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GEOMETRY OF GRUSHIN SPACES

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Abstract. We compare the Grushin geometry to Euclidean geometry, through quasisymmetric parametrization, bilipschitz parametrization and bilipschitz embedding, highlighting the role of the exponents and the fractal nature of the singular hyperplanes in Grushin geometry.

1. Introduction

Consider in $\mathbb{R}^n$ a system of diagonal vector fields

$$X_j = \lambda_j(x)\frac{\partial}{\partial x_j}, \quad j=1,\dots,n,$$

where

$$\lambda_1(x)=1 \quad \text{and} \quad \lambda_j(x)=\prod_{i=1}^{j-1}|x_i|^{\alpha_i}, \quad j=2,\dots,n$$

and $\alpha_i \in [0,\infty)$. These vector fields induce a metric on $\mathbb{R}^n$

$$d_{\mathbb{G}}(p,q) = \inf\Big\{T\ge 0\colon \exists\gamma\colon [0,T]\to\mathbb{R}^n, \text{ with } \gamma(0)=p, \gamma(T)=q, \text{ and }$$
$$\gamma'(t)=\sum_{j=1}^n b_j(t)X_j\big(\gamma(t)\big) \text{ with } \sum b_j(t)^2\le 1 \text{ a.e.}\Big\},$$

the infimum of the time to travel from p to q along absolutely continuous curves in $\mathbb{R}^n$ at unit speed. Since exponents $\alpha=(\alpha_1,\dots,\alpha_{n-1})$ are allowed to be real, Hörmander's rank condition may not be satisfied. Nevertheless, $d_{\mathbb{G}}$ is

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well-defined and equals the Carnot–Carathéodory distance

$$\inf_{\gamma} \int_0^1 \sqrt{\sum_{j=1}^{n} \frac{x_j'(t)^2}{\lambda_j(t)^2}}\, dt,$$

where the infimum is taken over all absolutely continuous curves $\gamma\colon [0,1] \to \mathbb{R}^n$ connecting $\gamma(0) = p$ to $\gamma(1) = q$.

The resulting space $\mathbb{G}_\alpha^n = (\mathbb{R}^n, d_{\mathbb{G}})$, which is Riemannian outside the hyperplanes $\prod_1^{n-1} x_j = 0$, is called a Grushin space, after the work of Grushin on the hypoelliptic operators [9]. For geodesics in Grushin spaces and metric properties, see Bellaïche [4] and Franchi and Lanconelli [7].

Grushin spaces are usually regarded as the simplest sub-Riemannian manifolds. Potential theoretic and PDE aspects of Grushin operators have been extensively investigated. Operators $\sum_1^n \lambda_j^2(x) \frac{\partial^2}{\partial x_j^2}$ associated to the vector fields studied in this note have been the focus in [6], [12], [13], [17], and [18]. Recent solutions of isoperimetric problems on Grushin planes by Monti and Morbidelli [16] and by Arcozzi and Baldi [3], and the isoperimetric profiles that they provide have generated new interest in Grushin geometry.

In this note, we compare the Grushin geometry to Euclidean geometry, highlighting the role of the exponents $\alpha = (\alpha_1, \ldots, \alpha_{n-1})$ and the fractal nature of the singular hyperplanes in Grushin geometry. We ask when a Grushin space $\mathbb{G}_\alpha^n$ is quasisymmetric to $\mathbb{R}^n$, when it is bilipschitzly homeomorphic to $\mathbb{R}^n$, and when $\mathbb{G}_\alpha^n$ can be bilipschitzly embedded in $\mathbb{R}^{n+1}$ in such a way that the embedded image has controlled quasiconformal geometry in $\mathbb{R}^{n+1}$.

Our answers to the parametrizability and the embeddability of $\mathbb{G}_\alpha^n$ in terms of the exponents α change abruptly from one range of α to another; they are based on a variety of theorems on quasiconformal mappings in Euclidean spaces, in particular, on snowflake embeddings by David and Toro [5] and quasisymmetric extensions by Tukia and Väisälä [24].

In the embedding question, we emphasize the embeddability into a Euclidean space of the lowest possible dimension. It is known that $\mathbb{G}_\alpha^n$ can always be embedded bilipschitzly into some Euclidean space (Seo [22]). Semmes [21] has observed that, on the contrary, the first Heisenberg group $\mathbb{H}$ when equipped with its Carnot metric does not admit bilipschitz embedding into Euclidean spaces, based on Pansu's Rademacher Differentiation Theorem [19].

In our embedding problem, the image is equipped with the ambient Euclidean metric. On the other hand, a theorem of Le Donne, extending a classical work of Nash, asserts that every sub-Riemannian manifold of topological dimension n can be embedded in $\mathbb{R}^{2n+1}$ path-isometrically [14]. Therefore, both the Grushin spaces $\mathbb{G}_\alpha^n$ and the Heisenberg groups can be so embedded.

We summarize our results in Section 2, and study the Grushin metrics in Section 3. We show that the Grushin spaces $\mathbb{G}_\alpha^n$ are not quasisymmetric to $\mathbb{R}^n$ for a certain range of exponents α in Section 4, and show however that $\mathbb{G}_\alpha^n$

can be quasisymmetrically parametrized by $\mathbb{R}^n$ for some other choices of α in Section 5. The questions on bilipschitz parametrization of $\mathbb{G}^n_\alpha$ by $\mathbb{R}^n$ and on bilipschitz embedding of $\mathbb{G}^n_\alpha$ into $\mathbb{R}^{n+1}$ are reduced to questions of snowflake embedding in Euclidean spaces in Section 6. Section 7 furnishes the theorems on Euclidean embedding needed in Section 6.

In what follows, constants will depend at most on n and α; we write $a \simeq b$ when a/b is bounded above and below by finite positive constants that depend at most on n and α, unless otherwise mentioned.

2. Summary

An embedding $f\colon X \to Y$ between two metric spaces is said to be η-*quasisymmetric* if there is a homeomorphism $\eta\colon [0,\infty) \to [0,\infty)$ so that

$$\big|f(a) - f(x)\big| \le \eta(t)\big|f(b) - f(x)\big|$$

whenever $a, b, x \in X$ with $|a - x| \le t|b - x|$, and is said to be L-*bilipschitz* if there exists $L \ge 1$ so that

$$L^{-1}|a - b| \le \big|f(a) - f(b)\big| \le L|a - b|$$

for all $a, b \in X$. Bilipschitz embeddings are quasisymmetric.

2.1. Dimension 2. In dimension 2, with a slight abuse of notation, we write $\mathbb{G} = \mathbb{G}^2_\alpha$ for the Grushin plane with exponent $\alpha \ge 0$, and observe that when $\alpha = 0$, $\mathbb{G}^2_0 = \mathbb{R}^2$.

The Grushin plane $\mathbb{G}^2_\alpha$ is quasisymmetric to $\mathbb{R}^2$ for every $\alpha > 0$ (Meyerson [15]).

The Grushin metric $d_{\mathbb{G}}$ on $\mathbb{G}^2_\alpha$ is Riemannian outside the singular line $\{x_1 = 0\}$ and it is the $\frac{1}{1+\alpha}$-snowflake of the Euclidean metric on $\{x_1 = 0\}$. So the Hausdorff dimension of the singular line is $1 + \alpha$.

In fact for each $\alpha \in (0, 1)$, there exists a bilipschitz homeomorphism from $\mathbb{G}^2_\alpha$ onto $\mathbb{R}^2$ that maps the singular line $\{x_1 = 0\}$ onto a von Koch-type snowflake curve of Hausdorff dimension $1+\alpha$; see Corollary 6.3 below. Therefore, all Grushin planes $\mathbb{G}^2_\alpha, 0 < \alpha < 1$, are bilipschitz to $\mathbb{R}^2$.

When $\alpha = 1$, the Grushin plane $\mathbb{G}^2_1$ is not bilipschitz to $\mathbb{R}^2$. However $\mathbb{G}^2_1$ can be bilipschitzly embedded in $\mathbb{R}^3$, and in such a way that the embedded image is a quasiplane in $\mathbb{R}^3$. The proof in [27] demonstrates an explicit embedding. Recall that a quasiplane is the image of $\mathbb{R}^2 \times \{0\}$ under a quasiconformal homeomorphism of $\mathbb{R}^3$.

Suppose α is a positive integer. A glance at the metric in [4] confirms that in the Grushin space $\mathbb{G}^2_\alpha$, every Grushin ball $B_{\mathbb{G}}((0, b), r)$ centered on the singular line $\{x_1 = 0\}$ meets $\{x_1 = 0\}$ on a segment of $(1+\alpha)$-dimensional (Hausdorff) Grushin measure $\simeq r^{1+\alpha}$. Furthermore, the ball $B_{\mathbb{G}}((0, b), r)$ contains a smaller Grushin ball of radius at least $\delta(\alpha)r$ outside the singular line, for some constant $0 < \delta(\alpha) < 1$. These facts, combined with a point of density argument, prevent any bilipschitz embedding of $\mathbb{G}^2_\alpha$ into $\mathbb{R}^\alpha$.

By generalizing the construction in [27], Romney and Vellis [20] recently showed that $\mathbb{G}^2_\alpha$ can be bilipschitzly embedded into $\mathbb{R}^{2+\lfloor\alpha\rfloor}$ for each $\alpha>0$, where $\lfloor\alpha\rfloor$ is the integer part of α.

2.2. Dimension 3. We now give an account of what can be proved in dimension 3; most of the statements below have higher dimensional counterparts.

Consider in $\mathbb{R}^3$ the vector fields

$$X_1=\frac{\partial}{\partial x_1},\qquad X_2=|x_1|^{\alpha_1}\frac{\partial}{\partial x_2},\qquad X_3=|x_1|^{\alpha_1}|x_2|^{\alpha_2}\frac{\partial}{\partial x_3}$$

with exponents $\alpha_1,\alpha_2\ge 0$, and let $\mathbb{G}=\mathbb{G}^3_\alpha$ be the Grushin space associated to $\alpha=(\alpha_1,\alpha_2)$. We assume the metric estimate (3.2) in the following discussion.

I. Suppose that $\alpha_1=0$. Then by (3.2), the Grushin metric

$$d_{\mathbb{G}}(x,y)\simeq|x_1-y_1|+|x_2-y_2|+\min\left\{|x_3-y_3|^{\frac{1}{1+\alpha_2}},\frac{|x_3-y_3|}{|x_2|^{\alpha_2}}\right\} \tag{2.1}$$

for all $x,y\in\mathbb{R}^3$. So as metric spaces, $\mathbb{G}^3_\alpha$ and $\mathbb{R}\times\mathbb{G}^2_{\alpha_2}$ are bilipschitzly equivalent.

When $0\le\alpha_2<1$, $\mathbb{G}^2_{\alpha_2}$ is bilipschitz to $\mathbb{R}^2$. Hence $\mathbb{G}^3_\alpha$, a product, is bilipschitzly homeomorphic to $\mathbb{R}^3$.

An interesting dichotomy occurs here.

When $\alpha_2\ge 1$, $\mathbb{G}^3_\alpha$ is not even quasisymmetric to $\mathbb{R}^3$.

In fact there exist patches on $\mathbb{G}^3_\alpha$ which resemble the product of an interval with a wrinkled surface. A method of Väisälä in [26] may be adapted to show that when $\alpha_2\ge 1$, the wrinkles are too severe to allow these patches to be quasisymmetrically embedded in $\mathbb{R}^3$; see Theorem 4.1 for details.

II. Suppose that $\alpha_2=0$. In this case, the Grushin metric

$$\begin{aligned}d_{\mathbb{G}}(x,y)\simeq|x_1-y_1|&+\min\left\{|x_2-y_2|^{\frac{1}{1+\alpha_1}},\frac{|x_2-y_2|}{|x_1|^{\alpha_1}}\right\}\\&+\min\left\{|x_3-y_3|^{\frac{1}{1+\alpha_1}},\frac{|x_3-y_3|}{|x_1|^{\alpha_1}}\right\},\end{aligned}$$

and the singular plane $\{x_1=0\}$ in $\mathbb{G}^3_\alpha$ is a snowflake surface which has Hausdorff dimension $2(1+\alpha_1)$.

When $\alpha_1\ge 0$, the Grushin space $\mathbb{G}^3_\alpha$ is quasisymmetric to $\mathbb{R}^3$; see Theorem 5.1.

When $0\le\alpha_1<1$, the Grushin space $\mathbb{G}^3_\alpha$ is bilipschitz homeomorphic to a codimension one quasiplane $\mathcal{P}_\alpha$ in $\mathbb{R}^4$; see Theorem 6.4 for a proof. Recall that a codimension one quasiplane in $\mathbb{R}^k$ is the image of $\mathbb{R}^{k-1}$ under a quasiconformal homeomorphism of $\mathbb{R}^n$.

Furthermore, when $0\le\alpha_1<c_0$, all Grushin spaces $\mathbb{G}^3_\alpha$ are bilipschitz to $\mathbb{R}^3$, where c_0 is a constant in $(0,1)$; see Theorem 6.2.

Under the assumption $\alpha_2=0$, the singular hyperplane $\{x_1=0\}$ is the product of two snowflakes of the Euclidean line, of equal exponent. Therefore questions about parametrizing and embedding Grushin spaces may be reduced to the existence of quasisymmetric extensions and snowflake embeddings in Euclidean spaces; results of this kind from [5], [24], [25], and [11] in Euclidean spaces may then be applied.

III. Suppose that $\alpha_1 > 0$ and $\alpha_2 \geq 1$. The space $\mathbb{G}^3_\alpha$ is not quasisymmetric to $\mathbb{R}^3$; again see Theorem 4.1

IV. In the case $\alpha_1 > 0$ and $0 < \alpha_2 < 1$, whether $\mathbb{G}^3_\alpha$ and $\mathbb{R}^3$ are quasisymmetrically equivalent remains unknown.

3. Grushin metrics

A comparison between Grushin balls and Euclidean boxes has been formulated by Franchi and Lanconelli in [7]. We recall this connection following the exposition in [7] (also [8] and [17]). Let $F_j\colon \mathbb{R}^n \times [0,\infty) \to [0,\infty)$ be functions

$$\begin{aligned} &F_1(x,r) = r, \\ &F_2(x,r) = r\lambda_2\big(|x_1| + F_1(x,r)\big), \qquad \dots \\ &F_j(x,r) = r\lambda_j\big(|x_1| + r, |x_2| + F_2(x,r), \dots, |x_{j-1}| + F_{j-1}(x,r)\big), \end{aligned}$$

that satisfy a recurrence relation

$$F_{j+1}(x,r) = F_j(x,r)\big(|x_j| + F_j(x,r)\big)^{\alpha_j}, \quad j = 1,\dots,n-1. \tag{3.1}$$

Note that λ_j depends only on $x_1,\dots,x_{j-1}$. Write

$$\varphi_j(x,\cdot) = F_j(x,\cdot)^{-1}$$

for their inverses; also write $B_{\mathbb{G}}(x,r)$ for balls in $\mathbb{G}^n_\alpha$, and

$$\mathrm{Box}(x,r) = \big\{x + h\colon |h_j| < F_j(x,r), j = 1,\dots,n\big\}$$

for Euclidean boxes.

Theorem 3.1 (Franchi and Lanconelli [7]). *There exists a constant $C > 1$ depending only on n and α such that*

$$\mathrm{Box}\big(x, C^{-1}r\big) \subset B_{\mathbb{G}}(x,r) \subset \mathrm{Box}(x,Cr) \quad \textit{for } x \in \mathbb{R}^n \textit{ and } r > 0,$$

and that

$$C^{-1} d_{\mathbb{G}}(x,y) \leq \sum_{j=1}^n \varphi_j\big(x, |x_j - y_j|\big) \leq C d_{\mathbb{G}}(x,y) \quad \textit{for } x,y \in \mathbb{R}^n.$$

The proof of Theorem 3.1 in [7] further suggests a short route from x to y:

$$\begin{aligned} d_{\mathbb{G}}(x,y) \simeq{}& d_{\mathbb{G}}\big(x, x + (y_1 - x_1)e_1\big) \\ &+ d_{\mathbb{G}}\big(x + (y_1 - x_1)e_1, x + (y_1 - x_1)e_1 + (y_2 - x_2)e_2\big) + \cdots \\ &+ d_{\mathbb{G}}\big(x + (y_1 - x_1)e_1 + \cdots + (y_{n-1} - x_{n-1})e_{n-1}, y\big). \end{aligned}$$

We deduce an explicit estimate of the Grushin metric from the simple fact:

$$\min\left\{t^{\frac{1}{1+a}}, t\xi^{-a}\right\} = \begin{cases} t^{\frac{1}{1+a}}, & \text{if } t \geq \xi^{1+a}, \\ t\xi^{-a}, & \text{if } t \leq \xi^{1+a}, \end{cases}$$

for all $t, \xi, a > 0$.

LEMMA 3.2. *Fix exponents $\alpha = (\alpha_1, \ldots, \alpha_{n-1})$. The inverses $\varphi_j(x, \cdot)$ of $F_j(x, \cdot), 1 \leq j \leq n$, satisfy a recurrence relation, $\varphi_1(x,t) = t$ and*

$$\varphi_{j+1}(x,t) = \varphi_j\big(x, g_j(x,t)\big) \quad \textit{for } 1 \leq j \leq n-1,$$

where

$$g_j(x,t) = \min\left\{t^{\frac{1}{1+\alpha_j}}, t|x_j|^{-\alpha_j}\right\}.$$

Consequently, the Grushin metric in $\mathbb{G} = \mathbb{G}^3_\alpha$ satisfies

$$\begin{aligned} d_{\mathbb{G}}(x,y) \simeq |x_1 - y_1| &+ \min\left\{|x_2 - y_2|^{\frac{1}{1+\alpha_1}}, \frac{|x_2 - y_2|}{|x_1|^{\alpha_1}}\right\} \\ &+ \min\left\{\left(\min\left\{|x_3 - y_3|^{\frac{1}{1+\alpha_2}}, \frac{|x_3 - y_3|}{|x_2|^{\alpha_2}}\right\}\right)^{\frac{1}{1+\alpha_1}}, \right. \\ &\left. \frac{\min\{|x_3 - y_3|^{\frac{1}{1+\alpha_2}}, \frac{|x_3-y_3|}{|x_2|^{\alpha_2}}\}}{|x_1|^{\alpha_1}}\right\}. \end{aligned} \tag{3.2}$$

Proof. Note first that

$$\begin{aligned} &F_1(x,r) = r, \qquad F_2(x,r) = r\big(|x_1| + r\big)^{\alpha_1}, \\ &F_3(x,r) = r\big(|x_1| + r\big)^{\alpha_1}\big(|x_2| + r\big(|x_1| + r\big)^{\alpha_1}\big)^{\alpha_2}, \qquad \ldots. \end{aligned}$$

Clearly, $\varphi_1(x,t) = t$.

We set $t = F_2(x,r)$, and consider the cases $r \geq |x_1|$ and $r \leq |x_1|$ separately. Straightforward calculation shows that

$$\varphi_2(x,t) \simeq \begin{cases} t^{\frac{1}{1+\alpha_1}}, & \text{if } t \geq |x_1|^{1+\alpha_1}, \\ t|x_1|^{-\alpha_1}, & \text{if } t \leq |x_1|^{1+\alpha_1}. \end{cases}$$

Hence,

$$\varphi_2(x,t) \simeq \min\left\{t^{\frac{1}{1+\alpha_1}}, t|x_1|^{-\alpha_1}\right\} = g_1(x,t).$$

Set $t = F_3(x,r)$ and $\tau = F_2(x,r)$. The recurrence relation (3.1) yields that $t = \tau(|x_2| + \tau)^{\alpha_2}$ and $\tau = r(|x_1| + r)^{\alpha_1}$. Hence,

$$\varphi_3(x,t) = g_1(x,\tau) = g_1\big(x, g_2(x,t)\big) = \varphi_2\big(x, g_2(x,t)\big).$$

The recurrence relation in the claim follows by induction.

The metric estimate 3.2 follows then from Theorem 3.1. □

REMARK 3.3. Write

$$\delta_\tau(p) = \big(p_1\tau, p_2\tau^{1+\alpha_1}, \dots, p_n\tau^{\prod_1^{n-1}(1+\alpha_j)}\big)$$

for $p \in \mathbb{R}^n$ and $\tau \geq 0$. The Grushin metric has the dilation property: for all $x, y \in \mathbb{R}^n$ and $\tau \geq 0$,

$$d_{\mathbb{G}}\big(\delta_\tau(x), \delta_\tau(y)\big) = \tau d_{\mathbb{G}}(x, y).$$

Moreover, the Grushin metric is translation invariant in x_n.

The x_j-coordinate axis in $\mathbb{G}^n_\alpha$ has Hausdorff dimension $\prod_0^{j-1}(1+\alpha_i)$, where $j = 1, \dots, n$, and $\alpha_0 = 0$.

4. Quasisymmetric non-parametrization

For certain choices of α, there exists a cube in $\mathbb{G}^3_\alpha = (\mathbb{R}^3, d_{\mathbb{G}})$ that resembles the product of an interval with a patch of wrinkled surface. The wrinkles on the patch are severe enough to prevent any quasisymmetric embedding of the cube into $\mathbb{R}^3$. To prove this, we apply a method of Väisälä in [26, Theorem 4.2], used there to show that the product of an interval and a non-rectifiable arc of finite diameter can never be quasisymmetrically embedded in $\mathbb{R}^2$. See also [2, Theorem 4.1].

THEOREM 4.1. *Suppose that $\alpha_1 \geq 0$ and $\alpha_2 \geq 1$. Then the Grushin space $\mathbb{G}^3_\alpha$ is not quasisymmetric to $\mathbb{R}^3$.*

Proof. Towards a contradiction, we assume that $f\colon \mathbb{G}^3_\alpha \to \mathbb{R}^3$ is a quasisymmetric homeomorphism.

Observe by (3.2) that for any $b \in \mathbb{R}$, the Grushin metric on the cube $Q = [1,2] \times [0,1] \times [b, b+1]$ can be estimated by

$$d_{\mathbb{G}}(x,y) \simeq |x_1 - y_1| + |x_2 - y_2| + \min\left\{|x_3 - y_3|^{\frac{1}{1+\alpha_2}}, \frac{|x_3 - y_3|}{|x_2|^{\alpha_2}}\right\}. \tag{4.1}$$

Hence, $(Q, d_{\mathbb{G}})$ is bilipschitz to the product of a Euclidean interval and a 2-cell in $\mathbb{G}^2_{\alpha_2}$.

Set $A = [1,2]$ and $D = [0,1] \times [b, b+1]$. Write

$$Q = A \times D,$$

and

$$\beta = \min\big\{\big|f(1,z) - f(2,z)\big| \colon z \in D\big\}.$$

Given an integer $m \geq 10$, we subdivide A into 2^m intervals $A_i = [a_{i-1}, a_i]$, $1 \leq i \leq 2^m$, of equal length; and partition D into Euclidean rectangles $\{D_j, 1 \leq j \leq N(m)\}$ as follows.

For every $0 \leq k \leq m-1$, subdivide the strip $[2^{-k-1}, 2^{-k}] \times [b, b+1]$ in D into $2^{m-k-1} \cdot \lfloor 2^{m+k\alpha_2} \rfloor$ rectangles of side lengths 2^{-m} and $\lfloor 2^{m+k\alpha_2}\rfloor^{-1}$. Here $\lfloor c \rfloor$ is the largest integer less than or equal to c. Subdivide also $[0, 2^{-m}] \times [b, b+1]$

into rectangles of side lengths 2^{-m} and $\lfloor 2^{m+m\alpha_2} \rfloor^{-1}$. Then, the total number $N(m)$ of rectangles in this subdivision has magnitude

$$N(m) \simeq \begin{cases} 2^{2m}, & \text{if } 0 \le \alpha_2 < 1, \\ 2^{2m} m, & \text{if } \alpha_2 = 1, \\ 2^{(1+\alpha_2)m}, & \text{if } \alpha_2 > 1. \end{cases}$$

In view of (4.1), each Euclidean box $Q_{ij} = A_i \times D_j$ contains a Grushin ball and is contained in another one, both of diameter comparable to 2^{-m} in the Grushin metric.

Fix for each j a point z_j in D_j having $\mathrm{dist}_{\mathbb{G}^2_{\alpha_2}}(z_j, \partial D_j) \simeq 2^{-m}$, and set

$$\beta_{ij} = \big| f(a_i, z_j) - f(a_{i-1}, z_j) \big|.$$

Then, by the quasisymmetry, $\beta_{ij}^3 \le c_1 \mathcal{H}^3(f(Q_{ij}))$ for some constant c_1 depending only on η. By Schwarz' inequality,

$$\beta^3 \le \left(\sum_{i=1}^{2^m} \beta_{ij} \right)^3 \le 2^{2m} \sum_{i=1}^{2^m} \beta_{ij}^3 \le c_1 2^{2m} \mathcal{H}^3 \left(f \left(\bigcup_{i=1}^{2^m} Q_{ij} \right) \right)$$

for each j. Summing over j, we get

$$\beta^3 \le c_1 N(m)^{-1} 2^{2m} \mathcal{H}^3 \big(f(Q) \big).$$

In the case $\alpha_2 \ge 1$, this leads to a contradiction as $m \to \infty$. $\square$

COROLLARY 4.2. *Suppose that* $\alpha_\ell \ge 1/(n-\ell)$ *for some* $\ell \in [2, n-1]$. *Then the Grushin space* $\mathbb{G}^n_\alpha$ *is not quasisymmetrically equivalent to* $\mathbb{R}^n$.

Proof. It follows from Lemma 3.2 that for any $b \in \mathbb{R}$, the Grushin metric on the cube $Q = [1,2]^{\ell-1} \times [0,1] \times [1,2]^{n-\ell-1} \times [b, b+1]$ can be estimated by

$$d_{\mathbb{G}}(x,y) \simeq \sum_{j=1}^{\ell-1} |x_j - y_j| + |x_\ell - y_\ell| + \sum_{j=\ell+1}^{n} \min \left\{ |x_j - y_j|^{\frac{1}{1+\alpha_\ell}}, \frac{|x_j - y_j|}{|x_\ell|^{\alpha_\ell}} \right\}.$$

The non-existence of quasisymmetric homeomorphism between $\mathbb{G}^n_\alpha$ and $\mathbb{R}^n$ can be established following the counting argument in the previous proof. $\square$

REMARK 4.3. Other conditions, in terms of $\alpha_1, \ldots, \alpha_{n-1}$, that prevent the quasisymmetric parametrization of $\mathbb{G}^n_\alpha$ by $\mathbb{R}^n$ may be established by considering the Grushin metric at other spots in $\mathbb{G}^n_\alpha$. Systematic studies have not been made.

5. Quasisymmetric parametrization

In this section, we show the existence of quasisymmetric parametrization for a particular class of $\mathbb{G}^n_\alpha$.

THEOREM 5.1. *When* $n \ge 2$, $\alpha_1 \ge 0$ *and* $\alpha_j = 0$ *for all* $j \in [2, n-1]$, *the Grushin space* $\mathbb{G}^n_\alpha$ *is quasisymmetric to* $\mathbb{R}^n$.

The theorem is due to Meyerson [15] in dimension 2.

Proof of Theorem 5.1. Assume as we may that $\alpha_1 > 0$; otherwise $\mathbb{G} = \mathbb{R}^n$.

We claim that the higher dimensional analogue of Meyerson's map

$$H(x) = \big(x_1|x_1|^{\alpha_1}, x_2, \dots, x_n\big) \tag{5.1}$$

maps $\mathbb{G}^n_\alpha$ quasisymmetrically onto $\mathbb{R}^n$.

Since $\alpha_j = 0$ for all $j \in [2, n-1]$, the Grushin metric

$$d_{\mathbb{G}}(x,y) \simeq |x_1 - y_1| + \sum_{j=2}^{n} \min\left\{ |x_j - y_j|^{\frac{1}{1+\alpha_1}}, \frac{|x_j - y_j|}{|x_1|^{\alpha_1}} \right\}. \tag{5.2}$$

Set $s = d_{\mathbb{G}}(x,y)$ and write $s_1 + \sum_{j=2}^n s_j$ for the right-hand side of the above, with the obvious identifications. Then

$$s \simeq \max\{s_j : 1 \le j \le n\}.$$

It is straightforward to check that

$$\begin{aligned}
\big|H(x) - H(y)\big| &\simeq \max\Big\{ \big|x_1|x_1\big|^{\alpha_1} - y_1\big|y_1\big|^{\alpha_1}\big|, |x_2 - y_2|, \dots, |x_n - y_n| \Big\} \\
&\simeq \max\Big\{ \max\big\{ s_j^{1+\alpha_1}, s_j|x_1|^{\alpha_1} \big\} : 1 \le j \le n \Big\} \\
&\simeq \begin{cases} s^{1+\alpha_1}, & \text{if } s \ge |x_1|, \\ s|x_1|^{\alpha_1}, & \text{if } s \le |x_1|. \end{cases}
\end{aligned}$$

Therefore, H is η-quasisymmetric with $\eta(t) = C\max\{t, t^{1+\alpha_1}\}$ and C a constant depending only on α_1. □

Remark 5.2. Recall from Section 2.2 that when $\alpha_1 = 0$ and $0 \le \alpha_2 < 1$, the Grushin space $\mathbb{G}^3_\alpha$ is bilipschitzly equivalent to $\mathbb{R} \times \mathbb{G}^2_{\alpha_2}$ which is in turn bilipschitz to $\mathbb{R}^3$.

The following question is left open by Theorems 4.1 and 5.1, and Remark 5.2.

Question 5.3. Is $\mathbb{G}^3_\alpha$ quasisymmetric to $\mathbb{R}^3$, when $\alpha_1 > 0$ and $0 < \alpha_2 < 1$?

6. Bilipschitz parametrization and embedding

Assume in this section that $n \ge 2$, $\alpha_1 > 0$, and $\alpha = (\alpha_1, 0, \dots, 0)$.

The corresponding Grushin operator

$$\frac{\partial^2}{\partial x_1^2} + \sum_{2}^{n} |x_1|^{2\alpha_1} \frac{\partial^2}{\partial x_j^2}$$

has been examined in several of the articles mentioned in the Introduction. The Grushin metric $d_{\mathbb{G}}$ associated to this operator has the form

$$d_{\mathbb{G}}(x,y) \simeq |x_1 - y_1| + \sum_{j=2}^{n} \min\left\{ |x_j - y_j|^{\frac{1}{1+\alpha_1}}, \frac{|x_j - y_j|}{|x_1|^{\alpha_1}} \right\}.$$

In this case, the singular hyperplane $\{x_1 = 0\}$ in $\mathbb{G}^n_\alpha$ is the product of $n-1$ snowflakes of the Euclidean line, of equal exponent. This fact enables us to reduce the questions about parametrizing and embedding Grushin spaces to questions on the existence of quasisymmetric extensions and snowflake embeddings in Euclidean spaces.

6.1. Reduction to snowflake embeddings in Euclidean spaces. For $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, we also write $x = (x_1, x')$ with $x' = (x_2, \dots, x_n)$.

THEOREM 6.1. *Assume that* $n \geq 2$, $\alpha_1 > 0$ *and* $\alpha = (\alpha_1, 0, \dots, 0)$. *Let* $H\colon \mathbb{G}^n_\alpha \to \mathbb{R}^n$ *be the quasisymmetric homeomorphism defined by* (5.1), *and let* $a = \frac{\alpha_1}{1+\alpha_1}$. *Suppose that* $m \geq n$ *and that* $\phi\colon \mathbb{R}^n \to \mathbb{R}^m$ *is an* η*-quasisymmetric embedding which is* $(1-a)$*-snowflake on* $\mathbb{R}^{n-1}$:

$$C^{-1}\big|x' - y'\big|^{1-a} \leq \big|\phi\big(x'\big) - \phi\big(y'\big)\big| \leq C\big|x' - y'\big|^{1-a} \quad \text{for all } x', y' \in \mathbb{R}^{n-1} \tag{6.1}$$

and some constant $C > 1$, *and has the property that*

$$|x_1|^a \phi|_{B(x,|x_1|/2)} \text{ are uniformly bilipschitz for all } x \in \mathbb{R}^n \setminus \big(\{0\} \times \mathbb{R}^{n-1}\big). \tag{6.2}$$

Then the composition $\phi \circ H\colon \mathbb{G}^n_\alpha \to \mathbb{R}^m$ *is a bilipschitz embedding, with a bilipschitz constant depending only on* n, a *and* η.

Note that the embedded hyperplane $\phi(\mathbb{R}^{n-1})$ has Hausdorff dimension $\frac{n-1}{1-a}$.

Proof of Theorem 6.1. Constants in the proof may also depend on η.

Note by (6.2) that

$$\big|\phi(x) - \phi(y)\big| \simeq \big(\max\big\{|x_1|, |y_1|\big\}\big)^{-a}|x - y| \tag{6.3}$$

for all $x, y \in \mathbb{R}^n$ with $|x - y| \leq \max\{|x_1|, |y_1|\}/2$.

Observe also that

$$\big|\phi(x) - \phi(y)\big| \simeq |x - y|^{1-a} \tag{6.4}$$

for all $x, y \in \mathbb{R}^n$ with $|x - y| \geq \max\{|x_1|, |y_1|\}/2$.

To check (6.4), we assume $|x_1| \geq |y_1|$. In the case $|x' - y'| \leq 10|x_1|$, we get $|x - y| \simeq |x_1|$. From (6.2) and the quasisymmetry, (6.4) follows. In the case $|x' - y'| > 10|x_1|$, we have $|x - y| \simeq |x' - y'|$. It follows from (6.1) and the quasisymmetry that

$$\big|\phi(x) - \phi(y)\big| \simeq \big|\phi(x) - \phi\big(0, y'\big)\big| \simeq \big|\phi\big(0, x'\big) - \phi\big(0, y'\big)\big| \simeq |x - y|^{1-a}.$$

The definition of H in (5.1) yields

$$|x_1|^{1+\alpha_1} = \max\big\{\operatorname{dist}\big(H(x), \{0\} \times \mathbb{R}^{n-1}\big), \operatorname{dist}\big(H(y), \{0\} \times \mathbb{R}^{n-1}\big)\big\}.$$

To prove that $\phi \circ H$ is bilipschitz, we consider four cases based on the relative locations of x and y in $\mathbb{R}^n$, and assume as we may that $|x_1| \geq |y_1|$. Since $\alpha_j = 0$ for all $j \in [2, n-1]$, the Grushin metric will be estimated by (5.2).

Case I. $|x_1|>0$, $|x_1-y_1|\le |x_1|/4$ and $|x'-y'|\le |x_1|^{1+\alpha_1}/4$. In this case, $|x_1|\simeq|y_1|$. By (5.2), the Grushin distance $d_{\mathbb{G}}(x,y)\simeq|x_1-y_1|+|x'-y'||x_1|^{-\alpha_1}$. And by (5.1), the Euclidean distance

$$\begin{aligned}\big|H(x)-H(y)\big| &\simeq |x_1|^{\alpha_1}|x_1-y_1|+\big|x'-y'\big|\\ &\le \max\big\{\operatorname{dist}\big(H(x),\{0\}\times\mathbb{R}^{n-1}\big),\operatorname{dist}\big(H(y),\{0\}\times\mathbb{R}^{n-1}\big)\big\}/2.\end{aligned}$$

In view of (6.3),

$$\big|\phi\circ H(x)-\phi\circ H(y)\big|\simeq \operatorname{dist}\big(H(x),\{0\}\times\mathbb{R}^{n-1}\big)^{\frac{-\alpha_1}{1+\alpha_1}}\big|H(x)-H(y)\big|\simeq d_{\mathbb{G}}(x,y).$$

Case II. $|x_1|>0$, $|x_1-y_1|\ge|x_1|/4$ and $|x'-y'|\le|x_1|^{1+\alpha_1}/4$. In this case, $|x_1-y_1|\simeq|x_1|$. So the Grushin distance $d_{\mathbb{G}}(x,y)\simeq|x_1-y_1|+|x'-y'|^{\frac{1}{1+\alpha_1}}\simeq|x_1|$. The Euclidean distance

$$\begin{aligned}\big|H(x)-H(y)\big| &\simeq |x_1|^{1+\alpha_1}+\big|x'-y'\big|\simeq|x_1|^{1+\alpha_1}\\ &\simeq \max\big\{\operatorname{dist}\big(H(x),\{0\}\times\mathbb{R}^{n-1}\big),\operatorname{dist}\big(H(y),\{0\}\times\mathbb{R}^{n-1}\big)\big\}.\end{aligned}$$

Estimates $|\phi\circ H(x)-\phi\circ H(y)|\simeq d_{\mathbb{G}}(x,y)$ may be deduced from (6.3) and (6.4).

Case III. $|x_1|>0$ and $|x'-y'|\ge|x_1|^{1+\alpha_1}/4$. In this case, $d_{\mathbb{G}}(x,y)\simeq|x'-y'|^{\frac{1}{1+\alpha_1}}$ and

$$\begin{aligned}\big|H(x)-H(y)\big| &\simeq \big|x'-y'\big|\\ &\gtrsim \max\big\{\operatorname{dist}\big(H(x),\{0\}\times\mathbb{R}^{n-1}\big),\operatorname{dist}\big(H(y),\{0\}\times\mathbb{R}^{n-1}\big)\big\}.\end{aligned}$$

Hence by (6.4), $|\phi\circ H(x)-\phi\circ H(y)|\simeq|H(x)-H(y)|^{\frac{1}{1+\alpha_1}}\simeq d_{\mathbb{G}}(x,y)$.

Case IV. $x_1=0$. $|\phi\circ H(x)-\phi\circ H(y)|\simeq d_{\mathbb{G}}(x,y)$ can be obtained by taking limits in Case III.

This completes the proof. ☐

The question on the bilipchitz parametrizability and the bilipschitz embeddability of the particular class of Grushin spaces considered in this section has now been reduced to a problem on the existence, in Euclidean spaces, of the kind of snowflake embeddings described in Theorem 6.1.

The discussion in the next subsection assumes the theorems on the existence of Euclidean embedding in Section 7.

6.2. Bilipschitz parametrization and embedding of Grushin spaces. Assume that $n\ge 2$, $\alpha_1>0$, and $\alpha=(\alpha_1,0,\dots,0)$. Then the Hausdorff dimension of the Grushin space

$$\dim\mathbb{G}^n_\alpha=\max\big\{n,(n-1)(1+\alpha_1)\big\}, \tag{6.5}$$

where $(n-1)(1+\alpha_1)$ is the Hausdorff dimension of the singular hyperplane $\{x_1=0\}$ in $\mathbb{G}^n_\alpha$.

When $\alpha_1 = \frac{1}{n-1}$, every Grushin ball centered on the singular hyperplane $\{x_1 = 0\}$ of radius r intersects $\{x_1 = 0\}$ on a set of n-dimensional (Hausdorff) Grushin measure $\simeq r^n$, and it contains a smaller Grushin ball of radius comparable to r outside the singular hyperplane. As in Section 2.1, a point of density argument yields that $\mathbb{G}^n_\alpha$ is not bilipschitz homeomorphic to $\mathbb{R}^n$.

Is $\mathbb{G}^n_\alpha$ bilipschitz homeomorphic to $\mathbb{R}^n$ whenever $\alpha_1 < \frac{1}{n-1}$, in other words, for the full range of α_1 allowed by the Hausdorff dimension?

Theorem 6.2. *For each integer $n \geq 2$ there exists $0 < c(n) \leq \frac{1}{n-1}$, such that if $0 < \alpha_1 < c(n)$ and $\alpha_j = 0$ for all $j \geq 2$ then $\mathbb{G}^n_\alpha$ is bilipschitz homeomorphic to $\mathbb{R}^n$. When $n = 2$, $c(2) = 1$.*

Proof. When $n = 2$, take $c(2) = 1$. Given $0 < \alpha_1 < c(2)$, let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the quasiconformal map on $\mathbb{R}^2$ associated to $a = \frac{\alpha_1}{1+\alpha_1}$ defined in Section 7.1, and let $H\colon \mathbb{G}^2_{\alpha_1} \to \mathbb{R}^2$ be the quasisymmetric map defined by (5.1). It follows from (7.1), (7.2), and Theorem 6.1 that $f \circ H\colon \mathbb{G}^2_{\alpha_1} \to \mathbb{R}^2$ is a bilipschitz homeomorphism.

For $n \geq 3$, take $a_1(n, \frac{1}{100})$ to be the number chosen in Proposition 7.2 and $c(n) = a_1(n, \frac{1}{100})(1 - a_1(n, \frac{1}{100}))^{-1}$. When $0 < \alpha_1 < c(n)$, $a := \frac{\alpha_1}{1+\alpha_1} < a_1(n, \frac{1}{100})$. Let, in this case, $\Psi\colon \mathbb{R}^n \to \mathbb{R}^n$ be the quasiconformal map associated to a in Proposition 7.2, and $H\colon \mathbb{G}^n_\alpha \to \mathbb{R}^n$ be the map in (5.1). Again by (7.5), (7.7), and Theorem 6.1, $\Psi \circ H\colon \mathbb{G}^n \to \mathbb{R}^n$ is bilipschitz. $\square$

Theorem 6.2 is sharp in dimension 2; the proof of Theorem 6.2, using only the von Koch-type snowflake embedding, is particularly simple. For special emphasis, we state it again.

Corollary 6.3. *For each $\alpha_1 \in (0, 1)$, there exists a bilipschitz homeomorphism from the Grushin plane $\mathbb{G}^2_\alpha$ onto $\mathbb{R}^2$ that maps the singular line $\{x_1 = 0\}$ onto a von Koch-type snowflake curve of Hausdorff dimension $1 + \alpha_1$. Here $\alpha = (\alpha_1)$.*

The existence of bilipschitz embedding of the Grushin plane $\mathbb{G}^2_\alpha$ into $\mathbb{R}^3$ has been proved in [27] and [20] for $0 < \alpha_1 < 2$; see the discussion in Section 2.1.

Consider next $n \geq 3$. In view of (6.5) and for a reason similar to that stated in the beginning of this subsection, $\mathbb{G}^n_\alpha$ can not be bilipschitzly embedded in $\mathbb{R}^{n+1}$ when $\alpha_1 \geq \frac{2}{n-1}$.

Can $\mathbb{G}^n_\alpha$ be bilipschitzly embedded in $\mathbb{R}^{n+1}$ as a codimension 1 quasiplane whenever $\alpha_1 < \frac{2}{n-1}$? The question asks about the embeddability of $\mathbb{G}^n_\alpha$ into an ambient space of precisely one topological dimension higher, for a full range of α_1 allowed by the Hausdorff dimension of $\mathbb{G}^n_\alpha$, and in such a way that the embedded image has a controlled quasiconformal geometry.

Recall that the image of $\mathbb{R}^{k-1}$ under a quasiconformal homeomorphism of $\mathbb{R}^k$ is called a *quasiplane* in $\mathbb{R}^k$ when $k \geq 3$, or a *quasiline* when $k = 2$.

THEOREM 6.4. *For each integer $n \geq 2$ there exists $0 < \beta(n) \leq \frac{2}{n-1}$ such that if $0 < \alpha_1 < \beta(n)$ and $\alpha_j = 0$ for all $j \geq 2$, then $\mathbb{G}^n_\alpha$ is bililipschitz homeomorphic to a codimension 1 quasiplane in $\mathbb{R}^{n+1}$. In fact, we have $\beta(2) = 2$ and $\beta(3) = 1$.*

Theorem 6.4 is sharp in dimensions 2 and 3; in other words, the full range of exponents allowed for bilipschitz embedding may be reached for dimensions 2 and 3. The roles of von Koch-type snowflake embeddings played in both cases do not seem to extend to higher dimensions.

Proof. The claim in dimension 2 has been proved in [27] and [20].

When $n = 3$, take $\beta(3) = 1$. Given $0 < \alpha_1 < \beta(3)$, let F be the quasiconformal homeomorphism of $\mathbb{R}^4$ associated to $a = \frac{\alpha_1}{1+\alpha_1}$ in Proposition 7.3, and $\phi = F|_{\mathbb{R}^3\times\{0\}}$. Let $H\colon \mathbb{G}^3_\alpha \to \mathbb{R}^3$ be the map in (5.1). Then it follows from Proposition 7.3 and Theorem 6.1 that $\phi \circ H\colon \mathbb{G}^3_\alpha \to \mathbb{R}^4$ is a bilipschitz embedding and $\phi \circ H(\mathbb{G}^3_\alpha) = F(\mathbb{R}^3 \times \{0\})$ is a codimension 1 quasiplane in $\mathbb{R}^4$.

When $n \geq 4$, let $a_2(n)$ be the constant fixed in Remark 7.6, and set $\beta(n) = a_2(n)/(1 - a_2(n))$. Given $0 < \alpha_1 < \beta(n)$, take $a := \frac{\alpha_1}{1+\alpha_1} < a_2(n)$, and let $\tilde{F}\colon \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be the quasiconformal map associated to a in Remark 7.6. Set $\phi = \tilde{F}|_{\mathbb{R}^n\times\{0\}}$ and let $H\colon \mathbb{G}^n_\alpha \to \mathbb{R}^n$ be the map in (5.1). Again by Remark 7.6 and Theorem 6.1, $\phi \circ H\colon \mathbb{G}^n_\alpha \to \mathbb{R}^{n+1}$ is a bilipschitz embedding and $\phi \circ H(\mathbb{G}^n_\alpha) = \tilde{F}(\mathbb{R}^n \times \{0\})$ is a codimension 1 quasiplane in $\mathbb{R}^{n+1}$. □

7. Snowflake embeddings of $\mathbb{R}^{n-1}$ in $\mathbb{R}^n$ with controlled quasiconformal geometry

We now discuss snowflake embeddings of Euclidean spaces into low dimensional target spaces. In particular, we establish the existence by applying known results on mixed-snowflake embeddings, local-snowflake embeddings, and von Koch-type snowflake embeddings.

7.1. Von Koch snowflake embeddings of $\mathbb{R}$ in $\mathbb{R}^2$. We first recall the construction of a snowflake embedding from $\mathbb{R}$ onto an infinite von Koch-type snowflake curve.

Given $0 < a < \frac{1}{2}$, let $\gamma = \gamma^a$ be the standard von Koch-type snowflake arc in $\mathbb{R}^2$, homeomorphic to the interval $[0, 1]$ and having endpoints $(0, 0), (1, 0)$, which consists of four self-similar pieces scaled by the factor

$$p = 4^{a-1} \in \left(\frac{1}{4}, \frac{1}{2}\right),$$

and has Hausdorff dimension $\frac{\log 4}{\log 1/p} = \frac{1}{1-a}$. More precisely, fix θ in $[0, \pi/2)$ with $p + p\cos\theta = \frac{1}{2}$, and define similarities $\omega_i \colon \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\omega_1(x) = px, \qquad \omega_2(x) = R_\theta(px) + pe_1,$$
$$\omega_3(x) = R_{-\theta}(px) + \frac{1}{2}e_1 + p\sin\theta e_2, \qquad \omega_4(x) = px + (1-p)e_1,$$

where R_θ and $R_{-\theta}$ are the rotations of $\mathbb{R}^2$ about the origin 0 by angles θ and $-\theta$, respectively. The von Koch snowflake arc γ^a is the unique compact subset of $\mathbb{R}^2$ verifying

$$\gamma^a = \bigcup_{i=1}^{4} \omega_i(\gamma^a).$$

Let $\varrho \colon [0,1] \to \gamma^a$ be the homeomorphism which maps 0 and 1 to the endpoints $(0,0)$ and $(1,0)$, respectively, and maps, for each fixed $k \geq 1$, all subintervals $[\frac{m-1}{4^k}, \frac{m}{4^k}]$, $1 \leq m \leq 4^k$, to congruent subarcs of γ^a. Observe that for $x \in [0,1]$,

$$\varrho(x) = p^{-1}\varrho(x/4),$$

and

$$\varrho(x) = \sigma \circ \varrho(1-x),$$

where σ is the reflection of $\mathbb{R}^2$ with respect to the line $\{\frac{1}{2}\} \times \mathbb{R}$.

The map ϱ induces a snowflake embedding $f \colon \mathbb{R} \to \mathbb{R}^2$ as follows. Set $Q := \varrho(\frac{1}{2}) = \frac{1}{2}e_1 + p\sin\theta e_2$, and define for each $k \geq 1$, an embedding f_k: $[-2 \cdot 4^{k-1}, 2 \cdot 4^{k-1}] \to \mathbb{R}^2$ by

$$f_k(x) = Q + p^{-k}\left(\varrho\left(\frac{x}{4^k} + \frac{1}{2}\right) - Q\right).$$

We claim that $\varrho(x + \frac{1}{2}) = Q + p^{-1}(\varrho(\frac{x}{4} + \frac{1}{2}) - Q)$ for all $x \in [-\frac{1}{2}, \frac{1}{2}]$.

When $x \in [0, \frac{1}{2}]$, the claim follows from the facts

$$\varrho\left(x + \frac{1}{2}\right) = R_{-\theta} \circ \varrho(x) + Q,$$

and

$$\varrho\left(\frac{x}{4} + \frac{1}{2}\right) = R_{-\theta} \circ \varrho\left(\frac{x}{4}\right) + Q = pR_{-\theta} \circ \varrho(x) + Q;$$

when $x \in [-\frac{1}{2}, 0]$, the claim follows from the symmetry,

$$\varrho\left(x + \frac{1}{2}\right) = \sigma \circ \varrho\left(-x + \frac{1}{2}\right) \quad \text{and} \quad \varrho\left(\frac{x}{4} + \frac{1}{2}\right) = \sigma \circ \varrho\left(-\frac{x}{4} + \frac{1}{2}\right),$$

and the previous reasoning.

Now we see that $f_j(x) = f_k(x)$ for all $x \in [-2 \cdot 4^{k-1}, 2 \cdot 4^{k-1}]$ and all $j \geq k$; and therefore the sequence (f_k) converges to an embedding $f \colon \mathbb{R} \to \mathbb{R}^2$. It is straightforward to check that f is $(1-a)$-snowflake:

$$C^{-1}|x-y|^{1-a} \leq \bigl|f(x) - f(y)\bigr| \leq C|x-y|^{1-a} \quad \text{for all } x, y \in \mathbb{R} \tag{7.1}$$

and an absolute constant $C > 1$, and hence η-quasisymmetric with $\eta(t) = C^2 t^{1-a}$. We call

$$f\colon \mathbb{R} \to \Sigma^a = f(\mathbb{R})$$

a snowflake embedding.

Identify now $\mathbb{R}$ with the line $\{0\} \times \mathbb{R}$ in the plane $\mathbb{R}^2$.

Following Ahlfors [1] and Tukia [23], we next extend f to a K-quasiconformal homeomorphism, again called f, of $\mathbb{R}^2$ which is continuously differentiable outside $\mathbb{R}$ and whose derivative at $x = (x_1, x_2) \in \mathbb{R}^2 \setminus (\{0\} \times \mathbb{R})$ satisfies

$$\begin{aligned} L^{-1} \frac{|f(x_2+x_1) - f(x_2-x_1)|}{2|x_1|} &\le \min_{|\xi|=1} \bigl|Df(x)\xi\bigr| \\ &\le \max_{|\xi|=1} \bigl|Df(x)\xi\bigr| \le L \frac{|f(x_2+x_1) - f(x_2-x_1)|}{2|x_1|}, \end{aligned}$$

where constants K and L depend only on a. It follows from the $(1-a)$-snowflake property on $\{0\} \times \mathbb{R}$ that

$$\bigl|Df(x)\xi\bigr| \simeq |x_1|^{-a} \quad \text{for all } x \in \mathbb{R}^2 \setminus \bigl(\{0\} \times \mathbb{R}\bigr) \text{ and } |\xi| = 1. \tag{7.2}$$

7.2. Snowflake embeddings of $\mathbb{R}^{n-1}$ in $\mathbb{R}^n$. Snowflake embedding of a Euclidean space of dimension greater than 1 into a low dimensional target space is much more difficult than the embedding of a line, even when the snowflake exponent is close to 1.

We first record a deep theorem of David and Toro [5] on the *mixed-snowflake embedding* of $\mathbb{R}^{n-1}$ into $\mathbb{R}^n$. To give a better overview, we also include some statements that will not be used in the sequel.

In what follows, $\mathbb{R}^{n-1}$ is identified with the hyperplane $\{0\} \times \mathbb{R}^{n-1}$ in $\mathbb{R}^n$.

THEOREM 7.1. *David–Toro* [5] *For each integer $n \ge 2$ there exists $0 < a(n) < \frac{1}{n}$ such that if $0 < a < a(n)$, then there is an embedding $\varphi\colon \mathbb{R}^{n-1} \to \mathbb{R}^n$ with the $(1-a,1)$-snowflake property*:

$$\frac{1}{C} \max\bigl\{|x-y|, |x-y|^{1-a}\bigr\} \le \bigl|\varphi(x) - \varphi(y)\bigr| \le C \max\bigl\{|x-y|, |x-y|^{1-a}\bigr\} \tag{7.3}$$

for all $x, y \in \mathbb{R}^{n-1}$ and some constant $C > 1$ depending only on n. The embedding φ can be extended to a K-quasiconformal homemorphism Φ of $\mathbb{R}^n$, which is smooth outside $\mathbb{R}^{n-1}$ and is equal to the identity far from $\mathbb{R}^{n-1}$, for some constant K depending only on n and a.

Furthermore, for every $\varepsilon > 0$ there exists $0 < a_1(n,\varepsilon) < a(n)$, so that if $0 < a < a_1(n,\varepsilon)$ then Φ may be chosen to have these additional properties: associated to each ball $B = B(x,r)$ in $\mathbb{R}^n$ there exist a number $m_B > 0$ and an affine map A_B of $\mathbb{R}^n$ so that

$$\frac{9}{10} m_B \le |DA_B \xi| \le \frac{11}{10} m_B \quad \textit{for all } \xi \textit{ in } \mathbb{R}^n \textit{ with } |\xi| = 1,$$

and

$$\big|\Phi(z) - A(z)\big| \le \varepsilon m_B r \quad \textit{for all } z \in B;$$

associated to every $B = B(x,r) \subset \mathbb{R}^n$ whose center is on $\mathbb{R}^{n-1}$, there is an isometry J_B of $\mathbb{R}^n$ so that Φ

$$\big|\Phi(z) - m_B J_B(z)\big| \le \varepsilon m_B r \quad \textit{for all } z \in B. \tag{7.4}$$

The first statement follows from Theorem 2.10, Corollary 2.19 and Remark 2.25 in David and Toro [5], and the second statement combines Lemma 13.12, its proof and the discussion thereafter.

Observe by (7.3) that for every $B = B(x,r)$ in $\mathbb{R}^n$ with $x \in \{0\} \times \mathbb{R}^{n-1}$ and $0 < r < 1$, the corresponding scaling factor m_B in (7.4) is in fact

$$m_B \simeq r^{-a};$$

and by (7.4) that

$$|x_1|^a \Phi|_{B(x,|x_1|/2)} \text{ are uniformly bilipschitz for all}$$
$$x = \big(x_1, x'\big) \in \mathbb{R}^n \setminus \big(\{0\} \times \mathbb{R}^{n-1}\big).$$

In view of the mixed exponents in (7.3), the embedded image $\Phi(\mathbb{R}^{n-1})$ in Theorem 7.1 resembles a snowflake hypersurface only in the small scale. We need however a hypersurface that is also snowflaking in the large scale; in other words, it must satisfy a global $(1-a)$-snowflake condition.

These surfaces could possibly be constructed by following the steps in [5]. However, in order to give a rigorous argument, a large number of estimates in [5] would need to be modified. To avoid a lengthy exposition, we apply a normal family argument.

Assume at the outset the existence of the number $a(n)$ and the function $a_1(n,\cdot) > 0$ in Theorem 7.1. Given $a \in (0, a(n))$, let φ and Φ be the mappings, associated to a, chosen in the first part of Theorem 7.1. Since Φ is K-quasiconformal on $\mathbb{R}^n$, it is η-quasisymmetric for some η depending only on K, therefore only on n and a.

Assume as we may that $\Phi(0) = 0$ and $\Phi(e_n) = e_n$. Define for every $k \ge 0$

$$\Phi_k(x) = 2^{k(1-a)}\Phi\big(2^{-k}x\big) \quad \text{in } \mathbb{R}^n;$$

these mappings are again η-quasisymmetric. Observe from (7.3) that

$$\big|\Phi_k(e_n) - \Phi_k(0)\big| = 2^{k(1-a)}\big|\Phi\big(2^{-k}e_n\big) - \Phi\big(2^{-k}0\big)\big| \simeq 1;$$

and from the quasisymmetry that for each fixed $x \in \mathbb{R}^n$,

$$\big|\Phi_k(x)\big| \simeq |x|^{1-a} \quad \text{for all } k \ge \frac{\ln|x|}{\ln 2}.$$

So $\{\Phi_k \colon k \ge 0\}$ is a normal family by Ascoli's theorem [10]. Hence, a subsequence of (Φ_k) converges uniformly on compact subsets of $\mathbb{R}^n$ to a quasisymmetric homeomorphism $\Psi \colon \mathbb{R}^n \to \mathbb{R}^n$, which again has the $(1-a)$-snowflake property on $\{0\} \times \mathbb{R}^{n-1}$.

PROPOSITION 7.2. *For each $n \geq 2$ there exists $0 < a(n) < \frac{1}{n}$ so that for every $a \in (0, a(n))$, there is a K-quasiconformal map $\Psi \colon \mathbb{R}^n \to \mathbb{R}^n$ which is $(1-a)$-snowflake on $\{0\} \times \mathbb{R}^{n-1}$:*

$$\frac{1}{C}|x-y|^{1-a} \leq \left|\Psi(x) - \Psi(y)\right| \leq C|x-y|^{1-a} \quad \text{for all } x, y \in \{0\} \times \mathbb{R}^{n-1}, \tag{7.5}$$

where constants $K > 1$ and $C > 1$ depend at most on n and a.

Moreover, for every $\varepsilon > 0$ there exists $a_1(n, \varepsilon) \in (0, a(n))$ so that when $0 < a < a_1(n, \varepsilon)$, the mapping Ψ may be modified so that it is close to a similarity on every ball $B \subset \mathbb{R}^n$ centered on $\mathbb{R}^{n-1}$:

$$\left|\Psi(z) - m_B J_B(z)\right| \leq \varepsilon m_B r \quad \text{for all } z \in B, \tag{7.6}$$

where J_B is an isometry of $\mathbb{R}^n$ and m_B is a scaling factor, and that

$$\begin{aligned} &|x_1|^a \Psi|_{B(x,|x_1|/2)} \text{ are uniformly bilipschitz for all} \\ &\qquad x = \left(x_1, x'\right) \in \mathbb{R}^n \setminus \left(\{0\} \times \mathbb{R}^{n-1}\right). \end{aligned} \tag{7.7}$$

Given $\varepsilon > 0$, suppose that φ and Φ satisfy the additional conditions in the second part of Theorem 7.1. Then properties (7.6) and (7.7) for Ψ may be checked by scaling.

7.3. Codimension 2 snowflake embeddings. With more room to place surfaces, codimension 2 snowflake embedding can be expected to exist for a larger range of exponents.

We identify $\mathbb{R}^2$ with $\{0\} \times \mathbb{R}^2$ in $\mathbb{R}^3$, and $\mathbb{R}^3$ with $\mathbb{R}^3 \times \{0\}$ in $\mathbb{R}^4$. The unusual identification prepares $\mathbb{R}^4$ to place the images of $\mathbb{G}^3_\alpha$ and the singular plane $\{x_1 = 0\}$ under H at $\mathbb{R}^3 \times \{0\}$ and $\{0\} \times \mathbb{R}^2 \times \{0\}$ respectively, and to select $\mathbb{R}^3 \times \{0\}$ as the domain of the mapping ϕ. Here H and ϕ are the maps in Theorem 6.1. Write

$$\mathcal{P} = \{0\} \times \mathbb{R}^2 \times \{0\}.$$

For embeddings from $\mathbb{R}^2$ into $\mathbb{R}^4$, we have the following.

PROPOSITION 7.3. *Given $0 < a < \frac{1}{2}$, let $f \colon \mathbb{R} \to \Sigma^a$ be the homeomorphism from $\mathbb{R}$ to the von Koch-type snowflake curve Σ^a defined in Section 7. Then the product map $F \colon \mathcal{P} = \{0\} \times \mathbb{R} \times \mathbb{R} \times \{0\} \to \mathbb{R}^2 \times \mathbb{R}^2$,*

$$F \colon (0, x_2, x_3, 0) \mapsto \left(f(x_2), f(x_3)\right),$$

is $(1-a)$-snowflake on $\mathcal{P}$:

$$C^{-1}|x-y|^{1-a} \leq \left|F(x) - F(y)\right| \leq C|x-y|^{1-a} \quad \text{for all } x, y \in \mathcal{P},$$

where $C > 1$ is an absolute constant. Moreover F may be extended to a K-quasiconformal homeomorphism of $\mathbb{R}^4$, which has the property that

$$\left(\operatorname{dist}(x, \mathcal{P})/2\right)^a F|_{B(x, \operatorname{dist}(x,\mathcal{P})/2)} \text{ are uniformly bilipschitz for all } x \in \mathbb{R}^4 \setminus \mathcal{P},$$

and where constant K depends only on a.

We now sketch the idea in Proposition 7.3.

Products of quasisymmetric maps are rarely quasisymmetric. An exception is the product of quasisymmetric maps which are snowflake maps of equal exponent. To extend $F\colon \mathcal{P} \to \mathbb{R}^4$ to a quasiconformal homeomorphism of $\mathbb{R}^4$, a hands-on construction would require much work; we choose to adapt a local-snowflake embedding theorem from [11].

In two papers by Tukia and Väisälä ([24] and [25]), they study extension of a quasisymmetric embedding $A \to \mathbb{R}^n$, defined on a subset A of $\mathbb{R}^n$, to a quasiconformal homeomorphism of $\mathbb{R}^n$. Their existence theorems apply to quasisymmetric embeddings that are locally uniformly close to similarities; the closeness is measured by a gauge conditioned on the source A. Their theorems further require the set A in the source to be either very flat or very thick (in order for the data on A to be carried over to Whitney-type sets in $\mathbb{R}^n \setminus A$) and to satisfy a form of boundedness (in order for the extension to behave properly at infinity).

'Close to similarities' leads naturally to the idea of factoring the von Koch-type snowflake embedding f in Proposition 7.3 into maps of small distortion, $f = f_k \circ \cdots \circ f_2 \circ f_1$. (In dimension 3 or higher, such factorization is known only for some quasiconformal maps.) After the factorization, methods of Tukia and Väisälä [24, Theorem 5.4] may be applied to extend $(f_j, f_j)\colon \mathbb{R}^2 \to \mathbb{R}^4, 1 \le j \le k$, to quasiconformal maps F_j of $\mathbb{R}^4$. Finally, the extensions may be reassembled into $F_k \circ \cdots \circ F_2 \circ F_1$.

Observe however that factors (f_j, f_j), with the exception of (f_1, f_1), are defined on intermediate snowflake surfaces which cannot be confined within a bounded distance from $\{0\} \times \mathbb{R}^2 \times \{0\}$. This gives rise to a stability problem in the extension process. We bypass this issue, instead appealing to [11], in which a *local-snowflake* counterpart has been proved following factoring-extending-reassembling. Equivariance, replacing boundedness, can be used to stabilize the extension.

To this end, we define an equivariant extension of the canonical map $\varrho\colon [0,1] \to \gamma^a$ from Section 7.1 by

$$g(x) = \big(\lfloor x \rfloor, 0\big) + \varrho\big(x - \lfloor x \rfloor\big) \quad \text{for } x \in \mathbb{R}, \tag{7.8}$$

where $\lfloor x \rfloor$ is the largest integer less than or equal to x. Now the image

$$\Gamma^a = g(\mathbb{R}) = \bigcup_{n \in \mathbb{Z}} \big(\gamma^a + n e_1\big)$$

is an equivariant snowflake curve contained in $[-2,2] \times \mathbb{R}^2 \times [-2,2]$.

Proposition 7.4 ([11, Proposition 6.1]). *Given $0 < a < \frac{1}{2}$, let $g\colon \mathbb{R} \to \Gamma^a$ be the homeomorphism in (7.8). Then the product map $G\colon \{0\} \times \mathbb{R} \times \mathbb{R} \times \{0\} \to \mathbb{R}^2 \times \mathbb{R}^2$ defined by*

$$G\colon (0, x_2, x_3, 0) \mapsto \big(g(x_2), g(x_3)\big),$$

has a local-snowflake property on $\mathcal{P}$:

$$C^{-1}|x-y|^{1-a} \le \big|G(x)-G(y)\big| \le C|x-y|^{1-a} \quad \textit{for all } x,y \in \mathcal{P}, |x-y| \le 1,$$

and an absolute constant $C>1$. The map G can be extended to a quasiconformal homeomorphism of $\mathbb{R}^4$ that is PL outside $\mathcal{P}$ and whose almost-everywhere derivative at $x \in \mathbb{R}^4 \setminus \mathcal{P}$ satisfies

$$\big|DG(x)\xi\big| \simeq \operatorname{dist}(x,\mathcal{P})^{-a} \quad \textit{for all } |\xi|=1.$$

This proposition is essentially Proposition 6.1 in [11]; to compare, the map G here corresponds to H^{-1} in [11]. A very long, final step in the proof of Proposition 6.1 in [11] is to ensure the extension is smooth outside the hyperplane in the source; smoothness is not needed here in Proposition 7.4.

Proposition 7.3 follows from Proposition 7.4 by a normal family argument, in the same way that was applied in the proof of Proposition 7.2. We leave the details to the reader. This completes our discussion on Proposition 7.4.

Remark 7.5. The method of taking the product of two snowflake maps on $\mathbb{R}$ to produce, for each $d \in (2,4)$, a quasiconformal map G on $\mathbb{R}^4$ which is snowflake on $\mathbb{R}^2$ with $\dim G(\mathbb{R}^2)=d$, is special when the target is of dimension 4. It yields less precise results when the target has higher dimension.

When taking the product of more than two snowflake maps, the codimension of the embedded image has to increase. In order to keep the codimension at 2, only two snowflake maps may be used; our procedure below requires the exponents of both maps to remain very close to 1.

Remark 7.6. Assume that $n \ge 4$ and $0<a<a(n-1)$, where $a(n-1)$ is the constant fixed in Proposition 7.2. Let $\Psi\colon \{0\}\times\mathbb{R}^{n-2} \to \mathbb{R}^{n-1}$ be the $(1-a)$-snowflake embedding chosen in Proposition 7.2, and $f\colon \mathbb{R}\to\Sigma^a$ be the $(1-a)$-snowflake map in Section 7.1. Thus the product $\tilde F\colon (\{0\}\times\mathbb{R}^{n-2})\times(\mathbb{R}\times\{0\}) \to \mathbb{R}^{n-1}\times\mathbb{R}^2$,

$$\tilde F\colon (0,x_2,\dots,x_n,0) \mapsto \big(\Psi(0,x_2,\dots,x_{n-1}), f(x_n)\big),$$

is also $(1-a)$-snowflake.

Before applying the method of Tukia and Väisälä in [24, Theorem 5.4] to extend $\tilde F$ to a quasiconformal map of $\mathbb{R}^{n+1}$, we need to show that the map $\tilde F$ is s-quasisymmetric (a quantitative way of measuring closeness to similarities locally uniformly) for some $s<s(n)$; here $s(n)$ may be considered as a gauge used to guarantee the extendability from $\mathbb{R}^{n-1}$ to $\mathbb{R}^{n+1}$.

It is straightforward to check the s-quasisymmetry when the snowflake exponent is sufficiently close to 1. As a consequence, there exists $a_2(n) \in (0,a(n-1))$ such that if $0<a<a_2(n)$ then $\tilde F$ can be extended to a K-quasiconformal map on $\mathbb{R}^{n+1}$ for some constant $K>1$ depending only on n and a. Moreover the extension, again called $\tilde F$, produced following Tukia and

Väisälä's procedure is PL outside $\{0\}\times\mathbb{R}^{n-1}\times\{0\}$, and its almost-everywhere derivative at $x\in\mathbb{R}^{n+1}\setminus(\{0\}\times\mathbb{R}^{n-1}\times\{0\})$ satisfies

$$\left|D\tilde{F}(x)\xi\right| \simeq \operatorname{dist}\bigl(x,\{0\}\times\mathbb{R}^{n-1}\times\{0\}\bigr)^{-a} \quad \text{for all } |\xi|=1.$$

Papers [24] and [25] present a variety of theorems and counterexamples on extendability of bilipschitz and quasisymmetric maps which are defined on subsets of Euclidean spaces. The definition of s-quasisymmetry can be found in [24, p. 155] and [25, p. 241]; the extension procedure relevant to Remark 7.6 is explained in detail in [24, Theorem 5.4].

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