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TWISTED REIDEMEISTER TORSION AND THE THURSTON NORM: GRAPH MANIFOLDS AND FINITE REPRESENTATIONS

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ABSTRACT. We show that the Thurston norm of any irreducible 3-manifold can be detected using twisted Reidemeister torsions corresponding to integral representations and also corresponding to representations over finite fields. In particular, our result holds for all graph manifolds, these are not covered by the earlier work of the first author and Vidussi.

1. Introduction

Define for an oriented surface Σ with components Σ_i the complexity

$$\chi_-(\Sigma) := \sum_i \max\bigl(-\chi(\Sigma_i), 0\bigr).$$

For a 3-manifold N, Thurston [Thu86] introduced a semi-norm on $H_2(N, \partial N; \mathbb{Z})$. This semi-norm, now called *Thurston norm*, is defined as

$$\|\sigma\|_T := \min\bigl\{\chi_-(\Sigma) : \Sigma \text{ oriented, embedded surface with } [\Sigma] = \sigma\bigr\},$$

where $[\Sigma]$ denotes the fundamental class of an oriented surface Σ. By Poincaré duality, we transfer this norm to $H^1(N;\mathbb{Z})$ and henceforth consider it only as a semi-norm on cohomology.

The cell complex $C_*(\widetilde{N};\mathbb{Z})$ of a universal cover $\widetilde{N}$ of N inherits the structure of a $\mathbb{Z}[\pi_1(N)]$-module from the deck transformations.

Fixing a field K, we can tensor this chain complex with a $(K(t), \mathbb{Z}[\pi_1(N)])$-bimodule A, obtaining $C_*(N;A) := A \otimes_{\mathbb{Z}[\pi_1(N)]} C_*(\widetilde{N};\mathbb{Z})$. For a $g \in \pi_1(N)$,

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the endomorphism $g_A\colon A \to A$ given by $g_A(v) := v \cdot g$ is a linear map of the $K(t)$-vector space A.

In Section 2, we recall the definition of the twisted Reidemeister torsion $\tau(N, A) \in K(t)$ where we assume the chain complex $C_*(N; A)$ to be acyclic. The Reidemeister torsion $\tau(N, A)$ is a unit in $K(t)$ and well-defined up to multiplication with $\pm \det g_A$ for a $g \in \pi_1(N)$. Given an element $p(t) \in K[t] \setminus \{0\}$ in the polynomial ring with $p(t) = \sum_{i=k}^{l} a_i t^i$ and a_l, a_k both non-zero, we define $\operatorname{width} p(t) := l - k$. We extend this assignment to any-non zero element in the quotient field $K(t)$ by declaring

$$\operatorname{width}\bigl(p(t)/q(t)\bigr) = \operatorname{width} p(t) - \operatorname{width} q(t),$$

for non-zero $q(t) \in K[t]$.

If the chain complex $C_*(N; A)$ is not acyclic, we adopt the convention that $\operatorname{width} \tau(N, A) = 0$. This differs from the convention in [Nag14].

Definition 1.1. (1) A *representation* V of a group G is a $(K, \mathbb{Z}[G])$-bimodule, i.e. a K-vector space with a linear right action by G.

(2) Let V be a representation of $\pi_1(N)$. For an element $\theta \in H^1(N; \mathbb{Z})$, we denote by V_θ the $(K(t), \mathbb{Z}[\pi_1(N)])$-bimodule with underlying $K(t)$-vector space $V_\theta := K(t) \otimes_K V$ and right $\pi_1(N)$-action given by

$$(z \otimes v) \cdot g := z t^{\langle \theta, g \rangle} \otimes v \cdot g,$$

where $\langle \theta, g \rangle$ is the evaluation of $\theta \in H^1(N; \mathbb{Z}) = \operatorname{Hom}(\pi_1(N), \mathbb{Z})$ on $g \in \pi_1(N)$.

It has been known for a very long time that Reidemeister torsions, or perhaps more precisely, its close cousin the Alexander polynomial, give a lower bound on the genus of a knot. This inequality was generalised in [FK06, Theorem 1.1] as follows.

Theorem 1.2. *Let N be a 3-manifold. Let $\theta \in H^1(N; \mathbb{Z})$ be a cohomology class. For every representation V of $\pi_1(N)$ the Thurston norm $\|\theta\|_T$ satisfies the inequality*

$$(\dim V) \cdot \|\theta\|_T \geq \operatorname{width} \tau(N, V_\theta).$$

In [FV15, Theorem 1.2] it was shown that for any irreducible 3-manifold that is not a closed graph manifold there exists a unitary representation such that the corresponding twisted Reidemeister torsions detect the Thurston norm. The proof of that result relies on the fact, as it was put in [AD15], that by the work of Agol [Ago13], Przytycki–Wise [PW12] and Wise [Wis12] such 3-manifolds are "full of cubulated goodness".

By the above theorem, the integer $\operatorname{width} \tau(N, V_\theta)$ is a lower bound on $(\dim V) \cdot \|\theta\|_T$. If we have equality $(\dim V) \cdot \|\theta\|_T = \operatorname{width} \tau(N, V_\theta)$, then we say the representation V of $\pi_1(N)$ *detects the Thurston norm* of $\theta \in H^1(N; \mathbb{Z})$.

The following theorem is the main result of this paper.

THEOREM 1.3. *Let N be an irreducible 3-manifold which is not $D^2 \times S^1$. For every $\theta \in H^1(N;\mathbb{Z})$ there is a representation V factoring through a finite group which detects the Thurston norm of θ, that is, such that*

$$(\dim V) \cdot \|\theta\|_T = \operatorname{width} \tau(N, V_\theta).$$

Additionally, the representation V can be chosen to be either

(1) *defined over the complex numbers and be integral, or*
(2) *defined over a finite field $\mathbb{F}_q$ for almost all primes q.*

Note that a complex representation which factors through a finite group can be made unitary.

Our main theorem extends [FV15, Theorem 1.2] in two ways: It extends the statement over to closed graph manifolds, that were excluded in [FV15] since these are in general "not full of cubulated goodness", see [Liu13]. This extension relies on recent work of the second author [Nag14, Theorem 2.15].

Secondly, our theorem gives a refined statement about which types of representations can detect the Thurston norm. In particular, the result that representations over finite fields can be used plays a critical role in the proof in [BF15] that the profinite completion of the knot group determines the knot genus.

We conclude this Introduction with an observation. By [FK06, Section 3], the degrees of twisted Reidemeister torsions of a 3-manifold N only depend on the fundamental group and on whether or not N has boundary. We thus obtain the following corollary.

COROLLARY 1.4. *The Thurston norm of an irreducible 3-manifold is an invariant of its fundamental group.*

For closed 3-manifolds that is of course also a consequence of the fact that irreducible 3-manifolds that are not lens spaces are determined by their fundamental groups, see [AFW15, Chapter 2.1] for detailed references. For 3-manifolds with boundary the statement is slightly less obvious, since there are non-homeomorphic irreducible 3-manifolds with non-trivial boundary which have isomorphic fundamental groups. It should not be hard though to prove Corollary 1.4 using the theory of Dehn flips introduced by Johannson, see [Joh79, Section 29] and [AFW15, Chapter 2.2] for details.

Conventions. A 3-manifold is understood to be connected, smooth, compact, orientable and having only toroidal boundary, which can be empty. A vector space is also understood to be finite dimensional.

2. Preliminaries

For this section, we fix an irreducible 3-manifold N with a CW-structure. Let K be a field.

A universal cover $\pi\colon \tilde{N} \to N$ inherits an induced CW-structure. The deck transformations act on $\tilde{N}$ from the left. With this left action the cellular chain complex of $\tilde{N}$ is a chain complex of left $\mathbb{Z}[\pi_1(N)]$-modules, which we denote by $C_*(\widetilde{N})$. For a subcomplex $Y \subset N$ the preimage $\pi^{-1}(Y)$ is a CW-subcomplex of $\tilde{N}$ and invariant under deck transformations. We define $C_*(\widetilde{Y \subset N})$ to be the cellular complex of $\pi^{-1}(Y)$. This is a complex of left $\mathbb{Z}[\pi_1(N)]$-modules. A lift of the cells of the CW-structure on N is called a *fundamental family* and determines a basis of each chain module $C_k(\widetilde{N})$.

We can tensor the complex above with a $(K(t), \mathbb{Z}[\pi_1(N)])$-bimodule A. We abbreviate the resulting modules by

$$C_*(N;A) := A \otimes_{\mathbb{Z}[\pi_1(N)]} C_*(\widetilde{N}),$$
$$C_*(Y \subset N;A) := A \otimes_{\mathbb{Z}[\pi_1(N)]} C_*(\widetilde{Y \subset N}).$$

Here $K(t)$ denotes the quotient field of the polynomial ring in one variable. If the chain complex $C_*(N;A)$ is not acyclic, then we define $\tau(N,A) := 0$. If the chain complex is acyclic, then its *Reidemeister torsion* $\tau(N,A) \in K(t) \setminus \{0\}$ is defined, see [Tur01] for an introduction. We quickly recall its construction. As the chain complex $C_*(N;A)$ is acyclic, we obtain exact sequences of the form

$$0 \to \operatorname{Im} \partial_{i+1} \to C_i(N;A) \to \operatorname{Im} \partial_i \to 0.$$

We fix a basis for each $\operatorname{Im} \partial_i$. From the basis of $\operatorname{Im} \partial_{i+1}$ and a lift of a basis of $\operatorname{Im} \partial_i$, we obtain a basis b_i of $C_i(N;A)$, which we will compare with the basis c_i of $C_i(N;A)$ given by a fundamental family and a basis of A. We denote the matrix expressing the basis b_i in terms of c_i by $[b_i/c_i]$. Define the Reidemeister torsion of $C(N;A)$ to be

$$\tau(N,A) := \prod_i \big(\det[b_i/c_i]\big)^{(-1)^{i+1}} \in K(t).$$

It is non-zero element in $K(t)$ well-defined up to multiplication with $\pm \det g_A$ for $g \in \pi_1(N)$.

For future reference, we mention the following elementary lemma.

Lemma 2.1. *If A and B are two $(K(t), \mathbb{Z}[\pi_1(N)])$-bimodules, then*

$$\tau(N, A \oplus B) = \tau(N,A) \cdot \tau(N,B).$$

Definition 2.2. (1) A representation V of $\pi_1(N)$ *detects the Thurston norm* of $\theta \in H^1(N;\mathbb{Z})$ if

$$(\dim V) \cdot \|\theta\|_T = \operatorname{width} \tau(N, V_\theta).$$

(2) Let $H \leq \pi_1(N)$ be a subgroup of finite index and V a representation of H. The *induced* representation of $\pi_1(N)$ is

$$\operatorname{Ind}_H^{\pi_1(N)} V := V \otimes_{\mathbb{Z}[H]} \mathbb{Z}\big[\pi_1(N)\big].$$

Analogously define the induced $(K(t), \mathbb{Z}[\pi_1(N)])$-bimodule $\operatorname{Ind}_H^{\pi_1(N)} A$ of a $(K(t), \mathbb{Z}[H])$-bimodule A.

(3) Given two representations V, W of $\pi_1(N)$, we can take the tensor product of the underlying vector spaces $V \otimes_K W$ and equip it with the diagonal action

$$(v \otimes w) \cdot g := (v \cdot g) \otimes (w \cdot g).$$

This defines a $(K, \mathbb{Z}[\pi_1(N)])$-bimodule denoted with $V \hat{\otimes} W$.

LEMMA 2.3. *Let H be a subgroup of G and U a representation of G and V a representation of H. Then the map*

$$\operatorname{Ind}_H^G(\operatorname{Res}_H U \hat{\otimes} V) \to U \hat{\otimes} \operatorname{Ind}_H^G V$$

induced by

$$(u \otimes v) \otimes g \mapsto u \cdot g \otimes (v \otimes g)$$

is an isomorphism of representations of G.

Proof. It is an isomorphism of vector spaces and it is equivariant with respect to the G-action. ☐

Gabai [Gab83, Corollary 6.13] proved that the Thurston norm is well-behaved under finite covers. Therefore, we are free to consider simpler finite covers. This is made precise in the lemma below.

LEMMA 2.4. *Let $p\colon M \to N$ be a connected finite cover and $\theta \in H^1(N;\mathbb{Z})$ a cohomology class. If V detects the Thurston norm of $p^*\theta$, then $\operatorname{Ind}_{\pi_1(M)}^{\pi_1(N)} V$ detects the Thurston norm of θ.*

Proof. By a result of Gabai [Gab83, Corollary 6.13] the Thurston norm fulfils the equality $\deg p \|\theta\|_T = \|p^*\theta\|_T$.

Note that $\operatorname{Res}_{\pi_1(M)} K_\theta = K_{p^*\theta}$. Recall that by definition $V_\theta := K_\theta \hat{\otimes} V$. With Lemma 2.3, we obtain

$$\begin{aligned}\big(\operatorname{Ind}_{\pi_1(M)}^{\pi_1(N)} V\big)_\theta &= K_\theta \hat{\otimes} \operatorname{Ind}_{\pi_1(M)}^{\pi_1(N)} V = \operatorname{Ind}_{\pi_1(M)}^{\pi_1(N)}\big((\operatorname{Res}_{\pi_1(M)} K_\theta) \hat{\otimes} V\big)\\ &= \operatorname{Ind}_{\pi_1(M)}^{\pi_1(N)}(K_{p^*\theta} \hat{\otimes} V) = \operatorname{Ind}_{\pi_1(M)}^{\pi_1(N)} V_{p^*\theta}\end{aligned}$$

as $(K(t), \mathbb{Z}[\pi_1(N)])$-bimodules. Let us abbreviate $V_{p^*\theta}$ with A and $\operatorname{Ind}_{\pi_1(M)}^{\pi_1(N)}$ with just Ind.

First, we prove that the chain complex $C_*(M;A)$ is acyclic if and only if $C_*(N;\operatorname{Ind} A)$ is acyclic. Choose the CW-structure on M which is induced of

N. Using that $\widetilde{N} = \widetilde{M}$, the following map is an isomorphism of chain complex of $K(t)$-vector spaces

$$\begin{aligned} A \otimes_{\mathbb{Z}[\pi_1(M)]} \mathbb{Z}\big[\pi_1(N)\big] \otimes_{\mathbb{Z}[\pi_1(N)]} C_*(\widetilde{N}) &\to A \otimes_{\mathbb{Z}[\pi_1(M)]} C_*(\widetilde{M}), \\ v \otimes g \otimes e &\mapsto v \otimes g \cdot e. \end{aligned}$$

Therefore one is acyclic if and only if the other is.

So let V detect the Thurston norm of $p^*\theta \in H^1(M;\mathbb{Z})$, i.e. we have

$$(\dim V) \cdot \big\|p^*\theta\big\|_T = \operatorname{width} \tau(M, A).$$

Note that we have the equality $\deg p \cdot \dim V = \dim \operatorname{Ind} V$.

We claim that equality $\operatorname{width} \tau(M, A) = \operatorname{width} \tau(N, A)$ holds. We pick representatives g_i of right cosets, so $\pi_1(N) = \coprod_i \pi_1(M) \cdot g_i$. Then we equip $\operatorname{Ind} A$ with the basis $\{v \otimes g_i\}$, where $\{v\}$ is a basis of A. Given a fundamental family $\{\widetilde{e}\}$ for N, we equip M with the fundamental family $\{g_i \cdot \widetilde{e}\}$. With these choices made, the isomorphism above also preserves the basis used for calculating the Reidemeister torsion. Thus, even $\tau(M, A)$ and $\tau(N, \operatorname{Ind} A)$ agree.

Combining the results obtained so far, we get

$$\begin{aligned} (\dim \operatorname{Ind} V) \cdot \|\theta\|_T = (\dim V)\big\|p^*\theta\big\|_T &= \operatorname{width} \tau(M, A) \\ &= \operatorname{width} \tau(N, \operatorname{Ind} A) \\ &= \operatorname{width} \tau\big(N, (\operatorname{Ind} V)_\theta\big). \end{aligned}$$

□

Definition 2.5. (1) Given a representation W of the group H and a group homomorphism $\alpha\colon G \to H$, we can let $\mathbb{Z}[G]$ act through $\mathbb{Z}[H]$ and obtain a representation $\operatorname{Res}_\alpha V$ of G.

(2) If a representation V of a group G is isomorphic to the restriction of a representation of a finite group, we say V *factors through a finite group.*

(3) A representation V of G over $\mathbb{C}$ is called *integral* if there is a $(\mathbb{Z}, \mathbb{Z}[G])$-bimodule W such that $V \cong \mathbb{C} \otimes_{\mathbb{Z}} W$.

Lemma 2.6. *Let $H \leq \pi_1(N)$ be a finite index subgroup and V be a representation of H. If V is integral, then also $\operatorname{Ind}_H^{\pi_1(N)} V$ is integral. If V factors through a finite group, then also $\operatorname{Ind}_H^{\pi_1(N)} V$ factors through a finite group.*

Proof. Let W be a $(\mathbb{Z}, \mathbb{Z}[\pi_1(N)])$-bimodule witnessing that V is an integral representation. We have

$$\operatorname{Ind}_H^{\pi_1(N)} V = V \otimes_{\mathbb{Z}[H]} \mathbb{Z}\big[\pi_1(N)\big] = \mathbb{C} \otimes_{\mathbb{Z}} \Big(W \otimes_{\mathbb{Z}[H]} \mathbb{Z}\big[\pi_1(N)\big]\Big)$$

and therefore $\operatorname{Ind}_H^{\pi_1(N)} V$ is integral as well.

Now we consider the second property. So suppose that the representation $H \to \operatorname{Aut}(V)$ factors through a finite group. This means that there exists a finite-index normal subgroup K of H such that the representation restricted to K is trivial. Note that K is a finite-index subgroup of $\pi_1(N)$. Since $\pi_1(N)$ is finitely generated the core $\Gamma := \bigcap_{g\in\pi_1(N)} gKg^{-1}$ of K is a finite-index normal subgroup of $\pi_1(N)$. It suffices to show that Γ acts trivially on $V \otimes_{\mathbb{Z}[H]} \mathbb{Z}[\pi_1(N)]$. This is indeed the case, since for $v \in V$ and $g \in \pi_1(N)$ and $\gamma \in \Gamma$ we have

$$(v \otimes g) \cdot \gamma = v \otimes g\gamma = v \otimes g\gamma g^{-1} g = v \cdot g\gamma g^{-1} \otimes g = v \otimes g.$$

Here in the last equality we used that $g\gamma g^{-1} \in \Gamma \subset K$. □

DEFINITION 2.7. For a character $\alpha\colon \pi_1(N) \to \mathbb{Z}/k\mathbb{Z}$ define the representation $\mathbb{C}^\alpha$ to be the representation with underlying $\mathbb{C}$-vector space $\mathbb{C}$ and right action given by

$$\begin{aligned} \mathbb{C}^\alpha \times \pi_1(N) &\to \mathbb{C}^\alpha, \\ (z, g) &\mapsto z\alpha(g), \end{aligned}$$

where we consider $\mathbb{Z}/k\mathbb{Z}$ embedded in $\mathbb{C}$ as the k roots of unity via $n + k\mathbb{Z} \mapsto \exp\left(\frac{2\pi i n}{k}\right)$.

REMARK 2.8. The representation $\mathbb{C}^\alpha$ factors through a finite group but is integral only for $k = 2$.

3. Closed graph manifolds

In this section, we only consider closed irreducible graph manifolds. Graph manifolds with boundary will be dealt with later in Theorem 4.1.

Recall that every irreducible 3-manifold N admits the JSJ-decomposition, a minimal collection $\mathcal{T}$ of embedded incompressible tori such that every component of $N|\mathcal{T}$ is either ateroidal or Seifert fibred, where $N|\mathcal{T}$ is the manifold N split along the tori.

DEFINITION 3.1. An irreducible 3-manifold N is called a *graph manifold* if all the pieces of its JSJ-decomposition are Seifert fibred.

We give a list of examples of graph manifolds below. The list is comprehensive in the sense that every graph manifold is finitely covered by a manifold contained in the list, see, for example, [Nag14, Proposition 2.9].

EXAMPLE 3.2. (1) The 3-sphere S^3,
(2) torus bundles,
(3) circle bundles,
(4) pieces of the form $\Sigma \times S^1$ glued together along their boundary tori, where the surface Σ always has negative Euler characteristic.

The Thurston norm in the first two examples vanishes. It also has a simple description for circle bundles and so we are mainly interested in understanding the last class of the list above. The manifolds in the last class have a finite cover that admits a graph structure, defined below.

Definition 3.3. (1) A *graph structure* for N consists of maps

$$\phi_+ \colon \coprod_{v \in I^+} \Sigma_v \times S^1 \to N,$$
$$\phi_- \colon \coprod_{v \in I^-} \Sigma_v \times S^1 \to N$$

such that N is the push-out of the following diagram

$$\begin{tikzcd} \coprod_{v \in I^+} \Sigma_v \times S^1 \arrow[rr, "\phi_+"] & & N \\ \coprod T_e \arrow[u, "i^+"] \arrow[rr, "i^-"'] & & \coprod_{v \in I^-} \Sigma_v \times S^1 \arrow[u, "\phi_-"'] \end{tikzcd}$$

where $i_\pm$ are identifications of $\coprod T_e$ with components of $\coprod \partial\Sigma_v \times S^1$. We denote by ϕ_v the composition

$$\phi_v \colon \Sigma_v \times S^1 \hookrightarrow \coprod_{\mu \in I^\pm} \Sigma_\mu \times S^1 \to N.$$

(2) For a manifold N with a graph structure, we refer to the homology classes $s_v := \phi_{v*}[*_v \times S^1]$ as the *class of the Seifert fibre* in the block v. Define a character $\alpha \colon \pi_1(N) \to \mathbb{Z}/k\mathbb{Z}$ to be *Seifert non-vanishing* if $\langle \alpha, s_v \rangle \neq 0$ for all $v \in I_\pm$. This is well-defined as $\mathbb{Z}/k\mathbb{Z}$ is an Abelian group.

Remark 3.4. Here, in comparison with the article [Nag14], we are more restrictive in what we consider to be a (composite) graph structure. Our graph structures here automatically have no self-pastings and the push-out diagram directly gives rise to a Mayer–Vietoris sequence, which we use below for the torsion computations.

Lemma 3.5. *Let N be a graph manifold which does not admit a Seifert fibred structure and is not a torus bundle. Then there is a finite cover $\widetilde{N}$ of N with a graph structure and on $\widetilde{N}$ there is for all but finitely many prime numbers p a character $\alpha \colon \pi_1(\widetilde{N}) \to \mathbb{Z}/p\mathbb{Z}$ which is Seifert non-vanishing.*

Proof. The manifold N admits a finite cover M with a collection $\mathcal{T}$ of embedded incompressible tori such that each component of $M|\mathcal{T}$ is of the form $\Sigma \times S^1$ with Σ of negative Euler characteristic, see [Nag14, Lemma 2.13]. To such a decomposition is associated the Bass–Serre graph which has vertices the components of $M|\mathcal{T}$ and edges the tori in $\mathcal{T}$.

If M were admitting a graph structure, then its Bass–Serre graph would be bipartite. We can achieve this using a further finite cover which is induced by the kernel of the map

$$\pi_1(M) \to \mathbb{Z}_2,$$
$$\gamma \mapsto \sum_{T \in \mathcal{T}} \gamma \cdot [T].$$

The existence of the character α in a further cover follows from [Nag14, Theorem 2.15]. □

Now we construct representations V which will detect the Thurston norm on a 3-manifold N with a graph structure and a Seifert non-vanishing character $\alpha\colon \pi_1(N) \to \mathbb{Z}/p\mathbb{Z}$.

DEFINITION 3.6. (1) A representation V of $\mathbb{Z}/p\mathbb{Z}$ is called *good* if the linear map $(1-g)_V : V \to V$ is invertible as an endomorphism of the K-vector space V for all non-trivial $g \in \mathbb{Z}/p\mathbb{Z}$.

(2) A representation V of $\pi_1(N)$ is called *good* if V is isomorphic to $\operatorname{Res}_\alpha W$ for a good representation W of $\mathbb{Z}/p\mathbb{Z}$ and a Seifert non-vanishing character α.

We give some examples of good representations for $\pi_1(N)$ in the list below. Recall that the representation $\mathbb{C}^\alpha$ was defined in Definition 2.7.

EXAMPLE 3.7. Let p, q be two different prime numbers with $q > 2$ and a Seifert non-vanishing character $\alpha\colon \pi_1(N) \to \mathbb{Z}/p\mathbb{Z}$.

(1) The representation $\mathbb{C}^\alpha$ is good.
(2) The augmentation ideal $I(\mathbb{C})$ of $\mathbb{C}[\mathbb{Z}/p\mathbb{Z}]$ is the kernel of the map

$$\mathbb{C}[\mathbb{Z}/p\mathbb{Z}] \to \mathbb{C},$$
$$\sum_{g \in \mathbb{Z}/p\mathbb{Z}} a_g g \mapsto \sum_{g \in \mathbb{Z}/p\mathbb{Z}} a_g.$$

It is a good representation of $\mathbb{Z}/p\mathbb{Z}$. Thus $I_\infty^\alpha := \operatorname{Res}_\alpha I(\mathbb{C})$ is a good representation of $\pi_1(N)$. Additionally, this representation is integral.
(3) The augmentation ideal $I(\mathbb{F}_q)$ of $\mathbb{F}_q[\mathbb{Z}/p\mathbb{Z}]$ is good and so is $I_q^\alpha := \operatorname{Res}_\alpha I(\mathbb{F}_q)$.

The following theorem is the main reason for considering good representations.

THEOREM 3.8. *Let N be a graph manifold with a graph structure. Every good representation V detects the Thurston norm of θ for every $\theta \in H^1(N;\mathbb{Z})$.*

Some of the calculations have already been discussed elsewhere. We only sketch these and refer to them [Nag14, Lemma 4.17, Proposition 4.18].

Proof of Theorem 3.8. Pick CW-structures such that the maps $\phi_\pm$ and $i_\pm$ are inclusion of subcomplexes. Let $\{T_e\}$ be the collection of graph tori. The graph structure of N gives rise to a short exact sequence

$$0 \to \bigoplus_{e} C_*(T_e \subset N; V_\theta) \xrightarrow{i^+ - i^-} \bigoplus_{v \in I_\pm} C_*\big(\Sigma_v \times S^1 \subset N; V_\theta\big) \to C_*(N; V_\theta) \to 0.$$

Note as the representation V is good, the action of a Seifert fibre s on V only depends on its homology class. Therefore, we consider the endomorphism $(1 - s_v)_V : V \to V$ for a Seifert fibre class s_v. As good representations come from Seifert fibre non-vanishing characters, the endomorphism $(1 - s_v)_V : V \to V$ is invertible. We abbreviate with $\mathrm{ev} : K(t) \to K$ the evaluation homomorphism $\mathrm{ev}\, p(t) = p(1)$. We have the equality

$$\mathrm{ev}\big(\det(1 - s_v)_{V_\theta}\big) = \det(1 - s_v)_V.$$

Consequently, we deduce from $\det(1 - s_v)_V \neq 0$ that $\det(1 - s_v)_{V_\theta} \neq 0$ as well. Therefore, the chain complex $C_*(\Sigma_v \times S^1 \subset N, V_\theta)$ is acyclic and its Reidemeister torsion is

$$\tau\big(\Sigma_v \times S^1 \subset N, V_\theta\big) = \det\big((1 - s_v)_{V_\theta}\big)^{-\chi(\Sigma_v)}.$$

This can be seen as follows. Because ϕ_v injects $\pi_1(\Sigma_v \times S^1)$ into $\pi_1(N)$, we can identify the chain complexes $C_*(\Sigma_v \times S^1 \subset N; V_\theta) \cong V_\theta \otimes \mathbb{C}[\pi_1(N)] \otimes C_*(\Sigma_v \times S^1)$. Now the Reidemeister torsion of the right-hand side can be calculated explicitly [Tur02, VII.5.2].

Similarly, we prove that the chain complex $C_*(T_e \subset N; V_\theta)$ is acyclic as well and $\tau(T_e \subset N; V_\theta) = 1$, see for example [KL99, Example 3.3]. For this, it is essential that 1 is not an eigenvalue of the endomorphism $(s_v)_V$. We obtain by the above short exact sequence and the multiplicativity of the torsion [Tur01, Theorem 1.5] that

$$\tau(N, V_\theta) = \prod_{v \in I_\pm} \det\big((1 - s_v)_{V_\theta}\big)^{-\chi(\Sigma_v)}.$$

We calculate $\mathrm{width} \det_{V_\theta}(1 - s_v) = \dim V |\langle \theta, s_v\rangle|$. Thus taking width in the equation above, we get the equalities

$$\begin{aligned} \mathrm{width}\, \tau(N, V_\theta) &= (\dim V) \sum_{v \in I_\pm} -\chi(\Sigma_v)\big|\langle \theta, s_v\rangle\big| \\ &= (\dim V) \sum_{v \in I_\pm} \big\|\phi_v^* \theta\big\|_T = (\dim V)\|\theta\|_T. \end{aligned}$$

The second equality is a calculation of the Thurston norm in $\Sigma_v \times S^1$, see, for example, [Nag14, Proposition 3.4]. The last equality holds by a result of Eisenbud–Neumann [EN85, Proposition 3.5]. ∎

Now we prove Theorem 1.3 for closed graph manifolds.

THEOREM 3.9. *Let N be a closed graph manifold. For every $\theta \in H^1(N;\mathbb{Z})$ there is a representation V factoring through a finite group which detects the Thurston norm of θ. Additionally, the representation V can be chosen to be either*

(1) *defined over the complex numbers and be integral, or*
(2) *defined over a finite field $\mathbb{F}_q$ with $q > 2$ prime.*

Proof. The statement is vacuous if $\|\theta\|_T = 0$. We can thus restrict ourselves to the case that $\|\theta\|_T > 0$. In particular, we can assume that N is neither covered by S^3 nor is it covered by a torus bundle [McM02, 7. Examples]. Also, the following claim shows that we can assume that N is not covered by a non-trivial circle bundle of a surface.

CLAIM. If N is covered by a non-trivial circle bundle $f\colon E \to B$, then the Thurston norm vanishes on N.

Since $f\colon E \to B$ is a non-trivial circle bundle, the Euler class is non-trivial. Using the Gysin sequence, we see that

$$H^1(B) \xrightarrow{f^*} H^1(E)$$

is surjective. Thus, we can represent every class in $H_2(E)$ by a multiple of the fundamental class of a torus. Therefore, the Thurston norm vanishes on E and so by [Gab83, Corollary 6.13] also on N. This concludes the proof of the claim.

If N is the trivial circle bundle, then the representation $\mathbb{C}^\alpha$, I^α_∞ and I^α_q will detect the Thurston norm of θ for any character $\alpha\colon \pi_1(\Sigma \times S^1) \to \mathbb{Z}/p\mathbb{Z}$ with prime p different from q which is non-zero on $\pi_1(S^1) \subset \pi_1(\Sigma \times S^1)$. If N is finitely covered by a trivial circle bundle, then the induced representations $\operatorname{Ind}^{\pi_1(N)} \mathbb{C}^\alpha$, $\operatorname{Ind}^{\pi_1(N)} I^\alpha_\infty$, and $\operatorname{Ind}^{\pi_1(N)} I^\alpha_q$ will detect the Thurston norm by Lemma 2.4.

So we are in the case that N does not admit a Seifert fibred structure. By Lemma 3.5 there is a finite cover $M \to N$ such that M admits a graph structure and a character $\alpha\colon \pi_1(N) \to \mathbb{Z}/p\mathbb{Z}$ which is Seifert fibre non-vanishing and with prime p being different from q. By Theorem 3.8, the representations $\mathbb{C}^\alpha$, I^α_∞ and I^α_q detect the Thurston norm of $p^*\theta$. By Lemma 2.4, the representations $\operatorname{Ind}^{\pi_1(N)} \mathbb{C}^\alpha$, $\operatorname{Ind}^{\pi_1(N)} I^\alpha_\infty$ and $\operatorname{Ind}^{\pi_1(N)} I^\alpha_q$ detect the Thurston norm of θ on N.

The representation $\operatorname{Ind}^{\pi_1(N)} I^\alpha_\infty$ is integral and $\operatorname{Ind}^{\pi_1(N)} I^\alpha_q$ is defined over $\mathbb{F}_q$. □

4. The proof of Theorem 1.3 for virtually fibred 3-manifolds

The goal of this section is to prove Theorem 1.3 for 3-manifolds that are not closed graph manifolds. More precisely, we will prove the following theorem.

Theorem 4.1. *Let N be an irreducible 3-manifold that is not a closed graph manifold and let $\theta \in H^1(N;\mathbb{Z})$. Then the following hold*:

(1) *there is an integral representation V factoring through a finite group which detects the Thurston norm*, *and*
(2) *for almost all primes q there is a representation V over the finite field $\mathbb{F}_q$ which detects the Thurston norm.*

The proof of Theorem 4.1 is, perhaps not surprisingly, a modification of the proof of the main theorem of [FV15]. In an attempt to keep the paper concise, we will only indicate which steps of [FV15] need to be modified.

A 3-manifold N is called *fibred* if it can be given the structure of a surface bundle over S^1. We say it is *virtually fibred* if N admits a finite cover which fibres.

Definition 4.2. A class $\theta \in H^1(N;\mathbb{Q})$ is called *fibred* if there exists a map $\rho\colon N \to S^1$ and a class $\tau \in H^1(S^1;\mathbb{Q})$ such that $\rho\colon N \to S^1$ is a fibre bundle and $\rho^*\tau = \theta$.

The following theorem is the key topological ingredient in the proof of Theorem 4.1. This theorem is a combination of the results of Agol [Ago08], [Ago13], Przytycki–Wise [PW12], [PW14] and Wise [Wis12]. We refer to [AFW15, Theorem 5.4.10 and (H.4)] for precise references.

Theorem 4.3. *Let N be an irreducible 3-manifold that is not a closed graph manifold. Then given a class $\theta \in H^1(N;\mathbb{Q})$ there exists a finite regular cover p of N such that the class $p^*\theta$ is in the closure of fibred classes.*

In the proof of Theorem 4.1 we will need the following lemma, which is a slight generalisation of [FV15, Lemma 5.7].

Lemma 4.4. *Let F be a free Abelian group and let $p_1,\dots,p_l \in \mathbb{Z}[F]$ be non-zero elements. Then there exists a prime q and a homomorphism $\alpha\colon F \to \mathbb{Z}/q\mathbb{Z}$ such that for any $j \in \{1,\dots,q-1\}$ the character*

$$\begin{aligned} \rho_j\colon \mathbb{Z}/q\mathbb{Z} &\to S^1, \\ a &\mapsto e^{2\pi i a j/q} \end{aligned}$$

has the property that $(\rho_j\circ\alpha)(p_1),\dots,(\rho_j\circ\alpha)(p_l)$ are non-zero elements of $\mathbb{C}$.

Proof. As in the proof of [FV15, Lemma 5.7] we first note that there exists a homomorphism $\Psi\colon F \to \mathbb{Z} = \langle s\rangle$ such that $\Psi(p_1),\dots,\Psi(p_l) \in \mathbb{Z}[\mathbb{Z}] = \mathbb{Z}[s^{\pm1}]$ are non-zero polynomials. Since the polynomials $\Psi(p_1),\dots,\Psi(p_l)$ have finitely many zeros it follows that there exists a prime q such that no primitive q-th root of unity is a zero of any $\Psi(p_i)$, $i=1,\dots,l$. The homomorphism $F \xrightarrow{\Psi} \mathbb{Z} \to \mathbb{Z}/q\mathbb{Z}$ has the desired property. $\square$

A *representation* V *of a group* G *over a ring* R is a $(R,\mathbb{Z}[G])$-bimodule such that V is a finitely generated free R-module. Given a representation V of G over $\mathbb{Z}$ we denote by $V^{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{Z}} V$ the corresponding complex representation of G and given a prime p we denote by $V^p = \mathbb{F}_p \otimes_{\mathbb{Z}} V$ the corresponding representation of G over the finite field $\mathbb{F}_p$.

LEMMA 4.5. *Let N be 3-manifold, let $\theta \in H^1(N;\mathbb{Z})$ be non-trivial and let V be an integral representation of $\pi_1(N)$. Then for all but finitely many primes p we have*

$$\mathrm{width}\big(\tau\big(N, V_\theta^p\big)\big) = \mathrm{width}\big(\tau\big(N, V_\theta^{\mathbb{C}}\big)\big).$$

Proof. We provide the proof in the case that N is closed. The case that N has non-empty boundary is proved completely analogously.

We write $\pi = \pi_1(N)$. It follows from [FV11, p. 49] that there exist $g, h \in \pi$ with $\theta(g) \neq 0$ and $\theta(h) \neq 0$ and a square matrix B over $\mathbb{Z}[\pi]$ such that for any representation W of π over a commutative ring R we have

$$\tau(N, W_\theta) = \det{}_W(B) \cdot \det{}_W(1-g)^{-1} \cdot \det{}_W(1-h)^{-1}.$$

Here, given a $k \times k$-matrix C over $\mathbb{Z}[\pi]$ we denote by $\det_W(C)$ the determinant of the homomorphism

$$W \otimes_{\mathbb{Z}[\pi]} \mathbb{Z}[\pi]^k \to W \otimes_{\mathbb{Z}[\pi]} \mathbb{Z}[\pi]^k$$

given by right multiplication by $\mathrm{id} \otimes C$.

Now let V be an integral representation of $\pi_1(N)$. Given a prime p we denote by $\rho_p \colon \mathbb{Z} \to \mathbb{F}_p$ the projection map. This map induces a map $\rho_p \colon \mathbb{Z}[t^{\pm 1}] \to \mathbb{F}_p[t^{\pm 1}]$ and also a map

$$\rho_p \colon \left\{ \frac{x(t)}{y(t)} \,\middle|\, x(t), y(t) \in \mathbb{Z}\big[t^{\pm 1}\big] \text{ with } \rho_p\big(y(t)\big) \neq 0 \right\} \to \mathbb{F}_p(t).$$

We let $x(t) = \det_V(B)$ and $y(t) = \det_V(1-g) \cdot \det_V(1-h)$. It follows easily from the above formula for Reidemeister torsions and the fact that taking determinants commutes with ring homomorphisms that

$$\tau\big(N, V_\theta^{\mathbb{C}}\big) = x(t) y(t)^{-1}$$

and that for any prime p we have

$$\tau\big(N, V_\theta^p\big) = \rho_p\big(x(t)\big) \cdot \rho_p\big(y(t)\big)^{-1}.$$

Thus, if p is coprime to the bottom and the top coefficients of $x(t)$ and $y(t)$ we have

$$\mathrm{width}\big(\tau\big(N, V_\theta^p\big)\big) = \mathrm{width}\big(\tau\big(N, V_\theta^{\mathbb{C}}\big)\big). \qquad \square$$

For the record, we recall the well-known elementary fact.

Lemma 4.6. *Given $q \in \mathbb{N}$ the augmentation ideal $I(\mathbb{C})$ of $\mathbb{C}[\mathbb{Z}/q\mathbb{Z}]$, viewed as a representation of $\mathbb{Z}/q\mathbb{Z}$, is isomorphic to the direct sum of the representations*

$$rcl\rho_j\colon \mathbb{Z}/q\mathbb{Z} \to \mathrm{Aut}(\mathbb{C}),$$
$$a \mapsto \left(e^{2\pi i j a/q}\right)$$

with $j \in \{1,\dots,q-1\}$.

Now we are in a position to prove Theorem 4.1.

Proof of Theorem 4.1. Let N be 3-manifold which is not a closed graph manifold and let $\theta \in H^1(N;\mathbb{Z})$.

By Theorem 4.3, there exists a finite k-fold regular cover $p\colon M \to N$ such that the class $p^*\theta$ is in the closure of fibred classes.

If we follow the proofs of Proposition 5.8 and Theorem 5.9 in [FV15], if we replace [FV15, Lemma 5.7] by Lemma 4.4, and if we apply Lemmas 2.1 and 4.6 then we see that there exists a prime q and a homomorphism $\alpha\colon \pi_1(M) \to \mathbb{Z}/q\mathbb{Z}$ such that

$$\mathrm{width}\big(\tau\big(M, \big(\mathrm{Res}_\alpha I(\mathbb{C})\big)_{p^*\theta}\big)\big) = (q-1)x_M\big(p^*\theta\big).$$

Put differently, the $(q-1)$-dimensional representation $\mathrm{Res}_\alpha I(\mathbb{C})$ detects the Thurston norm of $p^*\theta$. Now the first part of theorem is an immediate consequence of Lemma 2.6 together with Lemma 2.4.

The second part is a consequence of the first part together with Lemma 4.5. □

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