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LOOKING OUT FOR FROBENIUS SUMMANDS ON A BLOWN-UP SURFACE OF $\mathbb{P}^2$

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ABSTRACT. For an algebraic variety X in characteristic $p > 0$, the push-forward $F^e_* \mathcal{O}_X$ of the structure sheaf by an iterated Frobenius endomorphism F^e is closely related to the geometry of X. We study the decomposition of $F^e_* \mathcal{O}_X$ into direct summands when X is obtained by blowing up the projective plane $\mathbb{P}^2$ at four points in general position. We explicitly describe the decomposition of $F^e_* \mathcal{O}_X$ and show that there appear only finitely many direct summands up to isomorphism, when e runs over all positive integers. We also prove that these summands generate the derived category $D^b(X)$. On the other hand, we show that there appear infinitely many distinct indecomposable summands of iterated Frobenius push-forwards on a ten-point blowup of $\mathbb{P}^2$.

Throughout this paper, we work over an algebraically closed field k of characteristic $p > 0$. Let X be a smooth variety over k with the Frobenius morphism $F\colon X \to X$. Then the push-forward $F^e_* \mathcal{O}_X$ of the structure sheaf of X by the e-times iterate of the Frobenius is a locally free sheaf of rank $p^{e \dim X}$. When X is a projective curve, it is known that the structure of the vector bundle $F^e_* \mathcal{O}_X$ heavily depends on the genus g of X, as follows.

(1) If $X = \mathbb{P}^1$, then $F^e_* \mathcal{O}_X \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus p^e - 1}$.

(2) If X is an elliptic curve, then $F^e_* \mathcal{O}_X$ is semi-stable, and
  - (i) if X is ordinary, $F^e_* \mathcal{O}_X$ splits into p^e non-isomorphic p^e-torsion line bundles;
  - (ii) if X is supersingular, $F^e_* \mathcal{O}_X$ is isomorphic to Atiyah's indecomposable vector bundle $\mathcal{F}_{p^e}$ of degree zero and rank p^e ([At]).

(3) If $g \geq 2$, then $F^e_* \mathcal{O}_X$ is a stable vector bundle (Mehta–Pauly [MP]).

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Putting the detailed study of the case of genus $g \geq 2$ aside, the structure of the vector bundle $F^e_*\mathcal{O}_X$ is fairly well-understood in dimension one. So it is natural to ask about the surface case. In general, the Frobenius push-forward $F^e_*\mathcal{O}_X$ of a toric variety X splits into line bundles, and there are only finitely many isomorphism classes of line bundles that appear as a direct summand of $F^e_*\mathcal{O}_X$ for some $e \geq 0$ ([T], see also [A2], [OU]). This generalizes what happens in case (1) for curves. On the other hand, works by Sannai–Tanaka [ST], Kitadai–Sumihiro [KS] and Sun [Su] generalize cases (2) and (3) for curves to higher dimension, respectively.

In view of Frobenius splitting, toric varieties are globally F-regular (and so F-split), and ordinary abelian varieties (e.g., case (2)(i) above) are F-split but not globally F-regular. In these cases, any direct summand of $F^e_*\mathcal{O}_X$ remains to be a direct summand of $F^{e'}_*\mathcal{O}_X$ for all $e' \geq e$. We call such a vector bundle that appears as a direct summand of $F^e_*\mathcal{O}_X$ for some $e \geq 0$ a *Frobenius summand* on X and ask the following questions.

(a) Does $F^e_*\mathcal{O}_X$ splits into line bundles?
(b) Does there exist only finitely many isomorphism classes of indecomposable Frobenius summands on X?

If X satisfies the finiteness condition in (b), it is said to be of *globally finite F-representation type* (or *GFFRT*, for short). It follows from [T] and [ST] that question (a) is affirmative for toric and ordinary abelian varieties, but it is likely that (a) fails for non-toric rational surfaces; cf. Achinger [A2]. On the other hand, the GFFRT property, which breaks down for curves of genus $g \geq 1$, holds true for smooth quadric hypersurfaces of any dimension [A1]. Actually, this follows from the fact that Frobenius summands on a smooth quadric hypersurface are arithmetically Cohen–Macaulay (ACM) bundles and that an ACM bundle on a quadric is either a line bundle or a twisted spinor bundle.

In this paper, we will develop a method to study the structure of the Frobenius push-forward $F^e_*\mathcal{O}_X$ in the case where X is obtained by blowing up the projective plane $\mathbb{P}^2$ at n points $P_1, \dots, P_n$. Note that such a rational surface X is not toric if $n \geq 4$. As a result, we examine the GFFRT property of X in the simplest non-trivial case, that is, the case where $n = 4$. Our main result is the following.

Main Theorem. *Let $\pi\colon X \to \mathbb{P}^2$ be the blowup at four points $P_1, P_2, P_3, P_4 \in \mathbb{P}^2$ in general position. Let H be a line in $\mathbb{P}^2$ and $E_i = \pi^{-1}(P_i)$ the exceptional curve over P_i. Then any indecomposable Frobenius summand on the surface X coincides with one of the following vector bundles of rank ≤ 3.*

(1) *line bundles $\mathcal{O}_X$, $L_0 = \mathcal{O}_X(E_1 + E_2 + E_3 + E_4 - 2\pi^*H)$ and $L_i = \mathcal{O}_X(E_i - \pi^*H)$ for $i = 1, 2, 3, 4$;*

(2) *an indecomposable rank* 2 *bundle* $\mathcal{G}$ *given by a unique non-trivial extension*

$$0 \to \mathcal{O}_X\bigl(-\pi^*H\bigr) \to \mathcal{G} \to \mathcal{O}_X\bigl(E_1+E_2+E_3+E_4-2\pi^*H\bigr) \to 0;$$

(3) *an indecomposable rank* 3 *bundle* $\mathcal{B}$ *given by a non-trivial extension*

$$0 \to \mathcal{O}_X\bigl(E_1-\pi^*H\bigr) \oplus \mathcal{O}_X\bigl(E_2-\pi^*H\bigr) \to \mathcal{B} \to \mathcal{O}_X\bigl(E_3+E_4-\pi^*H\bigr) \to 0.$$

Furthermore, for any power $q=p^e$ *of the characteristic* p *with* $e \geq 1$ *one has*

$$F_*^e\mathcal{O}_X \cong \mathcal{O}_X \oplus \bigoplus_{i=0}^{4} L_i^{\oplus(q-2)} \oplus \mathcal{B} \oplus \mathcal{G}^{\oplus\frac{(q-2)(q-3)}{2}}.$$

We apply the above theorem to show that $F_*^e\mathcal{O}_X$ generates the derived category $D^b(X)$ in the sense of [OU]; see Proposition 6.4. In order to prove the theorem, we study the behavior of the Frobenius push-forward and Frobenius summands with respect to the blow-down morphism $\sigma\colon X \to Y$ of one of the exceptional curves, say, E_4. It follows that any Frobenius summand $\mathcal{F}$ on X sits in an exact sequence $0 \to \sigma^*\sigma_*\mathcal{F} \to \mathcal{F} \to \mathcal{O}_{E_4}(-1)^{\oplus n} \to 0$ with $n \leq \operatorname{rank}\mathcal{F}$ and $\sigma_*\mathcal{F}$ is a Frobenius summand on Y, which splits into a direct sum of copies of finitely many line bundles since Y is a toric surface. Then it turns out that $\mathcal{F}$ is obtained by successive extension of line bundles either of the form σ^*L or $\sigma^*(L) \otimes \mathcal{O}_X(E_4)$, where L is a rank one Frobenius summand on Y. Computing the involving extension classes we see that an indecomposable Frobenius summand on X is either one of those listed in (1)–(3) of the theorem. Finally, we can determine the multiplicity of each summand appearing in $F_*^e\mathcal{O}_X$ in terms of the Frobenius splitting method.

The GFFRT property of a blown-up surface of $\mathbb{P}^2$ would possibly break down as the number of the blown-up points gets larger than a certain bound. Actually, we can construct an example of a rational surface X obtained by blowing up $\mathbb{P}^2$ at more than ten points that is not GFFRT; see Proposition 6.2.

1. Preliminaries

We work over an algebraically closed field k of characteristic $p>0$. It is known that the Frobenius push-forward of any line bundle on a smooth toric variety splits into a direct sum of line bundles [T]; see also [A2], [OU].

1.1. Toric surfaces. We review a few examples for basic toric surfaces.

1.1a. The case $X=\mathbb{P}^2$. Let X be the projective plane with homogeneous coordinates x_0, x_1, x_2 and let $x = x_1/x_0, y = x_2/x_0$. Let $H \subset X$ be the line

$x_0 = 0$. For $q = p^e$ we choose a basis $\{x^{i/q}y^{j/q} | 0 \le i, j \le q-1\}$ of $K^{1/q}$ over the function field $K = k(x, y)$ of X. Then we have a decomposition

$$\begin{aligned} F^e_* \mathcal{O}_X &= \bigoplus_{0 \le i,j \le q-1} L_{ij} x^{i/q} y^{j/q} \\ &\cong \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus \frac{(q-1)(q+4)}{2}} \oplus \mathcal{O}_{\mathbb{P}^2}(-2)^{\oplus \frac{(q-1)(q-2)}{2}} \end{aligned}$$

with respect to the basis, where

$$L_{ij} = \begin{cases} \mathcal{O}_X & \text{if } i = j = 0; \\ \mathcal{O}_X(-H) & \text{if } 1 \le i + j \le q; \\ \mathcal{O}_X(-2H) & \text{if } q + 1 \le i + j \le 2q - 2. \end{cases}$$

1.1b. The blowup of $\mathbb{A}^2$ *at the origin.* Let $\pi \colon X \to Y = \mathbb{A}^2 = \operatorname{Spec} k[x, y]$ be the blowup at the origin $o \in \mathbb{A}^2$ with the exceptional curve $E = \pi^{-1}(o)$. Then for $q = p^e$ we have a decomposition

$$F^e_* \mathcal{O}_X = \bigoplus_{0 \le i,j \le q-1} L_{ij} x^{i/q} y^{j/q} \cong \mathcal{O}_X^{\oplus \frac{q(q+1)}{2}} \oplus \mathcal{O}_X(E)^{\oplus \frac{q(q-1)}{2}},$$

where

$$L_{ij} = \begin{cases} \mathcal{O}_X & \text{if } 0 \le i + j \le q - 1; \\ \mathcal{O}_X(E) & \text{if } q \le i + j \le 2q - 2. \end{cases}$$

In view of the trivial decomposition $F^e_* \mathcal{O}_Y \cong \bigoplus_{0 \le i,j \le q-1} \mathcal{O}_Y x^{i/q} y^{j/q}$ on $Y = \mathbb{A}^2$, we have the following exact sequence

$$0 \to \pi^* F^e_* \mathcal{O}_Y \to F^e_* \mathcal{O}_X \to \mathcal{N} \to 0, \tag{1.1}$$

where $\mathcal{N} = \bigoplus_{q \le i+j \le 2q-2} \mathcal{O}_E(E) x^{i/q} y^{j/q} \cong \mathcal{O}_E(-1)^{\oplus \frac{q(q-1)}{2}}$.

1.1c. The blowup of $\mathbb{P}^2$ *at a point.* Let $\pi \colon X \to \mathbb{P}^2$ be the blowup at a torus-invariant point $P \in \mathbb{P}^2$ and let $E = \pi^{-1}(P)$. If we choose $P = (1:0:0)$, then it follows from (1.1a) and (1.1b) that

$$\begin{aligned} F^e_* \mathcal{O}_X &= \bigoplus_{0 \le i,j \le q-1} L_{ij} x^{i/q} y^{j/q} \\ &\cong \mathcal{O}_X \oplus \mathcal{O}_X\big(-\pi^* H\big)^{\oplus \frac{q^2+q-2}{2}} \oplus \mathcal{O}_X\big(E - \pi^* H\big)^{\oplus (q-1)} \\ &\quad \oplus \mathcal{O}_X\big(E - 2\pi^* H\big)^{\oplus \frac{q^2-3q+2}{2}}, \end{aligned}$$

where

$$L_{ij} = \begin{cases} \mathcal{O}_X & \text{if } i = j = 0; \\ \mathcal{O}_X(-\pi^* H) & \text{if } 1 \le i + j \le q - 1; \\ \mathcal{O}_X(E - \pi^* H) & \text{if } i + j = q; \\ \mathcal{O}_X(E - 2\pi^* H) & \text{if } q + 1 \le i + j \le 2q - 2. \end{cases}$$

Note that the assignment of the summands $\mathcal{O}_X(-\pi^*H)$ and $\mathcal{O}_X(E-\pi^*H)$ to the pairs (i,j) with $1 \le i+j \le q$ depends on the choice of a torus-invariant point P. Namely:

(1) if $P=(1:0:0)$ then $L_{ij}=\mathcal{O}_X(E-\pi^*H)$ for $(i,j)=(1,q-1),(2,q-2),\dots,(q-1,1)$ as described above, whereas;
(2) if $P=(0:1:0)$ then $L_{ij}=\mathcal{O}_X(E-\pi^*H)$ for $(i,j)=(0,1),\dots,(0,q-1)$;
(3) if $P=(0:0:1)$ then $L_{ij}=\mathcal{O}_X(E-\pi^*H)$ for $(i,j)=(1,0),\dots,(q-1,0)$.

1.2. The blowup of $\mathbb{P}^2$ at more than three points. Let $\pi\colon X\to\mathbb{P}^2$ be the blowup of the projective plane $\mathbb{P}^2=\mathbb{P}^2_k$ at n points $P_1,\dots,P_n\in\mathbb{P}^2$ in general position and let $E_i=\pi^{-1}(P_i)$ be the exceptional curve over P_i. We may and will assume that $P_1=(1:0:0)$, $P_2=(0:1:0)$ and $P_3=(0:0:1)$. We factorize $\pi\colon X\to\mathbb{P}^2$ as $\pi=\sigma\circ\tau$, where $\tau\colon Y\to\mathbb{P}^2$ is the blowup at $n-1$ points $P_1,\dots,P_{n-1}$ and $\sigma=\sigma_n\colon X\to Y$ is the blowup at the point P_n. Then by (1.1) we have an exact sequence

$$0\to\sigma^*F^e_*\mathcal{O}_Y\to F^e_*\mathcal{O}_X\to\mathcal{O}_{E_n}(-1)^{\oplus\frac{q(q-1)}{2}}\to 0.$$

This implies that any direct summand $\mathcal{F}$ of $F^e_*\mathcal{O}_X$ sits in a similar exact sequence

$$0\to\sigma^*\mathcal{E}\to\mathcal{F}\to\mathcal{N}\to 0, \tag{1.2}$$

where $\mathcal{E}=\pi_*\mathcal{F}$ is a direct summand of $F^e_*\mathcal{O}_Y$ and $\mathcal{N}$ is a direct sum of copies of $\mathcal{O}_{E_n}(-1)$'s.

As for line bundle summands of $F^e_*\mathcal{O}_X$, we easily see the following: If a line bundle L is a direct summand of $F^e_*\mathcal{O}_X$, then $L'=\sigma_*L$ is also a direct summand on $F^e_*\mathcal{O}_Y$, and L sits in an exact sequence

$$0\to\sigma^*L'\to L\to\mathcal{N}\to 0$$

with $\mathcal{N}=0$ or $\mathcal{O}_{E_n}(-1)$. Hence $L\cong\sigma^*L'$ or $\sigma^*(L')\otimes\mathcal{O}_X(E_n)$.

We want to look out for all indecomposable *Frobenius summands* on X, that is, indecomposable vector bundles appearing as a direct summand of $F^e_*\mathcal{O}_X$ for some $e\ge 0$. This problem is closely related to F-splitting property: X is said to be *F-split* if the Frobenius map $F\colon\mathcal{O}_X\to F_*\mathcal{O}_X$ splits as an $\mathcal{O}_X$-module homomorphism, or equivalently, $F^e_*\mathcal{O}_X$ has a free direct summand $\mathcal{O}_X$ for all $e\ge 0$.[1]

The following are known for the F-splitting property of smooth del Pezzo surfaces.

Proposition 1.1 (Cf. [H, Example 5.5]). *Assume that $n\le 8$ under the notation of Section 1.2 above. Then X is F-split except for the following three cases.*

(1) $n=6$ *and* $p=2$;

[1] When X is projective over k, the maximal rank of a free Frobenius summand is at most 1, because $H^0(X,F^e_*\mathcal{O}_X)=k$.

(2) $n=7$ *and* $p=2$ *or* 3;
(3) $n=8$ *and* $p=2,3$ *or* 5.

Our main target in this paper is the case $n=4$, where X itself is a non-toric F-split surface but the surface Y downstairs, being the blowup of $\mathbb{P}^2$ at three points P_1,P_2,P_3, is toric. Summing up the results in (1.1c), we obtain the following lemma.

LEMMA 1.2. *Let* $\tau\colon Y\to\mathbb{P}^2$ *be the blowup at the three points* P_1,P_2,P_3 *in general position. Then* $F^e_*\mathcal{O}_Y$ *is isomorphic to*

$$\mathcal{O}_Y\oplus\mathcal{O}_Y\big(-\tau^*H\big)^{\oplus\frac{(q-1)(q-2)}{2}}\oplus\bigoplus_{i=1}^{3}\mathcal{O}_Y\big(E_i-\tau^*H\big)^{\oplus(q-1)}$$
$$\oplus\,\mathcal{O}_Y\big(E'-2\tau^*H\big)^{\oplus\frac{(q-1)(q-2)}{2}},$$

where $q=p^e$ *and* $E'=E_1+E_2+E_3$.

LEMMA 1.3. *Under the notation as in Lemma 1.2 we also have*

$$F^e_*\mathcal{O}_Y\big(-\tau^*H\big)\cong\mathcal{O}_Y\big(-\tau^*H\big)^{\oplus\frac{q(q+1)}{2}}\oplus\mathcal{O}_Y\big(E'-2\tau^*H\big)^{\oplus\frac{q(q-1)}{2}},$$
$$F^e_*\mathcal{O}_Y\big(E'-2\tau^*H\big)\cong\mathcal{O}_Y\big(-\tau^*H\big)^{\oplus\frac{q(q-1)}{2}}\oplus\mathcal{O}_Y\big(E'-2\tau^*H\big)^{\oplus\frac{q(q+1)}{2}}$$

and

$$F^e_*\mathcal{O}_Y\big(E_i-\tau^*H\big)$$
$$\cong\mathcal{O}_Y\big(-\tau^*H\big)^{\oplus\frac{q(q-1)}{2}}\oplus\mathcal{O}_Y\big(E_i-\tau^*H\big)^{\oplus q}\oplus\mathcal{O}_Y\big(E'-2\tau^*H\big)^{\oplus\frac{q(q-1)}{2}}$$

for $i=1,2,3$.

Proof. The proof is similar to Lemma 1.2 and left to the reader. □

Using the above lemmas, we can show that any Frobenius summand on a four-point blowup of $\mathbb{P}^2$ is obtained by successive extensions of line bundles. To analyze the extensions we need to know the dimension of the extension group $\mathrm{Ext}^1(L,M)$ for line bundles L,M. Since $\mathrm{Ext}^1(L,M)\cong H^1(X,L^{-1}\otimes M)$, its dimension is computed with Riemann–Roch. We collect a few such computations that will be used later; see also Lemma 4.3.

LEMMA 1.4. *Assume* $n=4$ *under the notation of Section 1.2 above, let* $E=E_1+E_2+E_3+E_4$ *and let* $i,j,l\in\{1,2,3,4\}$ *be any distinct indexes. Then the first cohomology groups*

$$H^1\big(X,\mathcal{O}_X\big(\pi^*H-E\big)\big),\qquad H^1\big(X,\mathcal{O}_X\big(E-2E_i-\pi^*H\big)\big),$$
$$H^1\big(X,\mathcal{O}_X(E_i-E_j-E_l)\big)$$

are one-dimensional k*-vector spaces. On the other hand,* $H^1(X,\mathcal{O}_X(D))=0$ *holds for* $D=\pm E_i$, E_i-E_j, $2\pi^*H-E$ *and* π^*H-E'' *with* $0\le E''\le E-E_i$.

Proof. Let $h^i = h^i(\mathcal{O}_X(\pi^*H - E))$ for $i = 0, 1, 2$. Then

$$\chi\big(\mathcal{O}_X\big(\pi^*H - E\big)\big) = h^0 - h^1 + h^2 = \frac{1}{2}\big(\pi^*H - E\big)\big(\pi^*H - E - K_X\big) + 1 = -1$$

by the Riemann–Roch formula. Here we used the fact that $K_X \sim E - 3\pi^*H$. On the other hand, $h^0 = 0$ since P_1, P_2, P_3, P_4 are not collinear, and $h^2 = h^0(\mathcal{O}_X(K_X - \pi^*H + E)) = 0$ by the Serre duality. Thus $h^1 = h^1(\mathcal{O}_X(\pi^*H - E)) = 1$, as required.

The dimensions of the other first cohomology groups are computed similarly. We just discuss the vanishing of $H^1(X, \mathcal{O}_X(\pi^*H - E''))$ from them. Let E'' be the sum of r distinct (-1)-curves with $0 \le r \le 3$. Then $\chi(\mathcal{O}_X(\pi^*H - E'')) = 3 - r$ by Riemann–Roch and $h^2(\mathcal{O}_X(\pi^*H - E'')) = 0$ by the Serre duality. It also follows that $h^0(\mathcal{O}_X(\pi^*H - E'')) = 3 - r$ from the condition of general position that no three of P_1, P_2, P_3, P_4 are collinear. Thus, $h^1(\mathcal{O}_X(\pi^*H - E'')) = 0$, as required. □

2. The behavior of Frobenius summands under point-blowups

Our strategy to study Frobenius summands on a four-point blowup X of $\mathbb{P}^2$ is to factorize $\pi\colon X \to \mathbb{P}^2$ as $\pi = \sigma \circ \tau$ as in Section 1.2 and look at what happens for the blowup $\sigma\colon X \to Y$ at P_4. For this purpose, we will work under the following setup in this section. Note that Frobenius summands on the blowup of $\mathbb{P}^2$ at four points satisfy conditions (a), (b) and (c) below; see Section 1 and especially (1.2).

2.1. Setup. Let $\sigma\colon X \to Y$ be the blowup at a point P on a smooth projective surface Y with exceptional curve $E = \sigma^{-1}(P)$. Let $\mathcal{F}$ be a vector bundle of rank r on X satisfying the following conditions:

(a) $\sigma_*\mathcal{F} \cong L_1 \oplus \cdots \oplus L_r$ for line bundles $L_1, \ldots, L_r$ on Y.

(b) $\mathcal{F}$ sits in an exact sequence

$$0 \to \sigma^*\sigma_*\mathcal{F} \to \mathcal{F} \to \mathcal{O}_E(-1)^{\oplus n} \to 0,$$

where n is an integer with $0 \le n \le r$.

(c) $H^i(X, \mathcal{F}) = 0$ for $i > 0$.

Under condition (b), condition (c) holds if and only if $H^i(X, \sigma^*\sigma_*\mathcal{F}) = 0$ for $i > 0$, or equivalently if $H^i(Y, L_j) = H^i(X, \sigma^*L_j) = 0$ for $i > 0$ and $1 \le j \le r$. Note that this vanishing holds if L_j's are Frobenius summands on a rational surface Y, since $H^i(Y, L_j) \subseteq H^i(Y, F^e_*\mathcal{O}_Y) = H^i(Y, \mathcal{O}_Y)$.

Let $\mathcal{F}$ be a vector bundle of rank r on X satisfying conditions (a)–(c) as above. Let $\mathcal{E} = \sigma_*\mathcal{F} \cong L_1 \oplus \cdots \oplus L_r$, $\mathcal{E}' = L_1 \oplus \cdots \oplus L_{r-1}$ and let $\mathcal{F}'$ be the saturation of $\sigma^*\mathcal{E}'$ in $\mathcal{F}$, that is, $\mathcal{F}'$ is the kernel of the natural map $\mathcal{F} \to (\mathcal{F}/\sigma^*\mathcal{E}')/\operatorname{tors}(\mathcal{F}/\sigma^*\mathcal{E}')$. Then $\mathcal{F}'$, which satisfies Serre's condition (S_2)

on a smooth surface, is a vector bundle of rank $r-1$, since $\mathcal{F}/\mathcal{F}'$ is a torsion-free $\mathcal{O}_X$-module of rank 1. We have the following commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \sigma^*\mathcal{E}' & \to & \sigma^*\mathcal{E} & \to & \sigma^*L_r & \to & 0 \\
 & & \downarrow^{\alpha} & & \downarrow & & \downarrow^{\beta} & & \\
0 & \to & \mathcal{F}' & \to & \mathcal{F} & \to & \mathcal{F}/\mathcal{F}' & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{N}' & \to & \mathcal{N} & \to & \mathcal{N}'' & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

with exact rows and columns, where $\mathcal{N} = \mathcal{O}_E(-1)^{\oplus n}$, and $\mathcal{N}'$ and $\mathcal{N}''$ are the cokernels of the maps α and β, respectively. Since $\mathcal{F}/\mathcal{F}'$ is isomorphic to σ^*L_r outside E, one has $(\mathcal{F}/\mathcal{F}') \otimes \sigma^*L_r^{-1} \cong \mathcal{I}_Z \otimes \mathcal{O}_X(mE)$ for an integer $m \geq 0$, where Z is a 0-dimensional closed subscheme of X supported in E. Actually, we must have $m = 0$ or 1 since $\mathcal{N}'' = \operatorname{Coker}(\beta)$ has an $\mathcal{O}_E$-module structure induced from the surjection $\mathcal{N} \twoheadrightarrow \mathcal{N}''$, and $m = 1$ if Z is non-empty.

CLAIM 2.1. $\mathcal{N}' \cong \mathcal{O}_E(-1)^{\oplus n-m}$, $\mathcal{N}'' \cong \mathcal{O}_E(-1)^{\oplus m}$ and $Z = \emptyset$.

Proof. Let $M = \sigma^*(L_r) \otimes \mathcal{O}_X(mE)$ and look at the exact sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{I}_Z \otimes M \to 0.$$

Then we have

$$\chi(\mathcal{F}) = \chi(\mathcal{F}') + \chi(\mathcal{I}_Z \otimes M) = \chi(\mathcal{F}') + \chi(M) - \deg Z.$$

The Euler characteristic on the left-hand side is computed as

$$\chi(\mathcal{F}) = h^0(\mathcal{F}) = h^0(\sigma^*\mathcal{E}) = h^0(\sigma^*\mathcal{E}') + h^0(\sigma^*L_r),$$

where the equalities follow from the vanishing (c), the exact sequence in (b) together with $h^0(\mathcal{N}) = 0$, and the splitting $\sigma^*\mathcal{E} \cong \sigma^*\mathcal{E}' \oplus \sigma^*L_r$, respectively. On the other hand, it follows that $\chi(M) = \chi(\sigma^*L_r) + m\chi(\mathcal{O}_E(-1)) = h^0(\sigma^*L_r)$ from the exact sequence $0 \to \sigma^*L_r \to M \to \mathcal{O}_E(-1)^{\oplus m} \to 0$ with $m = 0, 1$. We also have that $H^i(X, \sigma^*\mathcal{E}') = 0$ for $i > 0$, and all but the first cohomology groups of $\mathcal{N}'$ vanish because of the monomorphism $\mathcal{N}' \hookrightarrow \mathcal{N} = \mathcal{O}_E(-1)^{\oplus n}$. Looking at the exact sequence $0 \to \sigma^*\mathcal{E}' \to \mathcal{F}' \to \mathcal{N}' \to 0$, we know that $H^0(X, \sigma^*\mathcal{E}') \cong H^0(X, \mathcal{F}')$, $H^2(X, \mathcal{F}') = 0$ and that $\chi(\mathcal{F}') = h^0(\mathcal{F}') - h^1(\mathcal{F}') = h^0(\sigma^*\mathcal{E}') - h^1(\mathcal{F}')$. Summing up, we have $h^1(\mathcal{F}') + \deg Z = 0$, so that $h^1(\mathcal{F}') = \deg Z = 0$. Thus, $Z = \emptyset$ and $\mathcal{N}'' \cong \mathcal{O}_E(-1)^{\oplus m}$. It then follows that the exact sequence $0 \to \mathcal{N}' \to \mathcal{N} \to \mathcal{N}'' \to 0$ splits and $\mathcal{N}' \cong \mathcal{O}_E(-1)^{\oplus n-m}$. □

Thus, we are led to the following.

Lemma 2.1. *Let $\mathcal{F}$ be a vector bundle of rank r on X satisfying conditions* (a), (b) *and* (c) *as in Section* 2.1. *Then there exists a filtration $0=\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_r = \mathcal{F}$ of subbundles $\mathcal{F}_i$ with line bundle quotients $\mathcal{F}_i/\mathcal{F}_{i-1}$ such that the following hold for $1 \le i \le r$.*

(1) *$\mathcal{F}_i$ satisfies conditions* (a), (b) *and* (c) *with $\sigma_*\mathcal{F}_i \cong L_1 \oplus \cdots \oplus L_i$.*
(2) *$\mathcal{F}_i/\mathcal{F}_{i-1} \cong \sigma^* L_i$ or $\sigma^*(L_i) \otimes \mathcal{O}_X(E)$. In particular, if $\mathcal{F}_i$ is indecomposable, then $\mathcal{F}_i/\mathcal{F}_{i-1} \cong \sigma^*(L_r) \otimes \mathcal{O}_X(E)$.*

Proof. For each i let $\mathcal{F}_i$ be the saturation of $\sigma^*(L_1 \oplus \cdots \oplus L_i)$ in $\mathcal{F}$. Then $\mathcal{F}_i$ is a vector bundle of rank i with $\sigma_*\mathcal{F}_i \cong L_1 \oplus \cdots \oplus L_i$. Moreover, it follows from the argument above that assertions (1) for $i = r-1$ and (2) for $i = r$ hold. Since $\mathcal{F}_{r-1}$ satisfies conditions (a), (b) and (c), we obtain the whole statement (except for the case when $\mathcal{F}_i$ is indecomposable) by descending induction on i.

Finally, suppose that $\mathcal{F}_i$ is indecomposable and $\mathcal{F}_i/\mathcal{F}_{i-1} \cong \sigma^* L_i$. Then the splitting map $\sigma^* L_i \to \sigma^*\sigma_*\mathcal{F}_i$ composed with the inclusion map $\sigma^*\sigma_*\mathcal{F}_i \to \mathcal{F}_i$ gives a splitting of the map $\mathcal{F}_i \to \mathcal{F}_i/\mathcal{F}_{i-1}$, which contradicts to the indecomposability of $\mathcal{F}_i$. □

Corollary 2.2. *Let $\mathcal{F}$ be a vector bundle of rank r on X satisfying conditions* (a), (b) *and* (c). *For any subset $I = \{i_1, \ldots, i_s\} \subset \{1, \ldots, r\}$ with $J = \{1, \ldots, r\} \setminus I$, let $\mathcal{F}_I$ be the saturation of $\sigma^*(L_{i_1} \oplus \cdots \oplus L_{i_s})$ in $\mathcal{F}$ and let $\overline{\mathcal{F}}_J = \mathcal{F}/\mathcal{F}_I$. Then $\mathcal{F}_I$ and $\overline{\mathcal{F}}_J$ are vector bundles of rank s and $r-s$, respectively, satisfying conditions* (a), (b) *and* (c) *with $\sigma_*\mathcal{F}_I \cong \bigoplus_{i \in I} L_i$ and $\sigma_*\overline{\mathcal{F}}_J \cong \bigoplus_{j \in J} L_j$.*

Proof. We will prove our assertion by descending induction on $s = \sharp I$. The case $s = r$ is trivial. Suppose our assertion holds for s with $0 < s \le r$ and consider the following commutative diagram with exact rows and columns.[2]

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & \mathcal{F}_{I\setminus\{i_s\}} & = & \mathcal{F}_{I\setminus\{i_s\}} & \to & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{F}_I & \to & \mathcal{F} & \to & \overline{\mathcal{F}}_J & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & \overline{L_{i_s}} & \to & \overline{\mathcal{F}}_{J\cup\{i_s\}} & \to & \overline{\mathcal{F}}_J & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Since $\mathcal{F}_I$ and $\overline{\mathcal{F}}_J$ satisfy the required property by induction hypothesis, it follows from Lemma 2.1 that $\mathcal{F}_{I\setminus\{i_s\}}$ satisfies conditions (a)–(c) and that

[2] Note that the saturation of $\sigma^*(L_{i_1} \oplus \cdots \oplus L_{i_{s-1}})$ in $\mathcal{F}_I$ coincides with the saturation of $\mathcal{F}_{I\setminus\{i_s\}}$ in $\mathcal{F}$.

$\overline{L_{i_s}} \cong \sigma^* L_{i_s}$ or $\sigma^*(L_{i_s}) \otimes \mathcal{O}_X(E)$. Looking at the bottom exact sequence we see that $\overline{\mathcal{F}}_{J \cup \{i_s\}}$ also satisfies conditions (a)–(c). □

As for indecomposable bundles of ranks 2 and 3, we have the following corollary.

COROLLARY 2.3. *Under the notation of Section 2.1, assume that $\mathcal{F}$ is indecomposable of rank 2 such that $\sigma_* \mathcal{F} \cong L \oplus M$ for line bundles L and M on Y. Then we have an exact sequence*

$$0 \to \sigma^* L \to \mathcal{F} \to \sigma^*(M) \otimes \mathcal{O}_X(E) \to 0$$

given by a non-zero extension class $\varepsilon \in \operatorname{Ext}^1(\sigma^*(M) \otimes \mathcal{O}_X(E), \sigma^* L)$ sitting in

$$\operatorname{Ker}\bigl(\operatorname{Ext}^1\bigl(\sigma^*(M) \otimes \mathcal{O}_X(E), \sigma^* L\bigr) \to \operatorname{Ext}^1\bigl(\sigma^* M, \sigma^* L\bigr) \cong \operatorname{Ext}^1(M, L)\bigr) \cong k.$$

Proof. By Lemma 2.1, we have an exact sequence

$$0 \to \widetilde{L} \to \mathcal{F} \to \sigma^*(M) \otimes \mathcal{O}_X(E) \to 0$$

with $\widetilde{L} \cong \sigma^* L$ or $\sigma^*(L) \otimes \mathcal{O}_X(E)$. Suppose that $\widetilde{L} \cong \sigma^*(L) \otimes \mathcal{O}_X(E)$. Then the inclusion map $\mathcal{F} \to \sigma^* \sigma_*(\mathcal{F}) \otimes \mathcal{O}_X(E)$ composed with the map $\sigma^* \sigma_*(\mathcal{F}) \otimes \mathcal{O}_X(E) \to \sigma^*(L) \otimes \mathcal{O}_X(E)$, which is induced by the splitting map $\sigma_* \mathcal{F} \to L$, gives a splitting of the map $\widetilde{L} \to \mathcal{F}$. This absurdity implies that $\widetilde{L} \cong \sigma^* L$.

Next, we will show that the kernel of the natural map $\operatorname{Ext}^1(\overline{M}, \widetilde{L}) \xrightarrow{\rho} \operatorname{Ext}^1(M, L)$ is a vector space of dimension ≤ 1. To this end, we consider the exact sequence

$$0 \to \sigma^* M \to \sigma^*(M) \otimes \mathcal{O}_X(E) \to \mathcal{O}_E(-1) \to 0$$

and apply the functor $\operatorname{Ext}^\bullet(-, \sigma^* L)$ to obtain a long exact sequence

$$\operatorname{Ext}^1\bigl(\mathcal{O}_E(-1), \sigma^* L\bigr) \to \operatorname{Ext}^1\bigl(\sigma^*(M) \otimes \mathcal{O}_X(E), \sigma^* L\bigr) \xrightarrow{\rho} \operatorname{Ext}^1\bigl(\sigma^* M, \sigma^* L\bigr) \to 0.$$

It is sufficient to show that $\operatorname{Ext}^1(\mathcal{O}_E(-1), \sigma^* L)$ is 1-dimensional. But this follows because $\operatorname{Ext}^1(\mathcal{O}_E(-1), \sigma^* L) \cong H^0(X, \mathcal{E}xt^1(\mathcal{O}_E(-1), \sigma^* L))$ by the Leray spectral sequence and

$$\begin{aligned} \mathcal{E}xt^1\bigl(\mathcal{O}_E(-1), \sigma^* L\bigr) &\cong \mathcal{E}xt^1(\mathcal{O}_E, \omega_X) \otimes \omega_X(E)^{-1} \otimes \sigma^* L \\ &\cong \omega_E \otimes \omega_X(E)^{-1} \otimes \sigma^* L \\ &\cong \mathcal{O}_E. \end{aligned}$$

□

COROLLARY 2.4. *Under the notation of Section 2.1, assume that $\mathcal{F}$ is indecomposable of rank 3 with $\sigma_* \mathcal{F} \cong L_1 \oplus L_2 \oplus L_3$. Then we have a non-trivial extension*

$$0 \to \widetilde{L_1 \oplus L_2} \to \mathcal{F} \to \sigma^*(L_3) \otimes \mathcal{O}_X(E) \to 0,$$

and either one of the following two cases occurs.

(1) $\widetilde{L_1 \oplus L_2} = \sigma^*(L_1) \oplus \sigma^*(L_2)$ *with* $L_1 \not\cong L_2$.

(2) $\widetilde{L_1 \oplus L_2}$ *is indecomposable and* $\det(\widetilde{L_1 \oplus L_2}) \cong \sigma^*(L_1 \otimes L_2) \otimes \mathcal{O}_X(E)$.

Proof. The proof goes similarly as in Corollary 2.3. We just show the reason why $L_1 \ncong L_2$ in (1), as well as the indecomposability in (2).

Suppose that $L_1 \cong L_2$ in case (1) and let $\mathcal{F}$ be given by an extension class $\varepsilon = (\varepsilon_1, \varepsilon_2)$, where

$$\varepsilon_i \in K_i = \operatorname{Ker}\bigl(\operatorname{Ext}^1\bigl(\sigma^*(L_3) \otimes \mathcal{O}_X(E), \sigma^* L_i\bigr) \to \operatorname{Ext}^1(L_3, L_i)\bigr)$$

for $i = 1, 2$. Then $K_1 \cong K_2$ is a 1-dimensional vector space as we have seen in the proof of Lemma 2.3. It follows that an automorphism of $\sigma^* L_1 \oplus \sigma^* L_2$ transforms ε to an extension class of the form $(\varepsilon', 0)$. Then $\mathcal{F}$ splits as $\mathcal{F} \cong \mathcal{F}' \oplus \sigma^* L_2$, where $\mathcal{F}'$ is a rank 2 bundle given by an extension class ε'.

Suppose now that $\widetilde{L_1 \oplus L_2}$ is a decomposable bundle with determinant $\sigma^*(L_1 \otimes L_2) \otimes \mathcal{O}_X(E)$. It then follows from $\sigma_*(\widetilde{L_1 \oplus L_2}) \cong L_1 \oplus L_2$ that $\widetilde{L_1 \oplus L_2} \cong (\sigma^*(L_1) \otimes \mathcal{O}_X(E_4)) \oplus \sigma^* L_2$ or $\sigma^* L_1 \oplus (\sigma^*(L_2) \otimes \mathcal{O}_X(E_4))$. In each case $\mathcal{F}$ is given by an extension class of the form $(\varepsilon_1, 0)$ or $(0, \varepsilon_2)$, respectively. This contradicts the indecomposability of $\mathcal{F}$. □

3. Frobenius summands of rank 2 and 3 on a four-point blowup of $\mathbb{P}^2$

We will apply Corollaries 2.3 and 2.4 to the case $n = 4$ in the notation of Section 1.2. Namely, $\pi\colon X \to \mathbb{P}^2$ is the blowup of $\mathbb{P}^2$ at four points P_1, P_2, P_3, P_4 in general position. We factorize π as

$$\pi\colon X \xrightarrow{\sigma} Y \xrightarrow{\tau} \mathbb{P}^2,$$

where τ is the blowup at the 3 points P_1, P_2, P_3.

3.1. Rank 2 Frobenius summands on a four-point blowup. We will find out indecomposable rank 2 Frobenius summands on X with a non-trivial extension

$$0 \to \sigma^* L \to \mathcal{F} \to \sigma^*(M) \otimes \mathcal{O}_X(E_4) \to 0, \tag{3.1}$$

where L and M are non-trivial line bundles that are Frobenius summands on Y. In this case, Lemma 1.2 tells us that L and M are isomorphic to either $\mathcal{O}_Y(-\tau^* H)$, $\mathcal{O}_Y(E_i - \tau^* H)$ with $i = 1, 2, 3$, or $\mathcal{O}_Y(E' - 2\tau^* H)$, where H is a line on $\mathbb{P}^2$ and $E' = E_1 + E_2 + E_3$.

We give two candidates for rank 2 Frobenius summands in Examples 3.1 and 3.3, from which the latter one will be ruled out.

Example 3.1. Let $L = \mathcal{O}_Y(-\tau^* H)$ and $M = \mathcal{O}_Y(E' - 2\tau^* H)$. Then

$$\dim \operatorname{Ext}^1\bigl(\sigma^*(M) \otimes \mathcal{O}_X(E_4), \sigma^* L\bigr) = h^1\bigl(\mathcal{O}_X\bigl(\pi^* H - E\bigr)\bigr) = 1$$

by Lemma 1.4, where $E = E_1 + E_2 + E_3 + E_4 = \sigma^* E' + E_4$. Hence we have a unique non-trivial extension as (3.1) above that gives a vector bundle $\mathcal{G}$ of rank 2,

$$0 \to \mathcal{O}_X\bigl(-\pi^* H\bigr) \to \mathcal{G} \to \mathcal{O}_X\bigl(E - 2\pi^* H\bigr) \to 0. \tag{3.2}$$

Since $\operatorname{Ext}^1(M,L) \cong H^0(Y, \mathcal{O}_Y(\tau^*H - E')) \cong H^1(X, \mathcal{O}_X(\pi^*H - E + E_4)) = 0$ by Lemma 1.4, the natural map between the extension groups

$$\operatorname{Ext}^1\big(\sigma^*(M) \otimes \mathcal{O}_X(E_4), \sigma^*L\big) \to \operatorname{Ext}^1(M,L) = 0$$

is the zero map, and we do have $\sigma_*\mathcal{G} \cong L \oplus M$. To see that $\mathcal{G}$ is indecomposable, let $\sigma_i \colon X \to Y_i$ be the blow-down of E_i for $i = 1,2,3,4$. Then by symmetry we have

$$(\sigma_i)_*\mathcal{G} \cong \mathcal{O}_{Y_i}\big(-\pi_i^*H\big) \oplus \mathcal{O}_{Y_i}\big(E^{(i)} - 2\pi_i^*H\big),$$

where $E^{(i)} = (\sigma_i)_*E$ is the sum of three exceptional curves of $\pi_i \colon Y_i \to \mathbb{P}^2$. Suppose that $\mathcal{G}$ is decomposable. Then we can write $\mathcal{G} \cong \mathcal{O}_X(D_1 - \pi^*H) \oplus \mathcal{O}_X(D_2 - 2\pi^*H)$, where D_1 and D_2 are effective π-exceptional divisors with $D_1 + D_2 = E$. Pushing it out by σ_i and comparing with the splitting above, we see that $D_2 \geq E^{(i)}$ for all i. This implies that $D_1 = 0$ and $D_2 = E$, so that $\mathcal{G} \cong \mathcal{O}_X(-\pi^*H) \oplus \mathcal{O}_X(E - 2\pi^*H)$. However, this contradicts to the non-triviality of the extension (3.2).

Since $\mathcal{G}$ is indecomposable with $\sigma_*\mathcal{G} \cong \mathcal{O}_Y(-\tau^*H) \oplus \mathcal{O}_Y(E' - 2\tau^*H)$, we can replace L and M with each other to see that it sits in another unique non-trivial extension; cf. Lemma 1.4:

$$0 \to \mathcal{O}_X\big(E' - 2\pi^*H\big) \to \mathcal{G} \to \mathcal{O}_X\big(E_4 - \pi^*H\big) \to 0. \tag{3.3}$$

REMARK 3.2. We need the condition that the blown-up points P_1, P_2, P_3, P_4 are in general position for the symmetry used in Example 3.1, specifically for the vanishing $H^1(X, \mathcal{O}_X(\pi^*H - E + E_i)) = 0$ for $i = 1,2,3,4$; cf. Lemma 1.4.

The condition of general position allows us to assume that $P_1 = (1:0:0)$, $P_2 = (0:1:0), P_3 = (0:0:1)$ and $P_4 = (1:1:1)$. It then turns out that exact sequence (3.3) is the pullback of (3.2) by an automorphism on X. Indeed, the canonical Cremona transformation $(x_0 : x_1 : x_2) \mapsto (x_1x_2 : x_2x_0 : x_0x_1)$ on $\mathbb{P}^2$ lifts to an involution φ on X, which transforms $\pi^*(2H) - E'$ to π^*H and fixes E_4. Hence, (3.3) is the pullback of (3.2) by φ, so that $\varphi^*\mathcal{G} \cong \mathcal{G}$.

On the other hand, for $1 \leq i < j \leq 4$ one has a unique automorphism on $\mathbb{P}^2$ (i.e., an element of $\mathrm{PGL}(2,k)$) that permutes P_i and P_j and fixes the other two blown up points. It is easy to see that it lifts to an involution ψ_{ij} of X such that $\psi_{ij}^*\mathcal{G} \cong \mathcal{G}$.

EXAMPLE 3.3. Let $L = \mathcal{O}_Y(E_1 - \tau^*H)$ and $M = \mathcal{O}_Y(E_2 - \tau^*H)$. In this case we also have a unique non-trivial extension

$$0 \to \mathcal{O}_X\big(E_1 - \pi^*H\big) \to \mathcal{F} \to \mathcal{O}_X\big(E_2 + E_4 - \pi^*H\big) \to 0,$$

since $\dim \operatorname{Ext}^1(\mathcal{O}_X(E_2 + E_4), \mathcal{O}_X(E_1)) = 1$ by Lemma 1.4. This extension bundle $\mathcal{F}$ is indecomposable of rank 2 with $\sigma_*\mathcal{F} \cong L \oplus M$. Indeed, if $\mathcal{F}$ were decomposable, then the splitting type of $\mathcal{F}(\pi^*H)$ would be either $\mathcal{O}_X \oplus \mathcal{O}_X(E_1 + E_2 + E_4)$ or $\mathcal{O}_X(E_i) \oplus \mathcal{O}_X(E_j + E_k)$ with $\{i,j,k\} = \{1,2,4\}$. However, it follows from the above exact sequence that $\operatorname{Hom}(\mathcal{O}_X(E_1 + E_2 + E_4),$

$\mathcal{F}(\pi^*H)) = 0$, so that the former case cannot occur. In the latter case, we apply the functor $\operatorname{Ext}^\bullet(\mathcal{O}_X(E_j + E_k - \pi^*H), -)$ to the exact sequence above. If $i \neq 1$ then $\operatorname{Hom}(\mathcal{O}_X(E_j + E_k), \mathcal{F}(\pi^*H)) = 0$ and we get a contradiction. If $i = 1$ then we have the induced exact sequence

$$0 \to \operatorname{Hom}\big(\mathcal{O}_X(E_2 + E_4), \mathcal{F}\big(\pi^*H\big)\big) \to \operatorname{Hom}\big(\mathcal{O}_X(E_2 + E_4), \mathcal{O}_X(E_2 + E_4)\big)$$
$$\xrightarrow{\delta} \operatorname{Ext}^1\big(\mathcal{O}_X(E_2 + E_4), \mathcal{O}_X(E_1)\big)$$

with the non-zero connecting homomorphism δ. It follows that δ is injective and $\operatorname{Hom}(\mathcal{O}_X(E_2 + E_4), \mathcal{F}(\pi^*H)) = 0$. We get a contradiction as well.

Replacing L and M, we see that the indecomposable bundle $\mathcal{F}$ is given by another unique non-trivial extension

$$0 \to \mathcal{O}_X\big(E_2 - \pi^*H\big) \to \mathcal{F} \to \mathcal{O}_X\big(E_1 + E_4 - \pi^*H\big) \to 0.$$

Replacing the roles of E_4 by E_2 in the argument above we also have

$$0 \to \mathcal{O}_X\big(E_4 - \pi^*H\big) \to \mathcal{F} \to \mathcal{O}_X\big(E_1 + E_2 - \pi^*H\big) \to 0.$$

We denote this vector bundle $\mathcal{F}$ by $\mathcal{G}_3$. Similarly, we can define the indecomposable vector bundle $\mathcal{G}_i$ of rank 2 for $1 \leq i \leq 4$ by a unique non-trivial extension

$$0 \to \mathcal{O}_X\big(E_j - \pi^*H\big) \to \mathcal{G}_i \to \mathcal{O}_X\big(E_k + E_l - \pi^*H\big) \to 0$$

with any choice of j, k, l such that $\{i, j, k, l\} = \{1, 2, 3, 4\}$. Clearly the vector bundle $\mathcal{G}_i$ is stable under the automorphisms $\psi_{jk}, \psi_{kl}, \psi_{lj}$ introduced in Remark 3.2.

PROPOSITION 3.4. *Let X be the blowup of $\mathbb{P}^2$ at four points in general position, $\sigma\colon X \to Y$ the blow-down of an exceptional curve E_4 and let $\mathcal{F}$ be an indecomposable vector bundle of rank 2 on X. If $\sigma_*\mathcal{F}$ is a Frobenius summand on Y, then $\mathcal{F}$ is isomorphic either to the bundle $\mathcal{G}$ given in Example 3.1, or one of $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3$ given in Example 3.3.*

Proof. Since $\sigma_*\mathcal{F}$ is a Frobenius summand on a toric surface Y, it has a splitting $\sigma_*\mathcal{F} \cong L \oplus M$ by Lemma 1.2. We will examine the extension (3.1) with all possible types of such a splitting.

(i) Suppose first that $L \cong M$. Then

$$\operatorname{Ext}^1\big(\sigma^*(M) \otimes \mathcal{O}_X(E_4), \sigma^*L\big) \cong H^1\big(X, \mathcal{O}_X(-E_4)\big) = 0$$

by Lemma 1.4. Thus $\mathcal{F}$ splits into a direct sum of line bundles in this case.

(ii) Suppose $\sigma_*\mathcal{F}$ has a free summand, say, $L \cong \mathcal{O}_Y$. If $\mathcal{F}$ were indecomposable, then it would be given by a non-trivial extension

$$0 \to \mathcal{O}_X \to \mathcal{F} \to \sigma^*(M) \otimes \mathcal{O}_X(E_4) \to 0$$

by Corollary 2.3. But this is impossible, since $\mathrm{Ext}^1(\sigma^*(M)\otimes\mathcal{O}_X(E_4), \mathcal{O}_X)\cong H^1(X,\sigma^*(M^{-1})\otimes\mathcal{O}_X(-E_4))=0$ for all invertible Frobenius summands M on Y by Lemmas 1.2 and 1.4.

(iii) Suppose $\sigma_*\mathcal{F}\cong\mathcal{O}_Y(-\pi^*H)\oplus\mathcal{O}_Y(E_i-\tau^*H)$ with $i=1,2,3$. If $\mathcal{F}$ were indecomposable, then it would be given by a non-trivial extension

$$0\to\mathcal{O}_X\big(E_i-\pi^*H\big)\to\mathcal{F}\to\mathcal{O}_X\big(E_4-\pi^*H\big)\to 0$$

by Corollary 2.3. But this is impossible, since $\mathrm{Ext}^1(\mathcal{O}_X(E_4-\pi^*H), \mathcal{O}_X(E_i-\pi^*H))\cong H^1(X,\mathcal{O}_X(E_i-E_4))=0$ by Lemma 1.4.

(iv) Suppose $\sigma_*\mathcal{F}\cong\mathcal{O}_Y(E_i-\tau^*H)\oplus\mathcal{O}_Y(E'-2\tau^*H)$ with $i=1,2,3$. If $\mathcal{F}$ were indecomposable, then there would exist a non-trivial extension

$$0\to\mathcal{O}_X\big(E_i-\pi^*H\big)\to\mathcal{F}\to\mathcal{O}_X\big(E-2\pi^*H\big)\to 0$$

by Corollary 2.3. But this is impossible, since $\mathrm{Ext}^1(\mathcal{O}_X(E-2\pi^*H), \mathcal{O}_X(E_i-\pi^*H))\cong H^1(X,\mathcal{O}_X(\pi^*H+E_i-E))=0$ by Lemma 1.4.

In cases (i)–(iv) we have seen so far, any Frobenius summand $\mathcal{F}$ sitting in (3.1) splits into a direct sum of line bundles. In the remaining cases $\sigma_*\mathcal{F}$ is isomorphic either to $\mathcal{O}_Y(-\tau^*H)\oplus\mathcal{O}_Y(E'-2\tau^*H)$ or $\mathcal{O}_Y(E_i-\tau^*H)\oplus\mathcal{O}_Y(E_j-\tau^*H)$ with $1\le i<j\le 3$. These are exactly the cases treated in Examples 3.1 and 3.3, from which the assertion follows. ☐

Remark 3.5. It is not a sufficient condition for $\mathcal{F}$ to be a Frobenius summand on X that so is $\sigma_*\mathcal{F}$ on Y. Actually we will see that $\mathcal{G}$ is the only indecomposable Frobenius summand of rank 2 on X. The reason why $\mathcal{G}_i$ is not a Frobenius summand is as follows: Let $\rho=\sigma_i\colon X\to Y_i$ be the blowdown of E_i. Then $\mathcal{G}_i$ descends to a vector bundle $\mathcal{G}'_i=\rho_*\mathcal{G}_i$ on Y_i such that $\mathcal{G}_i=\rho^*\mathcal{G}'_i$. Since $\mathcal{G}_i$ is indecomposable, so is $\mathcal{G}'_i$, too. Thus, $\mathcal{G}'_i=\rho_*\mathcal{G}_i$ cannot be a Frobenius summand on the toric surface Y_i.

3.2. Rank 3 Frobenius summands on a 4-point blowup. We now apply Corollary 2.4 to give a candidate for an indecomposable Frobenius summands of rank 3.

Example 3.6. We consider case (1) of Corollary 2.4 for $L_i=\mathcal{O}_Y(E_i-\tau^*H)$ with $i=1,2,3$. Since $\mathrm{Ext}^1(\mathcal{O}_X(E_3+E_4),\mathcal{O}_X(E_i))$ is a one-dimensional vector space for $i=1,2$ by Lemma 1.4, we have an extension

$$\begin{aligned}(3.4)\qquad 0\to\mathcal{O}_X\big(E_1-\pi^*H\big)\oplus\mathcal{O}_X\big(E_2-\pi^*H\big)&\to\mathcal{B}\\ \to\mathcal{O}_X\big(E_3+E_4-\pi^*H\big)&\to 0\end{aligned}$$

given by an extension class $\varepsilon=(\varepsilon_1,\varepsilon_2)$ with $0\neq\varepsilon_i\in\mathrm{Ext}^1(\mathcal{O}_X(E_3+E_4), \mathcal{O}_X(E_i))\cong k$. It follows as in Example 3.1 that $\sigma_*\mathcal{B}\cong L_1\oplus L_2\oplus L_3$.

We claim that this vector bundle $\mathcal{B}$ of rank 3 is indecomposable. Suppose on the contrary that $\mathcal{B}$ is decomposable. Then it has a splitting $\mathcal{B}\cong\widetilde{L_i}\oplus\mathcal{B}'$ with $\widetilde{L_i}\cong\mathcal{O}_X(E_i-\pi^*H)$ or $\mathcal{O}_X(E_i+E_4-\pi^*H)$ for some $i=1,2,3$. The other summand $\mathcal{B}'$ must be isomorphic to $\mathcal{G}_i$ or $\mathcal{O}_X(D_1 - \pi^*H) \oplus \mathcal{O}_X(D_2 - \pi^*H)$, where D_1 and D_2 are effective π-exceptional divisors with $D_1 + D_2 \leq E - E_i$.

In case $i = 1$ or 2, we consider the composition map

$$\alpha\colon \mathcal{O}_X\big(E_i - \pi^*H\big) \xrightarrow{\imath} \mathcal{B} \twoheadrightarrow \widetilde{L_i},$$

where $\imath$ is the injective map induced by (3.4) and the second surjection is a splitting map. If $\alpha \in \mathrm{Hom}(\mathcal{O}_X(E_i - \pi^*H), \widetilde{L_i}) \cong k$ is a non-zero map, then $\mathcal{O}_X(E_i - \pi^*H)$ is a direct summand of $\mathcal{B}$ via the map $\imath$, contradicting to our assumption that $\varepsilon_i \neq 0$. If $\alpha = 0$, then there exists a non-zero map $\mathcal{O}_X(E_i - \pi^*H) \to \mathcal{B}'$ to the other summand $\mathcal{B}'$ of $\mathcal{B}$, but this is impossible, since $\mathrm{Hom}(\mathcal{O}_X(E_i - \pi^*H), \mathcal{B}') = 0$ in any case.

In the case where $\mathcal{B} \cong \widetilde{L_3} \oplus \mathcal{B}'$, we note that $\mathrm{Hom}(\mathcal{B}', \mathcal{O}_X(E_3 + E_4 - \pi^*H)) = 0$, so that a surjective map $\widetilde{L_3} \to \mathcal{O}_X(E_3 + E_4 - \pi^*H)$ is induced from exact sequence (3.4). Then $\widetilde{L_3} = \mathcal{O}_X(E_3 + E_4 - \pi^*H)$ and we consider the composition map

$$\beta\colon \widetilde{L_3} \hookrightarrow \mathcal{B} \twoheadrightarrow \mathcal{O}_X\big(E_3 + E_4 - \pi^*H\big)$$

instead of α, where the first injection is a splitting map and the second surjection is that in (3.4). Arguing as before, we obtain a contradiction as well. Thus, $\mathcal{B}$ is indecomposable.

REMARK 3.7. A rank 3 indecomposable bundle $\mathcal{B}$ given in Example 3.6 is uniquely determined up to isomorphism. Indeed, let $\varepsilon = (\varepsilon_1, \varepsilon_2)$ and $\varepsilon' = (\varepsilon_1', \varepsilon_2')$ be two extension classes with non-zero $\varepsilon_i, \varepsilon_i' \in \mathrm{Ext}^1(\mathcal{O}_X(E_3 + E_4 - \pi^*H), L_i) \cong k$ for $i = 1, 2$. Since ε_i and ε_i' live in the same one-dimensional k-vector space, one can consider their ratio $\varepsilon_i'/\varepsilon_i \in k^* \cong \mathrm{Aut}(L_i)$. Then the extension bundles corresponding to ε and ε' are identified via the automorphism $\varepsilon_1'/\varepsilon_1 \oplus \varepsilon_2'/\varepsilon_2 \in \mathrm{Aut}(L_1 \oplus L_2)$. On the other hand, if $\varepsilon_i = 0$ for $i = 1$ or 2, then the extension bundle has a direct summand $\sigma^* L_i$ and so decomposable. It follows from the symmetry with Corollary 2.4 that $\mathcal{B}$ is a unique indecomposable bundle sitting in an exact sequence

$$0 \to \mathcal{O}_X\big(E_i - \pi^*H\big) \oplus \mathcal{O}_X\big(E_j - \pi^*H\big) \to \mathcal{B} \to \mathcal{O}_X\big(E_k + E_l - \pi^*H\big) \to 0$$

with any choice of indexes i, j, k, l with $\{i, j, k, l\} = \{1, 2, 3, 4\}$.

4. The global FFRT property of a four-point blowup of $\mathbb{P}^2$

We continue to work on the blowup $\pi\colon X \to \mathbb{P}^2$ at four points P_1, P_2, P_3, P_4 on $\mathbb{P}^2$ in general position. Let $E_i = \pi^{-1}(P_i)$ be the exceptional curve over P_i, let $E = E_1 + E_2 + E_3 + E_4$ and let H be a line on $\mathbb{P}^2$.

This section is devoted to proving the following theorem.

THEOREM 4.1. *Any indecomposable Frobenius summand on the surface X coincides with one of the following vector bundles of rank ≤ 3.*

(1) *line bundles $\mathcal{O}_X$, $\mathcal{O}_X(E - 2\pi^*H)$ and $\mathcal{O}_X(E_i - \pi^*H)$ for $i = 1, 2, 3, 4$;*

(2) *an indecomposable rank* 2 *bundle* $\mathcal{G}$ *given by a unique non-trivial extension*

$$0 \to \mathcal{O}_X\big(-\pi^*H\big) \to \mathcal{G} \to \mathcal{O}_X\big(E - 2\pi^*H\big) \to 0;$$

(3) *an indecomposable rank* 3 *bundle* $\mathcal{B}$ *given by a non-trivial extension*

$$0 \to \mathcal{O}_X\big(E_1 - \pi^*H\big) \oplus \mathcal{O}_X\big(E_2 - \pi^*H\big) \to \mathcal{B} \to \mathcal{O}_X\big(E_3 + E_4 - \pi^*H\big) \to 0.$$

In particular, X is GFFRT.

REMARK 4.2. The rank 2 bundle $\mathcal{G}$ in (2) is the one given in Example 3.1, whereas the rank 3 bundle $\mathcal{B}$ is the one given in Example 3.6. Theorem 4.1 is suggested by our empirical computation made in characteristic $p = 2, 3$.

Proof of Theorem 4.1. We begin with the following setup. Let $\sigma_i\colon X \to Y_i$ be the blow-down of E_i for $i = 1, 2, 3, 4$, among which we specify

$$\sigma = \sigma_4\colon X \to Y = Y_4\text{: the blow-down of } E_4.$$

Let $\tau\colon Y \to \mathbb{P}^2$ be the blowup at P_1, P_2, P_3 and define the line bundles L_1, L_2, L_3 and M_1, M_2 on Y by

$$L_i = \mathcal{O}_Y\big(E_i - \tau^*H\big) \quad \text{for } i = 1, 2, 3;$$
$$M_1 = \mathcal{O}_Y\big(-\tau^*H\big), \qquad M_2 = \mathcal{O}_Y\big(E' - 2\tau^*H\big),$$

where $E' = E_1 + E_2 + E_3$. These are the non-trivial Frobenius summands on Y.

We first consider Frobenius summands of rank 1. We have the following line bundles that would be possibly non-trivial Frobenius summands on X; see Section 1.2 and Lemma 1.2:

$$\mathcal{O}_X\big(E' - 2\pi^*H\big), \mathcal{O}_X\big(E - 2\pi^*H\big), \mathcal{O}_X\big(-\pi^*H\big), \mathcal{O}_X\big(E_i - \pi^*H\big)$$
$$\text{and } \mathcal{O}_X\big(E_j + E_4 - \pi^*H\big),$$

where $i = 1, 2, 3, 4$ and $j = 1, 2, 3$. We can rule out $\mathcal{O}_X(E' - 2\pi^*H)$ from these candidates, since its push-forward by σ_j ($j = 1, 2, 3$) is not a Frobenius summand on Y_j. We can exclude $\mathcal{O}_X(E_j + E_4 - \pi^*H)$ in a similar way. The line bundle $\mathcal{O}_X(-\pi^*H)$ is also ruled out from the candidates, because there is an automorphism of X via which $\mathcal{O}_X(-\pi^*H)$ and $\mathcal{O}_X(E' - 2\pi^*H)$ correspond with each other.

In order to determine Frobenius summands of rank ≥ 2, we need the following results.

LEMMA 4.3. *Let $\widetilde{L}$ and $\widetilde{M}$ be line bundles on X such that the pair $(\widetilde{L}, \widetilde{M})$ is one of $(\sigma^*L_i, \sigma^*M_j)$, $(\sigma^*L_i, \sigma^*(M_j) \otimes \mathcal{O}_X(E_4))$ or $(\sigma^*(L_i) \otimes \mathcal{O}_X(E_4), \sigma^*M_j)$ for $1 \leq i \leq 3$ and $1 \leq j \leq 2$. Then $\operatorname{Ext}^1(\widetilde{L}, \widetilde{M}) = 0$.*

Proof. This is verified with one by one computation of $\dim \operatorname{Ext}^1(\widetilde{L}, \widetilde{M}) = h^1(\widetilde{L}^{-1} \otimes \widetilde{M})$ using Riemann–Roch; cf. Lemma 1.4. $\square$

PROPOSITION 4.4. *Let $\mathcal{F}$ be a vector bundle on X satisfying conditions* (a), (b) *and* (c) *as in Section 2.1 with $\sigma_*\mathcal{F} \cong \bigoplus_{i=1}^3 L_i^{\oplus a_i} \oplus \bigoplus_{j=1}^2 M_j^{\oplus b_j}$. Then one has a decomposition*

$$\mathcal{F} \cong \mathcal{F}_L \oplus \mathcal{F}_M$$

with vector bundles $\mathcal{F}_L$ and $\mathcal{F}_M$ satisfying conditions (a), (b) *and* (c) *such that $\sigma_*\mathcal{F}_L \cong \bigoplus_{i=1}^3 L_i^{\oplus a_i}$ and $\sigma_*\mathcal{F}_M \cong \bigoplus_{j=1}^2 M_j^{\oplus b_j}$, respectively.*

Proof. By Corollary 2.2, one has an exact sequence

$$0 \to \mathcal{F}_L \to \mathcal{F} \to \mathcal{F}_M \to 0$$

with vector bundles $\mathcal{F}_L$ and $\mathcal{F}_M$ satisfying conditions (a), (b) and (c) such that $\sigma_*\mathcal{F}_L \cong \bigoplus_{i=1}^3 L_i^{\oplus a_i}$ and $\sigma_*\mathcal{F}_M \cong \bigoplus_{j=1}^2 M_j^{\oplus b_j}$, respectively. Since $\mathcal{F}_L$ (resp. $\mathcal{F}_M$) is obtained by successive extensions with line bundles σ^*L_i and $\sigma^*(L_i)\otimes\mathcal{O}_X(E_4)$ (resp. σ^*M_j and $\sigma^*(M_j)\otimes\mathcal{O}_X(E_4)$) by Lemma 2.1, it follows inductively from Lemma 4.3 that $\operatorname{Ext}^1(\mathcal{F}_M, \mathcal{F}_L) = 0$, so that $\mathcal{F} \cong \mathcal{F}_L \oplus \mathcal{F}_M$. □

It follows that any indecomposable Frobenius summand of rank ≥ 2 on X is either of type $\mathcal{F}_L$ or $\mathcal{F}_M$ in Proposition 4.4. We first determine the structure of the Frobenius summand $\mathcal{F}_M$.

PROPOSITION 4.5. *Let $\mathcal{F}$ be an indecomposable Frobenius summand of rank $r = r_1 + r_2 \geq 2$ such that $\sigma_*\mathcal{F} \cong M_1^{\oplus r_1} \oplus M_2^{\oplus r_2}$. Then $r_1 = r_2 = 1$ and $\mathcal{F} \cong \mathcal{G}$.*

Proof. It is easy to see that $r_i > 0$ $(i = 1, 2)$ if $\mathcal{F}$ is indecomposable. Indeed, if $\sigma_*\mathcal{F} \cong M^{\oplus r}$ for a line bundle M, then $\mathcal{F}$ would decompose into line bundles σ^*M and $\sigma^*(M)\otimes\mathcal{O}_X(E_4)$. This follows inductively from Lemmas 2.1 and 1.4 as in case (i) in the proof of Proposition 3.4.

Let $\mathcal{F}_{i_1,i_2}$ denote the saturation of $\sigma^*M_1^{\oplus i_1} \oplus \sigma^*M_2^{\oplus i_2}$ in $\mathcal{F}$. Then we have an exact sequence

$$0 \to \mathcal{F}_{r_1,r_2-1} \to \mathcal{F} \to \mathcal{O}_X\big(E - 2\pi^*H\big) \to 0 \tag{4.1}$$

by Lemma 2.1. If $r_2 \geq 2$, then we also have

$$0 \to \mathcal{F}_{r_1,r_2-2} \to \mathcal{F}_{r_1,r_2-1} \to \overline{M_2} \to 0,$$

where $\overline{M_2} \cong \sigma^*M_2$ or $\sigma^*(M_2)\otimes\mathcal{O}_X(E_4) = \mathcal{O}_X(E - 2\pi^*H)$ with $E = E_1 + E_2 + E_3 + E_4$. If $\overline{M_2} \cong \sigma^*M_2$, then the above exact sequence splits, that is, $\mathcal{F}_{r_1,r_2-1} \cong \mathcal{F}_{r_1,r_2-2} \oplus \sigma^*M_2$. Thus exact sequence (4.1) turns out to be

$$0 \to \mathcal{F}_{r_1,r_2-2} \oplus \sigma^*M_2 \to \mathcal{F} \to \mathcal{O}_X\big(E - 2\pi^*H\big) \to 0.$$

However, since $\operatorname{Ext}^1(\mathcal{O}_X(E - 2\pi^*H), \sigma^*M_2) = H^1(X, \mathcal{O}_X(-E_4)) = 0$ by Lemma 1.4, $\mathcal{F}$ has a direct summand σ^*M_2 in this case, contradicting to the indecomposability of $\mathcal{F}$. Thus, $\overline{M_2} \cong \mathcal{O}_X(E - 2\pi^*H)$. Using a commutative diagram as in the proof of Corollary 2.2, we see that $\mathcal{F}$ sits in another exact sequence

$$0 \to \mathcal{F}_{r_1, r_2-2} \to \mathcal{F} \to \mathcal{O}_X\big(E - 2\pi^*H\big)^{\oplus 2} \to 0.$$

Repeating this procedure inductively, we obtain an exact sequence

$$0 \to \mathcal{F}_{r_1,0} \to \mathcal{F} \to \mathcal{O}_X\big(E - 2\pi^*H\big)^{\oplus r_2} \to 0.$$

As we have seen in the beginning of the proof, $\mathcal{F}_{r_1,0}$ is a direct sum of copies of line bundles $\sigma^* M_1 = \mathcal{O}_X(-\pi^*H)$ and $\mathcal{O}_X(E_4 - \pi^*H)$, and if $\mathcal{F}_{r_1,0}$ has a direct summand $\mathcal{O}_X(E_4 - \pi^*H)$, then so does $\mathcal{F}$, since $\mathrm{Ext}^1(\mathcal{O}_X(E - 2\pi^*H), \mathcal{O}_X(E_4 - \pi^*H)) = 0$ again by Lemma 1.4. Thus, we have an exact sequence

$$0 \to \mathcal{O}_X\big(-\pi^*H\big)^{\oplus r_1} \to \mathcal{F} \to \mathcal{O}_X\big(E - 2\pi^*H\big)^{\oplus r_2} \to 0.$$

Since $\mathrm{Ext}^1(\mathcal{O}_X(E - 2\pi^*H), \mathcal{O}_X(-\pi^*H))$ is a 1-dimensional vector space, the above exact sequence is given by an "extension matrix" ε_M of size $r_2 \times r_1$, to which we can make row and column elementary transformations without changing the isomorphism class of the extension bundle $\mathcal{F}$. If $r_1 \neq r_2$ or rank $\varepsilon_M < r_1 = r_2$, then $\mathcal{F}$ has a line bundle summand. If rank $\varepsilon_M = r_1 = r_2$, then ε_M is transformed to the identity matrix of size $r_1 = r_2 = r/2$ and $\mathcal{F} \cong \mathcal{G}^{\oplus r/2}$. It follows from the indecomposability of $\mathcal{F}$ that $r = 2$ and $\mathcal{F} \cong \mathcal{G}$. □

We now consider the structure of the Frobenius summand $\mathcal{F}_L$ in Proposition 4.4. We put $\mathcal{L} = \mathcal{O}_X(-\pi^*H) = \pi^*\mathcal{O}_{\mathbb{P}^2}(-1)$ and consider a vector bundle $\mathcal{F}$ of rank $r = a_1 + a_2 + a_3$ on X satisfying the following condition:

$$(*) \qquad \begin{aligned} \sigma_*\mathcal{F} &\cong L_1^{\oplus a_1} \oplus L_2^{\oplus a_2} \oplus L_3^{\oplus a_3} \quad \text{and} \\ \det \mathcal{F} &= \mathcal{L}^r(a_1E_1 + a_2E_2 + a_3E_3 + a_4E_4). \end{aligned}$$

Lemma 4.6. *Suppose that $\mathcal{F}$ is a Frobenius summand satisfying condition $(*)$ as above.*

(1) *$(\sum_{i=1}^4 a_i) - a_j \le r$ for all $j = 1, 2, 3, 4$.*
(2) *If $\mathcal{F}$ is indecomposable with $r \ge 2$, then $0 < a_4 \le a_j$ for all $j = 1, 2, 3$.*

Proof. (1) First note that condition $(*)$ implies $\pi_*\mathcal{F} \cong \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus r}$, so that its push-forward $(\sigma_j)_*\mathcal{F}$ by σ_j, which is a Frobenius summand on a three-point blowup Y_j, is isomorphic to a direct sum of r copies of line bundles $\mathcal{O}_{Y_j}(-\tau_j^*H)$ and $\mathcal{O}_{Y_j}(E_i - \tau_j^*H)$ with $i \neq j$. Since $\det(\sigma_j)_*\mathcal{F} = \mathcal{O}_{Y_j}(\sum_{i\neq j} a_iE_i - r\tau_j^*H)$, we see that $\sum_{i\neq j} a_i \le r$. Since this is an equality for $j = 4$, we have $a_j \ge a_4$ for $j = 1, 2, 3$.

(2) If $a_4 = 0$, then $\mathcal{F} = \sigma^*\sigma_*\mathcal{F} \cong \bigoplus_{i=1}^3 \mathcal{L}(E_i)^{\oplus a_i}$ splits into a direct sum of line bundles. □

REMARK 4.7. In order to prove and apply Lemma 4.6, we need the condition that the points P_1, P_2, P_3, P_4 are in general position. Indeed, it guarantees that Y_j and especially Y are all isomorphic to the projective plane blown up at three points that are not collinear, to which Lemma 1.2 is applicable.

PROPOSITION 4.8. *Suppose that $\mathcal{F}$ is an indecomposable Frobenius summand of rank $r \geq 2$ satisfying condition* $(*)$. *Then $\mathcal{F} \cong \mathcal{B}$.*

Proof. First, note that $0 < a_4 \leq a_3$ by Lemma 4.6. We denote by $\mathcal{F}_{i_1,i_2,i_3}$ the saturation of $\sigma^*(L_1^{\oplus i_1} \oplus L_2^{\oplus i_2} \oplus L_3^{\oplus i_3})$ in $\mathcal{F}$, and argue as in the proof of Proposition 4.5. First, we have an exact sequence

$$0 \to \mathcal{F}_{a_1,a_2,a_3-1} \to \mathcal{F} \to \mathcal{L}(E_3 + E_4) \to 0 \tag{4.2}$$

by Lemma 2.1. If $a_3 \geq 2$, then we also have

$$0 \to \mathcal{F}_{a_1,a_2,a_3-2} \to \mathcal{F}_{a_1,a_2,a_3-1} \to \overline{L_3} \to 0,$$

where $\overline{L_3} \cong \mathcal{L}(E_3)$ or $\mathcal{L}(E_3 + E_4)$. If $\overline{L_3} \cong \mathcal{L}(E_3)$, then the above exact sequence splits, i.e., $\mathcal{F}_{a_1,a_2,a_3-1} \cong \mathcal{F}_{a_1,a_2,a_3-2} \oplus \mathcal{L}(E_3)$. Thus exact sequence (4.2) turns out to be

$$0 \to \mathcal{F}_{a_1,a_2,a_3-2} \oplus \mathcal{L}(E_3) \to \mathcal{F} \to \mathcal{L}(E_3 + E_4) \to 0.$$

However, since $\operatorname{Ext}^1(\mathcal{L}(E_3 + E_4), \mathcal{L}(E_3)) = 0$ by Lemma 1.4, the extension bundle $\mathcal{F}$ obtained as above has a direct summand $\mathcal{L}(E_3)$, contradicting to the indecomposability of $\mathcal{F}$. Thus $\overline{L_3} \cong \mathcal{L}(E_3 + E_4)$. Using a commutative diagram as in the proof of Corollary 2.2, we see that $\mathcal{F}$ sits in another exact sequence

$$0 \to \mathcal{F}_{a_1,a_2,a_3-2} \to \mathcal{F} \to \mathcal{L}(E_3 + E_4)^{\oplus 2} \to 0.$$

Repeating this procedure inductively, we obtain an exact sequence

$$0 \to \mathcal{F}_{a_1,a_2,a_3-a_4} \to \mathcal{F} \to \mathcal{L}(E_3 + E_4)^{\oplus a_4} \to 0. \tag{4.3}$$

Since $\mathcal{F}' = \mathcal{F}_{a_1,a_2,a_3-a_4}$ is the saturation of $L_1^{\oplus a_1} \oplus L_2^{\oplus a_2} \oplus L_3^{\oplus a_3-a_4}$ in $\mathcal{F}$, $\sigma_*\mathcal{F}' = L_1^{\oplus a_1} \oplus L_2^{\oplus a_2} \oplus L_3^{\oplus a_3-a_4}$ and $\det \mathcal{F}' = \mathcal{L}^{r-a_4}(a_1E_1 + a_2E_2 + (a_3 - a_4)E_3)$. It then follows that $\mathcal{F}' = \sigma^*\sigma_*\mathcal{F}' \cong \mathcal{L}(E_1)^{\oplus a_1} \oplus \mathcal{L}(E_2)^{\oplus a_2} \oplus \mathcal{L}(E_3)^{\oplus a_3-a_4}$. Namely, exact sequence (4.3) turns out to be

$$0 \to \mathcal{L}(E_1)^{\oplus a_1} \oplus \mathcal{L}(E_2)^{\oplus a_2} \oplus \mathcal{L}(E_3)^{\oplus a_3-a_4} \to \mathcal{F} \to \mathcal{L}(E_3 + E_4)^{\oplus a_4} \to 0.$$

If $a_3 > a_4$, then $\mathcal{F}$ must have a line bundle summand $\mathcal{L}(E_3)$ again by the vanishing $\operatorname{Ext}^1(\mathcal{L}(E_3 + E_4), \mathcal{L}(E_3)) = 0$. Hence $a_3 = a_4$, and by the same reason $a_1 = a_2 = a_4$. Thus, letting $a = a_1 = a_2 = a_3 = a_4$ we have come to an exact sequence

$$0 \to \mathcal{L}(E_1)^{\oplus a} \oplus \mathcal{L}(E_2)^{\oplus a} \to \mathcal{F} \to \mathcal{L}(E_3 + E_4)^{\oplus a} \to 0.$$

Since $\mathrm{Ext}^1(\mathcal{L}(E_3+E_4),\mathcal{L}(E_i))$ is a 1-dimensional vector space for $i=1,2$, the above exact sequence is given by an "extension matrix"

$$\varepsilon_L = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix},$$

where ε_i is an $a\times a$-matrix with entries in $\mathrm{Ext}^1(\mathcal{L}(E_3+E_4),\mathcal{L}(E_i))\cong k$. If either one of ε_i is of rank $< a$, then $\mathcal{F}$ has a line bundle summand isomorphic to $\mathcal{L}(E_i)$, contradicting to the indecomposability of $\mathcal{F}$. Thus $\mathrm{rank}\,\varepsilon_1 = \mathrm{rank}\,\varepsilon_2 = a$ and both ε_1 and ε_2 are transformed to the identity matrix. It then follows that $\mathcal{F}\cong\mathcal{B}^{\oplus a}$, and the indecomposability of $\mathcal{F}$ implies that $a=1$. □

In conclusion, any indecomposable Frobenius summand on X is either one of those listed in Theorem 4.1. □

5. Determination of the Frobenius summands on the four-point blowup of $\mathbb{P}^2$

5.1. Observation. We again work on the blowup $\pi\colon X\to\mathbb{P}^2$ at four points in general position as in Section 4, but we will use the notation L_i in a different meaning in this section. Namely, let $L_0=\mathcal{O}_X(E_1+E_2+E_3+E_4-2\pi^*H)$ and $L_i=\mathcal{O}_X(E_i-\pi^*H)$ for $i=1,2,3,4$. We know that exactly one $\mathcal{O}_X$ appears as a direct summand of $F^e_*\mathcal{O}_X$ by Proposition 1.1, but we have not yet proven in Section 4 that L_i, $\mathcal{G}$ and $\mathcal{B}$ are really Frobenius summands on X. In this section, we will verify this to be true.

Thanks to Theorem 4.1, we can write

$$F^e_*\mathcal{O}_X \cong \mathcal{O}_X \oplus \bigoplus_{i=0}^{4} L_i^{\oplus a_e} \oplus \mathcal{B}^{\oplus b_e} \oplus \mathcal{G}^{\oplus c_e},$$

where $0\le a_e, b_e, c_e\in\mathbb{Z}$. (Note that the numbers of copies of L_i's are equal since they are transformed to each other via automorphisms of X.) Comparing the push-forwards by $\sigma=\sigma_4$ (or, ranks and the first Chern classes; note that the first Chern class of $F^e_*\mathcal{O}_X$ is $\frac{p^{2e}-p^e}{2}K_X$; see [KS]) of the both sides we deduce the following equalities.

$$a_e+b_e=p^e-1 \quad\text{and}\quad a_e+c_e=\frac{(p^e-1)(p^e-2)}{2}.$$

THEOREM 5.1. *In the notation as above, one has*

$$a_e=p^e-2,\qquad b_e=1\quad\textit{and}\quad c_e=\frac{(p^e-2)(p^e-3)}{2}$$

for all $e\ge 1$.

Proof. Let $q=p^e$ and let $L:=L_1=\mathcal{O}_X(E_1-\pi^*H)$. Our goal is to show that $a_e=q-2$, or equivalently that $F^e_*\mathcal{O}_X$ has exactly $q-2$ copies of L as its direct summands. To this end, we will look at the natural pairing

$$\operatorname{Hom}\big(L,F^e_*\mathcal{O}_X\big)\times\operatorname{Hom}\big(F^e_*\mathcal{O}_X,L\big)\to\operatorname{Hom}(L,L)=k. \tag{5.1}$$

Let $x=x_1/x_0, y=x_2/x_0$ be the affine coordinates at $P_1=(1:0:0)\in\mathbb{P}^2$. Then

$$\begin{aligned}\operatorname{Hom}\big(L,F^e_*\mathcal{O}_X\big)&\cong H^0\big(X,F^e_*\big(L^{-q}\big)\big)\\&=H^0\big(X,L^{-q}\big)=H^0\big(X,\mathcal{O}_X\big(q\pi^*H-qE_1\big)\big)\end{aligned}$$

is identified with the subspace of $H^0(\mathbb{P}^2,\mathcal{O}_{\mathbb{P}^2}(qH))=\langle x^iy^j|0\le i,j;i+j\le q\rangle$ consisting of polynomials of order $\ge q$ at the point P_1, which is exactly equal to

$$V=\big\langle x^q,x^{q-1}y,\dots,y^q\big\rangle.$$

On the other hand, it follows from the adjunction formula that

$$\begin{aligned}\mathcal{H}om\big(F^e_*\mathcal{O}_X,L\big)&\cong\mathcal{H}om\big(F^e_*\mathcal{O}_X,\omega_X\big)\otimes\omega_X^{-1}\otimes L\\&\cong F^e_*(\omega_X)\otimes\omega_X^{-1}\otimes L\cong F^e_*\big(\omega_X^{1-q}\otimes L^q\big)\\&\cong F^e_*\mathcal{O}_X\big((2q-3)\pi^*H+E_1-(q-1)(E_2+E_3+E_4)\big).\end{aligned}$$

Thus, $\operatorname{Hom}(F^e_*\mathcal{O}_X,L)\cong H^0(X,\mathcal{O}_X((2q-3)\pi^*H-(q-1)(E_2+E_3+E_4)))$ and this is identified with

$$W=\big\langle(x-1)^i(y-1)^j|i,j\le q-2,i+j\ge q-1\big\rangle$$

by a similar argument as above with respect to affine coordinates $x-1,y-1$ at $P_4=(1:1:1)\in\mathbb{P}^2$. The pairing (5.1) is identified with

$$\begin{aligned}H^0\big(X,F^e_*\big(L^{-q}\big)\big)\times H^0\big(X,F^e_*\big(\omega_X^{1-q}\otimes L^q\big)\big)&\xrightarrow{\mu}H^0\big(X,F^e_*\big(\omega_X^{1-q}\big)\big)\\&\xrightarrow{t_e}H^0(X,\mathcal{O}_X),\end{aligned}$$

where μ is the multiplication map and t_e is the map induced by the trace map (or the Cartier operator) $\operatorname{Tr}_e\colon F^e_*\omega_X\to\omega_X$. Furthermore, it is identified with the pairing $\langle -,-\rangle\colon V\times W\to k$ given by

$$\langle\varphi,\psi\rangle=\text{the coefficient of }\varphi\psi\text{ in }(xy)^{q-1}$$

for $\varphi\in V$ and $\psi\in W$; see, for example, [Sch, Observation 3.5]. Hence, L is a direct summand of $F^e_*\mathcal{O}_X$ via $\varphi\in V$ if and only if there exists $\psi\in W$ such that $\langle\varphi,\psi\rangle=1$. Now let

$$\begin{aligned}\operatorname{Ann}_V(W)&=\big\{\varphi\in V|\langle\varphi,\psi\rangle=0\text{ for all }\psi\in W\big\},\\\operatorname{Ann}_W(V)&=\big\{\psi\in W|\langle\varphi,\psi\rangle=0\text{ for all }\varphi\in V\big\}\end{aligned}$$

and let $\overline{V} = V/\operatorname{Ann}_V(W)$, $\overline{W} = W/\operatorname{Ann}_W(V)$. Then we have an induced perfect pairing $\overline{V} \times \overline{W} \to k$ and $\dim \overline{V} = \dim \overline{W}$ is equal to the rank of a $\frac{(q-1)(q-2)}{2} \times (q-1)$-matrix

$$A^{(e)} = \left[\binom{i}{l-1} \binom{j}{q-1-l} \right]_{(i,j)\in I, 1\le l\le q-1},$$

whose rows are indexed by $(i,j) \in I = \{(i,j) \in \mathbb{Z}^2 | 1 \le i,j \le q-2, i+j \ge q-1\}$ and columns by $l = 1, 2, \ldots, q-1$. Each entry of $A^{(e)}$ is a product of binomial coefficients that is equal to $\langle x^{q-l}y^l, (x-1)^i(y-1)^j\rangle \in \mathbb{F}_p$.

Recall that $a_e \ge 0$ is the maximum integer such that $L^{\oplus a_e}$ is isomorphic to a direct summand of $F_*^e\mathcal{O}_X$.

Claim 5.1. $a_e = \operatorname{rank} A^{(e)}$.

Proof. Let $r = \operatorname{rank} A^{(e)}$ and choose $\varphi_1, \ldots, \varphi_r \in V$ that reduce to a basis $\overline{\varphi}_1, \ldots, \overline{\varphi}_r$ of $\overline{V}$. Then $\varphi_1, \ldots, \varphi_r$ are linearly independent over $k(x^q, y^q)$, since $\overline{V}$ is spanned by images of $x^{q-1}y, \ldots, xy^{q-1}$. Hence, $\varphi_1, \ldots, \varphi_r$ give an injective $\mathcal{O}_X$-module homomorphism $\varphi\colon L^{\oplus r} \hookrightarrow F_*^e\mathcal{O}_X$. Next, choose $\psi_1, \ldots, \psi_r \in W$ that reduce to the dual basis $\overline{\varphi}_1^*, \ldots, \overline{\varphi}_r^*$ of $\overline{W} = \overline{V}^*$ and let $\psi\colon F_*^e\mathcal{O}_X \to L^{\oplus r}$ be the $\mathcal{O}_X$-module homomorphism given by them. Then one has $\langle \varphi_i, \psi_j\rangle = \langle \overline{\varphi}_i, \overline{\varphi}_j^*\rangle = \delta_{ij}$ for $1 \le i,j \le r$, so that $\psi \circ \varphi$ is the identity on $L^{\oplus r}$. Thus, $r \le a_e$.

To complete the proof of the claim, suppose to the contrary that $r < a := a_e$. Choose $\varphi_1, \ldots, \varphi_a \in V$ and $\psi_1, \ldots, \psi_a \in W$ so that the composition of the corresponding maps $\varphi\colon L^{\oplus a} \hookrightarrow F_*^e\mathcal{O}_X$ and $\psi\colon F_*^e\mathcal{O}_X \to L^{\oplus a}$ is the identity on $L^{\oplus a}$. Then one has that $\langle \overline{\varphi}_i, \overline{\psi}_j\rangle = \langle \varphi_i, \psi_j\rangle = \delta_{ij}$ for $1 \le i,j \le a$, that is, the $a \times a$-matrix with (i,j)-entries $\langle \overline{\varphi}_i, \overline{\psi}_j\rangle$ is the identity matrix. But this is impossible since $\overline{\varphi}_1, \ldots, \overline{\varphi}_a$ are linearly dependent. ☐

Thus, the theorem follows from the following lemma. ☐

Lemma 5.2. $\operatorname{rank} A^{(e)} = p^e - 2$.

Proof (Induction on e). Let $q = p^e$ as before. For $(i,j) \in I$, we denote the (i,j)th row vector of $A^{(e)}$ by

$$v_{i,j}^{(e)} = \left[\binom{i}{0}\binom{j}{q-2}, \binom{i}{1}\binom{j}{q-3}, \ldots, \binom{i}{l-1}\binom{j}{q-1-l}, \ldots, \binom{i}{q-2}\binom{j}{0} \right],$$

where we follow the convention on the binomial coefficients that $\binom{m}{n} = 0$ if $m < n$. We divide $A^{(e)}$ into $q-2$ row-blocks according to the value of $i+j = q-1, q, \ldots, 2q-4$. Then we have the following relation between rows in $(i+j)$- and $(i+j+1)$-blocks.

$$(i+1)v_{i,j}^{(e)} + jv_{i+1,j-1}^{(e)} = (i+j-q+3)v_{i+1,j}^{(e)} = (i+j+3)v_{i+1,j}^{(e)}.$$

Indeed, it is deduced from the following easy to check equality in $\mathbb{Z}$.

$$(i+1)\binom{i}{l-1}\binom{j}{q-1-l}+j\binom{i+1}{l-1}\binom{j-1}{q-1-l}$$
$$=(i+j-q+3)\binom{i+1}{l-1}\binom{j}{q-1-l}.$$

Since the coefficient of the right-hand side satisfies $2 \le i+j-q+3 \le q-1$, it is not zero in $\mathbb{F}_p$ if $q=p$ is prime. It follows that all the row vectors of $A^{(1)}$ are linear combinations (over $\mathbb{F}_p$) of $p-2$ linearly independent vectors

$$v^{(1)}_{1,p-2}=[1,p-2,0,\dots,0],$$
$$v^{(1)}_{2,p-3}=[0,2,p-3,0,\dots,0], \qquad \dots,$$
$$v^{(1)}_{p-2,1}=[0,\dots 0,p-2,1].$$

Thus, the claim holds true if $e=1$.

Now we consider the following subspaces of the k-vector space of all the row vectors of length p^e-1 with entries in k.

$V^{(e)}$: the subspace spanned by the row vectors of $A^{(e)}$.

$W^{(e)}$: the subspace consisting of the row vectors whose lpth entry is zero for all $l=1,\dots,p^{e-1}-1$.

(The notation $V^{(e)}$, $W^{(e)}$ here is independent of the notation V, W in the proof of Theorem 5.1.) Clearly, $\dim W^{(e)}=p^e-p^{e-1}$, and $\dim V^{(e-1)}=p^{e-1}-2$ by induction. Therefore, the lemma follows immediately from the following

Claim 5.2. Let $q=p^e$ with $e\ge 2$. Then

(1) $W^{(e)}=\langle v^{(e)}_{1,q-2}, v^{(e)}_{2,q-3},\dots,v^{(e)}_{q-2,1}\rangle \subset V^{(e)}$.

(2) $V^{(e)}/W^{(e)}\cong V^{(e-1)}$.

Assertion (1) of the claim follows from

$$v^{(e)}_{1,q-2}=[1,q-2,0,\dots,0],$$
$$v^{(e)}_{2,q-3}=[0,2,q-3,0,\dots,0], \qquad \dots,$$
$$v^{(e)}_{q-2,1}=[0,\dots 0,q-2,1].$$

To prove (2), we calculate the $(n,m;lp)$-entry

$$\binom{n}{lp-1}\binom{m}{p^e-1-lp} \bmod p$$

of $A^{(e)}$ for $l=1,\dots,p^{e-1}-1$. Looking at the rational expression of the binomial coefficient

$$\binom{n}{lp-1}=\frac{n(n-1)\cdots(n-lp+2)}{(lp-1)(lp-2)\cdots 2\cdot 1}$$

and comparing the numerator and the denominator with respect to the multiplicities of the prime divisor p and the factors prime to p, we see that

$$\binom{n}{lp-1} \equiv \begin{cases} \binom{i}{l-1} & \text{if } n = ip+p-1 \equiv -1 \\ 0 & \text{otherwise} \end{cases} \pmod p.$$[3]

Similarly,

$$\binom{m}{p^e-1-lp} \equiv \begin{cases} \binom{j}{p^{e-1}-1-l} & \text{if } m = jp+p-1 \equiv -1 \\ 0 & \text{otherwise} \end{cases} \pmod p.$$

Hence, the $(ip+p-1, jp+p-1; lp)$-entry of $A^{(e)}$ is equal to the $(i,j;l)$-entry of $A^{(e-1)}$ and all the other entries in the lpth column are zero. Thus, assertion (2) is proved. □

REMARK 5.3. The direct summand $L^{\oplus p^{e-1}-2}$ of $F^e_*\mathcal{O}_X$ corresponding to $V^{(e)}/W^{(e)} \cong V^{(e-1)}$ comes from the composition of the splitting injective maps $L^{\oplus p^{e-1}-2} \hookrightarrow F^{e-1}_*\mathcal{O}_X$ and $F^{e-1}_*\mathcal{O}_X \hookrightarrow F^e_*\mathcal{O}_X$.

COROLLARY 5.4. *Let the notation be as in Theorem 4.1. Then for any power $q = p^e$ of the characteristic p with $e \geq 1$ one has*

$$\begin{aligned} F^e_*\mathcal{O}_X &\cong \mathcal{O}_X \oplus \bigoplus_{i=0}^{4} L_i^{\oplus(q-2)} \oplus \mathcal{B} \oplus \mathcal{G}^{\oplus\frac{(q-2)(q-3)}{2}}, \\ F^e_*L_i &\cong L_i^{\oplus q} \oplus \mathcal{G}^{\oplus\frac{q(q-1)}{2}} \quad \textit{for } 0 \leq i \leq 4, \\ F^e_*\mathcal{B} &\cong \bigoplus_{i=0}^{4} L_i^{\oplus q} \oplus \mathcal{G}^{\oplus\frac{q(3q-5)}{2}} \quad \textit{and} \quad F^e_*\mathcal{G} \cong \mathcal{G}^{\oplus q^2}. \end{aligned}$$

Proof. The formula for $F^e_*\mathcal{O}_X$ is an immediate consequence of Theorems 4.1 and 5.1.

In order to show the formula for $F^e_*L_i$ we may assume that $i = 0$ by symmetry; cf. Remark 3.2. Since $\mathcal{O}_X$ and $\mathcal{B}$ are not direct summands of $F^e_*L_0$ by Theorem 5.1, we can write $F^e_*L_0 \cong L_0^{\oplus a} \oplus \bigoplus_{i=1}^4 L_i^{\oplus b} \oplus \mathcal{G}^{\oplus c}$ again by symmetry. Pushing it out by the blow-down $\sigma\colon X \to Y$ of E_4, one sees that

$$\begin{aligned} &F^e_*\mathcal{O}_Y\big(E' - 2\tau^*H\big) \\ &\quad \cong \mathcal{O}_Y\big(-\tau^*H\big)^{\oplus b+c} \oplus \bigoplus_{i=1}^{3} \mathcal{O}_Y\big(E_i - \tau^*H\big)^{\oplus b} \oplus \mathcal{O}_Y\big(E' - 2\tau^*H\big)^{\oplus a+c}. \end{aligned}$$

[3] The referee pointed out that this and the next congruences of binomial coefficients follow from Lucas' theorem; see, e.g., [F].

Comparing this decomposition with that in Lemma 1.3, we obtain $a=q$, $b=0$ and $c=q(q-1)/2$. The formulas for $F^e_*\mathcal{B}$ and $F^e_*\mathcal{G}$ follow similarly from Theorems 4.1, 5.1 and Lemma 1.3. □

6. Concluding remarks and questions

We are now at the starting point of the following.

QUESTION 6.1. Let X be the blowup of $\mathbb{P}^2$ at n points in general position. Does there exist an upper bound of n for which X is GFFRT? If it does, what is the effective bound?

Here is a positive evidence for the existence of an upper bound.

PROPOSITION 6.2. *Let C be an elliptic curve on $\mathbb{P}^2$ and let $P_1,\dots,P_{10}$ be distinct ten points on C. Assume that C and $P_1,\dots,P_{10}$ are defined over the algebraic closure $\overline{\mathbb{F}_p}$ of the prime field. Let X be the rational surface obtained by blowing up $\mathbb{P}^2$ at the points $P_1,\dots,P_{10}$. Then X is not GFFRT.*

Proof. Let $\pi\colon X\to\mathbb{P}^2$ be the blowup morphism and let $\widetilde{C}$ be the strict transform of C with respect to π. Then X and $\widetilde{C}$ are defined over $\overline{\mathbb{F}_p}$ and $\widetilde{C}^2=-1<0$. We choose a very ample divisor A on X (also defined over $\overline{\mathbb{F}_p}$) and let $L=A+(A\cdot\widetilde{C})\widetilde{C}$. Then L is a nef and big line bundle such that $L\widetilde{C}=0$ and $LD>0$ for any reduced curve $D\neq\widetilde{C}$. Since everything involving is defined over $\overline{\mathbb{F}_p}$, it follows from [K, Corollary 0.3] that L is semi-ample. Thus, we can choose a positive integer m such that mL gives a morphism $\Phi_{|mL|}\colon X\to\mathbb{P}^{h^0(mL)-1}$ birational onto its image $X'=\Phi_{|mL|}(X)$. Denoting the normalization of X' by Y, we obtain a birational morphism $\varphi\colon X\to Y$ via which $\widetilde{C}$ contracts to a simple elliptic singularity (Y,y).

In what follows, we will employ F-blowup introduced by Yasuda [Y] in order to avoid an argument involving completion of the local ring $\mathcal{O}_{Y,y}$; cf. [Y, Proposition 2.11]. It is easy to observe the following.

(1) If X is GFFRT, then the singularity (Y,y) is FFRT as defined in [SVdB], that is, there exists a finite set of $\mathcal{O}_{Y,y}$-modules such that for every $e\geq 0$, $F^e_*(\mathcal{O}_Y)_y\cong\mathcal{O}_{Y,y}^{1/p^e}$ is isomorphic to a direct sum of copies of those finitely many modules.
(2) If (Y,y) is FFRT, then the F-blowup sequence $\{\mathrm{FB}_e(Y)|e=1,2,\dots\}$ of (Y,y) stabilizes.

However, it follows from [HSY] that the F-blowup sequence of the simple elliptic singularity (Y,y) of type $\widetilde{E}_8$ does not stabilize in general. Thus, X is not GFFRT. □

REMARK 6.3. In the proposition above, if C is an ordinary elliptic curve, then X is F-split. On the other hand, there is a notion of global F-regularity

[Sm], which is stronger than F-splitting, and we do not have an example of a globally F-regular surface that is not GFFRT.

6.1. Interpretation via derived category. We interpret the structure of the Frobenius push-forward $F_*^e\mathcal{O}_X$ from a viewpoint of derived category. For a smooth projective variety X, we denote by $D^b(X)$ the bounded derived category of coherent sheaves on X. A coherent sheaf $\mathcal{F}$ on X is said to be *exceptional* if $\mathrm{Hom}(\mathcal{F},\mathcal{F}) = k$ and $\mathrm{Ext}^i(\mathcal{F},\mathcal{F}) = 0$ for all $i \neq 0$. An ordered collection of exceptional sheaves $\mathcal{F}_1,\dots,\mathcal{F}_n$ is called an *exceptional collection* if $\mathrm{Ext}^i(\mathcal{F}_k,\mathcal{F}_j) = 0$ for all integers i and for $1 \le j < k \le n$. An exceptional collection $\mathcal{F}_1,\dots,\mathcal{F}_n$ is said to be *strong* if $\mathrm{Ext}^i(\mathcal{F}_j,\mathcal{F}_k) = 0$ for $i > 0$ and $1 \le j,k \le n$. An exceptional collection is said to be *full* if it generates the derived category $D^b(X)$.

It turns out that if $p^e \ge 3$, then the eth Frobenius push-forward $F_*^e\mathcal{O}_X$ on the four-point blowup X of $\mathbb{P}^2$ generates the derived category $D^b(X)$ in the same sense as in Ohkawa–Uehara [OU], where results in the toric case are established.[4]

Proposition 6.4. *Let X be the blowup of $\mathbb{P}^2$ at four points in general position and let the notation be as in Section 5.1. Then Frobenius summands $\mathcal{G}, L_0, L_1, L_2, L_3, L_4, \mathcal{O}_X$ on X form a full strong exceptional collection.*

Proof. First, we will show that $\mathcal{G}, L_0, L_1, L_2, L_3, L_4, \mathcal{O}_X$ is an exceptional collection. We know that any line bundle on X is exceptional, $\mathrm{Ext}^i(\mathcal{O}_X,\mathcal{F}) = 0$ for any $i \in \mathbb{Z}$ and any indecomposable Frobenius summand $\mathcal{F} \not\cong \mathcal{O}_X$, and that $\mathrm{Ext}^i(L_j, L_k) = 0$ for any i and $0 \le j,k \le 4$ with $j \ne k$; cf. Proposition 3.4. Thus, it remains to show that $\mathcal{G}$ is exceptional and $\mathrm{Ext}^i(L_j,\mathcal{G}) = 0$ for all i and $0 \le j \le 4$. To this end, we look at an Ext long exact sequence induced by the non-trivial extension (3.2):

$$\begin{aligned} 0 &\to \mathrm{Hom}(L_0, \mathcal{O}_X(-\pi^*H)) \to \mathrm{Hom}(L_0,\mathcal{G}) \to \mathrm{Hom}(L_0,L_0) \\ &\xrightarrow{\delta} \mathrm{Ext}^1(L_0,\mathcal{O}_X(-\pi^*H)) \to \mathrm{Ext}^1(L_0,\mathcal{G}) \to \cdots . \end{aligned}$$

Since $\mathrm{Hom}(L_0,\mathcal{O}_X(-\pi^*H)) = \mathrm{Ext}^2(L_0,\mathcal{O}_X(-\pi^*H)) = \mathrm{Ext}^i(L_0,L_0) = 0$ for $i \ge 1$ and δ is an isomorphism of 1-dimensional vector spaces, we see that $\mathrm{Ext}^i(L_0,\mathcal{G}) = 0$ for all $i \in \mathbb{Z}$. This vanishing, together with a suitable Cremona automorphism, implies that $\mathrm{Ext}^i(L_j,\mathcal{G}) = 0$ holds for all i and $0 \le j \le 4$; cf. Example 3.1. Next, we note that $\mathrm{Ext}^i(\mathcal{O}_X(-\pi^*H), L_0) = H^i(X, \mathcal{O}_X(E - \pi^*H)) = 0$ for all i by Riemann–Roch and look at the long exact sequence of $\mathrm{Ext}(\mathcal{O}_X(-\pi^*H), -)$ applied to (3.2). It then follows that $\mathrm{Hom}(\mathcal{O}_X(-\pi^*H),\mathcal{G}) \cong k$ and $\mathrm{Ext}^i(\mathcal{O}_X(-\pi^*H),\mathcal{G}) = 0$ for $i \ge 1$. Now looking

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[4] Actually, the result on the generation of the derived category by the Frobenius push-forward is extended to the case of two-dimensional toric stacks as the title of the paper [OU] indicates.

at the long exact sequence of $\mathrm{Ext}(-,\mathcal{G})$ applied to (3.2) we conclude that $\mathrm{Hom}(\mathcal{G},\mathcal{G}) = k$ and $\mathrm{Ext}^i(\mathcal{G},\mathcal{G}) = 0$ for $i \geq 1$, i.e., $\mathcal{G}$ is exceptional.

We can verify similarly as above that $\mathrm{Ext}^i(L,\mathcal{O}_X) = \mathrm{Ext}^i(\mathcal{G},L) = 0$ for all $i > 0$ and $L = \mathcal{O}_X$ or L_j with $0 \leq j \leq 4$. Thus, the exceptional sequence is strong.

To prove that the exceptional collection is full, let $\mathcal{D} = \langle \mathcal{G}, L_i, \mathcal{O}_X | 0 \leq i \leq 4\rangle$ be the subcategory of $D^b(X)$ generated by $\mathcal{G}$, $L_0 = \mathcal{O}_X(E - 2\pi^*H)$, $L_i = \mathcal{O}_X(E_i - \pi^*H)$ $(i = 1,2,3,4)$ and $\mathcal{O}_X$. Since $\mathcal{G}$ and L_0 are objects of $\mathcal{D}$, it follows from (3.2) that $\mathcal{O}_X(-\pi^*H)$ is an object of $\mathcal{D}$. It then follows from the exact sequence

$$0 \to \mathcal{O}_X\big(-\pi^*H\big) \to \mathcal{O}_X\big(E_i - \pi^*H\big) \to \mathcal{O}_{E_i}(-1) \to 0$$

that $\mathcal{O}_{E_i}(-1)$ is an object of $\mathcal{D}$ for $i = 1,2,3,4$. Finally, it follows from the exact sequence

$$0 \to \mathcal{O}_X\big(-2\pi^*H\big) \to \mathcal{O}_X\big(E - 2\pi^*H\big) \to \bigoplus_{i=1}^{4} \mathcal{O}_{E_i}(-1) \to 0$$

that $\mathcal{O}_X(-2\pi^*H)$ is an object of $\mathcal{D}$. Since $D^b(\mathbb{P}^2) = \langle \mathcal{O}_{\mathbb{P}^2}(-2), \mathcal{O}_{\mathbb{P}^2}(-1), \mathcal{O}_{\mathbb{P}^2}\rangle$ and $D^b(X) = \langle \pi^*D^b(\mathbb{P}^2), \mathcal{O}_{E_i}(-1) | 1 \leq i \leq 4\rangle$ by [Hu, Proposition 11.18], we conclude that $\mathcal{D} = D^b(X)$, that is, the exceptional collection is full. □

Finally, we ask the following optimistic questions.

QUESTION 6.5. Let X be a smooth globally F-regular surface.

(1) Is X GFFRT?
(2) Is $D^b(X)$ generated by Frobenius summands on X?

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