel in $E(k)$, F is essential in $E(k)$. By construction, $F \cap S = \emptyset$ and so every loop on F links k zero times, contrary to our hypotheses. Thus, k must be free. □

Corollary 4. *Small knots, alternating knots, and Montesinos knots are free.*

Proof. Any such knot satisfies the hypotheses of Proposition 13 and so is free. This is obvious for small knots. On the other hand, a closed essential surface in the exterior of either an alternating knot or a Montesinos knot k contains a simple closed curve which is homotopic in $E(k)$ to a meridian of k [Me], [Oe], which implies the claim. □

Proposition 14. *Suppose that $k \geq k'$.*

(1) *If k is free, then k' is free.*
(2) *If k is free and $f: E(k) \to E(k')$ is a degree 1 map such that $g(S) = g(S')$ where S' and $S = f^{-1}(S')$ are incompressible Seifert surfaces for k', k, then $k = k'$.*
(3) *$\hat{g}(k) \geq \hat{g}(k')$, and if k is free with bounded $\hat{g}(k)$, then $\hat{g}(k) = \hat{g}(k')$ if and only if $k = k'$.*

Proof. (1) Let S' be an incompressible Seifert surface of k' with genus $g(k')$, and let $f: E(k) \to E(k')$ be a degree 1 map, transverse to S', which realizes the 1-domination $k \geq k'$. Since $E(k')$ is irreducible, f can be homotoped so that each component of $f^{-1}(S')$ is incompressible. Further, since f has degree 1, exactly one component S of $f^{-1}(S')$ is a Seifert surface of k and the remaining components are closed. Since k is a free knot, it follows that $S = f^{-1}(S')$.

Let $T(S) \subset E(k)$ be a tubular neighbourhood of S. Then f induces a proper degree 1 map

$$f| : H = \overline{E(k) \setminus T(S)} \to \overline{E\big(k'\big) \setminus T\big(S'\big)} = H'.$$

Consider a maximal compression body P' for $\partial H'$ in H'. There is a decomposition

$$H' = P' \cup V' = \big(\partial H' \times I\big) \cup \text{2-handles} \cup V',$$

where V' is a not necessarily connected, compact 3-manifold. Since P' is maximal, $\partial V'$ either has a component F' which is incompressible in V' or is a finite union of 2-spheres. In the former case, H' contains a closed incompressible surface F', and $f|H$ could be homotoped rel ∂H to a function g such that $g^{-1}(F)$ is a closed and essential in H, contrary to the hypothesis that H is a handlebody. Hence, the latter case arises and the incompressibility of S' and irreducibility of $E(k')$ implies that V' is a finite union of 3-balls. This shows that H' is a handlebody, and completes the proof of (1).

(2) By hypothesis, $f| : S \to S'$ is a proper degree 1 map between homeomorphic surfaces, and as such, homotopic to a homeomorphism. Thus, after a homotopy, f induces a proper degree 1 map

$$h = f| : H = \overline{E(k) \setminus T(S)} \to \overline{E\big(k'\big) \setminus T\big(S'\big)} = H',$$

where H and H' are handlebodies of genus $2g(S)$, such that $h| : \partial H \xrightarrow{\cong} \partial H'$ is a homeomorphism. The latter implies that $h_* : \pi_1(H) \to \pi_1(H')$ is surjective, and as $\pi_1(H) \cong \pi_1(H')$ are free, and therefore Hopfian, h_* is an isomorphism. Now apply Waldhausen's result (Theorem 13.6 of [He]) to conclude that h is homotopic rel ∂H to a homeomorphism. Consequently, the same conclusion holds for $f : E(k) \to E(k')$. Thus $k = k'$.

(3) The inequality $\hat{g}(k) \geq \hat{g}(k')$ follows from the equality of immersed and embedded genus (Corollary 6.18, [Ga]). Suppose then that k is free and $\hat{g}(k) = \hat{g}(k') < \infty$. Fix a proper degree 1 map $f : E(k) \to E(k')$ and an incompressible Seifert surface S' for k' with $g(S') = \hat{g}(k')$. The proof of part (1) shows that we can find an incompressible Seifert surface $S \subset E(k)$ for k and homotope f to be transverse to S and satisfy $S = f^{-1}(S')$. Then f induces a proper degree 1 map $S \to S'$ and hence, $\hat{g}(k) \geq g(S) \geq g(S') = \hat{g}(k') = \hat{g}(k)$. Thus, (2) implies that $k = k'$. □

An immediate consequence is the following corollary.

Corollary 5. *Suppose k_0 is a free knot and $\hat{g}(k_0)$ is bounded. Then for any 1-domination sequence $k_0 > k_1 > \cdots > k_n$, $n + \hat{g}(k_n) \leq \hat{g}(k_0)$. In particular the length of the sequence is at most $\hat{g}(k_0)$.*

6. Alexander invariant

The proof of the following proposition is styled on classic arguments [Br].

Proposition 15. *If $k_1 \geq k_2$, then $\Lambda_{k_1} = \Lambda_{k_2} \oplus \Lambda$ where Λ is a $\mathbb{Z}[t^{\pm 1}]$-module. In particular, Δ_{k_2} divides Δ_{k_1}.*

Proof. Let $\tilde{E}(k_i)$ be the infinite cyclic covering of $E(k_i)$ and t_i be the generator of the deck transformation group of the infinite cyclic covering. Then $f : E(k_1) \to E(k_2)$ lifts to a proper degree 1 map $\tilde{f} : \tilde{E}(k_1) \to \tilde{E}(k_2)$. We have induced homomorphisms $\tilde{f}_* : H_1(\tilde{E}(k_1); \mathbb{Q}) \to H_1(\tilde{E}(k_2); \mathbb{Q})$ and $\tilde{f}^* : H^1(\tilde{E}(k_2); \mathbb{Q}) \to H^1(\tilde{E}(k_1); \mathbb{Q})$.

Since knot complements have the homology of the circle, Assertion 5 of [Mi] shows that $H_*(\tilde{E}(k_1); \mathbb{Q})$ is finitely dimensional over $\mathbb{Q}$.

For each i, let u_i be the fundamental class of $H_2(E(k_i), \partial E(k_i); \mathbb{Q})$. There is a duality isomorphism $P_i = u_i \cap : H^1(\tilde{E}(k_i); \mathbb{Q}) \to H_1(\tilde{E}(k_i); \mathbb{Q})$, see [Mi, Assertion 9 and Section 4].

Let $\alpha : H_1(\tilde{E}(k_2); \mathbb{Q}) \to H_1(\tilde{E}(k_1); \mathbb{Q})$ be given by $\alpha(x) = u_1 \cap \tilde{f}^*(P_2^{-1}(x))$ for each $x \in H_1(\tilde{E}(k_2); \mathbb{Q})$. Then

$$\tilde{f}_*\alpha(x) = \tilde{f}_*(u_1) \cap \tilde{f}^*\big(P_2^{-1}(x)\big) = \tilde{f}^*(u_1) \cap \big(P_2^{-1}(x)\big) = u_2 \cap \big(P_2^{-1}(x)\big) = x,$$

Thus, $\tilde{f}_*\alpha$ is the identity on $H_1(\tilde{E}(k_2); \mathbb{Q})$. It follows that

$$H_1\big(\tilde{E}(k_1); \mathbb{Q}\big) \cong H_1\big(\tilde{E}(k_2); \mathbb{Q}\big) \oplus \ker \tilde{f}_*.$$

Next, we prove that an analogous splitting holds over $\mathbb{Z}$.

Since $H_1(\tilde{E}(k_i); \mathbb{Z})$ is torsion free, there is an inclusion $\tau_* : H_1(\tilde{E}(k_i); \mathbb{Z}) \to H_1(\tilde{E}(k_i); \mathbb{Q})$, and since both $\tilde{f}_*$ and α preserve integer homology, the restriction $\tilde{f}_*\alpha | H_1(\tilde{E}_2; \mathbb{Z})$ is the identity. It follows that

$$H_1\big(\tilde{E}(k_1); \mathbb{Z}\big) \cong H_1\big(\tilde{E}(k_2); \mathbb{Z}\big) \oplus \ker \tilde{f}_*|.$$

It is also easy to see that $\tilde{f}_* t_1 = t_2 \tilde{f}_*$ and $\alpha t_2 = t_1 \alpha$. Hence, the splitting above gives the desired splitting of $\mathbb{Z}[t^{\pm 1}]$ modules. $\square$

An immediate consequence of Proposition 15 is that $k_1 \geq k_2$ implies that Δ_{k_2} divides Δ_{k_1}. This follows from the fact that if k is a knot, then $H_1(\tilde{E}(k); \mathbb{Q}) \cong \Gamma/(p_1(t)) \oplus \cdots \oplus \Gamma/(p_n(t))$ where $p_1(t), \dots, p_n(t) \in \Gamma = \mathbb{Q}[t^{\pm 1}]$ and $p_1(t) \cdots p_n(t) = \Delta(k)$. Thus, if Δ_{k_1} and Δ_{k_2} have the same degree, then $\Delta_{k_1} = \pm \Delta_{k_2}$.

One might hope to use band-connected sum and Murasugi sum to produce examples of 1-dominance: see [Ka2] for definitions. The following direct application of Proposition 15 shows that this fails in general.

EXAMPLE 2. Figure 2 is a band connected sum k of the trefoil knot 3_1 and the trivial knot with $\Delta_k(t) = 1 - t^2 + t^4$, which does not have $\Delta_{3_1}(t) = 1 - t + t^2$ as a factor. It follows that band connected sum does not 1-dominate its factors in general.

EXAMPLE 3. Figure 3 is a Murasugi sum k of 5_2 and 4_1 with $\Delta_k(t) = 2 - 3t + 3t^2 - 3t^3 + 2t^4$, which contain neither $\Delta_{4_1}(t) = 1 - 3t + t^2$ nor $\Delta_{5_2}(t) = 2 - 3t + 2t^2$ as a factor. It follows that Murasugi sum does not, in general, 1-dominate its factors.

Referring to the definition of satellite knots in Section 2.1, if the winding number of k_p in V is ± 1, then there is a proper degree 1 map $g : V \setminus N(k_p) \to S^1 \times S^1 \times [0,1]$ which is a homeomorphism on the boundaries (see [Du]). This provides a 1-domination $k_s \geq k_c$. The next example shows that $k_s \geq k_c$ need not hold without the assumption of winding number ± 1, as Δ_{k_c} does not divide Δ_{k_s}.

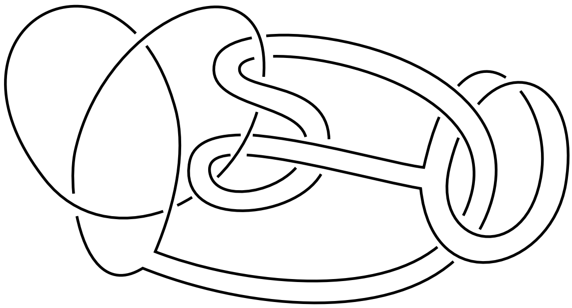

FIGURE 2. A band sum of a trefoil and trivial knot.

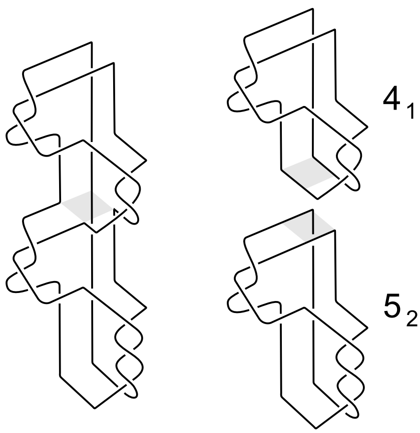

FIGURE 3. A Murasugi sum of 4_1 and 5_2.

EXAMPLE 4. Let k_s be the $(2,3)$ cable of the figure-eight knot $k_c = 4_1$. That is, k_s is the satellite of k_c defined by the pattern knot, which is a trefoil, $k_p = 3_1$ with winding number 2 in the solid torus V, as in the description above. The Alexander polynomials of these knots are

$$\Delta_{k_p} = 1 - t + t^2, \qquad \Delta_{k_c} = 1 - 3t + t^2,$$
$$\Delta_{k_s} = \left(1 - t - t^2\right)\left(1 - t + t^2\right)\left(1 + t - t^2\right).$$

REMARK 3. In [Wan], p. 463, it is asked whether Jones polynomials will provide an obstruction to 1-domination. In Example 4, we have Jones polynomial:

$$V_{k_s} = t^{-5} - t^{-4} + t + t^3 - t^4 - t^7 + t^8,$$

which is irreducible, and certainly does not have V_{k_p} as a factor, despite the 1-domination $k_s \geq k_p$. Therefore, the Jones polynomial does not reflect 1-dominance in a manner analogous to the Alexander or A-polynomials. The same may be said of the HOMFLYPT polynomial.

As a final topic in this section, we apply Gordon's approach to ribbon concordance [Go] to prove certain 1-domination rigidity results in terms of Alexander polynomials.

Let G be a group and let $H \subset G$ be a subgroup. Assume that p is a fixed integer either equal to 0 or a prime number. Define $G\natural H$ to be the subgroup of G generated by all the elements of the form $[x,y]z^p$ for $x \in G, y \in H$ and $z \in H$. The *lower p-central series* of G is defined as follows: $G_0 = G$, $G_{\alpha+1} = G\natural G_\alpha$ and $G_\beta = \bigcap_{\alpha<\beta} G_\alpha$ if β is a limit ordinal. We say that G is *transfinitely p-nilpotent* if $G_\alpha = \{1\}$ for some ordinal α. In particular the group is residually p-nilpotent (or residually p for short) if and only if $G_\omega = \{1\}$.

Definition 3 ([Go]). A knot $k \subset S^3$ is *transfinitely p-nilpotent* if its commutator subgroup $[\pi_1 E(k), \pi_1 E(k)]$ is transfinitely p-nilpotent.

The class of transfinitely p-nilpotent knots contains 2-bridge knots, fibred knots and, when $p > 0$, alternating knots k for which the leading coefficient of Δ_k is a power of p. Moreover, it has been observed by Gordon that the property of being transfinitely p-nilpotent is preserved by connected sum and cabling, see [Go].

For a polynomial P, we use $d^o(P)$ to denote the degree of P. The following proposition is essentially [Go, Lemma 3.4].

Proposition 16. *Let k_1 and k_2 be two knots in S^3 such that $k_1 \geq k_2$. If k_1 is transfinitely p-nilpotent for some p and $d^o(\Delta_{k_1}) = d^o(\Delta_{k_2})$, then $k_1 = k_2$.*

Proof. The proper degree 1 map $f : E(k_1) \to E(k_2)$ induces an epimorphism $f_* : \pi_1 E(k_1) \to \pi_1 E(k_2)$. It induces an epimorphism $\hat{f}_* : [\pi_1 E(k_1), \pi_1 E(k_1)] \to [\pi_1 E(k_2), \pi_1 E(k_2)]$.

For a knot $k \subset S^3$ it is well known that $H_1([\pi_1 E(k), \pi_1 E(k)]; \mathbb{Z})$ is torsion-free and $H_2([\pi_1 E(k), \pi_1 E(k)]; \mathbb{Z}) = 0$. Thus $H_2([\pi_1 E(k), \pi_1 E(k)]; \mathbb{F}_p) = \{0\}$ where $\mathbb{F}_p = \mathbb{Q}$ when $p = 0$ and $\mathbb{Z}/p\mathbb{Z}$ otherwise. It is also known that for a field $\mathbb{F}$, $\operatorname{rank}(H_1([\pi(k), \pi(k)]; \mathbb{F})) = d^o(\Delta_k)$. Therefore our hypotheses imply that the epimorphism $\hat{f}_*$ induces an isomorphism

$$\hat{f}_\sharp : H_1\big([\pi_1 E(k_1), \pi_1 E(k_1)]; \mathbb{F}_p\big) \to H_1\big([\pi_1 E(k_2), \pi_1 E(k_2)]; \mathbb{F}_p\big).$$

Stallings' theorem [St, Theorem 3.4] implies that for every ordinal α, $\hat{f}_*$ induces an isomorphism

$$\begin{aligned}&\big[\pi_1 E(k_1), \pi_1 E(k_1)\big] / \big[\pi_1 E(k_1), \pi_1 E(k_1)\big]_\alpha \\ &\quad \to \big[\pi_1 E(k_2), \pi_1 E(k_2)\big] / \big[\pi_1 E(k_2), \pi_1 E(k_2)\big]_\alpha.\end{aligned}$$

By hypothesis, we have $[\pi_1 E(k_1), \pi_1 E(k_1)]_\alpha = \{1\}$ for some ordinal α, so the epimorphism $\hat{f}_* : [\pi_1 E(k_1), \pi_1 E(k_1)] \to [\pi_1 E(k_2), \pi_1 E(k_2)]$ is in fact an isomorphism. Since f induces an isomorphism $f_\sharp : H_1(\pi_1 E(k_1); \mathbb{Z}) \to H_1(\pi_1 E(k_2); \mathbb{Z})$, it follows that the epimorphism $f_* : \pi_1 E(k_1) \to \pi_1 E(k_2)$ is an isomorphism. Finally, since this isomorphism preserves the peripheral structures of the two knots, the two knots are the same by Waldhausen [Wld], Chapter 13 and [GL]. □

Since alternating knots k for which the leading coefficient of Δ_k is a power of p are transfinitely p-nilpotent, the following are straightforward consequences of Propositions 15 and 16:

Corollary 6. *Suppose that $k_1 \geq k_2$ where k_1 is an alternating knot such that the leading coefficient of Δ_{k_1} is a power of a prime number and $d^o(\Delta_{k_1}) = d^o(\Delta_{k_2})$. Then $k_1 = k_2$.*

Corollary 7. *Suppose k_0 is an alternating knot such that the leading coefficient of Δ_{k_0} is a power of a prime number Then any 1-domination sequence $k_0 > k_1 > \cdots > k_n > \cdots$, contains at most $d^o(\Delta_{k_0})$ alternating knots.*

Question 4. Is each alternating knot transfinitely p-nilpotent?

Question 5. Suppose that k is alternating. Does $k \geq k'$ imply that k' is alternating?

Remark 4. (1) A positive answer to Question 4 implies that if k_1 is an alternating knot, $k_1 \geq k_2$, and $d^o(\Delta_{k_1}) = d^o(\Delta_{k_2})$, then $k_1 = k_2$.

(2) A positive answer to both Questions 4 and 5 implies that any 1-domination sequence of knots starting with an alternating knot k has length at most $d^o(\Delta_k)$.

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