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THE $SL(3,\mathbb{C})$-CHARACTER VARIETY OF THE FIGURE EIGHT KNOT

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Abstract. We give explicit equations that describe the character variety of the figure eight knot for the groups $\mathrm{SL}(3,\mathbb{C})$, $\mathrm{GL}(3,\mathbb{C})$ and $\mathrm{PGL}(3,\mathbb{C})$. For any of these G, it has five components of dimension 2, one consisting of totally reducible representations, another one consisting of partially reducible representations, and three components of irreducible representations. Of these, one is distinguished as it contains the curve of irreducible representations coming from $\mathrm{SL}(2,\mathbb{C})$. The other two components are induced by exceptional Dehn fillings of the figure eight knot. We also describe the action of the symmetry group of the figure eight knot on the character varieties.

1. Introduction

Since the foundational work of Thurston [46], [45] and Culler and Shalen [13], the varieties of representations and characters of three-manifold groups in $\mathrm{SL}(2,\mathbb{C})$ have been intensively studied, as they reflect geometric and topological properties of the three-manifold. In particular, they have been used to study knots $K \subset S^3$, by analysing the $\mathrm{SL}(2,\mathbb{C})$-character variety of the fundamental group of the knot complement $S^3 - K$ (these are called *knot groups*).

Much less is known for the character varieties of three-manifold groups in other Lie groups, notably for $\mathrm{SL}(r,\mathbb{C})$ with $r \geq 3$. There has been an increasing interest for those in the last years. For instance, inspired in the $\mathcal{A}$-coordinates in higher Teichmüller theory of Fock and Goncharov [20], some authors have used the so called Ptolemy coordinates for studying spaces of representations,

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based on subdivisions of ideal triangulations of the three-manifold. Among others, we mention the work of Dimofty, Gabella, Garoufalidis, Goerner, Goncharov, Thurston, and Zickert [17], [18], [22], [23], [21]. Geometric aspects of these representations, including volume and rigidity, have been addressed by Bucher, Burger, and Iozzi in [10], and by Bergeron, Falbel, and Guilloux in [3], who view these representations as holonomies of marked flag structures. We also recall the work Deraux and Falbel [14], [15], [16] to study CR and complex hyperbolic structures.

In a recent preprint, Falbel, Guilloux, Koseleff, Rouillier, and Thistlethwaite [19] compute the variety of characters of the figure eight knot in $\mathrm{SL}(3,\mathbb{C})$ using the ideal triangulation approach. We also compute this variety of characters in this paper, but with a completely different method and we obtain a different description. Here we describe it as an affine algebraic set with trace functions as coordinates.

The $\mathrm{SL}(3,\mathbb{C})$-character varieties for free groups have also been studied in [34], [35], [36], [49], and the $\mathrm{SL}(3,\mathbb{C})$-character variety of torus knot groups has been determined in [41]. Other results on the local geometry of $\mathrm{SL}(3,\mathbb{C})$-character varieties have been proved in [1], [4], [26], [40].

For Γ a finitely generated group, and for $G=\mathrm{SL}(r,\mathbb{C})$, $\mathrm{GL}(r,\mathbb{C})$, or $\mathrm{PGL}(r,\mathbb{C})$, the *variety of representations* is denoted by $R(\Gamma,G)$. It is an algebraic affine set, the action of G by conjugation is algebraic and the affine GIT quotient is naturally identified with the *variety of characters* $X(\Gamma,G)$, see [38]. Notice that both $R(\Gamma,G)$ and $X(\Gamma,G)$ can be reducible (hence not varieties in the usual sense), and their defining polynomial ideals may be non-radical (when this happens they are said to be scheme non-reduced), cf. [38]. When $G=\mathrm{SL}(r,\mathbb{C})$, points in $X(\Gamma,\mathrm{SL}(r,\mathbb{C}))$ are precisely characters of representations, that is, for $\rho\in R(\Gamma,G)$ its character is the map $\chi_\rho\colon\Gamma\to\mathbb{C}$ defined by $\chi_\rho(\gamma)=\mathrm{tr}(\rho(\gamma))$, $\forall\gamma\in\Gamma$.

Definition 1.1. A representation ρ is *reducible* if there exists some proper subspace $V\subset\mathbb{C}^r$ such that for all $g\in G$ we have $\rho(g)(V)\subset V$; otherwise ρ is *irreducible*. A *semisimple* representation is a direct sum of irreducible representations.

A representation $\rho\colon\Gamma\to\mathrm{SL}(3,\mathbb{C})$ is called *partially reducible* (respectively, *totally reducible*) if it is the sum of a one-dimensional and two-dimensional irreducible representation (respectively, a sum of three one-dimensional representations).

This paper focuses on the fundamental group of the complement of the figure eight knot, denoted by Γ. We consider the following two natural presentations:

$$\Gamma\cong\langle a,b,t\mid tat^{-1}=ab,tbt^{-1}=bab\rangle, \tag{1}$$

$$\cong\langle S,T\mid ST^{-1}S^{-1}TS=TST^{-1}S^{-1}T\rangle. \tag{2}$$

The former comes from the fibration of the three-manifold over the circle, the latter from knot theory [11], [43], as the generators are meridian loops. The presentations are related by

$$\begin{cases} S = t, \\ T = a^{-1}ta, \end{cases} \qquad \text{and} \qquad \begin{cases} t = S, \\ a = T^{-1}STS^{-1}, \\ b = TS^{-1}. \end{cases}$$

The figure eight knot exterior fibres over the circle, with fibre a punctured torus. Thus Γ is the split extension of the group of the fibre, the free group F_2, by a cyclic group, the group of the circle. This explains Presentation (1), as the group of the fiber is freely generated by a and b. This free group $F_2 = \langle a, b\rangle$ is the kernel of the Abelianisation $\varphi\colon \Gamma \to \Gamma_{\mathrm{ab}} \cong \mathbb{Z}$, which is given by

$$\varphi(S) = \varphi(T) = 1, \quad \text{and} \quad \varphi(t) = 1, \varphi(a) = \varphi(b) = 0, \tag{3}$$

respectively; so F_2 is also the commutator subgroup of Γ. Hence, for any representation $\rho\colon \Gamma \to G$ with $G = \mathrm{SL}(r,\mathbb{C})$, $\mathrm{GL}(r,\mathbb{C})$, or $\mathrm{PGL}(r,\mathbb{C})$, we have

$$\rho(F_2) = \big\langle \rho(a), \rho(b)\big\rangle \subset \mathrm{SL}(r,\mathbb{C}). \tag{4}$$

Presentation (2) is the usual presentation for a 2-bridge knot group, as the figure eight knot is the 2-bridge knot $\mathfrak{b}(5,3)$ in Schubert's notation (see [11, Section 12.A]). In particular, $X(\Gamma, \mathrm{SL}(3,\mathbb{C}))$ is a subvariety of the $\mathrm{SL}(3,\mathbb{C})$-character variety of the free group of rank 2 generated by S and T, and results of [34], [49] will apply, see Proposition 2.3. More precisely, we consider the algebraic map $X(\Gamma, \mathrm{SL}(3,\mathbb{C})) \to \mathbb{C}^8$ defined by

$$\chi \mapsto \big(y(\chi), \bar{y}(\chi), z(\chi), \bar{z}(\chi), \alpha(\chi), \bar{\alpha}(\chi), \beta(\chi), \bar{\beta}(\chi)\big),$$

where

$$\begin{aligned} &y(\chi) = \chi(S), \qquad \bar{y}(\chi) = \chi\big(S^{-1}\big), \qquad z(\chi) = \chi(ST), \\ &\bar{z}(\chi) = \chi\big(S^{-1}T^{-1}\big), \qquad \alpha(\chi) = \chi\big([T,S]\big), \qquad \bar{\alpha}(\chi) = \chi\big([S,T]\big), \\ &\beta(\chi) = \chi\big(S^{-1}T\big), \qquad \bar{\beta}(\chi) = \chi\big(ST^{-1}\big). \end{aligned} \tag{5}$$

We see in Proposition 2.3 that they define coordinates for $X(\Gamma, \mathrm{SL}(3,\mathbb{C}))$ as a subvariety of $\mathbb{C}^8$. Using Presentation (1), those coordinates are:

$$\begin{aligned} &\alpha(\chi) = \chi(a), \qquad \bar{\alpha}(\chi) = \chi\big(a^{-1}\big), \\ &\beta(\chi) = \chi(b), \qquad \bar{\beta}(\chi) = \chi\big(b^{-1}\big), \\ &y(\chi) = \chi(t), \qquad \bar{y}(\chi) = \chi\big(t^{-1}\big), \\ &z(\chi) = \chi\big(ta^{-1}ta\big) = \chi\big(t^2 b\big), \\ &\bar{z}(\chi) = \chi\big(a^{-1}t^{-1}at^{-1}\big) = \chi\big(b^{-1}t^{-2}\big). \end{aligned} \tag{6}$$

Throughout this paper, $\mu_3 = \{1, \varpi, \varpi^2\} \subset \mathbb{C}^*$, $\varpi = e^{2\pi i/3}$, denotes the group of the third roots of unity. We identify μ_3 with the center of $\mathrm{SL}(3,\mathbb{C})$, consisting of diagonal matrices, and for any knot group Γ, it acts on the spaces $R(\Gamma,\mathrm{SL}(3,\mathbb{C}))$ and $X(\Gamma,\mathrm{SL}(3,\mathbb{C}))$ via

$$(\varpi\rho)(\gamma)=\varpi^{\varphi(\gamma)}\rho(\gamma)\quad\text{and}\quad(\varpi\chi)(\gamma)=\varpi^{\varphi(\gamma)}\chi(\gamma),$$

respectively, where $\varphi\colon\Gamma\to\mathbb{Z}$ is the Abelianization map in (3). The action of the generator of the center $\varpi\in\mu_3$ in coordinates is

$$\varpi\cdot(y,\bar y,z,\bar z,\alpha,\bar\alpha,\beta,\bar\beta)=\big(\varpi y,\varpi^2\bar y,\varpi^2 z,\varpi\bar z,\alpha,\bar\alpha,\beta,\bar\beta\big).$$

The main result of this paper is the following.

THEOREM 1.2. *Let Γ be the group of the figure eight knot. The character variety $X(\Gamma,\mathrm{SL}(3,\mathbb{C}))\subset\mathbb{C}^8$ has five algebraic components. They are described in terms of the coordinates* (5) *as follows*:

(1) *The component X_{TR} corresponding to totally reducible representations is described by*:

$$\alpha=\bar\alpha=\beta=\bar\beta=3,\qquad(y,\bar y)\in\mathbb{C}^2,\qquad z=y^2-2\bar y,\qquad\bar z=\bar y^2-2y.$$

The component X_{TR} is smooth and isomorphic to $\mathbb{C}^2$.

(2) *The component X_{PR} corresponding to partially reducible representations is parametrized by the smooth variety*

$$\mathcal{P}=\left\{(v,w,x_1)\in\mathbb{C}\times\mathbb{C}^*\times\big(\mathbb{C}-\{1\}\big)\;\middle|\;\frac{x_1^2+x_1-1}{x_1-1}w=v^2\right\}.$$

More precisely, a parametrization $\Phi\colon\mathcal{P}\to X_{\mathrm{PR}}$ is given by:

$$\begin{aligned}&\alpha(v,w,x_1)=\bar\alpha(v,w,x_1)=x_1+1,\\&\beta(v,w,x_1)=\bar\beta(v,w,x_1)=\frac{x_1}{x_1-1}+1,\\&y(v,w,x_1)=v+\frac{1}{w},\qquad\bar y(v,w,x_1)=w+\frac{v}{w},\\&z(v,w,x_1)=w\alpha+\frac{1}{w^2},\qquad\bar z(v,w,x_1)=\frac{\alpha}{w}+w^2.\end{aligned}$$

The component X_{PR} is smooth except at the three points

$$(y,\bar y,z,\bar z,\alpha,\bar\alpha,\beta,\bar\beta)\in\mu_3\cdot(4,4,8,8,3,3,3,3).$$

There are three components V_0, V_1 and V_2 corresponding to irreducible representations.

(3) *The distinguished component V_0 is the zero set of the ideal generated by the following equations*:

$$\begin{aligned}\alpha=\bar\alpha,&\qquad\beta=\bar\beta,\\y\bar y&=(\alpha+1)(\beta+1),\\z\bar z&=2\alpha^2\beta+\alpha^2+1,\\y^3+\bar y^3&=\alpha^2\beta+\alpha\beta^2+6\alpha\beta+3\alpha+3\beta+2,\end{aligned}$$

$$\begin{aligned}
z^3+\bar z^3 &= \alpha^4\beta^2+10\alpha^2\beta+9\alpha^2-2\alpha^3-2,\\
yz+\bar y\bar z &= \alpha^2\beta+3\alpha\beta+3\alpha+1,\\
\bar y^2 z+y^2\bar z &= \alpha^2\beta^2+4\alpha^2\beta+2\alpha^2+4\alpha\beta+2\alpha+2\beta+1,\\
\bar y z^2+y\bar z^2 &= \alpha^3\beta^2+\alpha^3\beta+4\alpha^2\beta+3\alpha^2+5\alpha\beta+3\alpha-1.
\end{aligned}$$

(4) *The first non distinguished component V_1:*

$$\alpha=\bar\alpha=1,\qquad y\bar y=\beta+\bar\beta+2,$$
$$y^3+\bar y^3=\beta\bar\beta+5\beta+5\bar\beta+5,\qquad \bar z=y,\qquad z=\bar y.$$

(5) *The second non distinguished component V_2:*

$$\beta=\bar\beta=1,\qquad y\bar y=\alpha+\bar\alpha+2,$$
$$y^3+\bar y^3=\alpha\bar\alpha+5\alpha+5\bar\alpha+5,\qquad z=y^2-\bar y,\qquad \bar z=\bar y^2-y.$$

The components V_i, $i=0,1,2$, are smooth except at the three points: $\mu_3\cdot(2,2,2,2,1,1,1,1)$.

The component V_0 is called distinguished because it contains the composition of the holonomy representation of the complete hyperbolic structure with the irreducible representation $\mathrm{Sym}^2\colon \mathrm{PSL}(2,\mathbb{C})\to\mathrm{SL}(3,\mathbb{C})$. The components V_1 and V_2 factor through Dehn fillings of the knot, with respective slopes ±3, see Proposition 10.3. In particular, they do not contain faithful representations, and the volume of characters in those components vanishes.

Remark 1.3. The ideal generated by the equations in item (3) of Theorem 1.2 is not radical. Generators of the radical are given in Remark 7.2, those are the defining polynomials of the variety of characters, as we know that V_0 is scheme reduced, by Proposition 5.15.

The intersections of the components are as follows:

- $X_{\mathrm{TR}}\cap X_{\mathrm{PR}}$ is the curve $\alpha=\bar\alpha=\beta=\bar\beta=3$, $y^2\bar y^2-5y^3-5\bar y^3+28y\bar y-64=0$, $z=y^2-2\bar y$, and $\bar z=\bar y^2-2y$. This curve is smooth except at the three points $\mu_3\cdot(4,4,8,8,3,3,3,3)$.
- $X_{\mathrm{TR}}\cap V_1=X_{\mathrm{TR}}\cap V_2=\emptyset$.
- $X_{\mathrm{TR}}\cap V_0=\mu_3\cdot(4,4,8,8,3,3,3,3)$.
- The intersections $X_{\mathrm{PR}}\cap V_1=X_{\mathrm{PR}}\cap V_2=V_0\cap V_1\cap V_2=V_1\cap V_2$ consists of three points $\mu_3\cdot(2,2,2,2,1,1,1,1)$. These three points are singular points of V_i, $i=0,1,2$, and they are regular points on X_{PR}.
- $X_{\mathrm{PR}}\cap V_0$ is the curve given by the equations (2) of Theorem 1.2 and the equation $w^3-2vw+1=0$. This curve is nonsingular except at the three points
$$V_0\cap X_{\mathrm{PR}}\cap X_{\mathrm{TR}}=\mu_3\cdot(4,4,8,8,3,3,3,3).$$
- $V_0\cap V_1$ is the curve $\alpha=\bar\alpha=1$, $\beta=\bar\beta$, $y\bar y=2\beta+2$, $y^3+\bar y^3=\beta^2+10\beta+5$. This curve is nonsingular except at $\mu_3\cdot(2,2,2,2,1,1,1,1)$.

- $V_0 \cap V_2$ is the curve $\alpha = \bar{\alpha}$, $\beta = \bar{\beta} = 1$, $y\bar{y} = 2\alpha + 2$, $y^3 + \bar{y}^3 = \alpha^2 + 10\alpha + 5$. This curve is nonsingular except at $\mu_3 \cdot (2,2,2,2,1,1,1,1)$.

To obtain the irreducible components, we consider first the restriction of those characters to the group of the fiber $F_2 = \langle a, b\rangle$ (Presentation (1)) by considering the characters that are fixed by the action of the monodromy. Here we use Lawton's and Will's coordinates for $X(F_2, \mathrm{SL}(3,\mathbb{C}))$. This allows us to distinguish three components of irreducible characters, that are worked out explicitly.

The paper is organized as follows: Section 2 is devoted to generalities on character varieties of knot groups. Representations of Γ in $\mathrm{SL}(2,\mathbb{C})$, $\mathrm{GL}(2,\mathbb{C})$ and $\mathrm{PGL}(2,\mathbb{C})$ are discussed in Section 3, and reducible representations in $\mathrm{SL}(3,\mathbb{C})$, in Section 4. Section 5 is devoted to the description of the restriction to the variety of characters of F_2 as fixed points of the monodromy. Then the non-distinguished and the distinguished components are computed respectively in Sections 6 and 7. Section 8 is devoted to characters in $\mathrm{GL}(3,\mathbb{C})$ and $\mathrm{PGL}(3,\mathbb{C})$. In Section 9, we describe how the symmetry group of the figure eight knot acts on the variety of characters. In Section 10, we identify the non-distinguished components as induced by Dehn fillings on the knot. Finally, in Section 11 we discuss explicit representations that are relevant.

Some of the computations require software, either Sage [44] or Mathematica [31]. All worksheets and notebooks can be found in [27].

2. Character varieties of knot groups

Throughout this section, we let Γ denote any knot group (in the rest of the paper it denotes the figure eight knot exterior), and $\varphi\colon \Gamma \to \mathbb{Z}$ the Abelianization which maps the meridian of the knot to 1, see Equation (3). The center μ_r of $\mathrm{SL}(r,\mathbb{C})$ consists of diagonal matrices and it can be identified with the set of rth roots of unity $\{\varpi^k \mid k = 0, \dots, r-1\} \subset \mathbb{C}^*$. The center acts on $R(\Gamma, \mathrm{SL}(r,\mathbb{C}))$ and $X(\Gamma, \mathrm{SL}(r,\mathbb{C}))$ via multiplication, that is, for $\rho \in R(\Gamma, \mathrm{SL}(r,\mathbb{C}))$, $\chi \in X(\Gamma, \mathrm{SL}(r,\mathbb{C}))$, and $\varpi^k \in \mu_r$, we have for all $\gamma \in \Gamma$:

$$\varpi^k \cdot \rho(\gamma) = \varpi^{k\varphi(\gamma)}\rho(\gamma) \quad \text{and} \quad \varpi^k \cdot \chi(\gamma) = \varpi^{k\varphi(\gamma)}\chi(\gamma).$$

In what follows, we will use that $H^1(\Gamma, \mu_r) = \mathrm{Hom}(\Gamma, \mu_r) \cong \mu_r$, and $H^2(\Gamma, \mu_r) = 0$ for a knot group Γ. This follows from the universal coefficient theorem since $H_1(\Gamma, \mathbb{Z}) \cong H_1(C, \mathbb{Z}) \cong \mathbb{Z}$, and $H_2(C, \mathbb{Z}) = 0$ where C denotes the knot complement. Notice that the natural morphism $H_2(C, \mathbb{Z}) \to H_2(\Gamma, \mathbb{Z})$ is surjective (see [28, Lemma 3.1]).

Lemma 2.1. *Let Γ be a knot group and $\rho\colon \Gamma \to \mathrm{PSL}(r,\mathbb{C})$ be a representation. Then there exists a lift $\tilde{\rho}\colon \Gamma \to \mathrm{SL}(r,\mathbb{C})$ of ρ. Moreover, all lifts of ρ are given by $\mu_r \cdot \tilde{\rho}$.*

Proof. There is a short exact sequence

$$1 \to \mu_r \to \mathrm{SL}(r,\mathbb{C}) \to \mathrm{PSL}(r,\mathbb{C}) \to 1.$$

We associate to the representation $\rho\colon \Gamma \to \mathrm{PSL}(r,\mathbb{C})$ a second obstruction class $w_2 = w_2(\rho) \in H^2(\Gamma,\mu_r)$ defined as follows: choose any set-theoretic lift $f\colon \Gamma \to \mathrm{SL}(r,\mathbb{C})$ and define $w_2\colon \Gamma\times\Gamma \to \mu_r$ such that

$$\forall \gamma_1,\gamma_2 \in \Gamma, \quad f(\gamma_1\gamma_2) = f(\gamma_1)f(\gamma_2)w_2(\gamma_1,\gamma_2).$$

It is easy to see that $w_2 \in Z^2(\Gamma,\mu_r)$ is a cocyle. Moreover, the cohomology class represented by w_2 does not depend on the lift f. Now, w_2 represents the trivial cohomology class since $H^2(\Gamma,\mu_r)$ is trivial. Therefore, there exists a map $d\colon \Gamma \to \mu_r$ such that for all $\gamma_1,\gamma_2 \in \Gamma$,

$$w_2(\gamma_1,\gamma_2) = d(\gamma_1)\,d(\gamma_2)\,d(\gamma_1\gamma_2)^{-1}.$$

It is clear that $\tilde\rho\colon \Gamma \to \mathrm{SL}(r,\mathbb{C})$ given by $\tilde\rho(\gamma) = d(\gamma)f(\gamma)$ is a representation. Finally, d is unique up multiplication with a cocycle $h \in H^1(\Gamma,\mu_r) \cong \mathrm{Hom}(\Gamma,\mu_r) \cong \mu_r$. ☐

LEMMA 2.2. *Let Γ be a knot group. Then we have $X(\Gamma,\mathrm{PSL}(r,\mathbb{C})) \cong X(\Gamma,\mathrm{SL}(r,\mathbb{C}))/\mu_r$ and*

$$X\big(\Gamma,\mathrm{GL}(r,\mathbb{C})\big) \cong X\big(\Gamma,\mathrm{SL}(r,\mathbb{C})\big) \times_{\mu_r} \mathbb{C}^*.$$

Proof. The isomorphism $X(\Gamma,\mathrm{PSL}(r,\mathbb{C})) \cong X(\Gamma,\mathrm{SL}(r,\mathbb{C}))/\mu_r$ follows from Lemma 2.1.

Now the same proof as for Lemma 2.1 shows that for any homomorphism $h\colon \Gamma \to \mathbb{C}^*$ there exists a homomorphism $\tilde h\colon \Gamma \to \mathbb{C}^*$ such that $\tilde h^r = h$ (see [2, Lemma 2.1]). Therefore the map $R(\Gamma,\mathrm{SL}(r,\mathbb{C})) \times \mathbb{C}^* \to R(\Gamma,\mathrm{GL}(r,\mathbb{C}))$ given by $(\rho,\lambda) \mapsto \lambda^\varphi\rho$ is surjective, and (ρ,λ) and (ρ',λ') map to the same representation if and only if $(\rho',\lambda') \in \mu_r\cdot(\rho,\lambda)$. Hence,

$$\big(R\big(\Gamma,\mathrm{SL}(r,\mathbb{C})\big) \times \mathbb{C}^*\big)/\mu_r \cong R\big(\Gamma,\mathrm{GL}(r,\mathbb{C})\big).$$

The actions of μ_r and $\mathrm{GL}(r,\mathbb{C})$ on $R(\Gamma,\mathrm{SL}(r,\mathbb{C})) \times \mathbb{C}^*$ commute. Moreover, $\mathrm{GL}(r,\mathbb{C})$ acts trivially on the representations into the center $\mathbb{C}^*$. Hence,

$$\begin{aligned} R\big(\Gamma,\mathrm{GL}(r,\mathbb{C})\big)/\!\!/\,\mathrm{GL}(r,\mathbb{C}) &\cong \big(\big(R\big(\Gamma,\mathrm{SL}(r,\mathbb{C})\big) \times \mathbb{C}^*\big)/\!\!/\,\mathrm{GL}(r,\mathbb{C})\big)/\mu_r \\ &\cong \big(X\big(\Gamma,\mathrm{SL}(r,\mathbb{C})\big) \times \mathbb{C}^*\big)/\mu_r \\ &= X\big(\Gamma,\mathrm{SL}(r,\mathbb{C})\big) \times_{\mu_r} \mathbb{C}^*. \end{aligned}$$

☐

Distinguished component. For a hyperbolic knot group, there exists a unique one-dimensional component $X_0 \subset X(\Gamma, \mathrm{PSL}(2,\mathbb{C}))$, up to complex conjugation, which contains the character of the holonomy representation. Complex conjugation corresponds to changing the orientation of the three manifold, thus there is a unique $\mathrm{PSL}(2,\mathbb{C})$-character of the holonomy of an oriented knot exterior. The holonomy representation lifts to two representations $\rho\colon \Gamma \to \mathrm{SL}(2,\mathbb{C})$, and by composing any lift with the rational, irreducible, r-dimensional representation $\mathrm{Sym}^{r-1}\colon \mathrm{SL}(2,\mathbb{C}) \to \mathrm{SL}(r,\mathbb{C})$ we obtain an irreducible representation $\rho_r\colon \Gamma \to \mathrm{SL}(r,\mathbb{C})$. It follows from [40] that $\chi_{\rho_r} \in X(\Gamma, \mathrm{SL}(r,\mathbb{C}))$ is a smooth point contained in a unique $(r-1)$-dimensional component of $X(\Gamma, \mathrm{SL}(r,\mathbb{C}))$. We will call such a component a *distinguished* component of $X(\Gamma, \mathrm{SL}(r,\mathbb{C}))$. For *odd* r, as $\mathrm{Sym}^{r-1}\colon \mathrm{SL}(2,\mathbb{C}) \to \mathrm{SL}(r,\mathbb{C})$ factors through $\mathrm{PSL}(2,\mathbb{C})$, thus there is a unique distinguished component.

Totally reducible representations. Totally reducible representations are representations which split as a direct sum of one-dimensional representations. In particular they are representations of the Abelianization of a knot group Γ, which is $\mathbb{Z}$. Thus the restriction of a totally reducible representation to the commutator subgroup is trivial and it only depends on the image of a meridian, that is, a diagonal matrix.

If the image of a meridian is $\mathrm{diag}(\lambda_1, \dots, \lambda_r)$, then the space of parameters is $(\sigma_1, \dots, \sigma_r)$, where σ_i is the ith elementary symmetric polynomial on $\lambda_1, \dots, \lambda_r$ (hence $\sigma_r = 1$ for $\mathrm{SL}(r,\mathbb{C})$). Thus, for any knot group Γ

$$
\begin{aligned}
X_{\mathrm{TR}}\big(\Gamma, \mathrm{SL}(r,\mathbb{C})\big) &= \mathbb{C}^{r-1}, \\
X_{\mathrm{TR}}\big(\Gamma, \mathrm{GL}(r,\mathbb{C})\big) &= \mathbb{C}^{r-1} \times \mathbb{C}^*, \\
X_{\mathrm{TR}}\big(\Gamma, \mathrm{PGL}(r,\mathbb{C})\big) &= \big(\mathbb{C}^{r-1}\big)/\mu_r,
\end{aligned}
\tag{7}
$$

where $\varpi \cdot (\sigma_1, \dots, \sigma_{r-1}) = (\varpi\sigma_1, \varpi^2\sigma_2, \dots, \varpi^{r-1}\sigma_{r-1})$.

Now (and in the rest of the paper) we move to the specific case where Γ is the figure eight knot group. As the group Γ is generated by S and T (Presentation (2)), $X(\Gamma, \mathrm{SL}(3,\mathbb{C}))$ can be viewed as a subvariety of the variety of characters of a free group generated by S and T. Lawton and Will [34], [49] prove that the variety of characters of the free group on two generators S and T is an eight dimensional variety embedded in $\mathbb{C}^{10}$ with coordinates the traces of the elements:

$$S^{\pm 1}, T^{\pm 1}, S^{\pm 1}T^{\pm 1}, [S,T]^{\pm 1}.$$

This induces an embedding of the variety of characters of the group of the figure eight knot in $\mathbb{C}^{10}$, as it is generated by S and T. Using that S and T are conjugate, we reduce from ten to the the eight parameters in (5): $(y, \bar{y}, z, \bar{z}, \alpha, \bar{\alpha}, \beta, \bar{\beta})$. Thus, is the following proposition.

PROPOSITION 2.3. *For Γ the figure eight knot group, the character variety $X(\Gamma,\mathrm{SL}(3,\mathbb{C}))$ embeds into $\mathbb{C}^8$. The embedding is given by the parameters $(y,\bar{y},z,\bar{z},\alpha,\bar{\alpha},\beta,\bar{\beta})$ in (5).*

Of course the previous proposition applies to any two-bridge knot group, as it has a presentation similar to (2) with two generators represented by meridian curves.

3. Representations in $\mathrm{SL}(2,\mathbb{C})$, $\mathrm{GL}(2,\mathbb{C})$, and $\mathrm{PGL}(2,\mathbb{C})$

Let Γ be the knot group of the figure eight, in this section we analyze the space of representations $X(\Gamma,G)$ for $G=\mathrm{SL}(2,\mathbb{C})$, $\mathrm{GL}(2,\mathbb{C})$ and $\mathrm{PGL}(2,\mathbb{C})$. Reducible representations are totally reducible, hence they have been described in Section 2, and we discuss next *irreducible* representations.

To understand the irreducible representations of the figure eight knot group into $\mathrm{SL}(2,\mathbb{C})$, we follow [42]. For $G=\mathrm{SL}(2,\mathbb{C})$ or $\mathrm{GL}(2,\mathbb{C})$, let $\chi\in X(\Gamma,G)$ be a character. Consider the restriction map

$$\mathrm{res}\colon X(\Gamma,G)\to X\big(F_2,\mathrm{SL}(2,\mathbb{C})\big).$$

We use Fricke coordinates for $X(F_2,\mathrm{SL}(2,\mathbb{C}))$, given as

$$x_1(\chi)=\chi(a),\qquad x_2(\chi)=\chi(b),\qquad x_3(\chi)=\chi(ab),$$

which define an isomorphism $X(F_2,\mathrm{SL}(2,\mathbb{C}))\cong\mathbb{C}^3$. In those coordinates, conjugation by t induces a transformation given by:

$$(x_1,x_2,x_3)(\chi)\mapsto\big(\chi(ab),\chi(bab),\chi\big(ab^2ab\big)\big)=\big(x_3,x_2x_3-x_1,x_2x_3^2-x_1x_3-x_2\big).$$

Here we have used the basic identities for $Y,Z\in\mathrm{SL}(2,\mathbb{C})$,

$$\begin{aligned}\mathrm{tr}(YZ)&=\mathrm{tr}(Y)\,\mathrm{tr}(Z)-\mathrm{tr}\big(YZ^{-1}\big),\\ \mathrm{tr}(YZ)&=\mathrm{tr}(ZY),\\ \mathrm{tr}\big(Y^{-1}\big)&=\mathrm{tr}(Y).\end{aligned}\tag{8}$$

Thus

$$\begin{aligned}&\mathrm{res}\big(X(\Gamma,G)\big)\\ &\quad=\big\{(x_1,x_2,x_3)\in\mathbb{C}^3\mid x_1=x_3,x_2=x_2x_3-x_1,x_3=x_2x_3^2-x_1x_3-x_2\big\}\\ &\quad\cong\big\{(x_1,x_2)\in\mathbb{C}^2\mid x_1x_2=x_1+x_2\big\}\\ &\quad\cong\big\{(x_1,x_2)\in\mathbb{C}^2\mid(x_1-1)(x_2-1)=1\big\}\cong\mathbb{C}-\{1\}.\end{aligned}$$

The point with coordinates $(x_1,x_2)=(2,2)$ corresponds to the restriction of reducible representations to F_2, and the rest of the curve corresponds to irreducible representations.

To get $X(\Gamma,\mathrm{SL}(2,\mathbb{C}))$ we introduce a third variable:

$$y_0(\chi)=\chi(t)=\chi(ta)=\chi(tb)=\chi(tab)$$

(it is straightforward that t, ta, tb and tab are conjugate elements). The group Γ is generated by $t=S$ and $T=a^{-1}ta$. Notice that $b=[a^{-1},t]$ is conjugate to TS^{-1}, and therefore coordinates for $X(\Gamma,\mathrm{SL}(2,\mathbb{C}))$ are given by x_1, x_2 and y_0, by Fricke's theorem. By applying the identities (8) to the trace of $b=[a^{-1},t]$, we get $x_2=x_1^2-y_0^2(x_1-2)-2$ and $x_2=x_1/(x_1-1)$. Hence,

$$\begin{aligned} 0 &= x_1^3-x_1^2\big(y_0^2+1\big)+3x_1\big(y_0^2-1\big)+2\big(1-y_0^2\big)\\ &= (x_1-2)\big((-x_1+1)y_0^2+x_1^2+x_1-1\big)\\ &= (x_1-2)(x_1-1)\left(x_1+1-y_0^2+\frac{x_1}{x_1-1}\right). \end{aligned}$$

Hence, there are two components:

$$x_1=2, \qquad x_2=\frac{x_1}{x_1-1}=2, \qquad y_0\in\mathbb{C},$$

corresponding to reducible representations, and $(x_1-1)(x_2-1)=1$, $y_0^2=x_1+x_2+1$, corresponding to irreducible ones.

PROPOSITION 3.1. *The character variety $X(\Gamma,\mathrm{SL}(2,\mathbb{C}))$ has two irreducible components, written as follows, in terms of the coordinates $x_1=\chi(a)$, $x_2=\chi(b)$ and $y_0=\chi(t)$:*

- *The component corresponding to reducible representations*: $x_1=2$, $x_2=2$, $y_0\in\mathbb{C}$.
- *The component corresponding to irreducible representations*:

$$(x_1-1)(x_2-1)=1, \qquad y_0^2=x_1+x_2+1. \tag{9}$$

They intersect transversally in the two points: $x_1=x_2=2$, $y_0=\pm\sqrt{5}$.

From this, to get $\mathrm{PGL}(2,\mathbb{C})$-characters, we have to quotient by $\mu_2=\{\pm1\}$ acting on y_0, that is by involution $y_0\mapsto -y_0$ (see Section 2 and [24]).

PROPOSITION 3.2. *The character variety $X(\Gamma,\mathrm{PGL}(2,\mathbb{C}))$ has two irreducible components, written as follows, in terms of the coordinates $x_1=\chi(a)$, $x_2=\chi(b)$ and $z_0=y_0^2=\chi(t)^2$:*

- *The component corresponding to reducible representations*: $x_1=2$, $x_2=2$, $z_0\in\mathbb{C}$.
- *The component corresponding to irreducible representations*:

$$(x_1-1)(x_2-1)=1, \qquad z_0=x_1+x_2+1,$$

which is isomorphic to $\mathbb{C}-\{1\}$.

They intersect in one point: $x_1=x_2=2, z_0=5$.

To get the $\mathrm{GL}(2,\mathbb{C})$-representations, recall from Lemma 2.2 that

$$X\big(\Gamma,\mathrm{GL}(2,\mathbb{C})\big)=\big(X\big(\Gamma,\mathrm{SL}(2,\mathbb{C})\big)\times\mathbb{C}^*\big)/\mu_2.$$

This algebraic set has two components: the one consisting of characters of reducible representations is isomorphic to $(\mathbb{C}\times\mathbb{C}^*)/\mu_2$, and the one containing characters of irreducible representations that is isomorphic to $(E\times\mathbb{C}^*)/\mu_2$, where E is the curve defined by (9). The action of μ_2 on $\mathbb{C}\times\mathbb{C}^*$ generates the equivalence $(y_0,\lambda)\sim(-y_0,-\lambda)$. The ring of invariant functions of this action is generated by $u=y_0^2$, $v=y_0\lambda$, $w=\lambda^2$, and the algebraic relations between these functions are generated by $uw=v^2$. The variable u can be eliminated since $1/w$ is a regular function on $\mathbb{C}^*$. We obtain that $(\mathbb{C}\times\mathbb{C}^*)/\mu_2\cong\mathbb{C}\times\mathbb{C}^*$. It coincides with the component $X_{\mathrm{TR}}(\Gamma,\mathrm{GL}(2,\mathbb{C}))$ (see Equation (7)).

The product $E\times\mathbb{C}^*$ is parametrized by (x_1,y_0,λ), corresponding to characters satisfying the equations (9). In order to obtain $(E\times\mathbb{C}^*)/\mu_2$ we have to identify $(x_1,y_0,\lambda)\sim(x_1,-y_0,-\lambda)$. Now, the ring of invariant functions of this action is generated by $u=y_0^2$, $v=y_0\lambda$, $w=\lambda^2$ and x_1, and the algebraic relations between these functions are generated by $uw=v^2$. Hence, $(E\times\mathbb{C}^*)/\mu_2$ is isomorphic to

$$\left\{(v,w,x_1)\in\mathbb{C}\times\mathbb{C}^*\times\big(\mathbb{C}-\{1\}\big)\,\middle|\,\frac{x_1^2+x_1-1}{x_1-1}w=v^2\right\}.$$

The intersection of the two components is given by introducing the additional equation $x_1=2$, and therefore:

$$\big((E\times\mathbb{C}^*)/\mu_2\big)\cap\big((\mathbb{C}\times\mathbb{C}^*)/\mu_2\big)=\big\{\big(v,v^2/5,2\big)\mid v\in\mathbb{C}^*\big\}\cong\mathbb{C}^*.$$

Notice that for a representation $\rho_2\colon\Gamma\to\mathrm{GL}(2,\mathbb{C})$ with character $\chi_2:=\chi_{\rho_2}\in X(\Gamma,\mathrm{GL}(2,\mathbb{C}))$ we have:

$$\begin{aligned} v(\chi_2)&=\mathrm{tr}\big(\rho_2(t)\big)\in\mathbb{C}, \qquad w(\chi_2)=\det\big(\rho_2(t)\big)\in\mathbb{C}^*,\\ x_1(\chi_2)&=\mathrm{tr}\big(\rho_2(a)\big)\in\mathbb{C}-\{1\}.\end{aligned}$$

Proposition 3.3. *The character variety $X(\Gamma,\mathrm{GL}(2,\mathbb{C}))$ has two irreducible 2-dimensional components. More precisely,*

- *The component $X_{\mathrm{TR}}(\Gamma,\mathrm{GL}(2,\mathbb{C}))$ contains the characters of reducible representations, and it is isomorphic to $\mathbb{C}\times\mathbb{C}^*$:*

  $$X_{\mathrm{TR}}\big(\Gamma,\mathrm{GL}(2,\mathbb{C})\big)\cong\big\{(v,w,x_1)\in\mathbb{C}\times\mathbb{C}^*\times\big(\mathbb{C}-\{1\}\big)\mid x_1=2\big\}\cong\mathbb{C}\times\mathbb{C}^*.$$

- *The component X_2 which contains characters of irreducible representations:*

  $$X_2\cong\left\{(v,w,x_1)\in\mathbb{C}\times\mathbb{C}^*\times\big(\mathbb{C}-\{1\}\big)\;\middle|\;\frac{x_1^2+x_1-1}{x_1-1}w=v^2\right\}.$$

The intersection $X_2\cap X_{\mathrm{TR}}(\Gamma,\mathrm{GL}(2,\mathbb{C}))$ is isomorphic to $\mathbb{C}^*$:

$$\big\{(v,w,2)\in\mathbb{C}\times\mathbb{C}^*\times\big(\mathbb{C}-\{1\}\big)\mid 5w=v^2\big\}\cong\mathbb{C}^*.$$

4. Reducible representations in $\mathrm{SL}(3,\mathbb{C})$

We start by describing the *totally reducible* characters already given in Section 2, Equation (7), with the coordinates (5):

$$\alpha=\bar\alpha=\beta=\bar\beta=3,\qquad (y,\bar y)\in\mathbb{C}^2,\qquad z=y^2-2\bar y,\quad\text{and}\quad \bar z=\bar y^2-2y.$$

Here we have used that, by the Cayley–Hamilton theorem, for every $A \in \mathrm{SL}(3,\mathbb{C})$ the equality $\operatorname{tr}(A^2) = \operatorname{tr}(A)^2 - 2\operatorname{tr}(A^{-1})$ holds.

Now we move to *partially reducible* representations, that is, representations that are a direct sum of a 2-dimensional representation and a 1-dimensional representation. We shall use the explicit identification of Lemma 2.2:

$$R\big(\Gamma, \mathrm{GL}(2,\mathbb{C})\big) \cong \big(R\big(\Gamma, \mathrm{SL}(2,\mathbb{C})\big) \times \mathbb{C}^*\big)/\mu_2,$$

in particular a representation in $\mathrm{GL}(2,\mathbb{C})$ is written as a representation in $\mathrm{SL}(2,\mathbb{C})$ times a one-dimensional representation in $\mathbb{C}^*$. Let $\rho_2\colon \Gamma \to \mathrm{GL}(2,\mathbb{C})$ be irreducible. Then $\rho = \rho_2 \oplus (\det\rho_2)^{-1}$ is partially reducible, and if (v, w, x_1) denote the coordinates of the character $\chi_2 := \chi_{\rho_2}$ as in Proposition 3.3, then the coordinates of χ_ρ are functions of $\chi_2 = (v, w, x_1)$. More precisely:

$$\begin{aligned} &\alpha(\chi_2) = \bar\alpha(\chi_2) = x_1 + 1, && \beta(\chi_2) = \bar\beta(\chi_2) = x_2 + 1,\\ &y(\chi_2) = v + \frac{1}{w}, && \bar y(\chi_2) = \frac{v}{w} + w. \end{aligned}$$

In order to calculate $z(\chi_2)$, we will use that for all $Y, Z \in \mathrm{GL}(2,\mathbb{C})$ the identities

$$\operatorname{tr}(YZ) = \operatorname{tr}(Y)\operatorname{tr}(Z) - \det(Y)\operatorname{tr}\big(Y^{-1}Z\big) \quad \text{and} \quad \operatorname{tr}\big(Y^{-1}\big)\det(Y) = \operatorname{tr}(Y)$$

hold. This gives

$$z(\chi_2) = v^2 - wx_2 + \frac{1}{w^2} \quad \text{and} \quad \bar z(\chi_2) = \frac{v^2 - wx_2}{w^2} + w^2.$$

Now, we have

$$\frac{v^2}{w} = x_1 + x_2 + 1$$

and hence we obtain

$$\begin{aligned} &\alpha(\chi_2) = \bar\alpha(\chi_2) = x_1 + 1, && \beta(\chi_2) = \bar\beta(\chi_2) = x_2 + 1,\\ &y(\chi_2) = v + \frac{1}{w}, && \bar y(\chi_2) = \frac{v}{w} + w,\\ &z(\chi_2) = w(x_1+1) + \frac{1}{w^2}, && \bar z(\chi_2) = \frac{x_1+1}{w} + w^2. \end{aligned} \tag{10}$$

It follows that the component $X_{\mathrm{PR}}(\Gamma, \mathrm{SL}(3,\mathbb{C}))$ of partially reducible characters is parametrized by the component $X_2 = (E \times \mathbb{C}^*)/\mu_2 \subset X(\Gamma, \mathrm{GL}(2,\mathbb{C}))$ of irreducible characters. If $\rho_2, \rho_2' \in R(\Gamma, \mathrm{GL}(2,\mathbb{C}))$ are two semisimple representations then $\rho = \rho_2 \oplus (\det\rho_2)^{-1}$ and $\rho' = \rho_2' \oplus (\det\rho_2')^{-1}$ determine the same character in $X(\Gamma, \mathrm{SL}(3,\mathbb{C}))$ if and only if they are conjugate. Here we have used that each point of $X(\Gamma, \mathrm{SL}(3,\mathbb{C}))$ is the character of a semi-simple representation which is unique up to conjugation [38]. Hence, two characters $\chi_2, \chi_2' \in X_2$ can only give the same character of $X(\Gamma, \mathrm{SL}(3,\mathbb{C}))$ if ρ_2 and ρ_2' are reducible that is, if for the corresponding parameters (x_1, v, w) and (x_1', v', w') of χ_2 and χ_2', respectively the equations $x_1 = x_1' = 2$, $v^2 = 5w$, and $(v')^2 = 5w'$

hold. Therefore, if χ_2 and $\chi_2' \in X_2$ give the same character of $X(\Gamma,\mathrm{SL}(3,\mathbb{C}))$ then $y(\chi_2)=y(\chi_2')$ and $\bar y(\chi_2)=\bar y(\chi_2')$. This is equivalent to

$$v+\frac{5}{v^2}=v'+\frac{5}{(v')^2} \quad\text{and}\quad \frac{5}{v}+\frac{v^2}{5}=\frac{5}{v'}+\frac{(v')^2}{5}.$$

If $v=v'$, then $w=w'$, and $\chi_2=\chi_2'$ follows. If $v\neq v'$, then $(vv')^3=125$, and there are exactly three pairs of reducible characters which map to the same the character in $X(\Gamma,\mathrm{SL}(3,\mathbb{C}))$:

$$\left(\frac{5}{2}\pm\frac{\sqrt5}{2},\frac{3}{2}\pm\frac{\sqrt5}{2},2\right)\mapsto(4,4,8,8,3,3,3,3),$$
$$\left(\varpi\left(\frac{5}{2}\pm\frac{\sqrt5}{2}\right),\varpi^2\left(\frac{3}{2}\pm\frac{\sqrt5}{2}\right),2\right)\mapsto\varpi\cdot(4,4,8,8,3,3,3,3),$$
$$\left(\varpi^2\left(\frac{5}{2}\pm\frac{\sqrt5}{2}\right),\varpi\left(\frac{3}{2}\pm\frac{\sqrt5}{2}\right),2\right)\mapsto\varpi^2\cdot(4,4,8,8,3,3,3,3).$$

Recall that being reducible and being totally reducible are Zariski closed properties. In particular, the set of irreducible characters is Zariski open, and the set of partially reducible characters is also Zariski open in components that contain only reducible characters.

Proposition 4.1. *The locus of reducible representations of the character variety $X(\Gamma,\mathrm{SL}(3,\mathbb{C}))$ consists of two irreducible components*:

- *The component $X_{\mathrm{TR}}:=X_{\mathrm{TR}}(\Gamma,\mathrm{SL}(3,\mathbb{C}))$ contains only characters of totally reducible representations and it is isomorphic to $\mathbb{C}^2$*:

  $$\left\{(y,\bar y,z,\bar z,\alpha,\bar\alpha,\beta,\bar\beta)\in\mathbb{C}^8 \mid \alpha=\bar\alpha=\beta=\bar\beta=3, z=y^2-2\bar y, \bar z=\bar y^2-2y\right\}.$$

- *The component $X_{\mathrm{PR}}:=X_{\mathrm{PR}}(\Gamma,\mathrm{SL}(3,\mathbb{C}))$ contains only characters of (partially or totally) reducible representations, and it is parametrized by the component $X_2\subset X(\Gamma,\mathrm{GL}(2,\mathbb{C}))$ (see Proposition 3.3). A parametrization is given by*:

$$\begin{aligned}
\alpha(v,w,x_1)&=\bar\alpha(v,w,x_1)=x_1+1,\\
\beta(v,w,x_1)&=\bar\beta(v,w,x_1)=\frac{x_1}{x_1-1}+1,\\
y(v,w,x_1)&=v+\frac{1}{w}, \qquad \bar y(v,w,x_1)=w+\frac{v}{w},\\
z(v,w,x_1)&=w\alpha+\frac{1}{w^2}, \qquad \bar z(v,w,x_1)=\frac{\alpha}{w}+w^2.
\end{aligned}$$

The component X_{PR} is smooth except at the three points $\mu_3\cdot(4,4,8,8,3,3,3,3)$ which are contained in the intersection $X_{\mathrm{TR}}\cap X_{\mathrm{PR}}$. Moreover, $X_{\mathrm{TR}}\cap X_{\mathrm{PR}}$ is isomorphic to the plane curve with equations: $\alpha=\bar\alpha=\beta=\bar\beta=3$, $64-28y\bar y-y^2\bar y^2+5(y^3+\bar y^3)=0$, $z=y^2-2\bar y$, $\bar z=\bar y^2-2y$.

Remark 4.2. It can be checked that the parametrization of X_{PR} is an immersion and that the singularities are nodal, i.e. two branches of the parametrization are smooth and intersect transversely at each of the three singular points $\mu_3 \cdot (4,4,8,8,3,3,3,3)$.

To finish the section, we shall describe the set of reducible characters in $X(\Gamma, \mathrm{SL}(3,\mathbb{C}))$ that lie in the closure of the components of irreducible characters. Recall that the set of reducible characters $X_{\mathrm{red}} = X_{\mathrm{TR}} \cup X_{\mathrm{PR}}$ is Zariski-closed and its complement X_{irr} is Zariski-open.

Lemma 4.3. *The set* $X_{\mathrm{red}}(\Gamma, \mathrm{SL}(3,\mathbb{C})) \cap \overline{X_{\mathrm{irr}}(\Gamma, \mathrm{SL}(3,\mathbb{C}))}$ *is parametrized by a singular curve* $\mathcal{C} \subset X_2$ *given by*

$$\mathcal{C} = \left\{ (w, x_1) \in \mathbb{C}^* \times \big(\mathbb{C} - \{1\}\big) \mid w^6 - 2w^3 \frac{2x_1^2 + x_1 - 1}{x_1 - 1} + 1 = 0 \right\}.$$

More precisely, the curve $\mathcal{C}$ *has exactly three singular points* $\mu_3 \times \{0\}$*. The parametrization is given by restricting the parametrization of Proposition 4.1 to* $\mathcal{C}$*, i.e., by substituting* $v = (w^3+1)/2w$*. In addition, the intersection* $X_{\mathrm{red}}(\Gamma, \mathrm{SL}(3,\mathbb{C})) \cap \overline{X_{\mathrm{irr}}(\Gamma, \mathrm{SL}(3,\mathbb{C}))}$ *are smooth points of* $X_{\mathrm{red}}(\Gamma, \mathrm{SL}(3,\mathbb{C}))$ *and* $\overline{X_{\mathrm{irr}}(\Gamma, \mathrm{SL}(3,\mathbb{C}))}$ *respectively, with the exceptions of the six points* $\mu_3 \cdot (2,2,2,2,1,1,1,1)$ *and* $\mu_3 \cdot (4,4,8,8,3,3,3,3)$*.*

Proof. A reducible and semisimple representation $\rho\colon \Gamma \to \mathrm{SL}(3,\mathbb{C})$ with character χ_ρ can be written as $\rho = (\phi \otimes \varrho) \oplus \phi^{-2}$, with representations $\phi\colon \Gamma \to \mathbb{C}^*$ and $\varrho\colon \Gamma \to \mathrm{SL}(2,\mathbb{C})$. Namely, $\rho(\gamma) = \mathrm{diag}(\phi(\gamma)\varrho(\gamma), \phi(\gamma)^{-2})$, $\forall \gamma \in \Gamma$. Thus

$$v = \chi_\varrho(t)\phi(t) \quad \text{and} \quad w = \phi(t)^2. \tag{11}$$

By [29, Theorem 1.3], if $\chi_\rho \in \overline{X_{\mathrm{irr}}(\Gamma, \mathrm{SL}(3,\mathbb{C}))}$ is partially reducible, then $\xi = \phi(t)$ satisfies

$$\xi^6 - 2\chi_\varrho(t)\xi^3 + 1 = 0. \tag{12}$$

This uses that the Alexander polynomial twisted by ϱ is $\Delta^\varrho(x) = x^2 - 2\chi_\varrho(t)x + 1$. Kitano [33] has computed $\Delta^\varrho(1) = 2 - 2\chi_\varrho(t)$, and the same computation yields $\Delta^\varrho(x)$ as follows: using the fibration of the figure eight knot, we know that $\Delta^\varrho(x) = x^2 - \delta(\chi)x + 1$, for some regular function $\delta : X_{\mathrm{irr}}(\Gamma, \mathrm{SL}(2,\mathbb{C})) \to \mathbb{C}$; in addition δ is determined from $\Delta^\varrho(1) = 2 - 2\chi_\varrho(t)$. This implies that $\xi^3 + \xi^{-3} = 2\chi_\varrho(t)$. This condition is necessary for a partially reducible representation ρ, i.e. when ϱ is irreducible. The representation ϱ is reducible precisely when $\alpha = \beta = 3$ and the condition is also necessary in this case if we can show that

$$X_{\mathrm{TR}}\big(\Gamma, \mathrm{SL}(3,\mathbb{C})\big) \cap \overline{X_{\mathrm{irr}}\big(\Gamma, \mathrm{SL}(3,\mathbb{C})\big)} \subset \overline{X_{\mathrm{PR}}\big(\Gamma, \mathrm{SL}(3,\mathbb{C})\big)}. \tag{13}$$

To prove (13), the discussion in [30] using twisted cohomology yields that the ratio between two eigenvalues of $\rho(t)$ is a root of the untwisted Alexander polynomial t^2-3t+1. By [30], this is also a sufficient condition for a 2×2 block of ρ being approximated by irreducible representations in $\mathrm{GL}(2,\mathbb{C})$, because it is a simple root. Using (11) and (12), we get $w^3-2vw+1=0$, equivalently $v=\frac{w^3+1}{2w}$. By replacing the value of v in $\frac{x_1^2+x_1-1}{x_1-1}w=v^2$ we get the equation of the lemma.

Finally, we notice that when $(\alpha,\beta)\neq(1,1),(3,3)$, [29, Corollary 8.9] applies and the corresponding characters are smooth points of X_{PR} and $\overline{X_{\mathrm{irr}}}$ respectively, as the corresponding roots of the twisted Alexander polynomial are simple. $\square$

From Lemma 4.3 it follows that

$$w^6-2w^3\alpha\frac{2\alpha-3}{\alpha-2}+1=0 \tag{14}$$

on $X_{\mathrm{red}}(\Gamma,\mathrm{SL}(3,\mathbb{C}))\cap\overline{X_{\mathrm{irr}}(\Gamma,\mathrm{SL}(3,\mathbb{C}))}$. In addition, by Proposition 4.1, any $\chi\in X_{\mathrm{red}}(\Gamma,\mathrm{SL}(3,\mathbb{C}))$ satisfies $(\alpha-2)(\beta-2)=1$. Thus we get:

Corollary 4.4. *The fiber of the projection*

$$\begin{aligned}X_{\mathrm{red}}\big(\Gamma,\mathrm{SL}(3,\mathbb{C})\big)\cap\overline{X_{\mathrm{irr}}\big(\Gamma,\mathrm{SL}(3,\mathbb{C})\big)}&\\ \longrightarrow\big\{(\alpha,\beta)\in\mathbb{C}^2\mid(\alpha-2)(\beta-2)=1\big\}&\cong\mathbb{C}-\{2\}\end{aligned}$$

has six points except when $(\alpha,\beta)=(1,1)$, $(3,3)$, *or* $(\frac{1\pm\sqrt5}{2},\frac{1\mp\sqrt5}{2})$, *where it has three points.*

Proof. Notice that $\beta=(2\alpha-1)/(\alpha-2)$. each character in $X_{\mathrm{red}}(\Gamma,\mathrm{SL}(3,\mathbb{C}))\cap\overline{X_{\mathrm{irr}}(\Gamma,\mathrm{SL}(3,\mathbb{C}))}$ determines a unique parameter w with the exception of the three characters with coordinates $\mu_3\cdot(4,4,8,8,3,3,3,3,3)$. To such a singular point correspond two values of $w\in\mu_3\cdot(\frac{3}{2}\pm\frac{\sqrt5}{2})$. In this case $w^3=9\pm4\sqrt5$ and $\alpha=\beta=3$ follows from (14).

The discriminant of (14) is $4(\alpha^2-\alpha-1)(\alpha-1)^2$ and hence it vanishes if and only if $\alpha=\beta=1$ or $\alpha=\frac{1\pm\sqrt5}{2}$ and $\beta=\frac{1\mp\sqrt5}{2}$. $\square$

Remark 4.5. We will see later in Proposition 5.15 that $\mu_3\cdot(2,2,2,2,1,1,1,1)$ are the only singular points of the components of $\overline{X_{\mathrm{irr}}(\Gamma,\mathrm{SL}(3,\mathbb{C}))}$. We already saw that $\mu_3\cdot(2,2,2,2,1,1,1,1)$ are smooth points of X_{PR}. Hence, $\mu_3\cdot(4,4,8,8,3,3,3,3)$ are smooth points on $\overline{X_{\mathrm{irr}}(\Gamma,\mathrm{SL}(3,\mathbb{C}))}$ and X_{TR}, and they are singular on X_{PR}.

5. Restriction to the fibre

In this section and up to Section 8, we work with Presentation (1), corresponding to the fibration over the circle with fibre a punctured torus. In particular, the group of the fibre F_2 is the free group of rank 2 generated by

a and b. To compute $X_{\mathrm{irr}}(\Gamma, G)$ for $G = \mathrm{SL}(3,\mathbb{C})$, $\mathrm{PGL}(3,\mathbb{C})$, or $\mathrm{GL}(3,\mathbb{C})$, we look at the restriction:

$$\mathrm{res} : X(\Gamma, G) \to X\big(F_2, \mathrm{SL}(3,\mathbb{C})\big),$$

whose image does not depend on G. For $X(F_2, \mathrm{SL}(3,\mathbb{C}))$, we use Lawton's description in [34]. According to it, there is a two fold branched covering

$$\pi : X\big(F_2, \mathrm{SL}(3,\mathbb{C})\big) \to \mathbb{C}^8,$$

where the coordinates of $\mathbb{C}^8$ are the traces of

$$a, a^{-1}, b, b^{-1}, ab, b^{-1}a^{-1}, ab^{-1}, a^{-1}b. \tag{15}$$

The branched covering comes from a ninth coordinate, which is the trace of the commutator

$$[a,b] = aba^{-1}b^{-1}.$$

This trace satisfies a polynomial equation

$$x^2 - Px + Q = 0,$$

where P and Q are polynomials on the first eight variables (see [34] for the expression of P and Q). The solutions are precisely the trace of $[a,b]$ and the trace of its inverse.

We shall compute $X_{\mathrm{irr}}(\Gamma, \mathrm{SL}(3,\mathbb{C}))$ in several steps, first we start with $\pi(\mathrm{res}(X(\Gamma, G)))$, from this we describe $\mathrm{res}(X(\Gamma, G))$ as a 2:1 ramified covering, and then we prove that $X_{\mathrm{irr}}(\Gamma, \mathrm{SL}(3,\mathbb{C}))$ is a 3:1 ramified covering of $\mathrm{res}(X(\Gamma, G))$.

To compute $\pi(\mathrm{res}(X(\Gamma, G))) \subset \mathbb{C}^8$, first of all we reduce the eight coordinates to four by using conjugation identities:

$$\begin{aligned}
\alpha &= \chi(a) = \chi(ab),\\
\bar\alpha &= \chi\big(a^{-1}\big) = \chi\big(b^{-1}a^{-1}\big),\\
\beta &= \chi(b) = \chi\big(a^{-1}b\big),\\
\bar\beta &= \chi\big(b^{-1}\big) = \chi\big(ab^{-1}\big).
\end{aligned}$$

LEMMA 5.1. *The projection $\pi(\mathrm{res}(X(\Gamma, G)))$ has three components*:

$$\begin{aligned}
U_0 &= \big\{(\alpha, \bar\alpha, \beta, \bar\beta) \in \mathbb{C}^4 \mid \alpha = \bar\alpha, \beta = \bar\beta\big\},\\
U_1 &= \big\{(\alpha, \bar\alpha, \beta, \bar\beta) \in \mathbb{C}^4 \mid \alpha = \bar\alpha = 1\big\},\\
U_2 &= \big\{(\alpha, \bar\alpha, \beta, \bar\beta) \in \mathbb{C}^4 \mid \beta = \bar\beta = 1\big\}.
\end{aligned}$$

Proof. We mimic the proof for $\mathrm{SL}(2,\mathbb{C})$ in Section 3, that is, we look at the fixed points in $\mathbb{C}^8$ of the action of the monodromy (conjugation by t). Since for each character there is a unique semisimple representation, there is always a matrix that realizes the conjugation, and this can be taken as the image of t for extending the character to Γ. Four of the identities that we get by looking at the fixed points are equivalent to the reduction from eight to four coordinates. We discuss the other four identities.

The first identity comes from taking traces on the image of

$$tabt^{-1} = ab^2ab.$$

To express the trace of ab^2ab using Lawton's coordinates we use the $\mathrm{SL}(3,\mathbb{C})$ analog of the identities for $\mathrm{SL}(2,\mathbb{C})$ in (8). Denote the images in $\mathrm{SL}(3,\mathbb{C})$ by capital letters. From the characteristic polynomial identity, we have

$$(BA)^3 - \alpha(BA)^2 + \bar{\alpha}BA - \mathrm{Id} = 0.$$

Multiplying by A^{-1} we get

$$BABAB = \alpha BAB - \bar{\alpha}B + A^{-1}.$$

Thus, by taking traces we get the equation

$$\alpha = \alpha\beta - \bar{\alpha}\beta + \bar{\alpha}. \tag{16}$$

Similarly, for $t(ab)^{-1}t^{-1} = (ab^2ab)^{-1}$ we get

$$\bar{\alpha} = \bar{\alpha}\bar{\beta} - \alpha\bar{\beta} + \alpha. \tag{17}$$

The third identity comes from $tbt^{-1} = bab$. From the characteristic polynomial identity, we have

$$B^3 - \beta B^2 + \bar{\beta}B - \mathrm{Id} = 0.$$

Multiplying by $B^{-1}A$ we get

$$B^2A = \beta BA - \bar{\beta}A + B^{-1}A.$$

Thus, we get the equation

$$\beta = \beta\alpha - \bar{\beta}\alpha + \bar{\beta}. \tag{18}$$

Similarly, from $tb^{-1}t^{-1} = (bab)^{-1}$ we get

$$\bar{\beta} = \bar{\beta}\bar{\alpha} - \beta\bar{\alpha} + \beta. \tag{19}$$

The statement follows from identities (16)–(19). Notice that we do not need to compute more identities because of [34], and because $t[a,b]t^{-1} = [a,b]$. □

To get all the ambient coordinates we need a new variable:

$$\eta(\chi) = \chi\big([a,b]\big).$$

We know by [34] that

$$\eta^2 - P\eta + Q = 0, \tag{20}$$

for some polynomials $P, Q \in \mathbb{Z}[\alpha,\beta,\bar{\alpha},\bar{\beta}]$. Using Lemma 5.1 and by replacing the values of P and Q in [34], we obtain the following lemma.

LEMMA 5.2. *$W = \mathrm{res}(X(\Gamma,G))$ has three components W_0, W_1 and W_2, each W_i being a two-fold ramified covering of U_i according to* (20).

- *For W_0 the polynomials and the discriminant are*

$$\begin{aligned} Q &= \alpha^4\beta^2 + \alpha^2\beta^4 - 2\alpha^4\beta - 4\alpha^3\beta^2 - 4\alpha^2\beta^3 - 2\alpha\beta^4 + \alpha^4 \\ &\quad + 2\alpha^3\beta + 12\alpha^2\beta^2 + 2\alpha\beta^3 + \beta^4 + 4\alpha^3 + 4\beta^3 - 12\alpha^2 - 12\beta^2 + 9, \\ P &= \alpha^2\beta^2 - 2\alpha^2\beta - 2\alpha\beta^2 + 2\alpha^2 + 2\beta^2 - 3, \\ P^2 - 4Q &= \big(\alpha^2\beta^2 - 6\alpha\beta - 4\alpha - 4\beta - 3\big)(\alpha\beta - 2\alpha - 2\beta + 3)^2. \end{aligned}$$

- *For W_1, we have*

$$\begin{aligned} Q &= \beta^3 + \bar\beta^3 - 3\beta\bar\beta + 2, \\ P &= \beta\bar\beta - \beta - \bar\beta - 1, \\ P^2 - 4Q &= \beta^2\bar\beta^2 - 4\beta^3 - 4\bar\beta^3 - 2\beta^2\bar\beta - 2\beta\bar\beta^2 \\ &\quad + \beta^2 + \bar\beta^2 + 12\beta\bar\beta + 2\beta + 2\bar\beta - 7. \end{aligned}$$

- *For W_2, we have*

$$\begin{aligned} Q &= \alpha^3 + \bar\alpha^3 - 3\alpha\bar\alpha + 2, \\ P &= \alpha\bar\alpha - \alpha - \bar\alpha - 1, \\ P^2 - 4Q &= \alpha^2\bar\alpha^2 - 4\alpha^3 - 4\bar\alpha^3 - 2\alpha^2\bar\alpha - 2\alpha\bar\alpha^2 \\ &\quad + \alpha^2 + \bar\alpha^2 + 12\alpha\bar\alpha + 2\alpha + 2\bar\alpha - 7. \end{aligned}$$

We describe the image of the set of reducible characters.

LEMMA 5.3. *The image* $\operatorname{res}(X_{\mathrm{red}}(\Gamma, G)) \subset W$ *is the curve*

$$\big\{(\alpha,\beta,\eta) \in W_0 \mid \alpha\beta - 2\alpha - 2\beta + 3 = 0\big\}.$$

All characters in this curve are partially reducible, except at the point $\alpha = \beta = 3$, that corresponds to totally reducible representations.

In particular, the only point in W_1 and W_2 that is restriction of reducible representations is precisely their common intersection $W_1 \cap W_2$ which is the point given by $\alpha = \bar\alpha = \beta = \bar\beta = 1$ and $\eta = -1$. This point lies also on W_0.

Proof. We know from Section 4 that if x_1 and x_2 denote the traces in $\mathrm{SL}(2,\mathbb{C})$ of the images of a and b, then the image of $X(\Gamma, \mathrm{SL}(2,\mathbb{C}))$ in $X(F_2, \mathrm{SL}(2,\mathbb{C}))$ is given by the equation $x_1x_2 - x_1 - x_2 = 0$. Now the lemma follows from the identities $\alpha = \bar\alpha = x_1 + 1$, $\beta = \bar\beta = x_2 + 1$. □

REMARK 5.4. The discriminant locus of W_0 contains $\alpha\beta - 2\alpha - 2\beta + 3 = 0$. It contains a second component:

$$\alpha^2\beta^2 - 6\alpha\beta - 4\alpha - 4\beta - 3 = 0.$$

This is the set of characters of the representations obtained as composition of representations of Γ in $\mathrm{SL}(2,\mathbb{C})$ with $\mathrm{Sym}^2\colon\ \mathrm{SL}(2,\mathbb{C}) \to \mathrm{SL}(3,\mathbb{C})$. This may be seen with the same argument as in Lemma 5.3, but using that $\alpha = \bar\alpha = x_1^2 - 1$ and $\beta = \bar\beta = x_2^2 - 1$.

Corollary 4.4 yields the following.

COROLLARY 5.5. *The fiber of*

$$\begin{aligned}\operatorname{res}: X_{\mathrm{red}}\big(\Gamma, \mathrm{SL}(3,\mathbb{C})\big) \cap \overline{X_{\mathrm{irr}}\big(\Gamma, \mathrm{SL}(3,\mathbb{C})\big)} \\ \longrightarrow \big\{(\alpha,\beta,\eta)\in W_0 \mid \alpha\beta-2\alpha-2\beta+3=0\big\}\end{aligned}$$

has cardinality 6, *except for* $(\alpha,\beta)=(1,1)$, $(3,3)$, *or* $(\frac{1\pm\sqrt{5}}{2},\frac{1\mp\sqrt{5}}{2})$, *where it has cardinality* 3.

REMARK 5.6. The intersection of the two components of the discriminant (see Remark 5.4) is the pair of points $(\alpha,\beta)=\{(3,3),(\frac{1\pm\sqrt{5}}{2},\frac{1\mp\sqrt{5}}{2})\}$. The third point $(\alpha,\beta)=(1,1)$ in Corollary 5.5 yields singular points in V_0 (defined in Proposition 5.9).

A character of Γ that is irreducible may have a restriction to F_2 that is reducible.

LEMMA 5.7. *The characters of* $\operatorname{res}(X_{\mathrm{irr}}(\Gamma,G))$ *that are* F_2*-reducible are characters of representations* ρ *whose restriction to* F_2 *is totally reducible, and* $\rho(t)$ *acts as a cyclic permutation of the invariant subspaces of* $\rho(F_2)$. *In particular* ρ *is metabelian and* $\operatorname{tr}(\rho(t^{\pm1}))=0$.

Proof. Let $\rho\in R(\Gamma,G)$ be an irreducible representation whose restriction to F_2 is reducible. If $\rho|_{F_2}$ was partially reducible then, as the invariant subspaces would have different dimension, they should also be invariant by $\rho(t)$, and therefore ρ would be reducible. Thus, the only possibility is that $\rho|_{F_2}$ is totally reducible and $\rho(t)$ cyclically permutes its three invariant subspaces. □

Irreducible metabelian characters in $\mathrm{SL}(3,\mathbb{C})$ have been studied by Boden and Friedl in [6], [7], who prove that in $X(\Gamma,\mathrm{SL}(3,\mathbb{C}))$ there are five of them.

COROLLARY 5.8. *There are five characters in* $\operatorname{res}(X_{\mathrm{irr}}(\Gamma,G))$ *that are* F_2*-reducible: one in* W_0, $\alpha=\beta=-1$; *two in* W_1, $(\beta,\bar\beta)=(-1\pm2i,-1\mp2i)$; *and two in* W_2, $(\alpha,\bar\alpha)=(-1\pm2i,-1\mp2i)$. *In all cases* $\eta=3$.

Those yield precisely five characters in $X(\Gamma,G)$ for $G=\mathrm{SL}(3,\mathbb{C})$ and $G=\mathrm{PSL}(3,\mathbb{C})$ (they satisfy $y=\bar y=z=\bar z=0$) and five subvarieties for $G=\mathrm{GL}(3,\mathbb{C})$. To study further the fibre of res: $X_{\mathrm{irr}}(\Gamma,\mathrm{SL}(3,\mathbb{C}))\to W$, notice that the fibre of F_2-irreducible characters in W has precisely three points. Namely, if $\rho\in R(\Gamma,\mathrm{SL}(3,\mathbb{C}))$ satisfies that $\rho|_{F_2}$ is irreducible, the relations in (1) yield that $\rho(t)$ is unique up to the action of μ_3, the center of $\mathrm{SL}(3,\mathbb{C})$. It is straightforward to check that those three choices yield the same character if and only if ρ is a metabelian irreducible representation as in Lemma 5.7. Thus, the five characters in Corollary 5.8 are the ramification points of res: $X_{\mathrm{irr}}(\Gamma,\mathrm{SL}(3,\mathbb{C}))\to W$. Notice that those five characters also lie in the zero locus of the discriminant $P^2-4Q=0$, as they satisfy $\eta=3$.

In [7] Boden and Friedl prove that these representations are smooth points of $X(\Gamma,\mathrm{SL}(3,\mathbb{C}))$, so $\overline{X_{\mathrm{irr}}(\Gamma,\mathrm{SL}(3,\mathbb{C}))}$ has the same number of components as W. Summarizing, we have the following proposition.

PROPOSITION 5.9. *The set* $\overline{X_{\mathrm{irr}}(\Gamma,\mathrm{SL}(3,\mathbb{C}))}$ *has three components* V_0, V_1 *and* V_2, *that are respective* 3:1 *branched covers of* W_0, W_1 *and* W_2.

The branching points in $X_{\mathrm{irr}}(\Gamma,\mathrm{SL}(3,\mathbb{C}))$ *are the five metabelian irreducible characters in Corollary* 5.8.

If we add the variables

$$y(\chi)=\chi(t),\qquad \bar y(\chi)=\chi\big(t^{-1}\big),$$

then the variables $(\alpha,\bar\alpha,\beta,\bar\beta,\eta,y,\bar y)$ describe $X_{\mathrm{irr}}(\Gamma,\mathrm{SL}(3,\mathbb{C}))$, and the map

$$(\alpha,\bar\alpha,\beta,\bar\beta,\eta,y,\bar y)\mapsto(\alpha,\bar\alpha,\beta,\bar\beta,\eta)$$

is three-to-one except at $(y,\bar y)=(0,0)$. Actually, it is the quotient by the action of the center $(y,\bar y)\mapsto(\varpi y,\varpi^2\bar y)$, where $\varpi\in\mu_3$. Notice that in Lemma 4.3 we have shown that reducible characters in $\overline{X_{\mathrm{irr}}(\Gamma,\mathrm{SL}(3,\mathbb{C}))}$ are also determined by the values of y and $\bar y$. Thus, using Proposition 2.3, we get the following.

PROPOSITION 5.10. *The parameters* $(\alpha,\bar\alpha,\beta,\bar\beta,\eta,y,\bar y)$ *describe the set of irreducible characters* $X_{\mathrm{irr}}(\Gamma,\mathrm{SL}(3,\mathbb{C}))$ *pointwise* (*that is, they describe a variety homeomorphic to* $X_{\mathrm{irr}}(\Gamma,\mathrm{SL}(3,\mathbb{C}))$ *in the classical topology*). *The parameters* $(y,\bar y,z,\bar z,\alpha,\bar\alpha,\beta,\bar\beta)$, *with* $z=\chi(ta^{-1}ta)$ *and* $\bar z=\chi(a^{-1}t^{-1}at^{-1})$, *describe* $X_{\mathrm{irr}}(\Gamma,\mathrm{SL}(3,\mathbb{C}))$ *scheme-theoretically.*

REMARK 5.11. The set $X_{\mathrm{irr}}(\Gamma,\mathrm{SL}(3,\mathbb{C}))$ carries two topologies, the Zariski-topology and the classical topology. Following Lawton, we can identify the quasi-affine variety $X_{\mathrm{irr}}(\Gamma,\mathrm{SL}(3,\mathbb{C}))$ with a subset of $\mathbb{C}^8$ where the corresponding parameters are given by $(y,\bar y,z,\bar z,\alpha,\bar\alpha,\beta,\bar\beta)$ (see Proposition 2.3). Here we say *scheme-theoretically* because the natural defining ideal is a priori non-radical. Proposition 5.10 shows that $X_{\mathrm{irr}}(\Gamma,\mathrm{SL}(3,\mathbb{C}))$ is homeomorphic (classical topology) to a subset of $\mathbb{C}^7$ where the corresponding parameters are given by $(\alpha,\bar\alpha,\beta,\bar\beta,\eta,y,\bar y)$. Notice that by Lawton, η is a polynomial function of $(y,\bar y,z,\bar z,\alpha,\bar\alpha,\beta,\bar\beta)$. The restriction of the polynomial map $f\colon\mathbb{C}^8\to\mathbb{C}^7$ given by $f(y,\bar y,z,\bar z,\alpha,\bar\alpha,\beta,\bar\beta)=(\alpha,\bar\alpha,\beta,\bar\beta,\eta,y,\bar y)$ to $X_{\mathrm{irr}}(\Gamma,\mathrm{SL}(3,\mathbb{C}))$ is polynomial and bijective, but its inverse is not polynomial. More precisely, z and $\bar z$ are not polynomial functions in $(\alpha,\bar\alpha,\beta,\bar\beta,\eta,y,\bar y)$.

We consider the projection $V_0\to\mathbb{C}^2$ to the plane with coordinates (α,β), namely $\chi\mapsto(\chi(a),\chi(b))$. This is the composition of the projections $\mathrm{res}\colon V_0\to W_0$ and $\pi\colon W_0\to U_0\cong\mathbb{C}^2$. We will need later the computation of the cardinality of its fibre. From Corollary 5.5 (for reducible representations), Lemma 5.2 and Proposition 5.9, we have the following lemma.

LEMMA 5.12. *For $(\alpha,\beta)\in\mathbb{C}^2$, the cardinality of the fibre of $\pi\circ\mathrm{res}\colon V_0\to\mathbb{C}^2$ is*

$$\left|(\pi\circ\mathrm{res})^{-1}(\alpha,\beta)\right| = \begin{cases} 1 & \text{if } (\alpha,\beta)=(-1,-1),\\ 3 & \text{if } \alpha^2\beta^2-6\alpha\beta-4\alpha-4\beta-3=0 \text{ and}\\ & (\alpha,\beta)\neq(-1,-1), \text{ or if } (\alpha,\beta)=(1,1),\\ 6 & \text{otherwise.}\end{cases}$$

The Zariski tangent space at a character χ is denoted $T_\chi^{Zar}X(\Gamma,\mathrm{SL}(3,\mathbb{C}))$, without assuming that it is scheme-reduced.

LEMMA 5.13. *Let $\chi\in X(\Gamma,\mathrm{SL}(3,\mathbb{C}))$ be a character such that $\mathrm{res}(\chi)$ is irreducible. The restriction map induces an isomorphism*

$$T_\chi^{\mathrm{Zar}}X\big(\Gamma,\mathrm{SL}(3,\mathbb{C})\big)\cong T^{\mathrm{Zar}}_{\mathrm{res}(\chi)}\big(X\big(F_2,\mathrm{SL}(3,\mathbb{C})\big)^{\phi^*}\big),$$

where $\phi\colon F_2\to F_2$ denotes the action of the monodromy (conjugation by t), ϕ^* the induced map, and $X(F_2,\mathrm{SL}(3,\mathbb{C}))^{\phi^*}$ the fixed point set.

Proof. We have a natural isomorphism:

$$T^{\mathrm{Zar}}_{\mathrm{res}(\chi)}\big(X\big(F_2,\mathrm{SL}(3,\mathbb{C})\big)^{\phi^*}\big)\cong\big(T^{\mathrm{Zar}}_{\mathrm{res}(\chi)}X\big(F_2,\mathrm{SL}(3,\mathbb{C})\big)\big)^{\phi^*},$$

where ϕ^* still denotes the tangent map. By Weil's theorem, those Zariski tangent spaces (as schemes) are naturally isomorphic to cohomology groups:

$$T_\chi^{\mathrm{Zar}}X\big(\Gamma,\mathrm{SL}(3,\mathbb{C})\big)\cong H^1\big(\Gamma,\mathfrak{sl}(3,\mathbb{C})_{\mathrm{Ad}\,\rho}\big),$$
$$T^{\mathrm{Zar}}_{\mathrm{res}(\chi)}X\big(F_2,\mathrm{SL}(3,\mathbb{C})\big)\cong H^1\big(F_2,\mathfrak{sl}(3,\mathbb{C})_{\mathrm{Ad}\,\rho}\big),$$

for any representation ρ with character χ (cf. [38]), where $\mathrm{Ad}\,\rho$ denotes the adjoint representation. We let ϕ^* denote also the induced map in cohomology, as it corresponds to the tangent map. The Lyndon-Hochschild-Serre spectral sequence applied to $1\to F_2\to\Gamma\to\mathbb{Z}=\langle t\rangle\to 1$ yields the exact sequence:

$$H^1\big(\mathbb{Z},\mathfrak{sl}(3,\mathbb{C})^{F_2}_{\mathrm{Ad}\,\rho}\big)\to H^1\big(\Gamma,\mathfrak{sl}(3,\mathbb{C})_{\mathrm{Ad}\,\rho}\big)$$
$$\to H^1\big(F_2,\mathfrak{sl}(3,\mathbb{C})_{\mathrm{Ad}\,\rho}\big)^{\phi^*}\to H^2\big(\mathbb{Z},\mathfrak{sl}(3,\mathbb{C})^{F_2}_{\mathrm{Ad}\,\rho}\big),$$

cf. [48, 6.8.3]. Thus it suffices to show that the invariant subspace $\mathfrak{sl}(3,\mathbb{C})^{F_2}_{\mathrm{Ad}\,\rho}$ is trivial. By contradiction, assume that there exists a nonzero $\theta\in\mathfrak{sl}(3,\mathbb{C})$ which is F_2-invariant, namely $\mathrm{Ad}_{\rho(a)}(\theta)=\mathrm{Ad}_{\rho(b)}(\theta)=\theta$. This implies that $\exp(\lambda\theta)$ commutes with $\rho(a)$ and $\rho(b)$, for every $\lambda\in\mathbb{R}$, hence the restriction $\rho|_{F_2}$ is reducible. □

REMARK 5.14. The singular locus *as schemes* of the components of W is the set of F_2-reducible characters of W, as this is the singular locus of the discriminant in Lemma 5.2. Hence, it is the union of the five ramification points in Corollary 5.8 and the curve $\mathrm{res}(X_{\mathrm{red}}(\Gamma,\mathrm{SL}(3,\mathbb{C})))$ in Lemma 5.3.

In the previous remark, we say *as schemes* because during the computation of U_i and W_i we just use traces on group relations and Lawton's theorem, and we never compute radicals of the ideals.

PROPOSITION 5.15. *The components V_0, V_1 and V_2 are smooth (and scheme reduced) everywhere except possibly at $(\alpha,\beta)=(1,1)$.*

Scheme reduced at one point means that the local ring is reduced. This property, together with smoothness, holds when the dimension of the Zariski-tangent space equals the dimension of the irreducible component.

Proof. Let $\chi\in V_0\cup V_1\cup V_2$ be an irreducible character. If χ is one of the five ramification points of Corollary 5.8, then it is smooth and scheme reduced by [7]. Otherwise, $\chi|_{F_2}$ is irreducible, thus Lemma 5.13 applies and we get that $T^{\mathrm{Zar}}_{\chi}X(\Gamma,\mathrm{SL}(3,\mathbb{C}))$ is isomorphic to $T^{\mathrm{Zar}}_{\mathrm{res}(\chi)}(X(F_2,\mathrm{SL}(3,\mathbb{C}))^{\phi^*})$. In particular, if $\chi\notin V_0\cap(V_1\cup V_2)$, then $T^{\mathrm{Zar}}_{\chi}V_i$ is isomorphic to $T^{\mathrm{Zar}}_{\mathrm{res}(\chi)}W_i$ if $\chi\in V_i$, and by Remark 5.14 it is a two dimensional space. Hence, $\chi\in V_i$ is a smooth point.

For irreducible characters in $V_0\cap V_1$ or in $V_0\cap V_2$, the cohomology groups yield the tangent spaces to $V_0\cup V_1\cup V_2$ and $W=W_0\cup W_1\cup W_2$, thus we need to add linear conditions on the ambient coordinates to distinguish components ($\alpha=\bar\alpha$, $\beta=\bar\beta$; $\alpha=1$; or $\beta=1$). From those linear conditions, we easily get the dimension of the Zariski tangent space.

Reducible characters are discussed in Lemma 4.3, that yields smoothness for each reducible character in V_0 except for $\alpha=\beta=1$ and $\alpha=\beta=3$. Smoothness for this last point can be checked using the arguments of [28, Thm. 1.3]. □

6. Description of the non-distinguished components

Now we move to get equations for V_0,V_1,V_2. We name V_0 the *distinguished component* of the character variety $X(\Gamma,\mathrm{SL}(3,\mathbb{C}))$ since it is the one containing representations coming from $\mathrm{Sym}^2:\mathrm{SL}(2,\mathbb{C})\to\mathrm{SL}(3,\mathbb{C})$. Accordingly, V_1,V_2 will be called non-distinguished components.

In this section, we find explicit equations of the non-distinguished components V_1,V_2. We start with V_2. We keep on working with Presentation (1) and using capitals to denote the images of a, b and $t\in\Gamma$ in $\mathrm{SL}(3,\mathbb{C})$.

PROPOSITION 6.1. *The non-distinguished component V_2 is described as follows. Take coordinates $\alpha=\mathrm{tr}(A)$, $\bar\alpha=\mathrm{tr}(A^{-1})$, $\beta=\mathrm{tr}(B)$, $\bar\beta=\mathrm{tr}(B^{-1})$, $\eta=\mathrm{tr}([A,B])$, $y=\mathrm{tr}(T)$, $\bar y=\mathrm{tr}(T^{-1})$, $z=\mathrm{tr}(TA^{-1}TA)$, $\bar z=\mathrm{tr}(A^{-1}T^{-1}AT^{-1})$, the equations satisfied by V_2 are $\beta=\bar\beta=1$, $y\bar y=\alpha+\bar\alpha+2$, $y^3+\bar y^3=\alpha\bar\alpha+5\alpha+5\bar\alpha+5$. $\eta=\bar y^3-3(\alpha+\bar\alpha+1)$, $z=y^2-\bar y$, $\bar z=\bar y^2-y$.*

The only reducible representations are given by $(y,\bar y,z,\bar z,\alpha,\bar\alpha,\beta,\bar\beta)\in\mu_3\cdot(2,2,2,2,1,1,1,1)$ and are partially reducible.

Proof. By Lemma 5.2, W_2 is described as the double cover of the plane $(\alpha,\bar\alpha)$, ramified over the curve $\Delta=0$, where

$$\Delta=\alpha^2\bar\alpha^2-4\alpha^3-4\bar\alpha^3-2\alpha^2\bar\alpha-2\alpha\bar\alpha^2+\alpha^2+\bar\alpha^2+12\alpha\bar\alpha+2\alpha+2\bar\alpha-7.$$

Thus the ring of functions of W_2 is $\mathbb{Q}[\alpha,\bar\alpha][\sqrt{\Delta}]$. We have $\beta=\bar\beta=1$, so the matrix B has eigenvalues $1,i,-i$. We fix a basis that diagonalizes B, so that we can write

$$B=\begin{pmatrix}1&0&0\\0&i&0\\0&0&-i\end{pmatrix}.$$

Assume that the matrix A has non-zero entries $(2,1)$ and $(3,1)$. Rescaling the basis vectors, we can write

$$A=\begin{pmatrix}a&b&c\\1&d&e\\1&f&g\end{pmatrix}.$$

Solving the equations $\operatorname{tr}(A)=\operatorname{tr}(AB)$, $\operatorname{tr}(B)=\operatorname{tr}(BA^{-1})$, $\operatorname{tr}(B)=\operatorname{tr}(B^2A)$, the equations for the inverses $\operatorname{tr}(A^{-1})=\operatorname{tr}(A^{-1}B^{-1})$, $\operatorname{tr}(B^{-1})=\operatorname{tr}(B^{-1}A)$, $\operatorname{tr}(B^{-1})=\operatorname{tr}(B^{-2}A^{-1})$, together with $\det(A)=1$, $\alpha=\operatorname{tr}(A)$ and $\bar\alpha=\operatorname{tr}(A^{-1})$, we get the coefficients [27]

$$A=\begin{pmatrix}\frac{\alpha+1}{2}&\frac{1-i}{8}(\alpha^2-2\bar\alpha+1)&\frac{1+i}{8}(\alpha^2-2\bar\alpha+1)\\1&\frac{1-i}{4}(\alpha-1)&\frac{1+i}{4}\frac{(\alpha^3-\alpha^2-4\alpha\bar\alpha-4\alpha+5+2i\sqrt{\Delta})}{\alpha^2-2\bar\alpha+1}\\1&\frac{1-i}{4}\frac{(\alpha^3-\alpha^2-4\alpha\bar\alpha-4\alpha+5-2i\sqrt{\Delta})}{\alpha^2-2\bar\alpha+1}&\frac{1+i}{4}(\alpha-1)\end{pmatrix}.$$

This matrix is well-defined off the set $\alpha^2-2\bar\alpha+1=0$.

Now, solving the equations $TA=ABT$ and $TB=BABT$, we get a one-dimensional space of matrices T spanned by

$$T_0=\begin{pmatrix}
(\alpha^2-2\bar\alpha+1)(((1+i)\alpha+3-i)\sqrt{\Delta}-(2-2i)\bar\alpha^2-(6-6i)\alpha\bar\alpha+(2-2i)\bar\alpha-(1+i)\alpha^3+(-1+3i)\alpha^2+(3-i)\alpha+(7-3i)) & \frac14(\alpha^2-2\bar\alpha+1)^2((-1-i)\sqrt{\Delta}+2i\alpha^2+(1-i)\alpha+(1-i)\alpha\bar\alpha+(5-5i)\bar\alpha-(5-5i)) & \frac14(\alpha^2-2\bar\alpha+1)((-(1+i)\alpha^2-(4-4i)\alpha-(2+2i)\bar\alpha+(7-i))\sqrt{\Delta}+(6-6i)\alpha\bar\alpha^2+(-2-6i)\bar\alpha^2-(1-i)\alpha^3\bar\alpha+(3+5i)\alpha^2\bar\alpha+(1+15i)\alpha\bar\alpha-(3-11i)\bar\alpha+2\alpha^4+(-1-7i)\alpha^3+(-5-5i)\alpha^2+(11+13i)\alpha+(-3-19i)) \\
-2(\alpha^2-4i\alpha+2\bar\alpha-(3-4i))\sqrt{\Delta}-12i\alpha\bar\alpha^2-(8+4i)\bar\alpha^2+2i\alpha^3\bar\alpha+(-8+2i)\alpha^2\bar\alpha+(16+14i)\alpha\bar\alpha+(8+14i)\bar\alpha+(2-2i)\alpha^4+(-8-6i)\alpha^3+(-4+6i)\alpha^2+(1-12i)\alpha-(8-11i) & (\frac12(\alpha^2-2\bar\alpha+1)((2\alpha+2-4i)\sqrt{\Delta}+4i\bar\alpha^2+(8+4i)\alpha\bar\alpha-4i\bar\alpha-2\alpha^3+(2+4i)\alpha^2+(2-4i)\alpha-(10+4i)) & 4(\bar\alpha\alpha+\alpha+\bar\alpha-3)\sqrt{\Delta}+i\alpha^5+10i\alpha^3-15i\alpha+4i\bar\alpha^3+4i\alpha^2\bar\alpha^2+12i\alpha\bar\alpha^2+4i\bar\alpha^2-i\alpha^4\bar\alpha+4i\alpha^3\bar\alpha-6i\alpha^2\bar\alpha+36i\alpha\bar\alpha+17i\bar\alpha+32 \\
(\alpha^2-2\bar\alpha+1)(-2i\sqrt{\Delta}+(-2+2i)\alpha^2+2\alpha+2\alpha\bar\alpha+(6-4i)\bar\alpha-(8-2i)) & (\alpha^2-2\bar\alpha+1)^2(\alpha-\bar\alpha-4i) & \frac12(\alpha^2-2\bar\alpha+1)((2i\alpha+4+2i)\sqrt{\Delta}-2i\alpha^3+(-4+2i)\alpha^2+(-4+8i)\alpha\bar\alpha+(4+2i)\alpha+2i\bar\alpha^2-2i\bar\alpha-(5+2i))
\end{pmatrix}.$$

Let $y_0=\operatorname{tr}(T_0)$, $\bar y_0=\det(T_0)\operatorname{tr}(T_0^{-1})$, $d_0=\det(T_0)$. It is readily computed that

$$\begin{aligned}
y_0 = {}& 2\big(\alpha^2-2\bar\alpha+1\big)\big(((1+i)\alpha+3-i)\sqrt{\Delta}-(1+i)\alpha^3+(-1+3i)\alpha^2+(2+6i)\alpha\bar\alpha\\
&+(3-i)\alpha+(-2+2i)\bar\alpha^2+(2-2i)\bar\alpha-(7-7i)\big),\\
\bar y_0 = {}& 4\big(\alpha^2-2\bar\alpha+1\big)^2\big((-i\alpha^4-4\alpha^3+6i\alpha^2+(4i-4)\alpha^2\bar\alpha+12\alpha+(2+4i)4\alpha\bar\alpha\\
&+4i\bar\alpha^2+(4i-4)\bar\alpha-(33i+8))\sqrt{\Delta}+i\alpha^5\bar\alpha-3i\alpha^5+(4-3i)\alpha^4\bar\alpha\\
&-(12+11i)\alpha^4+(4-4i)\alpha^3\bar\alpha^2+(-8+6i)\alpha^3\bar\alpha+(-28+18i)\alpha^3\\
&-(20+4i)\alpha^2\bar\alpha^2+(16+46i)\alpha^2\bar\alpha+(20+2i)\alpha^2-(16+4i)\alpha\bar\alpha^3\\
&-(20+8i)\alpha\bar\alpha^2+(120-35i)\alpha\bar\alpha+(12-35i)\alpha+(-16+12i)\bar\alpha^3\\
&+(4-8i)\bar\alpha^2+(28+9i)\bar\alpha-(88-21i)\big),\\
d_0 = {}& 8(1-i)\big(\alpha^2-2\bar\alpha+1\big)^3\big((-\alpha^6+(6+6i)\alpha^5+(4+6i)\alpha^4\bar\alpha+(17-36i)\alpha^4\\
&+(8-24i)\alpha^3\bar\alpha-(84+20i)\alpha^3+(12-24i)\alpha^2\bar\alpha^2-(72-60i)\alpha^2\bar\alpha\\
&+(21+120i)\alpha^2-(72+72i)\alpha\bar\alpha^2+(136+104i)\alpha\bar\alpha+(134-122i)\alpha-8i\bar\alpha^3\\
&-(36-24i)\bar\alpha^2+(116+22i)\bar\alpha-(189+36i))\sqrt{\Delta}+\alpha^7\bar\alpha-7\alpha^7+(3+42i)\alpha^6\\
&-(1+6i)\alpha^6\bar\alpha+(-8-6i)\alpha^5\bar\alpha^2+(37+24i)\alpha^5\bar\alpha+(109-42i)\alpha^5\\
&+(-32+66i)\alpha^4\bar\alpha^2+(107-190i)\alpha^4\bar\alpha-(201+152i)\alpha^4+(4+24i)\alpha^3\bar\alpha^3\\
&+(76-12i)\alpha^3\bar\alpha^2-(369+104i)\alpha^3\bar\alpha-(97-396i)\alpha^3+(156-96i)\alpha^2\bar\alpha^3\\
&-(236+4i)\alpha^2\bar\alpha^2+(337+590i)\alpha^2\bar\alpha+(373-46i)\alpha^2+(48+8i)\alpha\bar\alpha^4\\
&-(132+200i)\alpha\bar\alpha^3-(260-530i)\alpha\bar\alpha^2+(395-704i)\alpha\bar\alpha-(309+426i)\alpha\\
&+(48-56i)\bar\alpha^4-(92-144i)\bar\alpha^3+(76+98i)\bar\alpha^2+(69-378i)\bar\alpha-(95-500i)\big).
\end{aligned}$$

The choices $T\in \mathrm{SL}(3,\mathbb{C})$ are of the form $\varpi^k d_0^{-1/3}T_0$, $\varpi=e^{2\pi i/3}$, $k=0,1,2$. Therefore, the coordinates $y=\operatorname{tr}(T)$, $\bar y=\operatorname{tr}(T^{-1})$ are given by $y=y_0 d_0^{-1/3}$, $\bar y=\bar y_0 d_0^{-2/3}$. A computer calculation gives [27]

$$\begin{aligned}
y\bar y &= \alpha+\bar\alpha+2,\\
y^3 &= \frac12(\alpha\bar\alpha+5\alpha+5\bar\alpha+5)+\frac12\sqrt{\Delta},\\
\bar y^3 &= \frac12(\alpha\bar\alpha+5\alpha+5\bar\alpha+5)-\frac12\sqrt{\Delta},\\
\eta=\operatorname{tr}\big([A,B]\big) &= \frac12(\alpha\bar\alpha-\alpha-\bar\alpha-1)-\frac12\sqrt{\Delta}.
\end{aligned}$$

Note that we only compute quantities that are μ_3-invariant. First, we note that $\eta=\bar y^3-3(\alpha+\bar\alpha+1)$.

The above equations describe a Zariski open set defined by $d_0\neq 0$, $\alpha^2-2\bar\alpha+1\neq 0$. When we approach a point on the set $d_0=0$, the matrix T_0 above becomes singular. But the normalized matrix $T=d_0^{-1/3}T_0$ has values

$y = \mathrm{tr}(T), \bar{y} = \mathrm{tr}(T^{-1})$ that tend to finite well-defined numbers. Moreover, the representations of F_2 given by (A, B) and (BA, BAB) are conjugated since they have the same Lawton coordinates. And as they are irreducible, the conjugation matrix T is uniquely defined (up to the action of μ_3). This gives well-defined $(y, \bar{y})$ up to μ_3, and by continuity, they satisfy the equations above.

For dealing with the set $\alpha^2 - 2\bar{\alpha} + 1 = 0$, we can argue by continuity as above. We can also parametrize the set of matrices A in which either the entry $(2,1)$ is zero or the entry $(3,1)$ is zero. This gives the matrices:

$$\begin{pmatrix} \frac{\alpha+1}{2} & 0 & \frac{1}{8}(-\alpha^3 - \alpha^2 - 3\alpha + 5) \\ 1 & (\frac{1}{4} - \frac{i}{4})(\alpha - 1) & \frac{1}{8}(-\alpha^2 - 2\alpha - 5) \\ 0 & 1 & (\frac{1}{4} + \frac{i}{4})(\alpha - 1) \end{pmatrix},$$

$$\begin{pmatrix} \frac{\alpha+1}{2} & \alpha - 1 & 0 \\ 0 & (\frac{1}{4} - \frac{i}{4})(\alpha - 1) & 1 \\ 1 & \frac{1}{8}(-\alpha^2 - 2\alpha - 5) & (\frac{1}{4} + \frac{i}{4})(\alpha - 1) \end{pmatrix}.$$

There are two matrices because of the double covering over the locus $\alpha^2 - 2\bar{\alpha} + 1 = 0$.

Finally, note that over the point $(\alpha, \bar{\alpha}) = (1, 1)$, we have partially reducible representations in $X_{\mathrm{PR}}(\Gamma, \mathrm{SL}(3, \mathbb{C}))$, and these form a curve, given as (see Proposition 4.1):

$$1 - (y\bar{y} + 3) + \left(y^3 - y\bar{y} + \bar{y}^3 + 3\right) - (y\bar{y} - 1)^2 = 0.$$

The intersection of the closure of W_2^{irr} is given by $y\bar{y} = 4$, $y^3 + \bar{y}^3 = 16$. These are the three points $(y, \bar{y}) = (2, 2), (2\varpi, 2\varpi^2), (2\varpi^2, 2\varpi)$, $\varpi = e^{2\pi i/3}$.

The parameters $(\alpha, \bar{\alpha}, y, \bar{y})$ describe point-wise the variety V_2. By Proposition 5.10, we must add the variables $z = \mathrm{tr}(TA^{-1}TA)$, $\bar{z} = \mathrm{tr}(A^{-1}T^{-1}AT^{-1})$, to describe V_2 scheme-theoretically. An easy computation with the above matrices A, B, T yields that [27]

$$z = y^2 - \bar{y},$$
$$\bar{z} = \bar{y}^2 - y.$$

Note that to describe V_2 we only need the variables $(\alpha, \bar{\alpha}, y, \bar{y})$ even scheme-theoretically. □

We can work out the component V_1 in a similar way and get the following.

Proposition 6.2. *The non-distinguished component* V_1 *is described as follows. Take coordinates* $\alpha = \mathrm{tr}(A)$, $\bar{\alpha} = \mathrm{tr}(A^{-1})$, $\beta = \mathrm{tr}(B)$, $\bar{\beta} = \mathrm{tr}(B^{-1})$, $\eta = \mathrm{tr}([A, B])$, $y = \mathrm{tr}(T)$, $\bar{y} = \mathrm{tr}(T^{-1})$, $z = \mathrm{tr}(TA^{-1}TA)$, $\bar{z} = \mathrm{tr}(A^{-1}T^{-1}AT^{-1})$, *the equations satisfied by* V_1 *are* $\alpha = \bar{\alpha} = 1$ *and* $y\bar{y} = \beta + \bar{\beta} + 2, y^3 + \bar{y}^3 = \beta\bar{\beta} + 5\beta + 5\bar{\beta} + 5$, $\eta = y^3 - 3(\beta + \bar{\beta} + 1)$, $z = \bar{y}$, $\bar{z} = y$.

The only reducible representations are given by $(y, \bar{y}, z, \bar{z}, \alpha, \bar{\alpha}, \beta, \bar{\beta}) \in \mu_3 \cdot (2, 2, 2, 2, 1, 1, 1, 1)$ *and are partially reducible.*

REMARK 6.3. One can check that the only singular points of V_2 are the points in $\mu_3\cdot(2,2,2,2,1,1,1,1)$. We check this as follows: V_2 is a six-fold branched covering of the plane $(\alpha,\bar\alpha)$, having three preimages over the curve $\Delta=0$ and one preimage over the points $(-1\pm 2i,-1\mp 2i)$. The singularities of $\Delta=0$ are at $(-1\pm 2i,-1\mp 2i)$, $(1,1)$, cf. Remark 5.14. We only have to check whether V_2 is smooth at those points, which can be done by hand.

Using the equations, one can easily see that the points $\mu_3\cdot(2,2,2,2,1,1,1,1)$ are ordinary double points (locally analytically isomorphic to the surface singularity $\{(u_1,u_2,u_3)\in\mathbb{C}^3 | u_1^2+u_2^2+u_3^2=0\}$).

The same happens for V_1.

7. Description of the distinguished component

The distinguished component is the component V_0, which is a triple covering of W_0.

PROPOSITION 7.1. *The distinguished component is parametrized as follows. Taking coordinates* $\alpha=\mathrm{tr}(A)$, $\bar\alpha=\mathrm{tr}(A^{-1})$, $\beta=\mathrm{tr}(B)$, $\bar\beta=\mathrm{tr}(B^{-1})$, $y=\mathrm{tr}(T)$, $\bar y=\mathrm{tr}(T^{-1})$, $z=\mathrm{tr}(TA^{-1}TA)$ *and* $\bar z=\mathrm{tr}(A^{-1}T^{-1}AT^{-1})$, *it has equations*:

$$
\begin{aligned}
\alpha=\bar\alpha, \quad \beta&=\bar\beta,\\
y\bar y&=(\alpha+1)(\beta+1),\\
z\bar z&=2\alpha^2\beta+\alpha^2+1,\\
y^3+\bar y^3&=\alpha^2\beta+\alpha\beta^2+6\alpha\beta+3\alpha+3\beta+2,\\
z^3+\bar z^3&=\alpha^4\beta^2+10\alpha^2\beta+9\alpha^2-2\alpha^3-2,\\
yz+\bar y\bar z&=\alpha^2\beta+3\alpha\beta+3\alpha+1,\\
\bar y^2 z+y^2\bar z&=\alpha^2\beta^2+4\alpha^2\beta+2\alpha^2+4\alpha\beta+2\alpha+2\beta+1,\\
\bar y z^2+y\bar z^2&=\alpha^3\beta^2+\alpha^3\beta+4\alpha^2\beta+3\alpha^2+5\alpha\beta+3\alpha-1.
\end{aligned}
$$

The intersection with the reducible locus is as follows:

- $V_0\cap X_{\mathrm{TR}}$ *is given by the three points* $(y,\bar y,z,\bar z,\alpha,\bar\alpha,\beta,\bar\beta)=\mu_3\cdot(4,4,8,8,3,3,3,3)$. *These points are smooth points of* V_0 *and* X_{TR} *respectively. We have* $\eta=\mathrm{tr}([A,B])=3$.
- $V_0\cap X_{\mathrm{PR}}$ *is given by a six-fold branched covering of the curve* $\alpha\beta-2\alpha-2\beta+3=0$, $\alpha\neq 3$, *ramified over the points* $(\alpha,\beta)=(1,1)$ *and* $(\frac{1\pm\sqrt5}{2},\frac{1\mp\sqrt5}{2})$, *where there are only three preimages.*

  For $(\alpha,\beta)=(1,1)$ *the preimages are* $(y,\bar y,z,\bar z)=\mu_3\cdot(2,2,2,2)$. *Those are the same three points as in* $V_j\cap X_{\mathrm{PR}}$, $j=1,2$. *In this case we have* $\eta=-1$.

  For $(\alpha,\beta)=(\frac{1\pm\sqrt5}{2},\frac{1\mp\sqrt5}{2})$ *the preimages are* $(y,\bar y,z,\bar z)=\mu_3\cdot(-1,-1,\frac{1\mp\sqrt5}{2},\frac{1\mp\sqrt5}{2})$. *Moreover,* $\eta=3$.

Proof. By Lemma 5.2, W_0 is described as the double cover of the plane (α,β) ramified over the curve $\Delta'=0$, where

$$\Delta' = \big(\alpha^2\beta^2 - 6\alpha\beta - 4\alpha - 4\beta - 3\big)(\alpha\beta - 2\alpha - 2\beta + 3)^2.$$

The locus $F=\alpha\beta-2\alpha-2\beta+3=0$ corresponds to reducible representations. Therefore, the ring of functions of $W_0\cap\operatorname{res}(X_{\mathrm{irr}}(\Gamma,G))$ is $\mathbb{Q}[\alpha,\beta][F^{-1},\sqrt{\Delta}]$, where

$$\Delta = \alpha^2\beta^2 - 6\beta\alpha - 4\alpha - 4\beta - 3.$$

We have $\beta=\bar\beta$, so the matrix B has one eigenvalue equal to 1. A slice of the set of such matrices is defined by

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \beta-1 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

Assume that the matrix A has non-zero entries $(2,1)$ and $(3,1)$. Rescaling the basis vectors, we can write

$$A = \begin{pmatrix} a & b & c \\ 1 & d & e \\ 1 & f & g \end{pmatrix}.$$

Solving the equations $\operatorname{tr}(A)=\operatorname{tr}(A^{-1})$, $\operatorname{tr}(A)=\operatorname{tr}(AB)$, $\operatorname{tr}(B)=\operatorname{tr}(BA^{-1})$, $\operatorname{tr}(AB^{-1})=\operatorname{tr}(B^{-1})$, $\operatorname{tr}(A^{-1})=\operatorname{tr}(A^{-1}B^{-1})$ and $\det(A)=1$, we get [27]

$$A = \begin{pmatrix} \frac{\alpha\beta-2\alpha-\beta}{\beta-3} & \frac{2(\alpha\beta-2\alpha-2\beta+3)(\beta-\alpha)}{(\beta-3)^2(\beta+1)} & \frac{(\alpha\beta-2\alpha-2\beta+3)(\beta-\alpha)(\beta-1)}{(\beta-3)^2(\beta+1)} \\ 1 & \frac{4\alpha^2-\alpha\beta^3+4\alpha\beta^2-9\alpha\beta-6\alpha+7\beta^2-6\beta-9+(\beta-3)(\beta+1)\sqrt{\Delta}}{2(\beta-3)(\beta+1)(\beta-\alpha)} & \frac{4\alpha\beta^3-\alpha\beta^4+5\beta^3-5\alpha\beta^2+2\alpha^2\beta-8\beta^2-2\alpha^2-2\alpha\beta-9\beta+(\beta-3)(\beta+1)(\beta-2)\sqrt{\Delta}}{2(\beta-3)(\beta+1)(\beta-\alpha)} \\ 1 & \frac{4\alpha^2-\alpha\beta^3+4\alpha\beta^2-9\alpha\beta-6\alpha+7\beta^2-6\beta-9-(\beta-3)(\beta+1)\sqrt{\Delta}}{2(\beta-3)(\beta+1)(\beta-\alpha)} & \frac{\alpha\beta^3+2\beta^3-8\alpha\beta^2+2\alpha^2\beta-5\beta^2-2\alpha^2+5\alpha\beta+6\alpha+6\beta+9-(\beta-3)(\beta+1)\sqrt{\Delta}}{2(\beta-3)(\beta+1)(\beta-\alpha)} \end{pmatrix}.$$

These matrices are well defined for $\beta\neq 3,-1,\alpha$.

Now, solving the equations $TA = ABT$ and $TB = BABT$, we get a one-dimensional space of matrices T spanned by

$$T_0 = \begin{pmatrix} -(\alpha-\beta)(\beta-3)^2\beta(\beta+1)^2(2\alpha\beta-\beta^2-4\alpha+2\beta-3)\sqrt{\Delta}+(\alpha-\beta)(\beta-3)(\beta+1)^2(2\alpha\beta-\beta^2-4\alpha+2\beta-3)(3\beta^2+\beta-\alpha\beta^3+3\alpha\beta^2-8\alpha-6) & (\alpha\beta-2\alpha-2\beta+3)(\alpha-\beta)(\beta-3)(\beta+1)(\alpha\beta^3-\alpha\beta^2-2\alpha\beta+4\alpha-\beta^3-2\beta^2-2\beta+3)\sqrt{\Delta}+(\alpha\beta-2\alpha-2\beta+3)(\alpha-\beta)(\beta+1)(\alpha^2\beta^5-\alpha\beta^5-4\alpha^2\beta^4-2\alpha\beta^4+\beta^4+\alpha^2\beta^3+14\alpha\beta^3+13\beta^3+10\alpha^2\beta^2-10\alpha\beta^2-10\beta^2+4\alpha^2\beta-17\alpha\beta-15\beta-16\alpha^2-24\alpha-9) & -(\alpha\beta-2\alpha-2\beta+3)(\alpha-\beta)(\beta-3)(\beta+1)(\beta^4-\alpha\beta^3-2\beta^3+\alpha\beta^2-\beta^2+2\alpha\beta+\beta+4\alpha+3)\sqrt{\Delta}+(\alpha\beta-2\alpha-2\beta+3)(\alpha-\beta)(\beta+1)(-\alpha\beta^6+\alpha^2\beta^5+5\alpha\beta^5+3\beta^5-4\alpha^2\beta^4-8\alpha\beta^4-3\beta^4+\alpha^2\beta^3-18\alpha\beta^3-17\beta^3+18\alpha^2\beta^2+53\alpha\beta^2+26\beta^2-20\alpha^2\beta+\alpha\beta+12\beta-16\alpha^2-24\alpha-9) \\ (\beta-3)^2(\beta+1)^2(8\alpha+\beta^4-3\beta^3-3\beta^2-\beta+6)\sqrt{\Delta}+(\beta-3)^2(\beta+1)^2\beta(-8\alpha^2+\alpha\beta^4-3\alpha\beta^3-3\alpha\beta^2+23\alpha\beta+6\alpha-3\beta^3-5\beta^2+15\beta+9) & (\beta-3)(\beta+1)(4\beta^5-16\beta^4+5\beta^3+8\beta^2-15\beta+\alpha^2\beta^5-8\alpha^2\beta^4+21\alpha^2\beta^3-14\alpha^2\beta^2-28\alpha^2\beta+32\alpha^2-\alpha\beta^6+7\alpha\beta^5-20\alpha\beta^4+30\alpha\beta^3+\alpha\beta^2-41\alpha\beta+48\alpha+18)\sqrt{\Delta}+(\beta-3)(\beta+1)\beta(2\beta^5-22\beta^4+28\beta^3+9\beta^2+\alpha^3\beta^5-8\alpha^3\beta^4+21\alpha^3\beta^3-14\alpha^3\beta^2-12\alpha^3\beta+16)-\alpha^2\beta^6+\alpha^2 7\beta^5-23\alpha^2\beta^4+52\alpha^2\beta^3-70\alpha^2\beta^2-33\alpha^2\beta+72\alpha^2+7\alpha\beta^5-37\alpha\beta^4+73\alpha\beta^3-30\alpha\beta^2-18\alpha\beta+81\alpha+27) & (\beta-3)(\beta+1)(\beta^6-8\beta^5+15\beta^4+3\beta^3-25\beta^2+3\beta-\alpha^2\beta^5+8\alpha^2\beta^4-21\alpha^2\beta^3+6\alpha^2\beta^2+36\alpha^2\beta-16\alpha^2+\alpha\beta^6-9\alpha\beta^5+28\alpha\beta^4-18\alpha\beta^3-33\alpha\beta^2+31\alpha\beta-24\alpha-9)\sqrt{\Delta}+(\beta-3)(\beta+1)(-7\beta^6+26\beta^5-23\beta^4-25\beta^3-3\beta^2-\alpha^3\beta^6+8\alpha^3\beta^5-21\alpha^3\beta^4+22\alpha^3\beta^3+4\alpha^3\beta^2-32\alpha^3\beta+9\beta+\alpha\beta^7-11\alpha\beta^6+54\alpha\beta^5-73\alpha\beta^4-15\alpha\beta^3+22\alpha\beta^2+18\alpha\beta+72\alpha+\alpha^2\beta^7-9\alpha^2\beta^6+31\alpha^2\beta^5-64\alpha^2\beta^4+46\alpha^2\beta^3+39\alpha^2\beta^2-16\alpha^2\beta+48\alpha^2+27) \\ (\beta-3)^2(\beta+1)^2(2\beta^3-\beta^2-8\beta-\alpha\beta^4+4\alpha\beta^3-\alpha\beta^2-10\alpha\beta+4\alpha+3)\sqrt{\Delta}+(\beta-3)^2(\beta+1)^2(2\beta^4-8\beta^3-5\beta^2+6\beta+5\alpha\beta^4-11\alpha\beta^3-13\alpha\beta^2+11\alpha\beta+24\alpha-\alpha^2\beta^5+4\alpha^2\beta^4-\alpha^2\beta^3-10\alpha^2\beta^2-4\alpha^2\beta-16\alpha^2+9) & (\beta-3)(\beta+1)(3\beta^5-14\beta^4+22\beta^3+7\beta^2-39\beta+2\alpha^2\beta^5-13\alpha^2\beta^4+26\alpha^2\beta^3-5\alpha^2\beta^2-34\alpha^2\beta+28\alpha^2-\alpha\beta^6+7\alpha\beta^5-25\alpha\beta^4+44\alpha\beta^3-3\alpha\beta^2-79\alpha\beta+33\alpha+9)\sqrt{\Delta}+(\beta-3)(\beta+1)(2\beta^6-17\beta^5+40\beta^4-20\beta^3-63\beta^2+27\beta+2\alpha^3\beta^6-13\alpha^3\beta^5+26\alpha^3\beta^4-5\alpha^3\beta^3-34\alpha^3\beta^2+12\alpha^3\beta+16\alpha^3-\alpha^2\beta^7+7\alpha^2\beta^6-31\alpha^2\beta^5+79\alpha^2\beta^4-55\alpha^2\beta^3-92\alpha^2\beta^2+105\alpha^2\beta+72\alpha^2+6\alpha\beta^6-37\alpha\beta^5+97\alpha\beta^4-73\alpha\beta^3-138\alpha\beta^2+108\alpha\beta+81\alpha+27) & (\beta-3)(\beta+1)(\alpha^2\beta^6-7\alpha^2\beta^5+16\alpha^2\beta^4-9\alpha^2\beta^3-11\alpha^2\beta^2+18\alpha^2\beta-20\alpha^2-\alpha\beta^7+7\alpha\beta^6-18\alpha\beta^5+17\alpha\beta^4+3\alpha\beta^3-13\alpha\beta^2+20\alpha\beta-39\alpha+3\beta^6-13\beta^5+8\beta^4+17\beta^3+\beta^2+6\beta-18)\sqrt{\Delta}+(\beta-3)(\beta+1)(\alpha^3\beta^7-7\alpha^3\beta^6+16\alpha^3\beta^5-9\alpha^3\beta^4-11\alpha^3\beta^3+2\alpha^3\beta^2+28\alpha^3\beta-48\alpha^3-\alpha^2\beta^8+7\alpha^2\beta^7-21\alpha^2\beta^6+36\alpha^2\beta^5-31\alpha^2\beta^4+13\alpha^2\beta^3-14\alpha^2\beta^2-23\alpha^2\beta+24\alpha^2+6\alpha\beta^7-34\alpha\beta^6+56\alpha\beta^5-6\alpha\beta^4-16\alpha\beta^3-86\alpha\beta^2-72\alpha\beta+9\alpha+2\beta^7-17\beta^6+29\beta^5+16\beta^4-13\beta^3-33\beta^2-36\beta) \end{pmatrix}.$$

The choices $T \in \mathrm{SL}(3,\mathbb{C})$ are of the form $\varpi^k d_0^{-1/3} T_0$, $\varpi = e^{2\pi i/3}$, $k = 0,1,2$, $d_0 = \det(T_0)$. Therefore, the coordinates $y = \mathrm{tr}(T)$, $\bar{y} = \mathrm{tr}(T^{-1})$, $z = \mathrm{tr}(TA^{-1}TA)$ and $\bar{z} = \mathrm{tr}(A^{-1}T^{-1}AT^{-1})$ can be computed using this explicit parametrization. Noting that only quantities that are μ_3-invariant can be computed, we get by explicit calculation [27]:

$$\begin{aligned} y\bar{y} &= (\alpha+1)(\beta+1), \\ 2y^3 &= \big(\alpha^2\beta + \alpha\beta^2 + 6\alpha\beta + 3\alpha + 3\beta + 2\big) - (\alpha-\beta)\sqrt{\Delta}, \end{aligned}$$

$$
(21)\qquad
\begin{aligned}
2\bar{y}^3 &= \big(\alpha^2\beta+\alpha\beta^2+6\alpha\beta+3\alpha+3\beta+2\big)+(\alpha-\beta)\sqrt{\Delta},\\
2z^3 &= \alpha^4\beta^2+10\alpha^2\beta+9\alpha^2-2\alpha^3-2-\big(-4\alpha+3\alpha^2+\alpha^3\beta\big)\sqrt{\Delta},\\
2\bar{z}^3 &= \alpha^4\beta^2+10\alpha^2\beta+9\alpha^2-2\alpha^3-2+\big(-4\alpha+3\alpha^2+\alpha^3\beta\big)\sqrt{\Delta},\\
z\bar{z} &= 1+\alpha^2+2\alpha^2\beta,\\
2yz &= \alpha^2\beta+3\alpha\beta+3\alpha+1+(1-\alpha)\sqrt{\Delta},\\
2\bar{y}\bar{z} &= \alpha^2\beta+3\alpha\beta+3\alpha+1-(1-\alpha)\sqrt{\Delta},\\
2(\alpha+1)(\beta+1)z &= \big(\alpha^2\beta+3\alpha\beta+3\alpha+1+(1-\alpha)\sqrt{\Delta}\big)\bar{y},\\
2(\alpha+1)(\beta+1)\bar{z} &= \big(\alpha^2\beta+3\alpha\beta+3\alpha+1-(1-\alpha)\sqrt{\Delta}\big)y,\\
2\eta=2\operatorname{tr}\big([A,B]\big) &= \big(\alpha^2\beta^2-2\alpha^2\beta-2\beta^2\alpha+2\alpha^2+2\beta^2-3\big)\\
&\qquad -(\alpha\beta-2\alpha-2\beta+3)\sqrt{\Delta}.
\end{aligned}
$$

This produces the equations in the statement. In order to see that these equations are sufficient, let Z be the zero set of those equations, it suffices to check that, for each $(\alpha,\beta)\in\mathbb{C}^2$, the fibre of the projection $Z\to\mathbb{C}^2$ has precisely the cardinality given in lemma 5.12, which is an elementary computation.

The intersection of V_0 with the reducible locus is as follows:

- With the totally reducible representations (i.e., $\alpha=\beta=3$), it is given by $(y,\bar{y},z,\bar{z})=(4,4,8,8)$, $(4\varpi,4\varpi^2,4\varpi^2,4\varpi)$, $(4\varpi^2,4\varpi,4\varpi,4\varpi^2)$, $\varpi=e^{2\pi i/3}$.
- With the partially reducible representations (i.e., $\alpha\beta-2\alpha-2\beta+3=0$, with $(\alpha,\beta)\neq(3,3)$), it is given by six points over each (α,β), if $\alpha\neq 1,3,\frac{1\pm\sqrt{5}}{2}$, defined by the six solutions for $(y,\bar{y})$, cf. Corollary 4.4.

  For $(\alpha,\beta)=(\frac{1\pm\sqrt{5}}{2},\frac{1\mp\sqrt{5}}{2})$, there are only three $(y,\bar{y},z,\bar{z})\in\mu_3\cdot(-1,-1,\frac{1\mp\sqrt{5}}{2},\frac{1\mp\sqrt{5}}{2})$, and in this case $\eta=3$.

  For $\alpha=\beta=1$, we have $\eta=-1$, and $(y,\bar{y},z,\bar{z})=\mu_3\cdot(2,2,2,2)$. These are the same three points as in $V_j\cap X_{\mathrm{PR}}$, $j=1,2$. □

Remark 7.2. We know by Proposition 5.15 that the component V_0 is scheme reduced. On the other hand, it is easy to see that the ideal $I\subset\mathbb{Q}[y,\bar{y},z,\bar{z},\alpha,\beta]$ generated by the equations in the statement of Proposition 7.1 is not a radical ideal. Now, computer supported calculations [27] produce generators of the radical $\operatorname{rad}(I)\subset\mathbb{Q}[y,\bar{y},z,\bar{z},\alpha,\beta]$. More precisely, $\operatorname{rad}(I)\subset\mathbb{Q}[y,\bar{y},z,\bar{z},\alpha,\beta]$ is generated by the following 18 polynomials:

$$
\begin{aligned}
&2yz+2\bar{y}\bar{z}-z\bar{z}+\alpha^2-6\alpha\beta-6\alpha-1,\\
&y\bar{y}-\alpha\beta-\alpha-\beta-1,\qquad 2\alpha^2\beta-z\bar{z}+\alpha^2+1,\\
&\bar{y}\alpha\beta+y^2-y\bar{z}+\bar{y}\alpha-z\beta-z,\qquad y\alpha\beta+\bar{y}^2-\bar{y}z+y\alpha-\bar{z}\beta-\bar{z},\\
&y\bar{z}\alpha-\bar{y}\alpha^2-\bar{z}^2+3\bar{y}\alpha-z\alpha-z,\qquad \bar{y}z\alpha-y\alpha^2-z^2+3y\alpha-\bar{z}\alpha-\bar{z},\\
&\bar{y}^2\alpha-\bar{y}z+y\alpha-\bar{z}\alpha+y-\bar{z},\qquad y^2\alpha-y\bar{z}+\bar{y}\alpha-z\alpha+\bar{y}-z,\\
&2\bar{y}\bar{z}^2-z\bar{z}^2+2y\alpha^2+\bar{z}\alpha^2-2\bar{z}\alpha\beta-4\bar{y}z+4z^2-8y\alpha-2\bar{z}\alpha+6y-\bar{z},
\end{aligned}
$$

$$\begin{aligned}
&\bar{y}z\bar{z}-\bar{y}\alpha^2-2z\alpha\beta+2y\bar{z}-2\bar{z}^2+4\bar{y}\alpha-2z\alpha-3\bar{y},\\
&\bar{y}^2\bar{z}+3y^2-2y\bar{z}-\bar{z}^2+2\bar{y}\alpha-2z\beta-2\bar{y}-2z,\\
&4\bar{y}^2z+4y^2\bar{z}-2z\bar{z}\beta-7z\bar{z}-\alpha^2-16\alpha\beta-8\alpha-6\beta+3,\\
&\bar{y}^3+y^2\bar{z}-\bar{y}\bar{z}\beta+\alpha\beta^2-\bar{y}\bar{z}-z\bar{z}-2\alpha\beta-\alpha-2\beta,\\
&2y^3-2y^2\bar{z}+2\bar{y}\bar{z}\beta-4\alpha\beta^2+2\bar{y}\bar{z}+z\bar{z}+\alpha^2-8\alpha\beta-4\alpha-2\beta-3,\\
&2z\bar{z}\alpha\beta-4\bar{y}z^2-4y\bar{z}^2+z\bar{z}\alpha-\alpha^3+8z\bar{z}+4\alpha^2+18\alpha\beta+11\alpha-12,\\
&z^2\bar{z}^2-2z\bar{z}\alpha^2+\alpha^4-4z^3-4\bar{z}^3-8\alpha^3+18z\bar{z}+18\alpha^2-27,\\
&4y^2\bar{z}^2-2z\bar{z}^2\beta-3z\bar{z}^2-\bar{z}\alpha^2+8z^2\beta\\
&\quad+36\bar{y}^2-32\bar{y}z+4z^2+16y\alpha-30\bar{z}\beta-29\bar{z}.
\end{aligned}$$

By Lemma 7.3, these generators are also generators of the vanishing ideal $I(V_0)\subset\mathbb{C}[y,\bar{y},z,\bar{z},\alpha,\beta]$.

Let k be a perfect field and k/K be a field extension. Notice that every field of characteristic zero is perfect. Consider the polynomial ring $k[x_1,\dots,x_n]$ and $K[x_1,\dots,x_n]\cong K\otimes_k k[x_1,\dots,x_n]$. Let $I\subset k[x_1,\dots,x_n]$ be an ideal. The exact sequence $0\to I\xrightarrow{\iota}k[x_1,\dots,x_n]\to k[x_1,\dots,x_n]/I\to 0$ gives an exact sequence

$$0\to K\otimes_k I\to K\otimes_k k[x_1,\dots,x_n]\to K\otimes_k \big(k[x_1,\dots,x_n]/I\big)\to 0.$$

Now there is a K-algebra isomorphism

$$K\otimes_k \big(k[x_1,\dots,x_n]/I\big)\cong K[x_1,\dots,x_n]/\big(I\cdot K[x_1,\dots,x_n]\big),$$

where $I\cdot K[x_1,\dots,x_n]$ denotes the image $(\mathrm{id}\otimes\iota)(K\otimes_k I)\subset K\otimes_k k[x_1,\dots,x_n]$ under the identification $K[x_1,\dots,x_n]\cong K\otimes_k k[x_1,\dots,x_n]$.

Lemma 7.3. *The ideal $I\subset k[x_1,\dots,x_n]$ is radical if and only if the ideal $I\cdot K[x_1,\dots,x_n]\subset K[x_1,\dots,x_n]$ is radical.*

Proof. We use that an ideal J in a ring R is radical if and only if R/J is reduced, that is, it does not admit nilpotent elements. The isomorphism

$$K\otimes_k \big(k[x_1,\dots,x_n]/I\big)\cong K[x_1,\dots,x_n]/\big(I\cdot K[x_1,\dots,x_n]\big)$$

implies at once that if $I\cdot K[x_1,\dots,x_n]$ is radical then $I\subset k[x_1,\dots,x_n]$ is radical.

On the other hand, if $k[x_1,\dots,x_n]/I$ is reduced then it follows from [9, V, §15, Theorem 3] that $K\otimes_k (k[x_1,\dots,x_n]/I)$ is also reduced. Here we use that k is perfect. □

Remark 7.4. By Proposition 5.15, the only singular points of V_0 are $\mu_3\cdot(2,2,2,2,1,1,1,1)$. Using equations (21), we can parametrize $(y,\bar{y},z,\bar{z})$ locally around $(\alpha,\beta)=(1,1)$. There are two branches, depending on the choice of sign for $\sqrt{\Delta}$. Both branches are smooth and intersect transversely at the point. So the singularities are simple nodes.

We have also characterized the partially reducible representations and the totally reducible representations that can deform into irreducible ones (see also the description in Lemma 4.3).

Remark 7.5. In Proposition 5.10, we have said that V_0 can be described pointwise by the parameters $(\alpha,\beta,\eta,y,\bar{y})$. The equations are [27]:

$$\begin{aligned}
&\eta^2 - P\eta + Q = 0, \quad \text{where } P,\ Q \text{ are given in Lemma 5.2},\\
&y\bar{y} = (\alpha+1)(\beta+1),\\
&y^3+\bar{y}^3 = \alpha^2\beta+\alpha\beta^2+6\alpha\beta+3\alpha+3\beta+2,\\
&(\alpha-\beta)\big(2\eta - \big(\alpha^2\beta^2-2\alpha^2\beta-2\beta^2\alpha+2\alpha^2+2\beta^2-3\big)\big)\\
&\quad = (\alpha\beta-2\alpha-2\beta+3)\big(2y^3 - \big(\alpha^2\beta+\alpha\beta^2+6\alpha\beta+3\alpha+3\beta+2\big)\big).
\end{aligned}$$

Remark 7.6. The covering $V_0 \to \mathbb{C}^2$ of Lemma 5.12 given by $\chi \mapsto (\chi(a),\chi(b))$ is regular. More precisely, the group of deck transformations $\mathcal{D}$ is generated by μ_3 and ι, where $\iota\colon X(\Gamma,\mathrm{SL}(3,\mathbb{C})) \to X(\Gamma,\mathrm{SL}(3,\mathbb{C}))$ is given by $\iota(\chi)(\gamma) = \chi(\gamma^{-1})$. If $\chi = \chi_\rho$ is the character of the representation $\rho\colon \Gamma \to \mathrm{SL}(3,\mathbb{C})$ then $\iota(\chi) = \chi_{\rho^*} : \Gamma \to \mathbb{C}$ is the character of the dual representation $\rho^* : \Gamma \to \mathrm{SL}(3,\mathbb{C})$ given by

$$\forall \gamma \in \Gamma, \quad \rho^*(\gamma) = \big(\rho(\gamma)^{-1}\big)^{\mathrm{t}},$$

where A^{t} denotes the transpose matrix for $A \in \mathrm{SL}(3,\mathbb{C})$. In the coordinates $(y,\bar{y},z,\bar{z},\alpha,\beta)$, the action of $\varpi \in \mu_3$ and ι is given by

$$\begin{aligned}
\varpi(y,\bar{y},z,\bar{z},\alpha,\beta) &= \big(\varpi y, \varpi^2\bar{y}, \varpi^2 z, \varpi\bar{z}, \alpha,\beta\big), \quad \text{and}\\
\iota(y,\bar{y},z,\bar{z},\alpha,\beta) &= (\bar{y},y,\bar{z},z,\alpha,\beta).
\end{aligned}$$

The ring of invariant functions of this action is generated by α, β and the following functions:

$$\begin{aligned}
&f_1 = y\bar{y}, \qquad f_2 = z\bar{z}, \qquad f_3 = y^3+\bar{y}^3, \qquad f_4 = z^3+\bar{z}^3,\\
&h_1 = yz+\bar{y}\bar{z}, \qquad h_2 = \bar{y}^2 z + y^2\bar{z}, \qquad h_3 = \bar{y}z^2 + y\bar{z}^2.
\end{aligned}$$

Therefore the quotient $V_0/\mathcal{D}$ embeds into $\mathbb{C}^7 \times \mathbb{C}^2$ and it follows from equations (21) that the image $q\colon V_0 \to \mathbb{C}^7 \times \mathbb{C}^2$ is isomorphic to $\mathbb{C}^2$ and given by

$$\begin{aligned}
&f_1 = (\alpha+1)(\beta+1), \qquad f_2 = 2\alpha^2\beta+\alpha^2+1,\\
&f_3 = \alpha^2\beta+\alpha\beta^2+6\alpha\beta+3\alpha+3\beta+2,\\
&f_4 = \alpha^4\beta^2+10\alpha^2\beta+9\alpha^2-2\alpha^3-2, \qquad h_1 = \alpha^2\beta+3\alpha\beta+3\alpha+1,\\
&h_2 = \alpha^2\beta^2+4\alpha^2\beta+2\alpha^2+4\alpha\beta+2\alpha+2\beta+1,\\
&h_3 = \alpha^3\beta^2+\alpha^3\beta+4\alpha^2\beta+3\alpha^2+5\alpha\beta+3\alpha+1.
\end{aligned}$$

Here $(f_1, f_2, f_3, f_4, h_1, h_2, h_3, \alpha, \beta)$ are the coordinates of $\mathbb{C}^7 \times \mathbb{C}^2$. Finally, $V_0 = q^{-1}(V_0/\mathcal{D})$ is the zero-locus of the equations stated in the proposition.

REMARK 7.7. Long and Reid give in [37] two one-parameter families of representations $\rho_k, \varrho_T \colon \Gamma \to \mathrm{SL}(3,\mathbb{C})$:

$$2\rho_k(t) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -k \\ 0 & 1 & -1-k \end{pmatrix}, \qquad \rho_k(a) = \begin{pmatrix} 1 & -2 & 3 \\ 0 & k & -1-2k \\ 0 & 1 & -2 \end{pmatrix},$$

$$\rho_k(b) = \begin{pmatrix} -2-k & -1 & 1 \\ -2-k & -2 & 3 \\ -1 & -1 & 2 \end{pmatrix},$$

$$\varrho_T(t) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & T^2 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \varrho_T(a) = \begin{pmatrix} -1+T^3 & -T & T^2 \\ 0 & -1 & 2T \\ -T & 0 & 1 \end{pmatrix},$$

$$\varrho_T(b) = \begin{pmatrix} -1 & 0 & 0 \\ -T^2 & 1 & -T \\ T & 0 & -1 \end{pmatrix}.$$

The characters of these two families are in the component V_0 and we have $(y, \bar{y}, z, \bar{z}, \alpha, \beta) = (-k-1, k, k^2-2, -2k+1, k-1, -k-2)$ and $(y, \bar{y}, z, \bar{z}, \alpha, \beta) = (0, -T^2, -T^2, T^4 - 2T, T^3 - 1, -1)$, respectively.

8. Character varieties for $\mathrm{PGL}(3,\mathbb{C})$ and $\mathrm{GL}(3,\mathbb{C})$

We use the descriptions of the various components of the $\mathrm{SL}(3,\mathbb{C})$-character variety given in Sections 4, 6 and 7 to give a similar description for the character varieties for $\mathrm{PGL}(3,\mathbb{C})$ and $\mathrm{GL}(3,\mathbb{C})$.

By Lemma 2.2, we have $X(\Gamma, \mathrm{PGL}(3,\mathbb{C})) \cong X(\Gamma, \mathrm{SL}(3,\mathbb{C}))/\mu_3$. One can get a coordinate description introducing new variables which are the generators of the ring $\mathbb{C}[y, \bar{y}, z, \bar{z}]^{\mu_3}$, namely:

$$\begin{aligned} &u_1 = y^3, && u_2 = \bar{y}^3, && u_3 = y\bar{y}, && u_4 = z^3, \\ &u_5 = \bar{z}^3, && u_6 = z\bar{z}, && u_7 = yz, && u_8 = \bar{y}\bar{z}, \\ &u_9 = y\bar{z}^2, && u_{10} = y^2\bar{z}, && u_{11} = \bar{y}z^2, && u_{12} = \bar{y}^2 z. \end{aligned} \tag{22}$$

These coordinates satisfy certain relations that can be found with a computer calculation in a standard way. Using the mathematical package Magma [8], we have found a set of 29 relations.

PROPOSITION 8.1. *The character variety* $X(\Gamma, \mathrm{PGL}(3,\mathbb{C}))$ *has five components.*

- *The component corresponding to totally reducible representations, isomorphic to* $\mathbb{C}^2/\mu_3$, μ_3 *acting by* $(y, \bar{y}) \mapsto (\varpi y, \varpi^2 \bar{y})$.

- *The component corresponding to partially reducible representations is parametrized by the smooth surface*

$$\{(v_1,v_2,v_3,x_1)\in\mathbb{C}\times\mathbb{C}^*\times\mathbb{C}\times\big(\mathbb{C}-\{1\}\big)\mid v_1v_2=v_3^3,\\ (x_1-1)v_3^2=\big(x_1^2+x_1-1\big)v_2\}.$$

  An explicit parametrization $u_j=u_j(v_1,v_2,v_3,x_1)$, $j=1,\dots,12$, is obtained from Equations (22) and Proposition 4.1 by using the new parameters $v_1=v^3$, $v_2=w^3$, $v_3=vw$.
- *Three components consisting of irreducible representations, which are $W_0-\{\alpha\beta-2\alpha-2\beta+3=0\}, W_1-\{\beta=\bar\beta=1\}, W_2-\{\alpha=\bar\alpha=1\}$ defined in Lemmas 5.2 and 5.3.*

Proof. The PGL$(3,\mathbb{C})$-character variety is obtained from the SL$(3,\mathbb{C})$ by taking the quotient by μ_3 acting on the coordinates y, $\bar y$, z and $\bar z$. The locus of reducible representations is determined in Proposition 4.1. For the partially reducible representations, we use the description in (10).

Finally, from the discussion at the end of Section 5, the irreducible locus $X_{\mathrm{irr}}(\Gamma,\mathrm{PGL}(3,\mathbb{C}))$ is isomorphic to the image under the restriction map res: $X_{\mathrm{irr}}(\Gamma,\mathrm{GL}(3,\mathbb{C}))\to X(F_2,\mathrm{SL}(3,\mathbb{C}))$. This is the set described in Lemmas 5.2 and 5.3. $\square$

To get the GL$(3,\mathbb{C})$-representations, recall that

$$X\big(\Gamma,\mathrm{GL}(3,\mathbb{C})\big)=\big(X\big(\Gamma,\mathrm{SL}(3,\mathbb{C})\big)\times\mathbb{C}^*\big)/\mu_3.$$

So the variables are $(y,\bar y,z,\bar z,\alpha,\bar\alpha,\beta,\bar\beta,\lambda)$ with μ_3 acting as

$$\varpi\cdot(y,\bar y,z,\bar z,\alpha,\bar\alpha,\beta,\bar\beta,\lambda)=\big(\varpi y,\varpi^2\bar y,\varpi^2 z,\varpi\bar z,\alpha,\bar\alpha,\beta,\bar\beta,\varpi\lambda\big).$$

Thus one can get a coordinate description introducing the variables u_j of Equations (22) and

$$\begin{aligned} &w_1=y\lambda^2, && w_2=y^2\lambda, && w_3=\bar y\lambda, && w_4=z\lambda,\\ &w_5=\bar z\lambda^2, && w_6=\bar z^2\lambda, && w_7=y\bar z\lambda, && w_8=\lambda^3.\end{aligned}$$

The coordinates u_j, w_k satisfy certain relations that can be found with a computer calculation in a standard way. Using the mathematical package Magma [8], we have found a set of 89 relations.

Substituting these variables into the statement of Theorem 1.2 gives the equations for the three components containing characters of irreducible representations.

For the totally reducible representations, we have a simpler description as $X_{\mathrm{TR}}(\Gamma,\mathrm{GL}(3,\mathbb{C}))=\mathbb{C}^2\times\mathbb{C}^*$. Also, for the partially reducible GL$(3,\mathbb{C})$-representations, they split as an irreducible GL$(2,\mathbb{C})$-representation and a one-dimensional representation. Therefore, in analogy to Proposition 4.1, the component $X_{\mathrm{PR}}(\Gamma,\mathrm{GL}(3,\mathbb{C}))$ can be parametrized by $X_2\times\mathbb{C}^*$. Here

$X_2 \subset X(\Gamma, \mathrm{GL}(2,\mathbb{C}))$ is the component of irreducible representations (see Proposition 3.3).

9. The symmetry group of the figure eight knot

The symmetry group $\mathrm{Sym}(S^3, K_8)$ of the figure eight knot K_8 is isomorphic to the outer automorphism group

$$\mathrm{Out}(\Gamma) = \mathrm{Aut}(\Gamma)/\mathrm{Inn}(\Gamma)$$

(see [32, 10.6]). The group $\mathrm{Out}(\Gamma)$ was calculated by Magnus [39] (see also [47]). It is isomorphic to the dihedral group D_4 of order eight

$$\mathrm{Out}(\Gamma) = \big\langle f, h \mid f^2 = h^4 = (fh)^2 = 1 \big\rangle \cong D_4, \tag{23}$$

where the elements f and h are represented by the following automorphisms (also denoted by f and h):

$$\begin{aligned} f(S) &= T^{-1} \\ f(T) &= S^{-1} \end{aligned} \quad \text{and} \quad \begin{aligned} h(S) &= ST^{-1}S^{-1} \\ h(T) &= TS^{-1}T^{-1}. \end{aligned}$$

They are also described by the action on $t, a, b \in \Gamma$ as follows:

$$\begin{aligned} f(t) &= T^{-1} = a^{-1}t^{-1}a \sim t^{-1}, \\ f(a) &= ST^{-1}S^{-1}T = a^{-1}, \\ f(b) &= S^{-1}T = ba^{-1} \sim b \end{aligned} \tag{24}$$

and

$$\begin{aligned} h(t) &= ST^{-1}S = ta^{-1}t^{-1}at^{-1} \sim t^{-1}, \\ h(a) &= TST^{-2} = Tb^{-1}T^{-1} = a^{-1}tab^{-1}a^{-1}t^{-1}a \sim b^{-1}, \\ h(b) &= TS^{-1}T^{-1}STS^{-1} = STS^{-1}T^{-1} = TaT^{-1} = a^{-1}tat^{-1}a \sim a. \end{aligned} \tag{25}$$

A peripheral system (m, ℓ) of the figure eight knot is given by

$$\begin{aligned} m &= S = t, \\ \ell &= T^{-1}STS^{-1}S^{-1}TST^{-1} = [a,b]. \end{aligned} \tag{26}$$

Notice that by (25), we obtain

$$\begin{aligned} h(m) &= ta^{-1}m^{-1}t^{-1}a, \\ h(\ell) &= h\big([a,b]\big) = a^{-1}ta\big[b^{-1}, a\big]a^{-1}t^{-1}a. \end{aligned}$$

Now the relation $t^{-1}a^{-1}t = ba^{-2}$ gives that the peripheral system $(h(m), h(\ell))$ is conjugated to (m^{-1}, ℓ). This reflects the amphicheirality of the figure eight knot.

The induced action on the varieties of representations are given in coordinates as follows:

$$\begin{aligned} &f^*(y) = \bar{y}, \qquad f^*(\bar{y}) = y, \qquad f^*(z) = \bar{z}, \qquad f^*(\bar{z}) = z, \\ &f^*(\alpha) = \bar{\alpha}, \qquad f^*(\bar{\alpha}) = \alpha, \qquad f^*(\beta) = \beta, \qquad f^*(\bar{\beta}) = \bar{\beta}, \end{aligned}$$

and

$$h^*(y)=\bar{y},\qquad h^*(\bar{y})=y,\qquad h^*(\alpha)=\bar{\beta},\qquad h^*(\bar{\alpha})=\beta,\qquad h^*(\beta)=\alpha,$$
$$h^*(\bar{\beta})=\bar{\alpha}.$$

LEMMA 9.1. *$h^*(z)=\bar{y}^2-\bar{z}$ and $h^*(\bar{z})=y^2-z$.*

Proof. We check this equality on each component. Away from V_0, z and $\bar{z}$ are functions on the other variables, and the proof is straightforward. For V_0, we compute $h^*(z)$ as follows: we have $ta^{-1}ta=ST$ and

$$h(ST)=ST^{-1}S^{-1}TS^{-1}T^{-1}=T^{-1}ST^{-1}S^{-1}\sim T^{-1}S^{-1}T^{-1}S.$$

Now, we proceed as in the proof of Lemma 5.1. Let $\rho\colon\Gamma\to\mathrm{SL}(3,\mathbb{C})$ be a representation. We put $\rho(S)=M$ and $\rho(T)=N$. By the Cayley–Hamilton theorem we have

$$\left(N^{-1}M^{-1}\right)^3=\mathrm{tr}\left(N^{-1}M^{-1}\right)\left(N^{-1}M^{-1}\right)^2-\mathrm{tr}(MN)\left(N^{-1}M^{-1}\right)+\mathrm{Id}\,.$$

Multiplying this identity by MNM^2 gives:

$$\begin{aligned}&\mathrm{tr}\left(N^{-1}M^{-1}N^{-1}M\right)\\&\quad=\mathrm{tr}\left(N^{-1}M^{-1}\right)\mathrm{tr}\left(N^{-1}M\right)-\mathrm{tr}(MN)\,\mathrm{tr}\left(M^2\right)+\mathrm{tr}\left(M^3N\right).\end{aligned}$$

Applying the same procedure to M^3N and M^2N, we obtain

$$\mathrm{tr}\left(M^3N\right)=\mathrm{tr}(M)\,\mathrm{tr}\left(M^2N\right)-\mathrm{tr}\left(M^{-1}\right)\mathrm{tr}(MN)+\mathrm{tr}(N)$$

and

$$\mathrm{tr}\left(M^2N\right)=\mathrm{tr}(M)\,\mathrm{tr}(MN)-\mathrm{tr}\left(M^{-1}\right)\mathrm{tr}(N)+\mathrm{tr}\left(M^{-1}N\right).$$

Now $S=t$, $T=a^{-1}ta$, $ta^{-1}ta=ST$ and $b=TS^{-1}$ gives $h^*(z)=\bar{z}\bar{\beta}+\bar{y}z-y^2\bar{y}+y\beta+y$. Using that on V_0, $\bar{\alpha}=\alpha$ and $\bar{\beta}=\beta$, $y\bar{y}=(1+\alpha)(1+\beta)$, and $y\alpha\beta+\bar{y}^2-\bar{y}z+y\alpha-\bar{z}\beta-\bar{z}=0$ (see Remark 7.2) the computation for $h^*(z)$ follows. The formula for $h^*(\bar{z})$ is proved in the same way. □

Thus we have the following proposition.

PROPOSITION 9.2. *f^* preserves the components of $X(\Gamma,\mathrm{SL}(3,\mathbb{C}))$ and h^* swaps V_1 and V_2.*

REMARK 9.3. If we consider also the action of μ_3, the center of $\mathrm{SL}(3,\mathbb{C})$, we realize that $y\bar{y}$ and $y^3+\bar{y}^3$ are invariant by both the symmetry group of the knot and the action of μ_3. This explains why $y\bar{y}$ and $y^3+\bar{y}^3$ are symmetric polynomials on α and β for V_0, on β and $\bar{\beta}$ for V_1 and on α and $\bar{\alpha}$ for V_2, as well as the symmetries on those variables (swapping V_1 and V_2). Similar considerations with the variables z and $\bar{z}$ can be made.

10. The non-distinguished components as Dehn fillings

In this section, we view the non-distinguished components as the variety of representations induced by an exceptional Dehn filling on the figure eight knot. Those representations also appear in work of Bing and Martin [5] in the proof of property P for twist knots. This is explained at the end of the section.

We recall that the figure eight knot has six slopes $s \in \mathbb{Q} \cup \{\infty\}$ whose Dehn filling $K(s)$ is a *small* Seifert fibered manifold; namely, a Seifert fibered manifold with basis orbifold a 2-sphere with three cone points of order p, q, and $r \geq 2$, $S^2(p,q,r)$. The precise coefficients are (cf. [25]):

$$\begin{aligned} &K(\pm 1) \text{ fibers over } S^2(2,3,7),\\ &K(\pm 2) \text{ fibers over } S^2(2,4,5),\\ &K(\pm 3) \text{ fibers over } S^2(3,3,4). \end{aligned}$$

The center of $\pi_1(K(\pm s))$, $s = 1,2,3$, is generated by a regular fibre. By Schur's lemma, any irreducible representation of $\pi_1(K(\pm s)) \to \mathrm{SL}(3,\mathbb{C})$, $s = 1,2,3$, maps the fibre to the center of $\mathrm{SL}(3,\mathbb{C})$. This motivates the study of representations of the orbifold fundamental groups $\pi_1^{\mathcal{O}}(S^2(p,q,r))$, that are isomorphic to the (orientable) triangle groups

$$\pi_1^{\mathcal{O}}\big(S^2(p,q,r)\big) \cong D(p,q,r) = \big\langle k, l \mid k^p, l^q, (kl)^r \big\rangle.$$

These surjections of $\pi_1(K(\pm s))$, $s = 1,2,3$, onto the corresponding triangle group are given by taking the quotient of the fundamental group of the small Seifert fibered manifold $\pi_1(K(\pm s))$ by its center.

In particular, using the Wirtinger Presentation (2), we have an epimorphism $\phi\colon \Gamma \to D(3,3,4) = \langle k, l \mid l^3, k^3, (kl)^4\rangle$ given by

$$\phi(S) = klk \quad \text{and} \quad \phi(T) = klklk.$$

With Presentation (1),

$$\phi(a) = k^{-1}l^{-1}kl, \qquad \phi(b) = \phi\big(TS^{-1}\big) = kl \quad \text{and} \quad \phi(t) = klk. \tag{27}$$

It satisfies $\phi(b)^4 = 1$ and $\phi(m^3\ell) = 1$. Notice that the surjection ϕ induces an injection

$$\phi^*\colon X\big(D(3,3,4), \mathrm{SL}(3,\mathbb{C})\big) \hookrightarrow X\big(\Gamma, \mathrm{SL}(3,\mathbb{C})\big).$$

Characters χ in V_1 satisfy $\chi(b^{\pm 1}) = 1$. In addition, by (27), $\phi(b) = kl$ has order 4. This motivates the following lemma.

LEMMA 10.1. *The variety $\overline{X^{\mathrm{irr}}(D(3,3,4), \mathrm{SL}(3,\mathbb{C}))}$ has a component $\mathcal{W}$ of dimension 2 and three isolated points. The variety $\mathcal{W}$ is isomorphic to the hypersurface in $\mathbb{C}^3$ given by the equation*

$$\zeta^2 - (\nu\bar\nu - 2)\zeta + \nu^3 + \bar\nu^3 - 5\nu\bar\nu + 5 = 0.$$

Here, the parameters are $\nu=\chi(k^{-1}l)$, $\bar\nu=\chi(kl^{-1})$ *and* $\zeta=\chi([k,l])$*. For every* $\chi\in\mathcal{W}$, $\chi(k^{\pm1})=\chi(l^{\pm1})=0$ *and* $\chi((kl)^{\pm1})=1$.

Moreover, all characters in $\mathcal{W}$ *are irreducible except for the three points* $(\nu,\bar\nu,\zeta)=(2,2,1)$, $(2\varpi,2\varpi^2,1)$, $(2\varpi^2,2\varpi,1)$, $\varpi=e^{2\pi i/3}$.

Proof. For an irreducible representation of $D(3,3,4)$, the eigenvalues of the image of elements of order three k and l are $\{1,\varpi,\varpi^2\}$ (otherwise the image of k or l would be central and the representation reducible). In particular $\chi(k^{\pm1})=\chi(l^{\pm1})=0$. For the image of kl there are three possible set of eigenvalues: $\{1,i,-i\}$, $\{-1,i,i\}$, $\{-1,-i,-i\}$, and $\{1,-1,-1\}$. We shall see that for $\{1,i,-i\}$ we get a two dimensional variety and for $\{-1,i,i\}$, $\{-1,-i,-i\}$, and $\{1,-1,-1\}$, isolated points.

First, assume that the eigenvalues of the image of kl are $\{1,i,-i\}$, namely $\chi(kl)=\chi((kl)^{-1})=1$. Then, by applying Lawton's theorem and by taking coordinates $\nu(\chi)=\chi(k^{-1}l)$, $\bar\nu(\chi)=\chi(kl^{-1})$, and $\zeta(\chi)=\chi([k,l])$, we get the hypersurface of $\mathbb{C}^3$,

$$\zeta^2-(\nu\bar\nu-2)t+\nu^3+\bar\nu^3-5\nu\bar\nu+5=0,$$

that we denote $\mathcal{W}$.

Next, we deal with the case where the eigenvalues of a representation of kl are $\{-1,i,i\}$, namely $\chi(kl)=-1+2i$ and $\chi((kl)^{-1})=-1-2i$. We apply Lawton's theorem again, but this is not sufficient to determine a representation, as the image of kl could not diagonalize. We need to impose further conditions that determine the value of the characters at kl^{-1} and $k^{-1}l$, which will imply that the dimension of the component of the character variety is zero. Namely, denote by K and L the respective images of k and l by a representation. As we require that KL is diagonalizable, we have

$$0=(KL+\mathrm{Id})(KL-i\,\mathrm{Id})=(KL)^2+(1-i)KL-i\,\mathrm{Id}\,.$$

Equivalently, $KL+(1-i)\,\mathrm{Id}-i(KL)^{-1}=0$. Multiplying with K^{-1}, $KLK^{-1}+(1-i)K^{-1}-iL^{-1}K^{-2}=0$, and since $\operatorname{tr}L=\operatorname{tr}K^{-1}=0$, we get $\operatorname{tr}L^{-1}K^{-2}=0$. In addition, since $K^{-2}=K$, $\operatorname{tr}KL^{-1}=\operatorname{tr}L^{-1}K^{-2}=0$. Similarly, $\operatorname{tr}K^{-1}L=0$, thus we get a zero dimensional variety. Lawton's formulas [34] yield that the trace of the commutator and its inverse are the same: $\operatorname{tr}[K,L]=\operatorname{tr}[K^{-1},L^{-1}]=1$. Thus, this is a single point in the character variety, by Lawton's coordinates. The case where the eigenvalues are $\{-1,-i,-i\}$ is precisely the same computation, by considering complex conjugation, and the case $\{1,-1,-1\}$ is completely analogous.

Finally, let ρ be a reducible semisimple representation with character χ_ρ in $\mathcal{W}$. Hence, up to conjugation we can assume that $\rho=\rho_1\oplus\rho_2$, where $\rho_1\colon D(3,3,4)\to\mathrm{GL}(2,\mathbb{C})$ is irreducible, and $\rho_2\colon D(3,3,4)\to\mu_3$ satisfies $\rho_2(g)\det(\rho_1(g))=1$ for all $g\in D(3,3,4)$. First, let us assume that $\det\circ\rho_1=\rho_2$ is trivial, i.e., $\rho_1\colon D(3,3,4)\to\mathrm{SL}(2,\mathbb{C})$. This implies that $\operatorname{tr}(\rho(g))=\operatorname{tr}(\rho_1(g))+1$ for all $g\in D(3,3,4)$. Hence, $K_1=\rho_1(k)$ and $L_1=\rho_1(l)$

are matrices of order three without common eigenspaces. Hence, up to conjugation, we can assume that

$$K_1 = \begin{pmatrix} \omega & 0 \\ a & \omega^{-1} \end{pmatrix} \quad \text{and} \quad L_1 = \begin{pmatrix} \omega & 1 \\ 0 & \omega^{-1} \end{pmatrix}.$$

Now, the condition $\mathrm{tr}(KL) = 1$ implies $\mathrm{tr}(K_1L_1) = 0$ and hence $a = 1$. This gives

$$K_1L_1^{-1} = \begin{pmatrix} 1 & -\omega \\ \omega^{-1} & 0 \end{pmatrix}, \qquad K_1^{-1}L_1 = \begin{pmatrix} 1 & \omega^{-1} \\ -\omega & 0 \end{pmatrix}, \quad \text{and}$$
$$[K_1, L_1] = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Hence, $\mathrm{tr}(\rho(kl^{-1})) = 2$, $\mathrm{tr}(\rho(k^{-1}l)) = 2$ and $\mathrm{tr}(\rho([k,l])) = 1$.

If $\lambda = \det \circ \rho_1$ is a non-trivial homomorphism, then $\lambda \cdot \rho \colon D(3,3,4) \to \mathrm{SL}(3,\mathbb{C})$ is still a reducible representation $\lambda \cdot \rho = (\lambda \cdot \rho_1) \oplus (\lambda \cdot \rho_2)$. Now, $\lambda \cdot \rho_2$ is trivial and the preceding argument applies to $\lambda \cdot \rho$. Finally, we have that $\mathrm{Hom}(D(3,3,4), \mu_3) \cong \mu_3$ and $\lambda(kl^{-1}) = \lambda(k^{-1}l)^2$ implies the result. □

Remark 10.2. Further details in the proof of Lemma 10.1 allow to describe those three isolated points. Composing with ϕ^*, they correspond to the points in $X(\Gamma, \mathrm{SL}(3,\mathbb{C}))$ with coordinates:

$$(\alpha, \bar\alpha, \beta, \bar\beta) = (1, 1, -1+2i, -1-2i), (1, 1, -1-2i, -1+2i), (-1, -1, -1, -1).$$

For those characters of Γ, $y = \bar y = z = \bar z = 0$. Those are precisely the three metabelian irreducible characters of Γ that do not lie in V_2, see Corollary 5.8.

Proposition 10.3. *The components V_1 and V_2 are characters of representations which factor through the surjections $\Gamma \twoheadrightarrow \pi_1(K(\pm 3))$ respectively. These components are isomorphic to the hypersurface*

$$\zeta^2 - (\nu\bar\nu - 2)\zeta + \nu^3 + \bar\nu^3 - 5\nu\bar\nu + 5 = 0.$$

Here, the parameters are

$$\nu = \begin{cases} \chi(t) & \text{for } V_2, \\ \chi(t^{-1}) & \text{for } V_1, \end{cases} \qquad \bar\nu = \begin{cases} \chi(t^{-1}) & \text{for } V_2, \\ \chi(t) & \text{for } V_1, \end{cases} \qquad \zeta = \begin{cases} \chi(a) & \text{for } V_2, \\ \chi(b^{-1}) & \text{for } V_1. \end{cases}$$

All characters are irreducible except for the three points $(\nu, \bar\nu, \zeta) = (2, 2, 1)$, $(2\varpi, 2\varpi^2, 1)$, $(2\varpi^2, 2\varpi, 1)$, with $\varpi = e^{2\pi i/3}$, that correspond to the intersection $V_1 \cap V_2 = V_0 \cap V_1 \cap V_2$. The intersection of $V_i \cap V_0$ is the zero locus of the discriminant on ζ:

$$\nu^2\bar\nu^2 - 4\nu^3 - 4\bar\nu^3 + 16\nu\bar\nu - 16 = 0.$$

The restriction map $X(\Gamma, \mathrm{SL}(3,\mathbb{C})) \to X(F_2, \mathrm{SL}(3,\mathbb{C}))$ maps the intersection $V_1 \cap V_2$ onto a single point $\alpha = \bar\alpha = \beta = \bar\beta = 1$.

Proof. By (27), the surjection $\phi\colon \Gamma \to D(3,3,4)$ maps b to kl, and a to a conjugate to $[k,l]$. Hence, for every character $\chi \in \mathcal{W}$ we obtain that $\mathrm{res} \circ \phi^*(\chi) = \mathrm{res}(\chi \circ \phi)\colon F_2 \to \mathbb{C}$ maps b and b^{-1} to 1 (recall that $\phi(b)^4 = 1$). Therefore, the map

$$\mathrm{res} \circ \phi^*\colon \mathcal{W} \to X\big(F_2, \mathrm{SL}(3,\mathbb{C})\big)$$

maps $\mathcal{W}$ onto the 2-dimensional component V_2 given by the equations $\beta = \bar\beta = 1$. On this component, we have $\alpha = \zeta$, and $\bar\alpha = (\nu\bar\nu - 2) - \zeta$ (i.e., α and $\bar\alpha$ are the solutions of the equation on ζ). In addition, the intersection with $\alpha = \bar\alpha$ corresponds to the two possible values of ζ (for fixed ν and $\bar\nu$) being equal, that is to the zero set of the discriminant of the quadratic equation on t. The parameters for $\chi \in \mathcal{W}$ are $\chi(k^{-1}l)$, $\chi(kl^{-1})$ and $\chi([k,l])$. Now, by (27) we have

$$\phi(t) = klk \sim k^2 l = k^{-1} l \quad \text{and} \quad \phi(a) = k^{-1}l^{-1}kl \sim [k,l],$$

and hence the parameters for $\phi^*\chi \in X(\Gamma, \mathrm{SL}(3,\mathbb{C}))$ are

$$\nu = \phi^*\chi(t), \qquad \bar\nu = \phi^*\chi\big(t^{-1}\big) \quad \text{and} \quad \zeta = \phi^*\chi(a).$$

We obtain the component V_2 by the same considerations and by replacing ϕ by $\phi \circ h$. Notice that by (25) we have $h(t) = t^{-1}$ and $h(a) = b^{-1}$. $\square$

The definition of the volume of a representation in $\mathrm{SL}(3,\mathbb{C})$ and its main properties can be found in [3], [10], [22]. Since characters in V_1 and V_2 factor through a closed Seifert fibered manifold, we have:

Corollary 10.4. *The volume of any representation in V_1 or V_2 vanishes.*

Remark 10.5. One may ask why the Dehn fillings $K(\pm 3)$ give new components of $X(\Gamma, \mathrm{SL}(3,\mathbb{C}))$, while $K(\pm 1)$ and $K(\pm 2)$ do not, even if all of them are small Seifert fibered orbifolds. The reason is that the groups of the base orbifolds are different and their varieties of representations have different dimension: $X(\pi_1^{\mathcal{O}}(S^2(3,3,4)), \mathrm{SL}(3,\mathbb{C}))$ has a component of dimension two, though $X(\pi_1^{\mathcal{O}}(S^2(2,q,r)), \mathrm{SL}(3,\mathbb{C}))$ $(q, r \geq 2)$ has dimension zero. This can be checked with the same argument as in the proof of Lemma 10.1.

11. Parametrizing representations

Similar representations of knot groups into $\mathrm{SL}(3,\mathbb{C})$ have been used in the literature before. In particular Bing and Martin used them to prove Property P for twist knots (see [5]). The study of representations of $D(3,q,r)$ to $\mathrm{SL}(3,\mathbb{C})$ goes back to [12]. Some of this is presented in [11, Section 15B].

We consider the following two matrices K and L of $\mathrm{SL}(3,\mathbb{C})$:

$$K = \begin{pmatrix} 0 & 0 & 1 \\ x_0 & 1 & x_1 \\ -1 & 0 & -1 \end{pmatrix} \quad \text{and} \quad L = \begin{pmatrix} 1 & y_0 & y_1 \\ 0 & -1 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Notice that K and L are of order three,

$$KL = \begin{pmatrix} 0 & 1 & 0 \\ x_0 & x_0y_0 + x_1 - 1 & x_0y_1 - 1 \\ -1 & -y_0 - 1 & -y_1 \end{pmatrix},$$

thus the characteristic polynomial of KL is given by

$$P_{KL}(t) = t^3 - (x_0y_0 + x_1 - y_1 - 1)t^2 + (x_0y_1 - x_1y_1 - x_0 - y_0 + y_1 - 1)t - 1.$$

Hence, $\mathrm{tr}(KL) = \mathrm{tr}(L^{-1}K^{-1}) = 1$ if and only if

$$x_0y_0 + x_1 - y_1 - 2 = 0 \quad \text{and} \quad x_0y_1 - x_1y_1 - x_0 - y_0 + y_1 - 2 = 0.$$

Now, define the ideal

$$I = (x_0y_0 + x_1 - y_1 - 2, x_0y_1 - x_1y_1 - x_0 - y_0 + y_1 - 2) \subset \mathbb{C}[x_0, x_1, y_0, y_1]$$

and $X = \mathbf{V}(I) \subset \mathbb{C}^4$ its zero set. For each point $(x_0, x_1, y_0, y_1) \in X$ we obtain a representation $\rho_{(x_0,x_1,y_0,y_1)} \colon D(3,3,4) \to \mathrm{SL}(3,\mathbb{C})$ mapping $\phi(b) = kl$ to a matrix B such that $\mathrm{tr}(B) = \mathrm{tr}(B^{-1}) = 1$.

An easy computation gives that X is a smooth irreducible variety, that we view as a subvariety of $R(D(3,3,4), \mathrm{SL}(3,\mathbb{C}))$. The following proposition says that we can view it as a birational slice.

PROPOSITION 11.1. *The projection*

$$R\big(D(3,3,4), \mathrm{SL}(3,\mathbb{C})\big) \to X\big(D(3,3,4), \mathrm{SL}(3,\mathbb{C})\big)$$

restricts to a birational map $X \to \mathcal{W}$.

Proof. We write the projection restricted to X as a regular map $f \colon X \to \mathcal{W} \subset \mathbb{C}^3$ given by

$$f(x_0, x_1, y_0, y_1) = \big(\nu(x_0, x_1, y_0, y_1), \bar{\nu}(x_0, x_1, y_0, y_1), \zeta(x_0, x_1, y_0, y_1)\big),$$

where we have used the parameters of Lemma 10.1,

$$\begin{aligned} \nu &= \mathrm{tr}\, \rho_{(x_0,x_1,y_0,y_1)}\big(k^{-1}l\big), \\ \bar{\nu} &= \mathrm{tr}\, \rho_{(x_0,x_1,y_0,y_1)}\big(kl^{-1}\big), \\ \zeta &= \mathrm{tr}\, \rho_{(x_0,x_1,y_0,y_1)}\big([k,l]\big). \end{aligned}$$

In the ambient coordinates the map f is given by

$$\begin{aligned} \nu(x_0, x_1, y_0, y_1) &= x_0y_0 - x_1y_0 + x_0 + y_1 - 2, \\ \bar{\nu}(x_0, x_1, y_0, y_1) &= x_0y_1 - x_1 + y_0 - y_1 + 1, \\ \zeta(x_0, x_1, y_0, y_1) &= x_0^2y_0y_1 - x_0x_1y_0y_1 - x_0^2y_0 + x_0y_0^2 - x_1y_0^2 + x_0x_1y_1 \\ &\quad - x_1^2y_1 - x_0y_0y_1 + x_1y_0y_1 + x_0y_1^2 - x_0x_1 + 2x_0y_0 \\ &\quad - x_1y_0 - 3x_0y_1 + 2x_1y_1 + y_0y_1 - y_1^2 + 4x_0 - x_1 - 2y_0 - 2. \end{aligned}$$

The rational inverse is the map $g\colon \mathcal{W} \to X$ given by

$$g(\nu,\bar\nu,\zeta) = \Big(\frac{\nu^2+\nu\bar\nu-2\bar\nu-\zeta-3}{\zeta-1}, \qquad \frac{\nu^2-\bar\nu^2+2\nu-2\bar\nu+\zeta-1}{\zeta-1},$$
$$\frac{\nu\bar\nu-\bar\nu^2+2\nu-2\zeta-2}{-\nu\bar\nu+\zeta+3}, \frac{-\nu^2+2\bar\nu-\zeta+1}{-\nu\bar\nu+\zeta+3}\Big).$$

The map g is defined off the algebraic set $Y = (X \cap \{\zeta = 1\}) \cup (X \cap \{\nu\bar\nu = \zeta+3\})$. The decomposition of $Y = Y_1 \cup \cdots \cup Y_6$ into irreducible components is obtained by computer supported calculations [27], and it is given by

$$Y_1 = \mathbf{V}(\zeta-1, \nu+\bar\nu+2), \qquad Y_2 = \mathbf{V}\big(\zeta-1, \nu+\eta^2\bar\nu-2\eta\big),$$
$$Y_3 = \mathbf{V}\big(\zeta-1, \nu-\eta\bar\nu+2\eta^2\big), \qquad Y_4 = \mathbf{V}\big(\nu+\bar\nu+2, \bar\nu^2+2\bar\nu+\zeta+3\big),$$
$$Y_5 = \mathbf{V}\big(\nu+\eta^2\bar\nu-2\eta, \bar\nu^2+2\eta^2\bar\nu-\eta\zeta-3\eta\big),$$
$$Y_6 = \mathbf{V}\big(\nu-\eta\bar\nu+2\eta^2, \bar\nu^2-2\eta\bar\nu+\eta^2\zeta+3\eta^2\big).$$

Each Y_i is isomorphic to an affine line and η is a primitive 6th root of unity. $\square$

This permits to give explicitly a representation $\Gamma \to \mathrm{SL}(3,\mathbb{C})$ which corresponds to a given point in $\mathcal{W} \setminus Y$.

Example 11.2. Let us compute all irreducible representations $\rho : \Gamma \to \mathrm{SL}(3,\mathbb{C})$ such that $\chi_\rho \in V_1$ and $\operatorname{tr}\rho(m) = \operatorname{tr}\rho(m^{-1}) = 3$, i.e., $\nu = \bar\nu = 3$. By Proposition 10.3, we obtain $\zeta^2 - 7\zeta + 14 = 0$ and hence $\zeta_\pm = 7/2 \pm i\sqrt{7}/2$. Now,

$$g(3,3,\zeta_\pm) = \Big(\frac{3}{2} \pm i\frac{\sqrt7}{2}+, 1, \frac{1}{2} \mp i\frac{\sqrt7}{2}, \frac{3}{2} \mp i\frac{\sqrt7}{2}\Big).$$

Hence,

$$\rho(S) = \begin{pmatrix} \frac{\sqrt7}{2}i+\frac32 & 1 & 1 \\ \frac{\sqrt7}{2}i+\frac52 & -\frac{\sqrt7}{2}i+\frac52 & 1 \\ -\frac{\sqrt7}{2}i-\frac52 & \frac{\sqrt7}{2}i-\frac32 & -1 \end{pmatrix},$$
$$\rho(T) = \begin{pmatrix} \frac{\sqrt7}{2}i+\frac52 & -\frac{\sqrt7}{2}i+\frac52 & 1 \\ 1 & -\frac{\sqrt7}{2}i+\frac32 & 1 \\ -\frac{\sqrt7}{2}i-\frac32 & \frac{\sqrt7}{2}i-\frac52 & -1 \end{pmatrix}.$$

Moreover, we obtain that

$$\rho(\ell) = \begin{pmatrix} 3i\sqrt7-2 & \frac{3\sqrt7}{2}i+\frac92 & \frac{3\sqrt7}{2}i+\frac32 \\ 3i\sqrt7+15 & -3i\sqrt7+10 & 9 \\ -\frac{3\sqrt7}{2}i-\frac{15}{2} & \frac{3\sqrt7}{2}i-\frac92 & -5 \end{pmatrix}$$

is also a unipotent matrix. These representations where previously studied by Deraux and Falbel in connection with spherical CR structures on the complement of the figure eight knot [14], [15], [16].

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