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ON A LINEARIZED p-LAPLACE EQUATION WITH RAPIDLY OSCILLATING COEFFICIENTS

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ABSTRACT. Related to a conjecture of Tom Wolff, we solve a singular Neumann problem for a linearized p-Laplace equation in the unit disk.

1. Introduction

Tom Wolff [20] constructed in 1984 a celebrated example of a bounded p-harmonic function u in the upper half-plane $\mathbb{R}^2_+ = \{(x,y) \in \mathbb{R}^2 \colon y > 0\}$ such that the set

$$\Big\{x \in \mathbb{R}\colon \lim_{y\to 0} u(x,y) \text{ exists}\Big\}$$

has 1-dimensional Lebesgue measure zero. Fatou's classical radial limit theorem [5], [17] states that any bounded harmonic function in a smooth Euclidean domain has nontangential limits almost everywhere on the boundary of the domain, so Wolff's construction demonstrates the failure of Fatou's theorem in the nonlinear case $p \neq 2$.

The most important ingredient in Wolff's argument (see [20, Lemma 1]) is the construction of a bounded p-harmonic function $\Phi = \Phi(x,y)$ in $\mathbb{R}^2_+$ such that Φ has period $\lambda = \lambda_p$ in the x variable, $\Phi(x,y) \to 0$ as $y \to \infty$ (uniformly in x), and

$$\int_0^\lambda \Phi(x,0)\,dx \neq 0.$$

This is essentially a failure of the mean value principle; no such construction can be done for harmonic functions. See Section 1.2 below for a description of how the failure of Fatou's theorem follows.

Wolff states [20, p. 372] that his argument *must generalize to other domains*, and that *the argument is easiest in a half-space since the p-Laplace*

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operator behaves well under Euclidean operations. It is particularly interesting whether a construction is possible in bounded domains such as the unit disk, because in general there are serious problems when trying to map an unbounded planar domain on a bounded one in the p-harmonic setting. Conformal invariance is lost, and there cannot exist any reasonable counterpart to the Kelvin transform when $p \neq 2$; see [11]. While many open problems for p-harmonic functions are resolved in two dimensions, boundary behavior is still far from well understood.

This paper is motivated by the problem of whether it is possible to construct a bounded p-harmonic function $u = u(r, \theta)$ in the unit disk such that the set

$$\left\{\theta \in [0, 2\pi) : \lim_{r \to 1} u(r, \theta) \text{ exists}\right\}$$

has measure zero. A closely related problem is to construct a sequence $(u_N)_{N=1}^{\infty}$ of bounded p-harmonic functions in the unit disk such that the Nth function has angular period $2\pi/N$ and for each function $u = u_N = u(r, \theta)$,

$$u(0) \neq \fint_0^{2\pi} u(1, \theta)\, d\theta.$$

The author claimed [19] to have constructed such a sequence, but the construction contained a gap that remains open and is explained below in Section 1.4. In this paper, we construct the corresponding sequence for a linearized equation.

1.1. Organization of the paper and statement of results. Section 2 contains preliminaries about p-harmonic functions, and Section 3 introduces the appropriate moving frame intrinsic to the unit disk. In Section 4, we start with a well-known sequence $(f_N)_{N=1}^{\infty}$ of p-harmonic functions in $\mathbb{R}^2$ with the polar form

$$f_N(r, \theta) = r^k a_N(\theta), \tag{1.1}$$

where $k = k(p, N) > 0$ and the function a_N is $2\pi/N$-periodic. Unfortunately for our purposes, each of these functions $f = f_N$ satisfy

$$f(0) = \fint_0^{2\pi} f(r, \theta)\, d\theta = 0 \quad \text{for each } r > 0,$$

so a suitable perturbation is called for. We let $v = v_N$ be a solution to the linearized p-Laplace equation

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \Delta_p(f + \varepsilon v) = 0, \tag{1.2}$$

where $f = f_N$ is as in (1.1). After introducing the appropriate weighted function spaces in Section 5 and proving a priori regularity in Section 6, our main result, proved in Section 7, is the following theorem.

THEOREM 1.1. *For a given $N \in \mathbb{N}$, there exists a solution $v = v_N$ to (1.2) such that*

$$\int_0^{2\pi} \frac{\partial v}{\partial n}(1,\theta)\, d\theta \neq 0, \tag{1.3}$$

where n denotes the outer unit normal.

As described below in Section 1.3, this would yield the desired failure of the mean value principle for p-harmonic functions, provided that the function $v = v_N$ in (1.3) satisfied $\nabla v \in L^p(\mathbb{D})$. We conjecture this to be true, but the best regularity we obtain is the following theorem (Section 6).

THEOREM 1.2. *Any solution $v = v(x)$ to (1.2) satisfies $v \in L^\infty(\mathbb{D})$ and $|x|\nabla v(x) \in L^\infty(\mathbb{D})$.*

We prove Theorem 1.2 by noticing that (1.2) is the Euler–Lagrange equation of a quadratic energy functional, and thereby we are able to utilize Wolff's results in the upper half-plane via a conformal map.

REMARK 1.3. The equation (1.2) works out to

$$\operatorname{div}(A\nabla v) = 0,$$

where

$$\begin{aligned} A(r,\theta) = {} & r^{(p-2)(k-1)}\big(a_\theta^2 + k^2a^2\big)^{\frac{p-4}{2}} \\ & \times \begin{pmatrix} a_\theta^2 + (p-1)k^2a^2 & (p-2)kaa_\theta \\ (p-2)kaa_\theta & k^2a^2 + (p-1)a_\theta^2 \end{pmatrix}, \end{aligned}$$

the function a is from (1.1), and a_θ denotes its derivative. The equation (1.2) is degenerate/singular at the origin, and in such cases regularity of solutions is not well understood, see, for example, [21].

The outline of our treatment if similar to that of [20, Section 3]. We give many additional details on various calculations that are only sketched in [20]. We use a moving frame, because using plain polar coordinates would render the Neumann problem in Section 7 difficult to solve.

1.2. How the failure of the mean value principle leads to the failure of Fatou's theorem. Having constructed the function $\Phi = \Phi(x, y)$ described above on p. 499, Wolff's argument in the half-plane continues roughly as follows:

- If $(T_j)_{j=1}^\infty$ is a suitably fast-growing sequence of positive real numbers and if $(L_j)_{j=1}^\infty$ is a suitable sequence of uniformly bounded Lipschitz functions, then the sequence

$$\sigma_k(x) = \sum_{j=1}^{k} \frac{1}{j} L_j(x)\Phi(T_jx, 0) \tag{1.4}$$

is uniformly bounded and diverges for almost every x as $k \to \infty$ ([20, Lemma 2.12]).

- For each $k \geq 1$, denote by $\widehat{\sigma}_k$ the unique p-harmonic function in $\mathbb{R}^2_+$ having boundary values σ_k. Lemma 1.6 in [20] enables one to fix the sequences $(T_j)_{j=1}^{\infty}$ and $(L_j)_{j=1}^{\infty}$, and to obtain a decreasing sequence of positive numbers $\beta_k \to 0$ along with the following estimates:

$$\left|\widehat{\sigma}_{k+1}(x,y) - \widehat{\sigma}_k(x,y)\right| < \frac{1}{2^k} \quad \text{when } y > \beta_k,$$
$$\left|\widehat{\sigma}_{k+1}(x,y) - \sigma_k(x)\right| < \frac{1}{k} \quad \text{when } y \leq \beta_k.$$

- It follows that the sequence $\widehat{\sigma}_k$ converges to a p-harmonic limit function G as $k \to \infty$, and that for a.e. x the limit $\lim_{y\to 0} G(x,y)$ does not exist; see [20, p. 385].

Remark. Denoting $\phi_j(x) = \Phi(T_j x, 0)$ one has $\widehat{\phi}_j(x,y) = \Phi(T_j x, T_j y)$, so in the upper half-plane it is enough to construct a single function failing the mean value principle. An analogous scaling with respect to the angular variable does not hold in the disk.

1.3. How the Neumann problem leads to the failure of the mean value principle. In Section 7, we find a solution $v \in Y_1$ to the linearized p-Laplace equation such that

$$\left|\int_0^{2\pi} \frac{\partial v}{\partial n}(1,\theta)\, d\theta\right| = M > 0. \tag{1.5}$$

By the fundamental theorem of calculus, there exists a radius r_0 close to one such that

$$\left|\int_0^{2\pi} v(r_0,\theta) - v(1,\theta)\, d\theta\right| > \frac{M}{2}(1-r_0),$$

especially

$$\int_0^{2\pi} v(r_0,\theta)\, d\theta \neq \int_0^{2\pi} v(1,\theta)\, d\theta. \tag{1.6}$$

Assuming v is continuous at the origin, both of the functions $v_1 = v(r,\theta)$ and $v_2 = v(r_0 r, \theta)$ have the same value at the origin. We conclude from (1.6) that

$$v_i(0) \neq \int_0^{2\pi} v_i(1,\theta)\, d\theta$$

for some $i \in \{1,2\}$.

Remark. Assuming $\nabla v \in L^p(\mathbb{D})$, a similar argument holds for the p-harmonic function $\widehat{f + \varepsilon v}$ for a small ε, even without the assumption that v is continuous at the origin. See [20, Lemmas 3.16–3.19], where $\nabla v \in L^\infty(\mathbb{R}^2_+)$ can be replaced by $\nabla v \in L^p(\mathbb{R}^2_+)$ but any weaker regularity does not seem to suffice. The calculations transfer verbatim to the disk case.

1.4. Statement of error. It was claimed by the author [19, Lemma 7.7] that $v \in C^0(\mathbb{D}) \cap W^{1,\infty}(\mathbb{D})$ holds for any solution to the linearized p-Laplace equation (1.2). The argument claimed uniform ellipticity in dyadic annuli near the origin, but in fact the gradients of the test functions in Caccioppoli-type inequalities do not stay bounded.

CONJECTURE. *We conjecture, based on numerical experiments, that for each $N \in \mathbb{N}$ there exists a solution $v = v_N$ to the linearized p-Laplace equation* (1.2) *such that* (1.5) *holds and such that $v \in C^0(\mathbb{D}) \cap W^{1,\infty}(\mathbb{D})$.*

2. Preliminaries

The p-Laplace equation $\Delta_p u = 0$, that is,

$$\operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big) = 0,$$

is the Euler–Lagrange equation for the variational integral

$$I(u) = \int_\Omega |\nabla u|^p \, dx, \tag{2.1}$$

where $1 < p < \infty$ and Ω is an Euclidean domain. A real-valued function $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is a solution of the p-Laplace equation if and only if

$$\int_\Omega \big\langle |\nabla u|^{p-2}\nabla u, \nabla\varphi\big\rangle \, dx = 0$$

for all $\varphi \in C_0^\infty(\Omega)$. A standard density argument (e.g., [7, Lemma 3.11]) yields that the class of test functions can be extended to $W_0^{1,p}(\Omega)$. In general p-harmonic functions are not twice continuously differentiable, but their first partial derivatives are locally Hölder continuous. A complete regularity characterization in the plane was given in [8]; in higher dimensions the optimal regularity is unknown. Outside critical points p-harmonic are real analytic [9].

For a given $g \in W^{1,p}(\Omega)$ there exists a unique p-harmonic function u in Ω such that $u - g \in W_0^{1,p}(\Omega)$. Equivalently, u is the unique minimizer of the p-Dirichlet integral $I(v)$ in (2.1) among functions $v \in W^{1,p}(\Omega)$ with $v - g \in W_0^{1,p}(\Omega)$; see, for example, [13].

There are two planar analogues between the cases $p = 2$ and $p \neq 2$ that we find particularly interesting:

(1) While harmonic functions are characterized by the asymptotic mean value property

$$u(x) = \fint_{B_r(x)} u(y) \, dy + o\big(r^2\big) \quad \text{as } r \to 0$$

for each $x \in \Omega$ and each ball $B_r(x) \subset \Omega$, p-harmonic functions are analogously characterized by

$$u(x) = \alpha \fint_{B_r(x)} u(y)\,dy + (1-\alpha)\frac{1}{2}\Big(\max_{\overline{B}_r(x)} u + \min_{\overline{B}_r(x)} u\Big) + o\big(r^2\big) \quad \text{as } r \to 0,$$

where $\alpha = 4/(p+2)$; see [3], [14].

(2) If u is harmonic in a simply connected planar domain, there exists a conjugate harmonic function v, unique up to a constant, such that

$$u_x = v_y, \qquad u_y = -v_x,$$

$\langle \nabla u, \nabla v\rangle = 0$, and such that the map $F = u + iv$ is conformal. Analogously, if u is p-harmonic in a simply connected domain $\Omega \subset \mathbb{R}^2$, there exists a conjugate q-harmonic[1] function v, unique up to a constant, such that

$$u_x = |\nabla v|^{q-2} v_y, \qquad u_y = -|\nabla v|^{q-2} v_x,$$

$\langle \nabla u, \nabla v\rangle = 0$, and such that the map $F = u + iv$ is locally quasiregular outside the isolated set $\{x \in \Omega \colon \nabla u = \nabla v = 0\}$; see [12].

For example, Wolff has $p > 2$ in [20], but Lewis [10] reduced the case $1 < p < 2$ to Wolff's result by using the conjugacy property (2) above. It may be worthwile to carry out our program in a complex setting with both of the conjugate pairs simultaneously present. Moreover, since the purpose of our work is to construct a p-harmonic function that fails the mean value principle in a specific way, the characterization 1) above could provide useful insights.

Throughout in what follows, we will assume $p > 2$; this property is used in Lemma 7.5. Our domain of interest will be the unit disk $\mathbb{D} = \{x \in \mathbb{R}^2 \colon |x| < 1\}$, and we denote $\mathbb{D}^* = \{x \in \mathbb{R}^2 \colon 0 < |x| < 1\}$. We will use the partial derivative notation (e.g., u_r and a_θ) also in the case of a single variable function.

3. A moving frame

Let (r, θ) denote the polar coordinates in the plane, and define a moving frame intrinsic to $\mathbb{D}^*$ by

$$e_r = \frac{\partial}{\partial r}, \qquad e_\theta = \frac{1}{r}\frac{\partial}{\partial \theta}.$$

Let f be a real-valued function defined outside the origin in the plane. The intrinsic gradient of f is formally defined as a vector

$$\nabla^\circ f(r,\theta) = \big(e_r f(r,\theta), e_\theta f(r,\theta)\big),$$

that is, $\nabla^\circ f = R(\theta)\nabla f$, where $R(\theta)$ is the rotation by θ and ∇f is the gradient in Cartesian coordinates.

[1] $1/p + 1/q = 1$.

The dual basis to $\{e_r, e_\theta\}$ is $\{dr, r\,d\theta\}$, and the volume element is $dA = r\,dr\,d\theta$. The adjoints e_r^* and e_θ^* are defined as

$$\int_{\mathbb{D}^*} e_r(u)v\,dA = \int_{\mathbb{D}^*} ue_r^*(v)\,dA \quad \text{for all } u, v \in C_0^\infty\big(\mathbb{D}^*\big),$$

and similarly for e_θ. The intrinsic divergence of a vector field $F = (f, g)$ is formally defined as

$$\operatorname{div}^\circ F = -\big(e_r^*(f) + e_\theta^*(g)\big).$$

Lemma 3.1. *Let $F = (f, g)$ be a differentiable vector field in $\mathbb{D}^*$. Then*

$$\operatorname{div}^\circ F = \frac{1}{r}e_r(rf) + e_\theta(g).$$

Proof. The claim is

$$e_r^*(f) = -\frac{1}{r}e_r(rf) \quad \text{and} \quad e_\theta^*(g) = -e_\theta(g).$$

Let $\varphi \in C_0^\infty(\mathbb{D}^*)$. Since $(\varphi rf)_r = \varphi_r rf + \varphi(rf)_r$, we have

$$\int f\varphi_r\,dA = -\int \varphi\frac{1}{r}(rf)_r\,dA,$$

that is, $e_r^*(f) = -\frac{1}{r}e_r(rf)$ as wanted. The e_θ case is similar: using $(g\varphi)_\theta = g\varphi_\theta + \varphi g_\theta$ leads to

$$\int \frac{1}{r^2}g\varphi_\theta\,dA = -\int \frac{1}{r^2}\varphi g_\theta\,dA,$$

that is, $e_\theta^*(g) = -e_\theta(g)$. □

One now easily verifies that $\Delta^\circ u := \operatorname{div}^\circ(\nabla^\circ u) = \Delta u$ by writing the Laplacian in polar coordinates, that is,

$$\Delta^\circ u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta},$$

and the same holds for the p-Laplacian.

Lemma 3.2. *The p-Laplace equation* $\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0$ *in* $\mathbb{D}^*$, $1 < p < \infty$, *can be written as*

$$\operatorname{div}^\circ\Big(\big|\nabla^\circ u\big|^{p-2}\nabla^\circ u\Big) = 0 \quad \textit{in } \mathbb{D}^*. \tag{3.1}$$

Proof. Let $u \in W^{1,p}(\mathbb{D}^*)$ be p-harmonic in $\mathbb{D}^*$ and let $\varphi \in C_0^\infty(\mathbb{D}^*)$. Since $|\nabla u| = |\nabla^\circ u|$ and since $\langle \nabla u, \nabla\varphi\rangle = \langle \nabla^\circ u, \nabla^\circ \varphi\rangle$, the p-Laplace equation can be rewritten as

$$\int_{\mathbb{D}^*} \big|\nabla^\circ u\big|^{p-2}\big\langle \nabla^\circ u, \nabla^\circ\varphi\big\rangle\,dA = 0,$$

which is the weak form of (3.1). □

4. A linearized p-Laplace equation

The following well-known lemma is adapted from Tkachev [18] and Aronsson [1], [2].

Lemma 4.1. *Given* $1 < p < \infty$ *and* $N \in \mathbb{N}$, *there exists a* $2\pi/N$*-periodic function* $a = a_{p,N} : \mathbb{R} \to \mathbb{R}$ *and a real number* $k = k(p, N) > 0$ *such that the function*

$$f = f_{p,N}(r, \theta) = r^k a(\theta)$$

is p-harmonic in $\mathbb{R}^2$. *Moreover, the following holds*:

(i) *The number* $k = k(p, N)$ *is determined from the quadratic equation*

$$(2N-1)(b+1)k^2 - 2\big(N^2 b + 2N - 1\big)k + N^2(1+b) = 0,$$

where $b = p/(p-2)$.

(ii) *The function a is characterized by the quasilinear ordinary differential equation*

$$a_{\theta\theta} + V(a, a_\theta)a = 0, \tag{4.1}$$

where

$$V(a, a_\theta) = \frac{((2p-3)k^2 - (p-2)k)a_\theta^2 + ((p-1)k^2 - (p-2)k)k^2 a^2}{(p-1)a_\theta^2 + k^2 a^2}.$$

(iii) *The equation* (4.1) *has a unique solution* $a \in C^\infty(\mathbb{R})$ *with given initial data* $a(0), a_\theta(0)$.

(iv) *Assume* $a(0) = 1$ *and* $a_\theta(0) = 0$, *and denote* $\lambda = \sqrt{k^2 - \frac{2k}{p/(p-2)+1}}$. *The function* $a = a(\theta)$ *admits the parametrization*

$$a = \big(t^2 + \lambda^2\big)^{(k-1)/2}\big(t^2 + k^2\big)^{-k/2},$$
$$\theta = \arctan\left(\frac{t}{k}\right) - \frac{k-1}{\lambda}\arctan\left(\frac{t}{\lambda}\right),$$

where $t \in \mathbb{R}$ *and* $\theta \in (-\frac{\pi}{2N}, \frac{\pi}{2N})$, *and for other values of* θ,

$$a(\theta) = -a(\pi/N - \theta), \qquad a(-\theta) = a(\theta).$$

(v) *The function* $f = f(x, y)$ *admits the parametrization*

$$f = h^{k(2N-1)}\cos(N\tau),$$
$$x = h^{2N-1}\big((k+\lambda)\cos\tau + (k-\lambda)\cos(2N-1)\tau\big),$$
$$y = h^{2N-1}\big((k+\lambda)\sin\tau - (k-\lambda)\sin(2N-1)\tau\big),$$

where $\tau \in [0, 2\pi], h > 0$.

The following lemma is analogous to [20, (3.13)].

LEMMA 4.2. *Let* $1 < p < \infty$, $N \in \mathbb{N}$, *and let* $f(r,\theta) = r^k a(\theta)$ *be as in Lemma* 4.1. *The expression*

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \Delta_p(f + \varepsilon v) = 0$$

reduces formally to

$$\operatorname{div}^\circ\big(A\nabla^\circ v\big) = 0,$$

where

$$A(r,\theta) = r^{(p-2)(k-1)}\big(a_\theta^2 + k^2a^2\big)^{\frac{p-4}{2}} \times \begin{pmatrix} a_\theta^2 + (p-1)k^2a^2 & (p-2)kaa_\theta \\ (p-2)kaa_\theta & k^2a^2 + (p-1)a_\theta^2 \end{pmatrix}. \tag{4.2}$$

Proof. Since

$$\begin{aligned}
&\frac{d}{d\varepsilon}\Big(\big|\nabla^\circ f + \varepsilon\nabla^\circ v\big|^{p-2}\big(\nabla^\circ f + \varepsilon\nabla^\circ v\big)\Big) \\
&\quad = \big|\nabla^\circ f + \varepsilon\nabla^\circ v\big|^{p-2}\nabla^\circ v \\
&\qquad + (p-2)\Big(\big\langle\nabla^\circ f, \nabla^\circ v\big\rangle + \varepsilon\big|\nabla^\circ v\big|^2\Big)\big|\nabla^\circ f + \varepsilon\nabla^\circ v\big|^{p-4}\big(\nabla^\circ f + \varepsilon\nabla^\circ v\big),
\end{aligned}$$

the expression

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \operatorname{div}^\circ\Big(\big|\nabla^\circ f + \varepsilon\nabla^\circ v\big|^{p-2}\big(\nabla^\circ f + \varepsilon\nabla^\circ v\big)\Big)$$

becomes

$$\begin{aligned}
&\operatorname{div}^\circ\Big(\big|\nabla^\circ f\big|^{p-2}\nabla^\circ v + (p-2)\big|\nabla^\circ f\big|^{p-4}\big\langle\nabla^\circ f, \nabla^\circ v\big\rangle\nabla^\circ f\Big) \\
&\quad = \operatorname{div}^\circ\Big(\big|\nabla^\circ f\big|^{p-2}\nabla^\circ v + (p-2)\big|\nabla^\circ f\big|^{p-4}\big(\nabla^\circ f \otimes \nabla^\circ f\big)\nabla^\circ v\Big) \\
&\quad = \operatorname{div}^\circ\big(A\nabla^\circ v\big),
\end{aligned}$$

where[2]

$$A = \big|\nabla^\circ f\big|^{p-4}\Big(\big|\nabla^\circ f\big|^2 I + (p-2)\big(\nabla^\circ f \otimes \nabla^\circ f\big)\Big).$$

Inserting

$$\big|\nabla^\circ f\big| = r^{k-1}\big(a_\theta^2 + k^2a^2\big)^{\frac{1}{2}}$$

yields (4.2). □

LEMMA 4.3. *Let* $1 < p < \infty$, $N \in \mathbb{N}$, *and let* A *be as in* (4.2). *Then*

$$r^{(p-2)(k-1)}\big(a_\theta^2 + k^2a^2\big)^{\frac{p-2}{2}}|\xi|^2 \le \langle A\xi, \xi\rangle \le (p-1)r^{(p-2)(k-1)}\big(a_\theta^2 + k^2a^2\big)^{\frac{p-2}{2}}|\xi|^2$$

for all $\xi \in \mathbb{R}^2$.

[2] If $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ are vectors in $\mathbb{R}^n$, their tensor product $a \otimes b$ is an $n \times n$ matrix with

$$(a \otimes b)_{ij} = a_i b_j.$$

It is straightforward to check that the formula $\langle a, b\rangle a = (a \otimes a)b$ holds for vectors $a, b \in \mathbb{R}^n$.

Proof. The eigenvalues μ of the matrix

$$\widetilde{A} = \begin{pmatrix} a_\theta^2 + (p-1)k^2a^2 & (p-2)kaa_\theta \\ (p-2)kaa_\theta & k^2a^2 + (p-1)a_\theta^2 \end{pmatrix}$$

are

$$\begin{aligned} \mu &= \frac{\operatorname{tr}\widetilde{A} \pm \sqrt{(\operatorname{tr}\widetilde{A})^2 - 4\det\widetilde{A}}}{2} \\ &= \frac{pa_\theta^2 + pk^2a^2 \pm \sqrt{p^2(a_\theta^2+k^2a^2)^2 - 4\det\widetilde{A}}}{2}. \end{aligned}$$

Since

$$\det\widetilde{A} = (p-1)\big(a_\theta^2 + k^2a^2\big)^2,$$

we obtain

$$\mu = \frac{(p \pm (p-2))(a_\theta^2 + k^2a^2)}{2},$$

and the claim follows. □

Remark. Denoting $\lambda \le \langle A\xi, \xi\rangle \le \Lambda$, we have $\Lambda/\lambda = p-1$. Moreover, the eigenvectors of A work out to $(-a_\theta, ka)$ and (ka, a_θ). These observations are not used in the present work.

5. A weighted Sobolev space

Let p and N be fixed, and let A be as in (4.2). We will look for weak solutions v to the equation

$$\operatorname{div}^\circ\big(A\nabla^\circ v\big) = 0 \quad \text{in } \mathbb{D}^*$$

in the weighted $W^{1,2}$ space of $2\pi/N$-angular periodic functions:

$$Y_1 = \left\{ v \in W^{1,2}_{\mathrm{loc}}(\mathbb{D}^*) : v\left(r, \theta + \frac{2\pi}{N}\right) = v(r,\theta), \int_{\mathbb{D}} |v|^2 r^{2\beta} + \big|\nabla^\circ v\big|^2 r^{2\alpha}\, dA < \infty \right\}.$$

Here $\alpha = (p-2)(k-1)/2$ and β is any number satisfying $\alpha - 1 < \beta < 2\alpha - 1$. The inequality $\alpha - 1 < \beta$ will be needed for the imbedding of Y_1 to the weighted L^2 space to be compact, and the inequality $\beta < 2\alpha - 1$ for continuously differentiable functions to be dense in Y_1. The negativity of $\alpha - 1$ is not an issue since we assume $p > 2$ and since we are ultimately interested in large values of the parameter k.

Define the weighted L^2 space as

$$Y_0 = \left\{ f \in L^2_{\mathrm{loc}}(\mathbb{D}^*) : f\left(r, \theta + \frac{2\pi}{N}\right) = f(r,\theta), \int_{\mathbb{D}} \big|f(r,\theta)\big|^2 r^{2\beta}\, dA < \infty \right\}.$$

The inner product in Y_0 is defined as

$$(f|g)_{Y_0} = \int_{\mathbb{D}} f(r,\theta) r^\beta g(r,\theta) r^\beta\, dA,$$

and the inner products in Y_1, and in

$$Y_0^* = \left\{ f \in L^2_{\mathrm{loc}}\big(\mathbb{D}^*\big) \colon \int_{\mathbb{D}} \big|f(r,\theta)\big|^2 r^{-2\beta}\, dA < \infty \right\}$$

are defined accordingly. The dual pairing between $f \in Y_0$ and $g \in Y_0^*$ is

$$\langle f | g \rangle = \int_{\mathbb{D}} f g \, dA.$$

In this section, we prove three lemmas about the spaces Y_0 and Y_1 that are omitted in the half-plane case of [20]. We prefer $|\nabla f|$ over $|\nabla^\circ f|$ in the notation, because the expressions are equal and the moving frame will not become apparent until in Section 7.

The first lemma will be used in Lemma 7.3.

LEMMA 5.1. *The imbedding $id \colon Y_1 \to Y_0$ is compact.*

Proof. Let $\varepsilon > 0$ be small, and denote $id = id_0 + id_1$, where id_0 and id_1 denote the restrictions of id to functions in Y_1 defined on the annuli

$$A_0 = \big\{ (r,\theta) \colon 0 < r < \varepsilon, 0 \le \theta < 2\pi \big\}$$

and

$$A_1 = \big\{ (r,\theta) \colon \varepsilon \le r < 1, 0 \le \theta < 2\pi \big\},$$

respectively. The imbedding id_1 is compact for each ε, because the imbedding $W^{1,2} \to L^2$ is compact and we are away from the origin. It suffices to show for $u \in Y_1$ that

$$\|u\|^2_{Y_0(A_0)} := \int_{A_0} u^2 r^{2\beta}\, dA \le C_\varepsilon \|u\|^2_{Y_1},$$

where $C_\varepsilon \to 0$ as $\varepsilon \to 0$. Since the imbeddings id_1 are compact, this yields (see, e.g., [6, Theorem 0.34]) that id itself is compact.

Let $u \in Y_1$, $0 < r < 1/2$, and $0 \le \theta < 2\pi$. Since

$$u(r,\theta) \le u(R,\theta) + \int_r^R \big|\nabla u(\rho,\theta)\big|\, d\rho$$

for each $R \in (r,1)$, we estimate

$$\big|u(r,\theta)\big| \le \fint_{(1/2,1)} \big|u(\rho,\theta)\big|\, d\rho + \int_r^1 \big|\nabla u(\rho,\theta)\big|\, d\rho.$$

The Cauchy–Schwarz inequality yields

$$\begin{aligned}\int_r^1 \big|\nabla u(\rho,\theta)\big|\, d\rho &\le \left(\int_r^1 \big|\nabla u(\rho,\theta)\big|^2 \rho^{2\alpha+1}\, d\rho \right)^{\frac12} \left(\int_r^1 \rho^{-(2\alpha+1)}\, d\rho \right)^{\frac12} \\ &\le C_\alpha r^{-\alpha} \left(\int_r^1 \big|\nabla u(\rho,\theta)\big|^2 \rho^{2\alpha+1}\, d\rho \right)^{\frac12},\end{aligned}$$

so we obtain for $u \in Y_1$ that

$$\begin{aligned}
(5.1) \qquad & \int_{A_0} u^2 r^{2\beta}\, dA \\
&\le C \int_{A_0} \Bigg(\bigg(\fint_{(1/2,1)} \big| u(\rho,\theta) \big|\, d\rho \bigg)^2 \\
&\qquad + r^{-2\alpha} \int_r^1 \big| \nabla u(\rho,\theta) \big|^2 \rho^{2\alpha+1}\, d\rho \Bigg) r^{2\beta}\, dA.
\end{aligned}$$

The first term on the right-hand side of (5.1) is estimated using $r < 1$ and the Cauchy–Schwarz inequality:

$$\begin{aligned}
&\int_{A_\varepsilon} \bigg(\fint_{(1/2,1)} \big| u(\rho,\theta) \big|\, d\rho \bigg)^2 r^{2\beta}\, dA \\
&\le \int_{A_\varepsilon} \bigg(\fint_{(1/2,1)} \big| u(\rho,\theta) \big|\, d\rho \bigg)^2 dA \\
&\le \int_{A_\varepsilon} \bigg(\int_{1/2}^1 \big| u(\rho,\theta) \big|^2 \rho^{2\beta} \rho\, d\rho \bigg) \bigg(\int_{1/2}^1 \rho^{-2\beta}\, d\rho \bigg) dA \\
&\le C \|u\|_{Y_1}^2.
\end{aligned}$$

The second term on the right-hand side of (5.1) is estimated with Fubini's theorem as

$$\begin{aligned}
&\int_{A_\varepsilon} r^{-2\alpha} \int_r^1 \big| \nabla u(\rho,\theta) \big|^2 \rho^{2\alpha+1}\, d\rho r^{2\beta}\, dA \\
&\quad = \int_0^\varepsilon r^{2\beta-2\alpha} \int_0^{2\pi} \int_r^1 \big| \nabla u(\rho,\theta) \big|^2 \rho^{2\alpha} \rho\, d\rho\, d\theta r\, dr \\
&\quad \le \|u\|_{Y_1}^2 \int_0^\varepsilon r^{2\beta-2\alpha+1}\, dr \le \varepsilon^{2(\beta-\alpha+1)} \|u\|_{Y_1}^2.
\end{aligned}$$

Thus we obtain

$$\int_{A_\varepsilon} u^2 r^{2\beta}\, dA \le C \big(\varepsilon^{2(\beta-\alpha+1)} + \varepsilon^2 \big) \|u\|_{Y_1}^2 \xrightarrow[\varepsilon\to 0]{} 0$$

as wanted, since $\beta > \alpha - 1$. □

The next lemma is omitted in [20, p. 390] and is added here for completeness.

Lemma 5.2. *Let $u \in Y_1$ be continuous. For $k \in \mathbb{N}$ define $r_k = 2^{-k}$ and*

$$m_k = \min_\theta u(r_k,\theta).$$

Then

$$\limsup_{k\to\infty} r_k^\alpha m_k \le 0. \tag{5.2}$$

Proof. Let us first assume that u is a radial function. We will assume that (5.2) is false and prove that

$$\int_0^1 |u_r(r)| r^{2\alpha} r\, dr = \infty,$$

which contradicts the fact that $u \in Y_1$.

So let

$$\limsup_{k\to\infty} r_k^\alpha u(r_k) = \varepsilon_0 > 0.$$

Then there exists a subsequence of r_k (still call it r_k) such that $r_k^\alpha u(r_k) \geq \varepsilon_0/2$ for each k. Fix r_1 from the subsequence, and choose r_2 from the subsequence small enough such that $u(r_2) \geq 2u(r_1)$. We have, by the fundamental theorem of calculus (for Sobolev functions), that

$$M := \int_{r_1}^{r_2} u_r(r)\, dr = u(r_2) - u(r_1) \geq u(r_2)/2 \geq r_2^{-\alpha}\varepsilon_0/4.$$

Let $v(r) = u_r(r) r^{\alpha+1/2}$, so that

$$M = \int_{r_1}^{r_2} v(r) r^{-(\alpha+1/2)}\, dr = \left\langle v, r^{-(\alpha+1/2)} \right\rangle.$$

By elementary Hilbert space geometry, the smallest value of

$$\|v\|_2^2 := \|v\|^2_{L^2((r_1,r_2))} = \int_{r_1}^{r_2} v(r)^2\, dr$$

under the condition $\langle v, g\rangle = M$, is attained when v is parallel to g, that is, $v = gM/\|g\|_2$. In our case, $g(r) = r^{-(\alpha+1/2)}$ and

$$\begin{aligned}\|v\|_2^2 &\geq \frac{M^2}{\|g\|_2^2} = \frac{M^2}{\int_{r_1}^{r_2} r^{-(2\alpha+1)}\, dr} = \frac{M^2}{r_2^{-2\alpha} - r_1^{-2\alpha}} \\ &\geq \frac{\varepsilon_0^2}{16} \cdot \frac{r_2^{-2\alpha}}{r_2^{-2\alpha} - r_1^{-2\alpha}} = \frac{\varepsilon_0^2}{16} \cdot \frac{1}{1 - (\frac{r_2}{r_1})^{2\alpha}} \geq C_\alpha \varepsilon_0^2,\end{aligned}$$

since $r_2/r_1 \leq 1/2$ by definition.

Now choose r_3 from the subsequence such that $u(r_3) \geq 2u(r_2)$ and repeat the process above. Continuing in a similar fashion and summing over the chosen radii r_j, we have

$$\|u\|_{Y_1}^2 \geq \sum_{j=1}^\infty \|v\|^2_{L^2_{((r_j, r_{j+1}))}} = \infty,$$

which concludes the radial case.

When u is not radial, denote

$$\widetilde{u}(r) = \frac{1}{2\pi}\int_0^{2\pi} u(r,\theta)\, d\theta$$

and repeat the proof above as follows: Choose r_2 from the subsequence such that

$$\min_\theta u(r_2,\theta) \geq 2\max_\theta u(r_1,\theta).$$

Then

$$M := \int_{r_1}^{r_2} \widetilde{u}_r(r)\,dr \geq \frac{1}{2}\min_\theta u(r_2,\theta) \geq r_2^{-\alpha}\varepsilon_0/4.$$

As before, we obtain

$$\int_{r_1}^{r_2} \left|\widetilde{u}_r(r)\right|^2 r^{2\alpha} r\,dr \geq C_\alpha \varepsilon_0^2.$$

Finally, since

$$\left|\widetilde{u}_r(r)\right|^2 \leq \frac{1}{2\pi}\int_0^{2\pi} \left|u_r(r,\theta)\right|^2 d\theta,$$

we have, by Fubini's theorem, that

$$\begin{aligned}
\int_{\mathbb{D}} |\nabla u|^2 r^{2\alpha}\,dA &\geq \int_0^{2\pi}\int_0^1 \left|u_r(r,\theta)\right|^2 r^{2\alpha} r\,dr\,d\theta \\
&= \int_0^1\int_0^{2\pi} \left|u_r(r,\theta)\right|^2 d\theta r^{2\alpha+1}\,dr \\
&\geq 2\pi\int_0^1 \left|\widetilde{u}_r(r)\right|^2 r^{2\alpha+1}\,dr \\
&\geq 2\pi\sum_{j=1}^{\infty}\int_{r_j}^{r_{j+1}} \left|\widetilde{u}_r(r)\right|^2 r^{2\alpha+1}\,dr = \infty,
\end{aligned}$$

contradicting $u \in Y_1$. □

The result below is used in the proof of Lemma 7.7.

LEMMA 5.3. *The space $C^1(\mathbb{D})$ of continuously differentiable functions in $\mathbb{D}$ is dense in Y_1, i.e. Y_1 is the closure of $C^1(\mathbb{D})$ under the norm*

$$\|f\|_{Y_1} = \left(\int_{\mathbb{D}} \left|f(r,\theta)\right|^2 r^{2\beta} + \left|\nabla f(r,\theta)\right|^2 r^{2\alpha}\,dA\right)^{\frac{1}{2}}.$$

Proof. Let $u \in Y_1$ and $\varepsilon > 0$. We are looking for a function $v = v_\varepsilon \in C^1(\mathbb{D})$ such that $\|u - v\|_{Y_1} \to 0$ when $\varepsilon \to 0$. By truncating u, we may assume that u is bounded. We may also assume that u is radial; the general case follows as in the proof of Lemma 5.2.

Let $\varphi = \varphi_\varepsilon \in C^\infty(0,1)$ be such that $\varphi(r) = 0$ when $0 < r < \varepsilon$, $\varphi(r) = 1$ when $2\varepsilon < r < 1$, and $0 \leq \varphi_r \leq C/\varepsilon$. We will show below that $u\varphi \in Y_1$ satisfies $\|u - u\varphi\|_{Y_1} \to 0$ as $\varepsilon \to 0$. Thereafter, the desired function v is obtained from $u\varphi$ by a standard convolution approximation left to the reader.

We start with

$$\begin{aligned}\|u-u\varphi\|_{Y_1}^2 &= \big\|u(1-\varphi)\big\|_{Y_1}^2\\ &= \int_0^1 \big|u(1-\varphi)\big|^2 r^{2\beta+1}\,dr + \int_0^1 \big|\big(u(1-\varphi)\big)_r\big|^2 r^{2\alpha+1}\,dr.\end{aligned}$$

The first integral is bounded above by

$$C\int_0^{2\varepsilon} |u|^2 r^{2\beta+1}\,dr$$

and goes to zero with ε, since $u\in Y_0$. For the second integral, we estimate

$$\begin{aligned}&\int_\varepsilon^{2\varepsilon} \big|\big(u(1-\varphi)\big)_r\big|^2 r^{2\alpha+1}\,dr\\ &\quad\le 2\int_\varepsilon^{2\varepsilon} |1-\varphi|^2|u_r|^2 r^{2\alpha+1}\,dr + 2\int_\varepsilon^{2\varepsilon} |u|^2\big|(1-\varphi)_r\big|^2 r^{2\alpha+1}\,dr.\end{aligned}$$

Here the first integral goes to zero with ε, since $u\in Y_1$. Since $|(1-\varphi)_r| = |\varphi_r|\le C/\varepsilon$, the second integral is estimated using the Cauchy–Schwarz inequality:

$$\begin{aligned}\frac{C}{\varepsilon^2}\int_\varepsilon^{2\varepsilon} |u|^2 r^{2\alpha+1}\,dr &\le \frac{C}{\varepsilon^2}\left(\int_\varepsilon^{2\varepsilon} |u|^2 r^{2\beta+1}\,dr\right)^{\frac12}\left(\int_\varepsilon^{2\varepsilon} |u|^2 r^{4\alpha-2\beta+1}\,dr\right)^{\frac12}\\ &\le \frac{C}{\varepsilon^2}\|u\|_{Y_0}\left(\int_\varepsilon^{2\varepsilon} |u|^2 r^{4\alpha-2\beta+1}\,dr\right)^{\frac12}\\ &\le \frac{C}{\varepsilon^2}\|u\|_{Y_0}\|u\|_\infty\left(\int_\varepsilon^{2\varepsilon} r^{4\alpha-2\beta+1}\,dr\right)^{\frac12}\\ &= C\varepsilon^{2\alpha-\beta-1},\end{aligned}$$

which goes to zero when $\beta<2\alpha-1$. □

6. Regularity

We proceed to prove a priori regularity of solutions to the linearized p-Laplace equation (1.2). Since the coefficients of the matrix A in (4.2) are in the class $C^\infty(\overline{\mathbb{D}}\setminus\{0\})$, also the solutions are in this class by standard linear regularity theory. The question of interest is regularity at the origin.

6.1. The half-plane case. We start by quoting some results from [20, pp. 387–390]. Fix arbitrary constants $\lambda>0$, $\beta>0$ and $0<\alpha<\beta$, denote $S^\lambda=[0,\lambda)\times(0,\infty)\subset\mathbb{R}^2_+$, and consider the Hilbert space $\widetilde{Y}_1$ defined via

$$\widetilde{Y}_0 = \left\{ f\in L^2_{\mathrm{loc}}\big(\mathbb{R}^2_+\big) : f(x+\lambda,y)=f(x,y), \int_{S^\lambda}\big|f(x,y)\big|^2 e^{-2\beta y}\,dx\,dy<\infty\right\},$$

$$\widetilde{Y}_1 = \widetilde{Y}_0\cap\left\{ f\in L^2_{\mathrm{loc}}\big(\mathbb{R}^2_+\big) : \nabla f\in L^2_{\mathrm{loc}}\big(\mathbb{R}^2_+\big), \int_{S^\lambda}\big|\nabla f(x,y)\big|^2 e^{-2\alpha y}\,dx\,dy<\infty\right\}.$$

Fix $\widetilde{A}\colon \overline{\mathbb{R}^2_+} \to 2\times 2$ real symmetric matrices. Assume $\widetilde{A}$ is C^∞ on $\overline{\mathbb{R}^2_+}$, $\widetilde{A}(x+\lambda,y)=\widetilde{A}(x,y)$ and that there exists a constant $C>0$ such that

$$C^{-1}e^{-2\alpha y}|\xi|^2 \le \big\langle \widetilde{A}(x,y)\xi,\xi\big\rangle \le Ce^{-2\alpha y}|\xi|^2. \tag{6.1}$$

Then the following two results hold:

THEOREM 6.1 ([20, Lemma 3.8]). *If $u\in \widetilde{Y}_1\cap C^\infty(\overline{\mathbb{R}^2_+})$ satisfies $\operatorname{div}(\widetilde{A}\nabla u)=0$ in $\mathbb{R}^2_+$, then u is bounded. In fact, $u(x,y)\le \max_t u(t,0)$ for all $(x,y)\in\mathbb{R}^2_+$.*

THEOREM 6.2 ([20, Lemma 3.12]). *If $u\in \widetilde{Y}_1\cap C^\infty(\overline{\mathbb{R}^2_+})$ satisfies $\operatorname{div}(\widetilde{A}\nabla u)=0$ in $\mathbb{R}^2_+$, then there are $\gamma>0$ and $\mu\in\mathbb{R}$ such that $|u(x,y)-\mu|\le 2e^{-\gamma y}\|u\|_\infty$ for all $(x,y)\in\mathbb{R}^2_+$. Consequently $\nabla u\in L^q(S^\lambda)$ for all $q\in(0,\infty]$.*

6.2. The disk case. The weak form of $\operatorname{div}(\widetilde{A}\nabla u)=0$ in $\mathbb{R}^2_+$, where $\widetilde{A}$ satisfies (6.1), is the Euler–Lagrange equation for minimizing

$$\int_{S^\lambda}\langle\widetilde{A}\nabla u,\nabla u\rangle\,dx\,dy$$

among functions in $\widetilde{Y}_1$. Analogously the weak form of the equation $\operatorname{div}^\circ(A\nabla^\circ v)=0$ in $\mathbb{D}^*$, where A satisfies (4.2), is the Euler–Lagrange equation for minimizing

$$\int_{\mathbb{D}^*}\big\langle A\nabla^\circ v,\nabla^\circ v\big\rangle r\,dr\,d\theta \tag{6.2}$$

among functions in Y_1.

THEOREM 6.3. *Let $v\in Y_1$ minimize (6.2) with a given boundary data. Map the strip $S=\{(x,y)\in\mathbb{R}^2_+ : -\pi<x\le\pi\}$ to $\mathbb{D}^*$ via the map $G\colon\mathbb{C}\to\mathbb{C}$, $G(z)=e^{iz}$. Then the composed function $u=v\circ G$ minimizes*

$$\int_S c(x,y)\big|\nabla u(x,y)\big|^2 e^{-2\alpha y}\,dx\,dy, \tag{6.3}$$

where $\alpha=((p-2)(k-1)+2)/2$ and $C^{-1}\le c(x,y)\le C$ for some $C>0$.

Proof. The expression (6.2) has the form

$$\int_{\mathbb{D}^*}\big(a_\theta^2+k^2a^2\big)^{\frac{p-2}{2}} r^{(p-2)(k-1)}\big|\nabla v(r,\theta)\big|^2 r\,dr\,d\theta. \tag{6.4}$$

In Cartesian coordinates $(\widetilde{x},\widetilde{y})\in\mathbb{D}$, the expression (6.4) reads

$$\int_{\mathbb{D}^*} c(\widetilde{x},\widetilde{y})\big(\widetilde{x}^2+\widetilde{y}^2\big)^{(p-2)(k-1)/2}\big|\nabla v(\widetilde{x},\widetilde{y})\big|^2\big(\widetilde{x}^2+\widetilde{y}^2\big)^{1/2}\,d\widetilde{x}\,d\widetilde{y},$$

where $c(\widetilde{x},\widetilde{y}) = (a_\theta^2 + k^2a^2)^{\frac{p-2}{2}}$. A change of variables $(\widetilde{x},\widetilde{y}) = G(x,y)$ yields

$$\begin{aligned}\int_{\mathbb{D}^*} \left(a_\theta^2 + k^2a^2\right)^{\frac{p-2}{2}} r^{(p-2)(k-1)}\left|\nabla v(r,\theta)\right|^2 r\,dr\,d\theta \\ = \int_S c(x,y)\left|\nabla u(x,y)\right|^2 e^{-2\alpha y}\,dx\,dy,\end{aligned} \tag{6.5}$$

because $r^2 = \widetilde{x}^2 + \widetilde{y}^2 = e^{-2y}$ and because the Jacobian of G is $r = e^{-y}$. A minimizer of (6.2) minimizes the left-hand side in (6.5); hence also the right-hand side. Since $c(x,y) = a_\theta^2 + k^2a^2$ is bounded away from zero and infinity, the claim follows. □

THEOREM 6.4. *If $u \in Y_1$ satisfies $\operatorname{div}^\circ(A\nabla^\circ u) = 0$ in $\mathbb{D}^*$, then $u \in L^\infty$ and $u(r,\theta) \le \max_\theta u(1,\theta)$ in $\mathbb{D}^*$. Moreover, the expression $\sqrt{x^2+y^2}\nabla u(x,y)$ stays bounded in $\mathbb{D}^*$.*

Proof. Consider a solution v to the linearized p-Laplacian (1.2) in $\mathbb{D}^*$. When S is mapped to $\mathbb{D}^*$ via G, the function $u = v \circ G$ on S minimizes a quadratic functional that belongs to the class (6.1). By Theorems 6.1 and 6.2, both u and ∇u stay bounded in S. □

7. The oblique derivative problem

The main result of this paper is the following that corresponds to [20, Lemma 3.15].

THEOREM 7.1. *Let $p > 2$, let A be as in (4.2), and let $M > 0$. There exists a solution $v \in Y_1$ to*

$$Tv := -\operatorname{div}^\circ\left(A\nabla^\circ v\right) = 0 \quad \text{in } \mathbb{D}^*$$

such that

$$\int_0^{2\pi} \frac{dv}{dn}(1,\theta)\,d\theta = M,$$

where n denotes the outer normal vector on $\partial\mathbb{D}$.

We prove Theorem 7.1 via a series of lemmas. The first step is to transform the problem to an oblique derivative problem.

LEMMA 7.2. *Denote $n^* = An$. Assume that there exists a function $\psi\colon \partial\mathbb{D} \to \mathbb{R}$ and a solution $v \in Y_1$ to*

$$\begin{cases} Tv = 0 & \text{in } \mathbb{D}^*, \\ \frac{\partial v}{\partial n^*} + \tau\frac{\partial v}{\partial\theta} = \frac{\psi}{q} & \text{on } \partial\mathbb{D}, \end{cases} \tag{7.1}$$

such that

$$\int_0^{2\pi} \psi(\theta)\,d\theta = M. \tag{7.2}$$

Then Theorem 7.1 holds.

Proof. In our moving frame the outer normal is $n=(1,0)$, and the conormal on $\partial\mathbb{D}$ with respect to T is

$$n^*(\theta):=An=A(1,\theta)\begin{pmatrix}1\\0\end{pmatrix}=\bigl(a^2+k^2a_\theta^2\bigr)^{\frac{p-4}{2}}\begin{pmatrix}a_\theta^2+(p-1)k^2a^2\\(p-2)kaa_\theta\end{pmatrix}.$$

With

$$\omega=\bigl(a_\theta^2+k^2a^2\bigr)^{\frac{p-4}{2}},$$

we have

$$n^*=\omega\bigl(a_\theta^2+(p-1)k^2a^2\bigr)\begin{pmatrix}1\\0\end{pmatrix}+\omega(p-2)kaa_\theta\begin{pmatrix}0\\1\end{pmatrix},$$

so that

$$\begin{aligned}n=\begin{pmatrix}1\\0\end{pmatrix}&=\frac{1}{\omega(a_\theta^2+(p-1)k^2a^2)}\left(n^*-(p-2)\omega kaa_\theta\begin{pmatrix}0\\1\end{pmatrix}\right)\\&=q\left(n^*+\tau\begin{pmatrix}0\\1\end{pmatrix}\right),\end{aligned}$$

where

$$\begin{aligned}q(\theta)&=\frac{(a_\theta^2+k^2a^2)^{\frac{4-p}{2}}}{a_\theta^2+(p-1)k^2a^2},\\\tau(\theta)&=-\bigl(a_\theta^2+k^2a^2\bigr)^{\frac{p-4}{2}}(p-2)kaa_\theta.\end{aligned}\tag{7.3}$$

It follows that

$$\frac{\partial}{\partial n}=q\left(\frac{\partial}{\partial n^*}+\tau\frac{\partial}{\partial\theta}\right),\tag{7.4}$$

and in particular that the equation

$$\frac{\partial v}{\partial n}=\psi$$

for some $\psi\colon\partial\mathbb{D}\to\mathbb{R}$, is equivalent to the equation

$$\frac{\partial v}{\partial n^*}+\tau\frac{\partial v}{\partial\theta}=\frac{\psi}{q}.\tag{7.5}$$

□

The following corresponds to [20, Lemma 3.7].

LEMMA 7.3. *Let ψ be a function on $\partial\mathbb{D}$, and let q and τ be as in (7.3). Let E be the set of all admissible boundary values $f(\theta)=F(1,\theta)$ of solutions $F\in Y_1$ to*

$$\begin{cases}TF=0 & \text{in } \mathbb{D}^*,\\ \frac{\partial F}{\partial n^*}-\frac{\partial}{\partial\theta}(\tau F)=0 & \text{on } \partial\mathbb{D}.\end{cases}\tag{7.6}$$

Then E is finite-dimensional, and the oblique derivative problem (7.1) *has a solution $v \in Y_1$ if $\frac{\psi}{q} \perp E$, i.e. if*

$$\int_0^{2\pi} \frac{\psi(\theta)}{q(\theta)} f(\theta)\, d\theta = 0 \quad \textit{for all } f \in E. \tag{7.7}$$

Proof. The strategy is to first consider the problem of finding a function $u \in Y_1$ such that

$$\begin{cases} Tu = g & \text{in } \mathbb{D}, \\ \frac{\partial u}{\partial n^*} + \tau \frac{\partial u}{\partial \theta} = 0 & \text{on } \partial\mathbb{D}, \end{cases} \tag{7.8}$$

that is, to find a suitable condition for $g \in Y_1^*$ such that the problem (7.8) admits a solution. Thereafter, the problem (7.1) is reduced to the problem (7.8).

Following [6, Chapter 7], we start by constructing a suitable Dirichlet form.[3] Our form $D\colon Y_1 \times Y_1 \to \mathbb{R}$ should satisfy

$$D(v,u) - \langle v | Tu \rangle = \int_{\partial\mathbb{D}} v \left(\frac{\partial u}{\partial n^*} + \tau \frac{\partial u}{\partial \theta} \right) d\theta, \tag{7.9}$$

so that the condition

$$D(v,u) = \langle v | g \rangle \quad \text{for all } v \in Y_1$$

guarantees that $u \in Y_1$ is a weak solution to (7.8).

Let $B = A + \mathcal{C}$, where

$$\mathcal{C} = \begin{pmatrix} 0 & -c \\ c & 0 \end{pmatrix}, \tag{7.10}$$

and $c \in C^\infty(\overline{\mathbb{D}})$ is any function such that $-c(1,\theta) = \tau(\theta)$ and $c(r,\theta) = 0$ for $r < 1/2$. Our Dirichlet form is defined such that

$$\int_{\partial\mathbb{D}} v \frac{\partial u}{\partial n_B^*}\, d\theta = \int_{\mathbb{D}} v \operatorname{div}^\circ\big(A \nabla^\circ u\big)\, dA + D(v,u), \tag{7.11}$$

i.e.

$$D(v,u) = \int_{\mathbb{D}} \big\langle \nabla^\circ v, B \nabla^\circ u \big\rangle\, dA + \int_{\mathbb{D}} v \big\{ e_\theta(c_{21}) e_r(u) + e_r(c_{12}) e_\theta(u) \big\}\, dA;$$

see Section 7.1 (especially Lemma 7.8) below. Because our Dirichlet form is coercive and the injection $Y_1 \to Y_0$ is compact, the standard Fredholm–Riesz–Schauder theory ([6, Thm. 7.21]) yields that the space

$$\begin{aligned} \mathcal{W} &= \big\{ u \in Y_1 \colon D^*(v,u) = 0 \text{ for each } v \in Y_1 \big\} \\ &= \big\{ v \in Y_1 \colon D(v,u) = 0 \text{ for each } u \in Y_1 \big\} \end{aligned}$$

[3] The notation in [6] is different from the notation in [20, Lemma 3.7]; our notation corresponds to that in [6].

is finite-dimensional in Y_0, and that the problem (7.8) admits a solution whenever

$$\langle g|v\rangle = \int_{\mathbb{D}} gv\,dA = 0$$

for each $v \in \mathcal{W}$. The remaining step is to solve the problem (7.1) with the additional condition (7.7).

Let $\psi\colon \partial\mathbb{D} \to \mathbb{R}$ be continuous and such that

$$\int_0^{2\pi} \frac{\psi(\theta)}{q(\theta)} v(1,\theta)\,d\theta = 0 \tag{7.12}$$

holds for each $v \in \mathcal{W}$, and let $h \in Y_1$ be such that

$$\frac{\partial h}{\partial n^*}(1,\theta) + \tau(\theta)\frac{\partial h}{\partial \theta}(1,\theta) = \frac{\psi(\theta)}{q(\theta)}.$$

We claim that

$$\int_{\mathbb{D}} vTh\,dA = 0 \quad \text{for all } v \in \mathcal{W}. \tag{7.13}$$

Indeed, let $v \in \mathcal{W}$, i.e. $D(v,u) = 0$ for each $u \in Y_1$. By (7.9),

$$D(v,h) - \int_{\mathbb{D}} vTh\,dA = \int_0^{2\pi} v\left(\frac{\partial h}{\partial n^*} + \tau\frac{\partial h}{\partial \theta}\right)d\theta,$$

so (7.13) follows by (7.12). Since (7.13) holds, we can solve the problem (7.8) with $g = -Th$, obtaining a weak solution u to

$$\begin{cases} Tu = -Th & \text{in } \mathbb{D}, \\ \frac{\partial u}{\partial n^*} + \tau\frac{\partial u}{\partial \theta} = 0 & \text{on } \partial\mathbb{D}, \end{cases}$$

and the function $w = u + h$ solves (7.1). □

Remark 7.4. Any solution $v \in Y_1$ to (7.1) has angular period $2\pi/N$, because we have chosen to include the periodicity in the space Y_1. Without the inclusion, the periodicity of v would follow from a periodic boundary function ψ below. However, the argument below is non-constructive, and constructing a quantitative solution of the Neumann problem with specific values of N is beyond the scope of this paper.

Proof of Theorem 7.1. By Lemma 7.3, what we need is

$$\int_0^{2\pi} \frac{\psi}{q} g\,d\theta = 0 \quad \text{for all } g \in E, \quad \text{and} \quad \int_0^{2\pi} \psi\,d\theta = M. \tag{7.14}$$

Writing $\varphi = \psi/q$ and using the bracket notation, (7.14) reads

$$\langle \varphi, g\rangle = 0 \quad \text{for all } g \in E, \quad \text{and} \quad \langle \varphi, q\rangle = M.$$

A necessary condition clearly is $q \notin E$, but it is also sufficient: if $q \notin E$, we write $q = q_E + q_\perp$, where $q_E \in E$ and $q_\perp \in E^\perp$. Then

$$\langle \varphi, q\rangle = \langle \varphi, q_\perp\rangle,$$

so if $q \notin E$, we can choose any function ψ such that $\varphi = \frac{\psi}{q} \notin E$ in order to have $\langle \varphi, q \rangle \neq 0$, and then multiply by a constant to obtain $\langle \varphi, q \rangle = M$.

In order to finish the proof, we need to show that $q \notin E$. Suppose on the contrary that $q \in E$, that is, $q(\theta) = F(1,\theta)$ for some solution $F \in Y_1 \cap C^\infty(\overline{\mathbb{D}} \setminus \{0\})$ to

$$\begin{cases} TF = 0 & \text{in } \mathbb{D}, \\ \frac{\partial F}{\partial n^*}(1,\theta) = \frac{d}{d\theta}(\tau(\theta) q(\theta)) & \text{on } \partial\mathbb{D}. \end{cases}$$

Let θ_0 be a global minimum point of q. (Such a point exists since $q \in C^\infty(\partial\mathbb{D})$.) By Lemma 6.4, $(1,\theta_0)$ is a minimum point of F on $\overline{\mathbb{D}}$. By (7.4),

$$\frac{\partial F}{\partial n^*}(1,\theta) = \frac{1}{q(\theta)} \frac{\partial F}{\partial n}(1,\theta) - \tau(\theta) \frac{\partial F}{\partial \theta}(1,\theta).$$

At the minimum point $(1,\theta_0)$, the last term on the right-hand side equals zero, and the outer normal derivative of F has to be nonpositive. Since $q > 0$, we obtain

$$\frac{\partial F}{\partial n^*}(1,\theta_0) \le 0,$$

that is,

$$\left.\frac{d}{d\theta}\right|_{\theta=\theta_0} \big(\tau(\theta) q(\theta)\big) \le 0,$$

by (7.6). But this is impossible by the following essential result that corresponds to [20, Lemma 3.5] and that finishes the proof. $\square$

LEMMA 7.5. *Let $p > 2$, $k \ge 2$, let q and τ be as in (7.3), and let θ_0 be a minimum point of q. Then*

$$\left.\frac{d}{d\theta}\right|_{\theta=\theta_0} \big(\tau(\theta) q(\theta)\big) > 0.$$

Proof. Recall that

$$q = \big(a_\theta^2 + k^2 a^2\big)^{\frac{4-p}{2}} \big(a_\theta^2 + (p-1)k^2 a^2\big)^{-1},$$
$$\tau q = -\big(a_\theta^2 + (p-1)k^2 a^2\big)^{-1} (p-2) k a a_\theta,$$

and that a satisfies $a_{\theta\theta} = -Va$, where

$$V = \frac{((2p-3)k^2 - (p-2)k) a_\theta^2 + ((p-1)k^2 - (p-2)k) k^2 a^2}{(p-1) a_\theta^2 + k^2 a^2}.$$

We start with the simpler case $p = 4$, where

$$q = \big(a_\theta^2 + 3k^2 a^2\big)^{-1},$$
$$\tau q = -\big(a_\theta^2 + 3k^2 a^2\big)^{-1} 2k a a_\theta,$$
$$V = \frac{(5k^2 - 2k) a_\theta^2 + (3k^2 - 2k) k^2 a^2}{3a_\theta^2 + k^2 a^2}.$$

Now

$$q_\theta = -\big(a_\theta^2 + 3k^2a^2\big)^{-2} 2\big(a_\theta a_{\theta\theta} + 3k^2 a a_\theta\big)$$
$$= -2aa_\theta\big(a_\theta^2 + 3k^2a^2\big)^{-2}\big(3k^2 - V\big),$$

which is zero only when $a = 0$ or $a_\theta = 0$ or $V = 3k^2$. The last alternative reads

$$\big(5k^2 - 2k\big)a_\theta^2 + \big(3k^2 - 2k\big)k^2a^2 = 3k^2\big(3a_\theta^2 + k^2a^2\big)$$

and simplifies to

$$a_\theta^2\big(4k^2 + 2k\big) = k^2a^2(-2k),$$

which is impossible. Next, consider the case $a = 0$. Denote $A = 2(a_\theta^2 + 3k^2a^2)^{-2}$, so that $q_\theta = -Aaa_\theta(3k^2 - V)$, and

$$q_{\theta\theta} = \big(-A\big(3k^2 - V\big)\big)_\theta aa_\theta - A\big(3k^2 - V\big)(aa_\theta)_\theta.$$

When $a = 0$, this equals

$$-a_\theta^2 A\big(3k^2 - V\big),$$

and thus has the same sign as $V - 3k^2$. But when $a = 0$, we have $V = (5k^2 - 2k)/3$, and $3k^2 - V > 0$. Hence, $q_{\theta\theta} < 0$ when $a = 0$, that is, points where $a = 0$ are local maxima for q. Hence, a local minimum of q can occur only when $a_\theta = 0$. At such a point, since $a_{\theta\theta} = -Va$,

$$(\tau q)_\theta = -\big(a_\theta^2 + 3k^2a^2\big)^{-2}\big(2k\big(a_\theta^2 + 3k^2a^2\big)(aa_\theta)_\theta - 2kaa_\theta\big(a_\theta^2 + 3k^2a^2\big)_\theta\big)$$
$$= -\big(3k^2a^2\big)^{-2} 3k^2a^2 \cdot 2kaa_{\theta\theta} = \frac{2V}{3k} > 0.$$

Now consider the case $p \neq 4$. First, we consider the sign of $(\tau q)_\theta$. With $B := a_\theta^2 + (p-1)k^2a^2 > 0$, we have $\tau q = -(p-2)kB^{-1}aa_\theta$. Disregarding $(p-2)k > 0$, the sign of $(\tau q)_\theta$ is the same as the sign of

$$B^{-2}\big(B_\theta aa_\theta - B(aa_\theta)_\theta\big).$$

We disregard $B^{-2} > 0$, and since $(aa_\theta)_\theta = a_\theta^2 - Va^2$, we have

$$\operatorname{sgn}\big((\tau q)_\theta\big) = \operatorname{sgn}\big(B_\theta aa_\theta - B\big(a_\theta^2 - Va^2\big)\big).$$

Next, we calculate

$$B_\theta = 2a_\theta a_{\theta\theta} + 2(p-1)k^2aa_\theta = 2aa_\theta\big((p-1)k^2 - V\big), \tag{7.15}$$

so that

$$\operatorname{sgn}\big((\tau q)_\theta\big) = \operatorname{sgn}\big(2a^2a_\theta^2\big((p-1)k^2 - V\big) - B\big(a_\theta^2 - Va^2\big)\big).$$

Inserting B yields that $(\tau_q)_\theta$ has the same sign as the expression

$$2a^2a_\theta^2\big((p-1)k^2 - V\big) - \big(a_\theta^2 + (p-1)k^2a^2\big)\big(a_\theta^2 - Va^2\big),$$

which simplifies to

$$(p-1)k^2Va^4 + \big((p-1)k^2 - V\big)a^2a_\theta^2 - a_\theta^4,$$

and factorizes to

$$\big((p-1)k^2a^2 - a_\theta^2\big)\big(Va^2 + a_\theta^2\big).$$

Thus, we conclude that $(\tau q)_\theta$ is positive only when $a_\theta^2 < (p-1)k^2a^2$.

Next, consider q_θ. With $A := a_\theta^2 + k^2a^2$ and again $B := a_\theta^2 + (p-1)k^2a^2$, we have $q = A^{(4-p)/2}B^{-1}$ and

$$\begin{aligned} q_\theta &= \frac{4-p}{2}A^{\frac{2-p}{2}}A_\theta B^{-1} - A^{\frac{4-p}{2}}B^{-2}B_\theta \\ &= B^{-2}A^{\frac{2-p}{2}}\left(\frac{4-p}{2}A_\theta B - AB_\theta\right). \end{aligned} \tag{7.16}$$

We already calculated B_θ in (7.15), and similarly $A_\theta = 2(k^2 - V)aa_\theta$, so in (7.16),

$$\begin{aligned} &\frac{4-p}{2}A_\theta B - AB_\theta \\ &\quad = (4-p)\big(k^2 - V\big)aa_\theta\big(a_\theta^2 + (p-1)k^2a^2\big) - 2\big(a_\theta^2 + k^2a^2\big)\big((p-1)k^2 - V\big)aa_\theta \\ &\quad = aa_\theta\big[(4-p)\big(k^2-V\big)\big(a_\theta^2 + (p-1)k^2a^2\big) - 2\big((p-1)k^2 - V\big)\big(a_\theta^2 + k^2a^2\big)\big] \\ &\quad =: aa_\theta \cdot C. \end{aligned}$$

Let us simplify the bracket term C above. The coefficient of a_θ^2 is

$$k^2\big((4-p) - 2(p-1)\big) + V\big(2 - (4-p)\big) = (p-2)\big(V - 3k^2\big),$$

and the coefficient of k^2a^2 is

$$\begin{aligned} &(4-p)\big(k^2 - V\big)(p-1) - 2\big((p-1)k^2 - V\big) \\ &\quad = k^2\big((4-p)(p-1) - 2(p-1)\big) + V\big(2 - (4-p)(p-1)\big) \\ &\quad = k^2(p-1)(2-p) + V(p-2)(p-3). \end{aligned}$$

We factor out $(2-p)$ to obtain $C = (2-p)D$, where

$$D := \big(3k^2 - V\big)a_\theta^2 + \big((p-1)k^2 - (p-3)V\big)k^2a^2. \tag{7.17}$$

Hence, (7.16) reads

$$q_\theta = B^{-2}A^{\frac{2-p}{2}}(2-p)aa_\theta D, \tag{7.18}$$

and we deduce that the extremal points of q are the points where $a = 0$ or $a_\theta = 0$ or $D = 0$.

Differentiating (7.18) yields

$$q_{\theta\theta} = (2-p)\Big(aa_\theta\big(B^{-2}A^{\frac{2-p}{2}}D\big)_\theta + (aa_\theta)_\theta B^{-2}A^{\frac{2-p}{2}}D\Big).$$

When $a = 0$ or $a_\theta = 0$, we have

$$q_{\theta\theta} = (2-p)\big(a_\theta^2 - Va^2\big)B^{-2}A^{\frac{2-p}{2}}D,$$

and we conclude: $q_{\theta\theta}$ has the sign of $-D$ when $a = 0$, and $q_{\theta\theta}$ has the sign of $+D$ when $a_\theta = 0$.

Next, we insert the formula for V in (7.17). We denote

$$V=\frac{\beta a_\theta^2+\gamma k^2a^2}{(p-1)a_\theta^2+k^2a^2},$$

and calculate in (7.17)

$$\begin{aligned}3k^2-V&=3k^2-\frac{\beta a_\theta^2+\gamma k^2a^2}{(p-1)a_\theta^2+k^2a^2}\\&=\frac{3k^2((p-1)a_\theta^2+k^2a^2)-\beta a_\theta^2-\gamma k^2a^2}{(p-1)a_\theta^2+k^2a^2}\\&=\frac{a_\theta^2(3k^2(p-1)-\beta)+k^2a^2(3k^2-\gamma)}{(p-1)a_\theta^2+k^2a^2},\end{aligned}$$

and

$$\begin{aligned}&(p-1)k^2-(p-3)V\\&\quad=(p-1)k^2-(p-3)\frac{\beta a_\theta^2+\gamma k^2a^2}{(p-1)a_\theta^2+k^2a^2}\\&\quad=\frac{(p-1)k^2((p-1)a_\theta^2+k^2a^2)-(p-3)(\beta a_\theta^2+\gamma k^2a^2)}{(p-1)a_\theta^2+k^2a^2}\\&\quad=\frac{a_\theta^2((p-1)^2k^2-(p-3)\beta)+k^2a^2((p-1)k^2-(p-3)\gamma)}{(p-1)a_\theta^2+k^2a^2}.\end{aligned}$$

Further, inserting $\beta=(2p-3)k^2-(p-2)k$ and $\gamma=(p-1)k^2-(p-2)k$ in the nominator yields

$$\begin{aligned}3k^2(p-1)-\beta&=p\big(k^2+k\big)-2k,\\3k^2-\gamma&=p\big(-k^2+k\big)+4k^2-2k,\\(p-1)^2k^2-(p-3)\beta&=p^2\big(-k^2+k\big)+p\big(7k^2-5k\big)+\big(-8k^2+6k\big),\\(p-1)k^2-(p-3)\gamma&=p^2\big(-k^2+k\big)+p\big(5k^2-5k\big)+\big(-4k^2+6k\big).\end{aligned}$$

Disregarding the positive denominator, we have that D in (7.17) has the same sign as the expression

$$\begin{aligned}&a_\theta^4\big(p\big(k^2+k\big)-2k\big)\\&\quad+a_\theta^2k^2a^2\big(p^2\big(-k^2+k\big)+p\big(6k^2-4k\big)+\big(-4k^2+4k\big)\big)\\&\quad+k^4a^4\big(p^2\big(-k^2+k\big)+p\big(5k^2-5k\big)+\big(-4k^2+6k\big)\big),\end{aligned}\tag{7.19}$$

which factorizes to

$$\begin{aligned}&\big(\big[p\big(k^2+k\big)-2k\big]a_\theta^2+\big[p^2\big(-k^2+k\big)+p\big(5k^2-5k\big)+\big(-4k^2+6k\big)\big]k^2a^2\big)\\&\quad\times\big(a_\theta^2+k^2a^2\big).\end{aligned}$$

Modifying further, this becomes

$$\begin{aligned}(7.20)\qquad &\big((k+1)p-2\big)a_\theta^2+\big((-k+1)p^2+(5k-5)p-4k+6\big)k^2a^2\\ &=\big((k+1)p-2\big)a_\theta^2+\big((-k+1)(p-p_1)(p-p_2)\big)k^2a^2,\end{aligned}$$

where

$$p_{1,2}=\frac{5}{2}\pm\frac{1}{2}\sqrt{\frac{9k-1}{k-1}}$$

with the convention $p_1<p_2$. Factoring out $k-1$ in (7.20) finally yields that D has the same sign as

$$\left(\frac{k+1}{k-1}p-\frac{2}{k-1}\right)a_\theta^2-(p-p_1)(p-p_2)k^2a^2.$$

We note that the coefficient of a_θ^2 is positive and that $p_1<1$. Thus, the sign of D is positive whenever $p<p_2$. In this case, we observe that the local minimum of q occurs when $a_\theta=0$, and we have $(\tau q)_\theta>0$ as desired.

The remaining case to check is $p\geq p_2$ and $D\leq 0$, where $D\leq 0$ reads

$$\big((k+1)p-2\big)a_\theta^2\leq\big((k-1)p^2+(-5k+5)p+(4k-6)\big)k^2a^2,$$

that is,

$$a_\theta^2\leq\frac{(k-1)p^2+(-5k+5)p+(4k-6)}{(k+1)p-2}k^2a^2.$$

Denote the fraction on the right-hand side by F. It suffices to show (for $k\geq 2$) that $F<p-1$ when $p\geq p_2$, since $a_\theta^2<(p-1)k^2a^2$ yielded $(\tau q)_\theta>0$ as desired. Now $F<p-1$ precisely when

$$p^2+(2k-4)p-2k+4>0,$$

in particular whenever

$$p>\sqrt{k^2-2k}-k+2.$$

But for $k\geq 2$ we have $\sqrt{k^2-2k}-k+2<4$, while $p\geq p_2>4$. This completes the proof. ☐

7.1. Calculations for the Dirichlet form. In this subsection, we provide the calculations missing from the proof of Lemma 7.3 above.

LEMMA 7.6. *Let A be as in (4.2) and write $A=\left(\begin{smallmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{smallmatrix}\right)$. Then*

$$\begin{aligned}&\operatorname{div}^\circ\big(A\nabla^\circ u\big)\\ &\quad=a_{11}e_re_r(u)+(a_{12}+a_{21})e_\theta e_r(u)+a_{22}e_\theta e_\theta(u)\\ &\qquad+\left(e_r(a_{11})+\frac{1}{r}a_{11}+e_\theta(a_{21})\right)e_r(u)+\big(e_r(a_{12})+e_\theta(a_{22})\big)e_\theta(u).\end{aligned}$$

Proof. We calculate

$$\begin{aligned}
\operatorname{div}^\circ\big(A\nabla^\circ u\big) &= \frac{1}{r}e_r\big(r\big[A\nabla^\circ u\big]_1\big) + e_\theta\big(\big[A\nabla^\circ u\big]_2\big) \\
&= a_{11}e_re_r(u) + e_r(a_{11})e_r(u) + a_{12}e_re_\theta(u) + e_r(a_{12})e_\theta(u) \\
&\quad + \frac{1}{r}a_{11}e_r(u) + \frac{1}{r}a_{12}e_\theta(u) \\
&\quad + a_{21}e_\theta e_r(u) + e_\theta(a_{21})e_r(u) + a_{22}e_\theta e_\theta(u) + e_\theta(a_{22})e_\theta(u) \\
&= a_{11}e_re_r(u) + a_{12}e_re_\theta(u) + a_{21}e_\theta e_r(u) + a_{22}e_\theta e_\theta(u) \\
&\quad + \left(e_r(a_{11}) + \frac{1}{r}a_{11} + e_\theta(a_{21})\right)e_r(u) \\
&\quad + \left(e_r(a_{12}) + \frac{1}{r}a_{12} + e_\theta(a_{22})\right)e_\theta(u).
\end{aligned}$$

Moreover,

$$\begin{aligned}
e_re_\theta(u) &= e_r\left(\frac{1}{r}\frac{\partial u}{\partial\theta}\right) = -\frac{1}{r^2}\frac{\partial u}{\partial\theta} + \frac{1}{r}e_r\left(\frac{\partial u}{\partial\theta}\right) \\
&= -\frac{1}{r^2}e_\theta(u) + \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial\theta}\right) = -\frac{1}{r^2}e_\theta(u) + \frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{\partial u}{\partial r}\right) \\
&= -\frac{1}{r^2}e_\theta(u) + e_\theta e_r(u),
\end{aligned}$$

so

$$a_{12}e_re_\theta(u) = a_{12}e_\theta e_r(u) - \frac{1}{r}a_{12}e_\theta(u),$$

and the claim follows. □

LEMMA 7.7. *Let A be as in (4.2), let $\mathcal{C}$ be as in (7.10), and let $B = A + \mathcal{C}$. Denote the conormal derivative with respect to B on $\partial\mathbb{D}$ of a function u by*

$$\frac{\partial u}{\partial n_B^*} = \left\langle B\begin{pmatrix}1\\0\end{pmatrix}, \nabla^\circ u\right\rangle.$$

Then

$$\int_{\partial\mathbb{D}} v\frac{\partial u}{\partial n_B^*}\,d\theta = \int_{\mathbb{D}} v\operatorname{div}^\circ\big(B\nabla^\circ u\big)\,dA + \int_{\mathbb{D}}\big\langle\nabla^\circ v, B\nabla^\circ u\big\rangle\,dA \tag{7.21}$$

for each $u, v \in Y_1$.

Proof. By Lemma 5.3, we may assume that $u, v \in C^1(\mathbb{D})$. By definition,

$$\int_{\mathbb{D}} v\operatorname{div}^\circ U\,dA = -\int_{\mathbb{D}}\big\langle U, \nabla^\circ v\big\rangle\,dA \tag{7.22}$$

for each $U \in C^1(\mathbb{D};\mathbb{R}^2)$ and $v \in C_0^\infty(\mathbb{D})$. When v is not compactly supported, we multiply it by φ_ε, a standard radial function in $C_0^\infty(\mathbb{D})$ satisfying $\varphi_\varepsilon \to \chi_{\mathbb{D}}$

as $\varepsilon \to 0$. Then, by (7.22),

$$\begin{aligned}\int_{\mathbb{D}} \varphi_\varepsilon v \operatorname{div}^\circ U\, dA &= -\int_{\mathbb{D}} \big\langle U, \nabla^\circ(\varphi_\varepsilon v)\big\rangle\, dA \\ &= -\int_{\mathbb{D}} \big\langle U, v\nabla^\circ\varphi_\varepsilon\big\rangle\, dA - \int_D \big\langle U, \varphi_\varepsilon \nabla^\circ v\big\rangle\, dA.\end{aligned}$$

Letting $\varepsilon \to 0$ yields, since $\nabla^\circ\varphi_\varepsilon \to (-\delta_1, 0)$ as $\varepsilon \to 0$,

$$\int_{\mathbb{D}} v \operatorname{div}^\circ U\, dA = \int_{\partial\mathbb{D}} U_1 v\, d\theta - \int_{\mathbb{D}} \big\langle U, \nabla^\circ v\big\rangle\, dA.$$

With $U = B\nabla^\circ u$, we have $U_1 = b_{11}e_r(u) + b_{12}e_\theta(u)$, and

$$\frac{\partial u}{\partial n_B^*} = \left\langle B\begin{pmatrix}1\\0\end{pmatrix}, \nabla^\circ u\right\rangle = U_1,$$

which finishes the proof. □

Lemma 7.8. *The Dirichlet form* (7.11) *satisfies* (7.9).

Proof. Replacing the divergence term in (7.21) by Lemma 7.6 yields

$$\begin{aligned}\int_{\partial\mathbb{D}} v\frac{\partial u}{\partial n_B^*}\, d\theta & \\ = \int_{\mathbb{D}} &\nabla^\circ v \cdot B\nabla^\circ u\, dA \\ &+ \int_{\mathbb{D}} v\big\{b_{11}e_re_r(u) + (b_{12}+b_{21})e_\theta e_r(u) + b_{22}e_\theta e_\theta(u)\big\}\, dA \\ &+ \int_{\mathbb{D}} v\bigg\{\Big(e_r(b_{11}) + \frac{1}{r}b_{11} + e_\theta(b_{21})\Big)e_r(u) \\ &+ \big(e_r(b_{12}) + e_\theta(b_{22})\big)e_\theta(u)\bigg\}\, dA,\end{aligned} \tag{7.23}$$

and replacing the middle term on the right-hand side of (7.23) by

$$\begin{aligned}\operatorname{div}^\circ\big(A\nabla^\circ u\big) &= b_{11}e_re_r(u) + (b_{12}+b_{21})e_\theta e_r(u) + b_{22}e_\theta e_\theta(u) \\ &\quad + \Big(e_r(a_{11}) + \frac{1}{r}a_{11} + e_\theta(a_{21})\Big)e_r(u) + \big(e_r(a_{12}) + e_\theta(a_{22})\big)e_\theta(u).\end{aligned}$$

yields

$$\begin{aligned}&\int_{\partial\mathbb{D}} v\frac{\partial u}{\partial n_B^*}\, d\theta \\ &\ = \int_{\mathbb{D}} \nabla^\circ v \cdot B\nabla^\circ u\, dA + \int_{\mathbb{D}} v \operatorname{div}^\circ\big(A\nabla^\circ u\big)\, dA \\ &\quad + \int_{\mathbb{D}} v\bigg\{\Big(e_r(b_{11}) + \frac{1}{r}b_{11} + e_\theta(b_{21})\Big)e_r(u) + \big(e_r(b_{12}) + e_\theta(b_{22})\big)e_\theta(u)\bigg\}\, dA \\ &\quad - \int_{\mathbb{D}} v\bigg\{\Big(e_r(a_{11}) + \frac{1}{r}a_{11} + e_\theta(a_{21})\Big)e_r(u) + \big(e_r(b_{12}) + e_\theta(b_{22})\big)e_\theta(u)\bigg\}\, dA.\end{aligned}$$

Thus, we obtain (since $c_{ij} = b_{ij} - a_{ij}$, $c_{11} = c_{22} = 0$, and $c_{21} = -c_{12} = c$)

$$\int_{\partial\mathbb{D}} v \frac{\partial u}{\partial n_B^*} \, d\theta = \int_{\mathbb{D}} \nabla^\circ v \cdot B \nabla^\circ u \, dA + \int_{\mathbb{D}} v \operatorname{div}^\circ \bigl(A \nabla^\circ u\bigr) \, dA$$
$$+ \int_{\mathbb{D}} v \bigl\{ e_\theta(-c) e_r(u) + e_r(c) e_\theta(u) \bigr\} \, dA.$$

We want

$$\frac{\partial u}{\partial n_B^*} = \frac{\partial u}{\partial n_A^*} + \tau(\theta) \frac{\partial u}{\partial \theta}.$$

Since

$$\frac{\partial u}{\partial n_B^*} = (A + \mathcal{C})^t \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \nabla^\circ u = \frac{\partial u}{\partial n_A^*} + \mathcal{C}^t \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \nabla^\circ u,$$

and since

$$\mathcal{C}^t \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \nabla^\circ u = \begin{pmatrix} 0 & c \\ -c & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} e_r(u) \\ e_\theta(u) \end{pmatrix} = -c e_\theta(u) = -c \frac{\partial u}{\partial \theta},$$

we choose c to be any function in $C^\infty(\overline{\mathbb{D}})$ such that $-c(1,\theta) = \tau(\theta)$. The additional condition $c(r,\theta) = 0$ for $r < 1/2$ is needed both for Lemma 7.7 above and for the coercivity estimate below. ☐

LEMMA 7.9. *There exist constants C_1 and C_2, independent of u, such that*

$$\bigl|D(u,u)\bigr| \geq C_1 \|u\|_{Y_1}^2 - C_2 \|u\|_{Y_0}^2$$

for each $u \in Y_1$.

Proof. We estimate

$$\bigl|D(u,u)\bigr| \geq \left| \int_{\mathbb{D}} \nabla^\circ u \cdot B \nabla^\circ u \, dA \right| - \left| \int_{\mathbb{D}} u \bigl\{ e_\theta(c_{21}) e_r(u) + e_r(c_{12}) e_\theta(u) \bigr\} \, dA \right|. \tag{7.24}$$

Since $\nabla^\circ u \cdot B \nabla^\circ u = \nabla^\circ u \cdot A \nabla^\circ u$ and since $A\xi \cdot \xi \geq C r^{2\alpha} |\xi|^2$, we have

$$\int_{\mathbb{D}} \nabla^\circ u \cdot B \nabla^\circ u \, dA \geq C \int_{\mathbb{D}} r^{2\alpha} \bigl|\nabla^\circ u\bigr|^2 \, dA = C \bigl(\|u\|_{Y_1}^2 - \|u\|_{Y_0}^2 \bigr).$$

For the second term on the right-hand side of (7.24), we have

$$\int_{\mathbb{D}} u \bigl\{ e_\theta(c_{21}) e_r(u) + e_r(c_{12}) e_\theta(u) \bigr\} \, dA$$
$$\leq C \int_{\{r \geq \frac{1}{2}\}} \bigl| u \bigl(e_r(u) + e_\theta(u) \bigr) \bigr| \, dA,$$

since the functions c_{ij} are in the class $C^\infty(\overline{\mathbb{D}})$ and are supported in the annulus $\{r \geq 1/2\}$. By Young's inequality,

$$\begin{aligned}\int_{\{r\geq\frac{1}{2}\}} \left|u\big(e_r(u)+e_\theta(u)\big)\right| dA &\leq C\int_{\{r\geq\frac{1}{2}\}} \left|u\nabla^\circ u\right| dA \\ &\leq C\varepsilon \int_{\{r\geq\frac{1}{2}\}} \left|\nabla^\circ u\right|^2 dA + C\frac{1}{\varepsilon}\int_{\{r\geq\frac{1}{2}\}} |u|^2\, dA \\ &\leq C\varepsilon \|u\|_{Y_1}^2 + C\frac{1}{\varepsilon}\|u\|_{Y_0}^2.\end{aligned}$$

Choose $\varepsilon > 0$ small enough such that $C\varepsilon \leq 1/2$ to obtain

$$\begin{aligned}\left|D(u,u)\right| &\geq \left|\int_{\mathbb{D}} \nabla^\circ u\cdot B\nabla^\circ u\, dA\right| - \left|\int_{\mathbb{D}} u\big\{e_\theta(c_{21})e_r(u)+e_r(c_{12})e_\theta(u)\big\}\, dA\right| \\ &\geq C\|u\|_{Y_1}^2 - \frac{1}{2}\|u\|_{Y_1}^2 - C_2\|u\|_{Y_0}^2 = C_1\|u\|_{Y_1}^2 - C_2\|u\|_{Y_0}^2,\end{aligned}$$

as wanted. □

Finally, we prove a result about the adjoint Dirichlet form $D^*(v,u) = D(u,v)$ that is needed in the proof of Lemma 7.3.

LEMMA 7.10. *If $u \in Y_1$ satisfies, for some $f \in Y_0^*$,*

$$D^*(v,u) = \langle v|f\rangle \quad \textit{for all } v \in Y_1,$$

then u is a weak solution to the boundary value problem

$$\begin{cases} Tu = f & \textit{in } \mathbb{D}, \\ \frac{\partial u}{\partial n^*} - \frac{\partial}{\partial\theta}(\tau u) = 0 & \textit{on } \partial\mathbb{D}. \end{cases} \tag{7.25}$$

Proof. Since A is symmetric, Lemma 7.7 yields

$$\int_{\mathbb{D}} vTu - uTv\, dA = \int_0^{2\pi} u(1,\theta)\frac{\partial v}{\partial n^*}(1,\theta) - v(1,\theta)\frac{\partial u}{\partial n^*}(1,\theta)\, d\theta. \tag{7.26}$$

By definition of $D(v,u)$,

$$D(v,u) = \int_{\mathbb{D}} vTu\, dA + \int_0^{2\pi} v(1,\theta)\left(\frac{\partial u}{\partial n^*}(1,\theta) + \tau(\theta)\frac{\partial u}{\partial\theta}(1,\theta)\right) d\theta,$$

and

$$D^*(v,u) = \int_{\mathbb{D}} uTv\, dA + \int_0^{2\pi} u(1,\theta)\left(\frac{\partial v}{\partial n^*}(1,\theta) + \tau(\theta)\frac{\partial v}{\partial\theta}(1,\theta)\right) d\theta,$$

so combined with (7.26),

$$D^*(v,u) - D(v,u) = \int_0^{2\pi} u(1,\theta)\tau(\theta)\frac{\partial v}{\partial n^*}(1,\theta) - v(1,\theta)\tau(\theta)\frac{\partial u}{\partial n^*}(1,\theta)\, d\theta.$$

Thus $D^*(v,u)$ and $D(v,u)$ differ only on the boundary, and the boundary condition for D^* is

$$\int_0^{2\pi} v\frac{\partial u}{\partial n^*} + u\tau\frac{\partial v}{\partial \theta}\,d\theta = \int_0^{2\pi} v\left(\frac{\partial}{\partial n^*} - \frac{\partial}{\partial \theta}(\tau u)\right)d\theta. \qquad \square$$

8. Related problems

We close with some problems listed by Wolff in [20], where progress has since been made:

(1) *Are there bounded p-harmonic functions with bad behavior at every point on the boundary and if not, is a Fatou theorem true if one interprets "almost everywhere" using a finer measure?* These questions were answered by Manfredi and Weitsman [15] in 1988, see also [4]. The Hausdorff dimension of the set (on the boundary of a smooth Euclidean domain) where radial limits exist is bounded below with a positive constant that depends only on the number p and the dimension of the underlying space. No estimates for this constant are known even in the plane.

(2) *What can be said about radial limits of quasiregular mappings?* Wolff states [20, p. 373] that this question was the main motivation for his work. Progress was made by K. Rajala [16]: If a quasiregular mapping $\mathbb{B}^n \to \mathbb{R}^n$ is a local homeomorphism, then radial limits exist at infinitely many boundary points. Apart from this result, the question seems to be open.

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