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2-LOCAL TRIPLE DERIVATIONS ON VON NEUMANN ALGEBRAS

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Abstract. We prove that every (not necessarily linear nor continuous) 2-local triple derivation on a von Neumann algebra M is a triple derivation, equivalently, the set $\text{Der}_t(M)$, of all triple derivations on M, is algebraically 2-reflexive in the set $\mathcal{M}(M) = M^M$ of all mappings from M into M.

1. Introduction

Let X and Y be Banach spaces. According to the terminology employed in the literature (see, for example, [4]), a subset $\mathcal{D}$ of the Banach space $B(X,Y)$, of all bounded linear operators from X into Y, is called *algebraically reflexive* in $B(X,Y)$ when it satisfies the property:

$$T \in B(X,Y) \quad \text{with } T(x) \in \mathcal{D}(x), \forall x \in X \Rightarrow T \in \mathcal{D}. \tag{1.1}$$

Algebraic reflexivity of $\mathcal{D}$ in the space $L(X,Y)$, of all linear mappings from X into Y, a stronger version of the above property not requiring continuity of T, is defined by:

$$T \in L(X,Y) \quad \text{with } T(x) \in \mathcal{D}(x), \forall x \in X \Rightarrow T \in \mathcal{D}. \tag{1.2}$$

In 1990, Kadison proved that (1.1) holds if $\mathcal{D}$ is the set $\text{Der}(M,X)$ of all (associative) derivations on a von Neumann algebra M into a dual M-bimodule X [18]. Johnson extended Kadison's result by establishing that the set $\mathcal{D} = \text{Der}(A,X)$, of all (associative) derivations from a C*-algebra A into a Banach A-bimodule X satisfies (1.2) [17].

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Algebraic reflexivity of the set of local triple derivations on a C*-algebra and on a JB*-triple have been studied in [24], [9], [12] and [14]. More precisely, Mackey proves in [24] that the set $\mathcal{D}=\mathrm{Der}_t(M)$, of all triple derivations on a JBW*-triple M satisfies (1.1). The result has been supplemented in [12], where Burgos, Fernández-Polo and the third author of this note prove that for each JB*-triple E, the set $\mathcal{D}=\mathrm{Der}_t(E)$ of all triple derivations on E satisfies (1.2).

Hereafter, *algebraic reflexivity* will refer to the stronger version (1.2) which does not assume the continuity of T.

In [6], Brešar and Šemrl proved that the set of all (algebra) automorphisms of $B(H)$ is algebraically reflexive whenever H is a separable, infinite-dimensional Hilbert space. Given a Banach space X, a linear mapping $T:X\to X$ satisfying the hypothesis at (1.2) for $\mathcal{D}=\mathrm{Aut}(X)$, the set of automorphisms on X, is called a *local automorphism*. Larson and Sourour showed in [22] that for every infinite dimensional Banach space X, every surjective local automorphism T on the Banach algebra $B(X)$, of all bounded linear operators on X, is an automorphism.

Motivated by the results of Šemrl in [31], references witness a growing interest in a subtle version of algebraic reflexivity called *algebraic 2-reflexivity* (cf. [1], [2], [10], [11], [21], [23], [25], [26] and [29]). A subset $\mathcal{D}$ of the set $\mathcal{M}(X,Y)=Y^X$, of all mappings from X into Y, is called *algebraically 2-reflexive* when the following property holds: for each mapping T in $\mathcal{M}(X,Y)$ such that for each $a,b\in X$, there exists $S=S_{a,b}\in\mathcal{D}$ (depending on a and b), with $T(a)=S_{a,b}(a)$ and $T(b)=S_{a,b}(b)$, then T lies in $\mathcal{D}$. A mapping $T:X\to Y$ satisfying the property that for each $a,b\in X$, there exists $S=S_{a,b}\in\mathcal{D}$ (depending on a and b), with $T(a)=S_{a,b}(a)$ and $T(b)=S_{a,b}(b)$ will be called a 2-local $\mathcal{D}$-mapping. If we assume that every mapping $S\in\mathcal{D}$ is r-homogeneous (that is, $S(ta)=t^rS(a)$ for every $t\in\mathbb{R}$ or $\mathbb{C}$) with $0<r$, then every 2-local $\mathcal{D}$-mapping $T:X\to Y$ is r-homogeneous. Indeed, for each $a\in X$, $t\in\mathbb{C}$ take $S_{a,ta}\in\mathcal{D}$ satisfying $T(ta)=S_{a,ta}(ta)=t^rS_{a,ta}(a)=t^rT(a)$.

Šemrl establishes in [31] that for every infinite-dimensional separable Hilbert space H, the sets $\mathrm{Aut}(B(H))$ and $\mathrm{Der}(B(H))$, of all (algebra) automorphisms and associative derivations on $B(H)$, respectively, are algebraically 2-reflexive in $\mathcal{M}(B(H))=\mathcal{M}(B(H),B(H))$. Ayupov and the first author of this note proved in [1] that the same statement remains true for general Hilbert spaces (see [20] for the finite dimensional case). Actually, the set $\mathrm{Hom}(A)$, of all homomorphisms on a general C*-algebra A, is algebraically 2-reflexive in the Banach algebra $B(A)$, of all bounded linear operators on A, and the set *-$\mathrm{Hom}(A)$, of all *-homomorphisms on A, is algebraically 2-reflexive in the space $L(A)$, of all linear operators on A (cf. [27]).

In recent contributions, Burgos, Fernández-Polo and the third author of this note prove that the set *-$\mathrm{Hom}(M)$ (respectively, $\mathrm{Hom}_t(M)$), of all *-homomorphisms (respectively, triple homomorphisms) on a von Neumann algebra (respectively, on a JBW*-triple) M, is an algebraically 2-reflexive subset of $\mathcal{M}(M)$ (cf. [10], [11], respectively), while Ayupov and the first author of this note establish that the set $\mathrm{Der}(M)$ of all derivations on M is algebraically 2-reflexive in $\mathcal{M}(M)$ (see [2]).

In this paper, we consider the set $\mathrm{Der}_t(A)$ of all triple derivations on a C*-algebra A. We recall that every C*-algebra A can be equipped with a ternary product of the form

$$\{a,b,c\} = \frac{1}{2}\big(ab^*c + cb^*a\big).$$

When A is equipped with this product it becomes a JB*-triple in the sense of [19]. A linear mapping $\delta : A \to A$ is said to be a *triple derivation* when it satisfies the (triple) Leibnitz rule:

$$\delta\{a,b,c\} = \big\{\delta(a),b,c\big\} + \big\{a,\delta(b),c\big\} + \big\{a,b,\delta(c)\big\}.$$

It is known that every triple derivation is automatically continuous (cf. [3]). We refer to [3], [15] and [28] for the basic references on triple derivations. According to the standard notation, 2-local $\mathrm{Der}_t(A)$-mappings from A into A are called *2-local triple derivations*.

The goal of this note is to explore the algebraic 2-reflexivity of $\mathrm{Der}_t(A)$ in $\mathcal{M}(A)$. Our main result proves that every (not necessarily linear nor continuous) 2-local triple derivation on an arbitrary von Neumann algebra M is a triple derivation (hence, linear and continuous) (see Theorem 2.14), equivalently, $\mathrm{Der}_t(M)$ is algebraically 2-reflexive in $\mathcal{M}(M)$.

2. 2-local triple derivations on von Neumann algebras

We start by recalling some generalities on triple derivations. Let A be a C*-algebra. For each $b \in A$, we shall denote by M_b the Jordan multiplication mapping by the element b, that is $M_b(x) = b \circ x = \frac{1}{2}(bx + xb)$. Following standard notation, given elements a, b in A, we denote by $L(a,b)$ the operator on A defined by $L(a,b)(x) = \{a,b,x\} = \frac{1}{2}(ab^*x + xb^*a)$. It is known that the mapping $\delta(a,b) : A \to A$, given by

$$\delta(a,b)(x) = L(a,b)(x) - L(b,a)(x),$$

is a triple derivation on A (cf. [3], [15]). A triple derivation which is a finite linear combination of derivations of the form $\delta(a,b)$ is called an *inner triple derivation*.

Let $\delta : A \to A$ be a triple derivation on a unital C*-algebra. By [15, Lemmas 1 and 2], $\delta(\mathbf{1})^* = -\delta(\mathbf{1})$, $M_{\delta(\mathbf{1})} = \delta(\frac{1}{2}\delta(\mathbf{1}),\mathbf{1})$ is an inner triple derivation on A, and the difference $D = \delta - \delta(\frac{1}{2}\delta(\mathbf{1}),\mathbf{1})$ is a Jordan *-derivation on A, more concretely,

$$D(x \circ y) = D(x) \circ y + x \circ D(y), \quad \text{and} \quad D\big(x^*\big) = D(x)^*,$$

for every $x, y \in A$. By [3, Corollary 2.2], δ (and hence D) is a continuous operator. A widely known result, due to B. E. Johnson, states that every bounded Jordan derivation from a C*-algebra A to a Banach A-bimodule is an associative derivation (cf. [16]). Therefore, D is an associative *-derivation in the usual sense. When $A = M$ is a von Neumann algebra, we can guarantee that D is an inner derivation, that is, there exists $\widetilde{a} \in A$ satisfying $D(x) = [\widetilde{a}, x] = \widetilde{a}x - x\widetilde{a}$, for every $x \in A$ (cf. [30, Theorem 4.1.6]). Further, from the condition $D(x^*) = D(x)^*$, for every $x \in A$, we deduce that $(\widetilde{a}^* + \widetilde{a})x = x(\widetilde{a}^* + \widetilde{a})$. Thus, taking $a = \frac{1}{2}(\widetilde{a} - \widetilde{a}^*)$, it follows that $[a, x] = [\widetilde{a}, x]$, for every $x \in M$. We have therefore shown that for every triple derivation δ on a von Neumann algebra M, there exist skew-Hermitian elements $a, b \in M$ satisfying

$$\delta(x) = [a, x] + b \circ x,$$

for every $x \in M$.

Our first lemma is a direct consequence of the above arguments (see [15, Lemmas 1 and 2]).

Lemma 2.1. *Let $T : A \to A$ be a (not necessarily linear nor continuous) 2-local triple derivation on a unital C*-algebra. Then*

(a) $T(\mathbf{1})^* = -T(\mathbf{1})$;
(b) $M_{T(\mathbf{1})} = \delta(\frac{1}{2}T(\mathbf{1}), \mathbf{1})$ *is an inner triple derivation on A*;
(c) $\widehat{T} = T - \delta(\frac{1}{2}T(\mathbf{1}), \mathbf{1})$ *is a 2-local triple derivation on A with $\widehat{T}(\mathbf{1}) = 0$.*

In what follows, we denote by A_{sa} the Hermitian elements of the C*-algebra A.

Lemma 2.2. *Let $T : A \to A$ be a (not necessarily linear nor continuous) 2-local triple derivation on a unital C*-algebra satisfying $T(\mathbf{1}) = 0$. Then $T(x) = T(x)^*$ for all $x \in A_{sa}$.*

Proof. Let $x \in A_{sa}$. By assumptions,

$$\begin{aligned} T(x)^* &= \{\mathbf{1}, T(x), \mathbf{1}\} = \{\mathbf{1}, \delta_{x,\mathbf{1}}(x), \mathbf{1}\} = \delta_{x,\mathbf{1}}\{\mathbf{1}, x, \mathbf{1}\} - 2\{\delta_{x,\mathbf{1}}(\mathbf{1}), x, \mathbf{1}\} \\ &= \delta_{x,\mathbf{1}}(x^*) - 2\{T(\mathbf{1}), x, \mathbf{1}\} = \delta_{x,\mathbf{1}}(x) = T(x). \end{aligned}$$

The proof is complete. □

Lemma 2.3. *Let $T : M \to M$ be a (not necessarily linear nor continuous) 2-local triple derivation on a von Neumann algebra satisfying $T(\mathbf{1}) = 0$. Then for every $x, y \in M_{sa}$ there exists a skew-Hermitian element $a_{x,y} \in M$ such that*

$$T(x) = [a_{x,y}, x], \quad \textit{and} \quad T(y) = [a_{x,y}, y].$$

Proof. For every $x, y \in M_{sa}$ we can find skew-Hermitian elements $a_{x,y}, b_{x,y} \in M$ such that

$$T(x) = [a_{x,y}, x] + b_{x,y} \circ x \quad \text{and} \quad T(y) = [a_{x,y}, y] + b_{x,y} \circ y.$$

Taking into account that $T(x)=T(x)^*$ (see Lemma 2.2), we obtain

$$\begin{aligned}[a_{x,y},x]+b_{x,y}\circ x &= T(x)=T(x)^*=[a_{x,y},x]^*+(b_{x,y}\circ x)^*\\ &= \left[x,a_{x,y}^*\right]+x\circ b_{x,y}^*=[x,-a_{x,y}]-x\circ b_{x,y}\\ &= [a_{x,y},x]-b_{x,y}\circ x,\end{aligned}$$

that is, $b_{x,y}\circ x=0$, and similarly $b_{x,y}\circ y=0$. Therefore, $T(x)=[a_{x,y},x]$, $T(y)=[a_{x,y},y]$, and the proof is complete. □

We state now an observation which plays a useful role in our study.

LEMMA 2.4. *Let a and b be skew-Hermitian elements in a C^*-algebra A. Suppose $x\in A$ is self-adjoint with $[a,x]+2b\circ x=0$. Then $[a,x]=0$ and $b\circ x=0$.*

Proof. Since $0=ax-xa+bx+xb$, by passing to the adjoint, we obtain $ax-xa-(bx+xb)=0$. The conclusion is reached by adding and subtracting these two equalities. □

Let M be a von Neumann algebra. If $x\in M_{sa}$, we denote by $s(x)$ the support projection of x, that is, the projection onto $(\ker(x))^{\perp}=\overline{\operatorname{ran}(x)}$. We say that x *has full support* if $s(x)=\mathbf{1}$ (equivalently, $\ker(x)=\{0\}$).

LEMMA 2.5. *Let M be a von Neumann algebra. Suppose $u\in M_+$ has full support, $c\in M$ is self-adjoint, and $\sigma(c^2u)\cap(0,\infty)=\emptyset$. Then $c=0$. Consequently, if u and c are as above, and $uc+cu=0$ (or $c^2u=-cuc\leq 0$), then $c=0$.*

Proof. For the first statement of the lemma, suppose $\sigma(c^2u)\cap(0,\infty)=\emptyset$. Note that

$$\sigma\left(c^2u\right)\cup\{0\}=\sigma(c\cdot cu)\cup\{0\}=\sigma(cuc)\cup\{0\}.$$

However, cuc is positive, hence $\sigma(cuc)\subset[0,\|cuc\|]$, with $\max_{\lambda\in\sigma(cuc)}=\|cuc\|$. Thus, $cu^{1/2}u^{1/2}c=cuc=0$, which means that $cu^{1/2}=u^{1/2}c=0$ and hence $s(c)\leq 1-s(u^{1/2})=1-s(u)=0$, which leads to $c=0$.

To prove the second part, we have $c^2u=-cuc\leq 0$, hence in particular, $\sigma(c^2u)\subset(-\infty,0]$. The proof is complete. □

In [2, Lemma 2.2], Ayupov and the first author of this note prove that for every (not necessarily linear nor continuous) 2-local derivation on a von Neumann algebra $\Delta: M\to M$, and every self-adjoint element $z\in M$, there exists $a\in M$ satisfying

$$\Delta(x)=[a,x],$$

for every $x\in\mathcal{W}^*(z)$, where $\mathcal{W}^*(z)=\{z\}''$ denotes the Abelian von Neumann subalgebra of M generated by the element z, and the unit element and $\{z\}''$ denotes the bicommutant of the set $\{z\}$. We prove next a ternary version of this result.

LEMMA 2.6. *Let $T: M \to M$ be a (not necessarily linear nor continuous) 2-local triple derivation on a von Neumann algebra. Let $z \in M$ be a self-adjoint element and let $\mathcal{W}^*(z) = \{z\}''$ be the Abelian von Neumann subalgebra of M generated by the element z and the unit element. Then there exist skew-Hermitian elements $a_z, b_z \in M$, depending on z, such that*

$$T(x) = [a_z, x] + b_z \circ x = a_z x - x a_z + \frac{1}{2}(b_z x + x b_z)$$

for all $x \in \mathcal{W}^*(z)$. In particular, T is linear on $\mathcal{W}^*(z)$.

Proof. We can assume that $z \neq 0$. Note that the Abelian von Neumann subalgebras generated by $\mathbf{1}$ and z and by $\mathbf{1}$ and $\mathbf{1} + \frac{z}{2\|z\|}$ coincide. So, replacing z with $\mathbf{1} + \frac{z}{2\|z\|}$ we can assume that z is an invertible positive element.

By definition, there exist skew-Hermitian elements $a_z, b_z \in M$ (depending on z) such that

$$T(z) = [a_z, z] + b_z \circ z.$$

Define a mapping $T_0 : M \to M$ given by $T_0(x) = T(x) - ([a_z, x] + b_z \circ x)$, $x \in M$. Clearly, T_0 is a 2-local triple derivation on M. We shall show that $T_0 = 0$ on $\mathcal{W}^*(z)$. Let $x \in \mathcal{W}^*(z)$ be an arbitrary element. By assumptions, there exist skew-Hermitian elements $c_{z,x}, d_{z,x} \in M$ such that

$$T_0(z) = [c_{z,x}, z] + d_{z,x} \circ z, \quad \text{and} \quad T_0(x) = [c_{z,x}, x] + d_{z,x} \circ x.$$

Since $0 = T_0(z) = [c_{z,x}, z] + d_{z,x} \circ z$, we get $[c_{z,x}, z] + d_{z,x} \circ z = 0$.

Taking into account that z is a Hermitian element and Lemma 2.4 we get $c_{z,x} z = z c_{z,x}$ and $d_{z,x} z = -z d_{z,x}$.

Since z has full support, and $d_{z,x}^2 z = -d_{z,x} z d_{z,x}$, Lemma 2.5 implies that $d_{z,x} = 0$. Further

$$c_{z,x} \in \{z\}' = \{z\}''' = \mathcal{W}^*(z)',$$

that is, $c_{z,x}$ commutes with any element in $\mathcal{W}^*(z)$. Therefore $T_0(x) = [c_{z,x}, x] + d_{z,x} \circ x = 0$, for all $x \in \mathcal{W}^*(z)$. The proof is complete. □

2.1. Complete additivity of 2-local derivations and 2-local triple derivations on von Neumann algebras. Let $\mathcal{P}(M)$ denote the lattice of projections in a von Neumann algebra M. Let X be a Banach space. A mapping $\mu : \mathcal{P}(M) \to X$ is said to be *finitely additive* when

$$\mu\left(\sum_{i=1}^{n} p_i\right) = \sum_{i=1}^{n} \mu(p_i), \tag{2.1}$$

for every family $p_1, \ldots, p_n$ of mutually orthogonal projections in M.

A mapping $\mu : \mathcal{P}(M) \to X$ is said to be *bounded* when the set

$$\left\{ \left\| \mu(p) \right\| : p \in \mathcal{P}(M) \right\}$$

is bounded. The celebrated Bunce–Wright–Mackey–Gleason theorem ([7], [8]) states that if M has no summand of type I_2, then every bounded finitely additive mapping $\mu:\mathcal{P}(M)\to X$ extends to a bounded linear operator from M to X.

According to the terminology employed in [32] and [13], a completely additive mapping $\mu:\mathcal{P}(M)\to\mathbb{C}$—that is, (2.1) holds for an arbitrary set I; see (2.2) below—is called a *charge*. The Dorofeev–Sherstnev theorem ([32, Theorem 29.5] or [13, Theorem 2]) states that any charge on a von Neumann algebra with no summands of type I_n $(n<\infty)$ is bounded.

We shall use the Dorofeev–Shertsnev theorem in Corollary 2.8 in order to be able to apply the Bunce-Wright-Mackey-Gleason theorem in Proposition 2.9. To this end, we need Proposition 2.7, which is implicitly applied in [2, proof of Lemma 2.3] for 2-local associative derivations. A proof is included here for completeness reasons.

First, we recall some facts about the strong* topology. For each normal positive functional φ in the predual of a von Neumann algebra M, the mapping

$$x\mapsto\|x\|_\varphi=\left(\varphi\left(\frac{xx^*+x^*x}{2}\right)\right)^{\frac{1}{2}}\quad(x\in M)$$

defines a prehilbertian seminorm on M. The *strong* topology* of M is the locally convex topology on M defined by all the seminorms $\|\cdot\|_\varphi$, where φ runs over the set of all positive functionals in M_* (cf. [30, Definition 1.8.7]). It is known that the strong* topology of M is compatible with the duality (M,M_*), that is a functional $\psi:M\to\mathbb{C}$ is strong* continuous if and only if it is weak* continuous (see [30, Corollary 1.8.10]). if and only if it is weak* continuous. We also recall that multiplication in every von Neumann algebra is jointly strong* continuous on bounded sets (see [30, Proposition 1.8.12]).

Suppose $X=W$ is another von Neumann algebra, and let τ denote the norm, the weak* or the strong* topology of W. The mapping μ is said to be *τ-completely additive* (respectively, *countably or sequentially τ-additive*) when

$$\mu\left(\sum_{i\in I}p_i\right)=\tau\text{-}\sum_{i\in I}\mu(p_i)\tag{2.2}$$

for every family (respectively, sequence) $\{p_i\}_{i\in I}$ of mutually orthogonal projections in M.

It is known that every family $(p_i)_{i\in I}$ of mutually orthogonal projections in a von Neumann algebra M is summable with respect to the weak* topology of M and $p=\text{weak}^*\text{-}\sum_{i\in I}p_i$ is a projection in M (cf. [30, Definition 1.13.4]). Further, for each normal positive functional ϕ in M_* and every finite set $F\subset I$, we have

$$\left\|p-\sum_{i\in F}p_i\right\|_\phi^2=\phi\left(p-\sum_{i\in F}p_i\right),$$

which implies that the family $(p_i)_{i\in I}$ is summable with respect to the strong* topology of M with the same limit, that is, $p=\text{strong}^*\text{-}\sum_{i\in I}p_i$.

Proposition 2.7. *Let $T : M \to M$ be a (not necessarily linear nor continuous) 2-local triple derivation on a von Neumann algebra. Then the following statements hold:*

(a) *The restriction $T|_{\mathcal{P}(M)}$ is sequentially strong* additive, and consequently sequentially weak* additive;*

(b) *$T|_{\mathcal{P}(M)}$ is weak* completely additive, that is,*

$$T\left(\textit{weak}^*\text{-}\sum_{i\in I} p_i\right) = \textit{weak}^*\text{-}\sum_{i\in I} T(p_i) \tag{2.3}$$

for every family $(p_i)_{i\in I}$ of mutually orthogonal projections in M.

Proof. (a) Let $(p_n)_{n\in\mathbb{N}}$ be a sequence of mutually orthogonal projections in M. Let us consider the element $z = \sum_{n\in\mathbb{N}} \frac{1}{n} p_n$. By Lemma 2.6, there exist skew-Hermitian elements $a_z, b_z \in M$ such that $T(x) = [a_z, x] + b_z \circ x$ for all $x \in \mathcal{W}^*(z)$. Since $\sum_{n=1}^{\infty} p_n$, $p_m \in \mathcal{W}^*(z)$, for all $m \in \mathbb{N}$, and the multiplication in M is jointly strong* continuous, we obtain that

$$\begin{aligned} T\left(\sum_{n=1}^{\infty} p_n\right) &= \left[a_z, \sum_{n=1}^{\infty} p_n\right] + b_z \circ \left(\sum_{n=1}^{\infty} p_n\right) \\ &= \sum_{n=1}^{\infty} [a_z, p_n] + \sum_{n=1}^{\infty} b_z \circ p_n = \sum_{n=1}^{\infty} T(p_n), \end{aligned}$$

that is, $T|_{\mathcal{P}(M)}$ is a countably or sequentially strong* additive mapping.

(b) Let φ be a positive normal functional in M_*, and let $\|\cdot\|_\varphi$ denote the prehilbertian seminorm given by $\|z\|_\varphi^2 = \frac{1}{2}\varphi(zz^* + z^*z)$ $(z \in M)$. Let $\{p_i\}_{i\in I}$ be an arbitrary family of mutually orthogonal projections in M. For every $n \in \mathbb{N}$, define

$$I_n = \left\{i \in I : \left\|T(p_i)\right\|_\varphi \geq 1/n\right\}.$$

We claim, that I_n is a finite set for every natural n. Otherwise, passing to a subset if necessary, we can assume that there exists a natural k such that I_k is infinite and countable. In this case, the series $\sum_{i\in I_k} T(p_i)$ does not converge with respect to the semi-norm $\|\cdot\|_\varphi$. On the other hand, since I_k is a countable set, by (a), we have

$$T\left(\sum_{i\in I_k} p_i\right) = \text{strong}^*\text{-}\sum_{i\in I_k} T(p_i),$$

which is impossible. This proves the claim.

We have shown that the set

$$I_0 = \left\{i \in I : \left\|T(p_i)\right\|_\varphi \neq 0\right\} = \bigcup_{n\in\mathbb{N}} I_n$$

is a countable set, and $\|T(p_i)\|_\varphi = 0$, for every $i \in I \backslash I_0$.

Set $p=\sum_{i\in I\setminus I_0} p_i \in M$. We shall show that $\varphi(T(p))=0$. Let q denote the support projection of φ in M (see [30, 1.14.2]). Having in mind that $\|T(p_i)\|_\varphi^2=0$, for every $i\in I\setminus I_0$, we deduce that $T(p_i)\perp q$ for every $i\in I\setminus I_0$.

Replacing T with $\widehat{T}=T-\delta(\frac{1}{2}T(\mathbf{1}),\mathbf{1})$ we can assume that $T(\mathbf{1})=0$ (cf. Lemma 2.1) and $T(x)=T(x)^*$, for every $x\in M_{sa}$ (cf. Lemma 2.2). By Lemma 2.3, for every $i\in I\setminus I_0$ there exists a skew-Hermitian element $a_i=a_{p,p_i}\in M$ such that

$$T(p)=a_ip-pa_i \quad \text{and} \quad T(p_i)=a_ip_i-p_ia_i.$$

Since $T(p_i)\perp q$ we get $(a_ip_i-p_ia_i)q=q(a_ip_i-p_ia_i)=0$, for all $i\in I\setminus I_0$. Thus, since $pa_ip_iq=p_ia_iq$,

$$\begin{aligned}\big(T(p)p_i\big)q&=(a_ip-pa_i)p_iq=a_ip_iq-pa_ip_iq\\&=a_ip_iq-p_ia_iq=(a_ip_i-p_ia_i)q=0,\end{aligned}$$

and similarly

$$q\big(p_iT(p)\big)=0,$$

for every $i\in I\setminus I_0$. Consequently,

$$\big(T(p)p\big)q=T(p)\Bigg(\sum_{i\in I\setminus I_0} p_i\Bigg)q=0=q\Bigg(\sum_{i\in I\setminus I_0} p_i\Bigg)T(p)=q\big(pT(p)\big). \tag{2.4}$$

Therefore,

$$\begin{aligned}T(p)&=\delta_{p,\mathbf{1}}(p)=\delta_{p,\mathbf{1}}\{p,p,p\}=2\big\{\delta_{p,\mathbf{1}}(p),p,p\big\}+\big\{p,\delta_{p,\mathbf{1}}(p),p\big\}\\&=2\big\{T(p),p,p\big\}+\big\{p,T(p),p\big\}=pT(p)+T(p)p+pT(p)^*p\\&=pT(p)+T(p)p+pT(p)p,\end{aligned}$$

which implies that

$$\begin{aligned}\varphi\big(T(p)\big)&=\varphi\big(pT(p)+T(p)p+pT(p)p\big)\\&=\varphi\big(qpT(p)q\big)+\varphi\big(qT(p)pq\big)+\varphi\big(qpT(p)pq\big)=\big(\text{by } (2.4)\big)=0.\end{aligned}$$

Finally, by (a) we have

$$T\Bigg(\sum_{i\in I_0} p_i\Bigg)=\|.\|_\varphi\text{-}\sum_{i\in I_0} T(p_i).$$

Two more applications of (a) give:

$$\begin{aligned}\varphi\Bigg(T\Bigg(\sum_{i\in I} p_i\Bigg)\Bigg)&=\varphi\Bigg(T\Bigg(p+\sum_{i\in I_0} p_i\Bigg)\Bigg)=\varphi\Bigg(T(p)+T\Bigg(\sum_{i\in I_0} p_i\Bigg)\Bigg)\\&=\varphi\big(T(p)\big)+\varphi\Bigg(T\Bigg(\sum_{i\in I_0} p_i\Bigg)\Bigg)=\sum_{i\in I_0}\varphi\big(T(p_i)\big).\end{aligned}$$

By the Cauchy–Schwarz inequality, $0 \leq |\varphi T(p_i)|^2 \leq \|T(p_i)\|_\varphi^2 = 0$, for every $i \in I \backslash I_0$, and hence $\sum_{i\in I_0} \varphi(T(p_i)) = \sum_{i\in I} \varphi(T(p_i))$. The arbitrariness of φ shows that $T(\text{weak}^*\text{-} \sum_{i\in I} p_i) = \text{weak}^*\text{-} \sum_{i\in I} T(p_i)$. □

Let ϕ be a normal functional in the predual of a von Neumann algebra M. Our previous Proposition 2.7 assures that for every (not necessarily linear nor continuous) 2-local triple derivation $T : M \to M$ the mapping $\phi \circ T|_{\mathcal{P}(M)} : \mathcal{P}(M) \to \mathbb{C}$ is a completely additive mapping or a charge on M. Under the additional hypothesis of M being a continuous von Neumann algebra or, more generally, a von Neumann algebra with no Type I_n $(1 < n < \infty)$ direct summands, the Dorofeev–Sherstnev theorem ([32, Theorem 29.5] or [13, Theorem 2]) imply that $\phi \circ T|_{\mathcal{P}(M)}$ is a bounded charge, that is, the set $\{|\phi \circ T(p)| : p \in \mathcal{P}(M)\}$ is bounded. The uniform boundedness principle gives:

Corollary 2.8. *Let M be a von Neumann algebra with no Type I_n direct summands $(1 < n < \infty)$ and let $T : M \to M$ be a (not necessarily linear nor continuous) 2-local triple derivation. Then the restriction $T|_{\mathcal{P}(M)}$ is a bounded weak* completely additive mapping.*

2.2. Additivity of 2-local triple derivations on Hermitian parts of von Neumann algebras. Suppose now that M is a von Neumann algebra with no Type I_n direct summands $(1 < n < \infty)$, and $T : M \to M$ is a (not necessarily linear nor continuous) 2-local triple derivation. By Corollary 2.8 combined with the Bunce-Wright-Mackey-Gleason theorem [7], [8], there exists a bounded linear operator $G : M \to M$ satisfying $G(p) = T(p)$, for every projection $p \in M$.

Let z be a self-adjoint element in M. By Lemma 2.6, there exist skew-Hermitian elements $a_z, b_z \in M$ such that $T(x) = [a_z, x] + b_z \circ x$, for every $x \in \mathcal{W}^*(z)$. Since $G|_{\mathcal{W}^*(z)}, T|_{\mathcal{W}^*(z)} : \mathcal{W}^*(z) \to M$ are bounded linear operators, which coincide on the set of projections of $\mathcal{W}^*(z)$, and since every self-adjoint element in $\mathcal{W}^*(z)$ can be approximated in norm by finite linear combinations of mutually orthogonal projections in $\mathcal{W}^*(z)$, it follows that $T(x) = G(x)$ for every $x \in \mathcal{W}^*(z)$, and hence

$$T(a) = G(a), \quad \text{for every } a \in M_{sa},$$

in particular, T is additive on M_{sa}.

The above arguments prove the following result.

Proposition 2.9. *Let $T : M \to M$ be a (not necessarily linear nor continuous) 2-local triple derivation on a von Neumann algebra with no Type I_n-factor direct summands $(1 < n < \infty)$. Then the restriction $T|_{M_{sa}}$ is additive.*

Corollary 2.10. *Let $T : M \to M$ be a (not necessarily linear nor continuous) 2-local triple derivation on a properly infinite von Neumann algebra. Then the restriction $T|_{M_{sa}}$ is additive.*

Next, we shall show that the conclusion of the above corollary is also true for a finite von Neumann algebra.

First, we show that every 2-local triple derivation on a von Neumann algebra "intertwines" central projections.

LEMMA 2.11. *If T is a (not necessarily linear nor continuous) 2-local triple derivation on a von Neumann algebra M, and p is a central projection in M, then $T(Mp) \subset Mp$. In particular, $T(px) = pT(x)$ for every $x \in M$.*

Proof. If $x \in Mp$, then $x = pxp = \{x,p,p\}$. Since T coincides with a triple derivation $\delta_{x,p}$ on the set $\{x,p\}$,

$$T(x) = \delta_{x,p}(x) = \delta_{x,p}\{x,p,p\} = \big\{\delta_{x,p}(x),p,p\big\} + \big\{x,\delta_{x,p}(p),p\big\} + \big\{x,p,\delta_{x,p}(p)\big\}$$

lies in Mp.

For the final statement, fix $x \in M$, and consider skew-Hermitian elements $a_{x,xp}$, $b_{x,xp} \in M$ satisfying

$$T(x) = [a_{x,xp},x] + b_{x,xp} \circ x, \quad \text{and} \quad T(xp) = [a_{x,xp},xp] + b_{x,xp} \circ (xp).$$

The assumption p being central implies that $pT(x) = T(px)$. □

PROPOSITION 2.12. *Let $T : M \to M$ be a (not necessarily linear nor continuous) 2-local triple derivation on a finite von Neumann algebra. Then the restriction $T|_{M_{sa}}$ is additive.*

Proof. Since M is finite there exists a faithful normal semi-finite trace τ on M. We shall consider the following two cases.

Case 1. Suppose τ is a finite trace. Replacing T with $\widehat{T} = T - \delta(\frac{1}{2}T(\mathbf{1}),\mathbf{1})$ we can assume that $T(\mathbf{1}) = 0$ (cf. Lemma 2.1) and $T(x) = T(x)^*$, for every $x \in M_{sa}$ (cf. Lemma 2.2). By Lemma 2.3, for every $x,y \in M_{sa}$ there exists a skew-Hermitian element $a_{x,y} \in M$ such that $T(x) = [a_{x,y},x]$ and $T(y) = [a_{x,y},y]$. Then

$$T(x)y + xT(y) = [a_{x,y},x]y + x[a_{x,y},y] = [a_{x,y},xy],$$

that is,

$$[a_{x,y},xy] = T(x)y + xT(y).$$

Further

$$0 = \tau\big([a_{x,y},xy]\big) = \tau\big(T(x)y + xT(y)\big),$$

that is, $\tau(T(x)y) = -\tau(xT(y))$, for every $x,y \in M_{sa}$. For arbitrary $u,v,w \in M_{sa}$, set $x = u+v$, and $y = w$. The above identity implies

$$\begin{aligned}\tau\big(T(u+v)w\big) &= -\tau\big((u+v)T(w)\big) = \\ &= -\tau\big(uT(w)\big) - \tau\big(vT(w)\big) = \tau\big(T(u)w\big) + \tau\big(T(v)w\big) \\ &= \tau\big(\big(T(u)+T(v)\big)w\big),\end{aligned}$$

and so

$$\tau\big(\big(T(u+v) - T(u) - T(v)\big)w\big) = 0$$

for all $u, v, w \in M_{sa}$. Take $w = T(u+v) - T(u) - T(v)$. Then $\tau(ww^*) = 0$. Since the trace τ is faithful it follows that $ww^* = 0$, and hence $w = 0$. Therefore, $T(u+v) = T(u) + T(v)$.

Case 2. Suppose now that τ is a semi-finite trace. As in *Case* 1, we may assume $T(\mathbf{1}) = 0$. Since M is finite there exists a family of mutually orthogonal central projections $\{z_i\}$ in M such that z_i has finite trace for all i and $\bigvee z_i = \mathbf{1}$ (cf. [30, §2.2 or Corollary 2.4.7]). By Lemma 2.11, for each i, T maps $z_i M$ into itself. From Case 1, $T|_{z_i M} : z_i M \to z_i M$ is additive. Furthermore,

$$z_i T(x+y) = T|_{z_i M}(z_i x + z_i y) = T|_{z_i M}(z_i x) + T|_{z_i M}(z_i y) = z_i T(x) + z_i T(y),$$

for every $x, y \in M$ and every i. Therefore,

$$\begin{aligned} T(x+y) &= \Big(\sum_i z_i\Big) T(x+y) = \sum_i z_i T(x+y) = \sum_i \big(z_i T(x) + z_i T(y)\big) \\ &= \Big(\sum_i z_i\Big) T(x) + \Big(\sum_i z_i\Big) T(y) = T(x) + T(y), \end{aligned}$$

for every $x, y \in M$. The proof is complete. □

Let $T : M \to M$ be a (not necessarily linear nor continuous) 2-local triple derivation on an arbitrary von Neumann algebra. In this case, there exist orthogonal central projections $z_1, z_2 \in M$ with $z_1 + z_2 = \mathbf{1}$ such that:

- $z_1 M$ is a finite von Neumann algebra;
- $z_2 M$ is a properly infinite von Neumann algebra,

(cf. [30, §2.2]).

By Lemma 2.11, for each $k = 1, 2$, $z_k T$ maps $z_k M$ into itself. By Corollary 2.10 and Proposition 2.12 both $z_1 T$ and $z_2 T$ are additive on M_{sa}. So $T = z_1 T + z_2 T$ also is additive on M_{sa}.

We have thus proved the following result.

Proposition 2.13. *Let $T : M \to M$ be a* (*not necessarily linear nor continuous*) 2-*local triple derivation on an arbitrary von Neumann algebra. Then the restriction $T|_{M_{sa}}$ is additive.*

2.3. Main result. We can state now the main result of this paper.

Theorem 2.14. *Let M be an arbitrary von Neumann algebra and let $T : M \to M$ be a* (*not necessarily linear nor continuous*) 2-*local triple derivation. Then T is a triple derivation* (*hence linear and continuous*). *Equivalently, the set* $\mathrm{Der}_t(M)$, *of all triple derivations on M, is algebraically* 2-*reflexive in the set $\mathcal{M}(M) = M^M$ of all mappings from M into M.*

We need the following two lemmata.

Lemma 2.15. *Let $T : M \to M$ be a* (*not necessarily linear nor continuous*) 2-*local triple derivation on a von Neumann algebra with $T(\mathbf{1}) = 0$. Then there* *exists a skew-Hermitian element $a \in M$ such that $T(x) = [a, x]$, for all $x \in M_{sa}$.*

Proof. Let $x \in M_{sa}$. By Lemma 2.3, there exists a skew-Hermitian element $a_{x,x^2} \in M$ such that $T(x) = [a_{x,x^2}, x]$, $T(x^2) = [a_{x,x^2}, x^2]$.

Thus,

$$T\left(x^2\right) = \left[a_{x,x^2}, x^2\right] = [a_{x,x^2}, x]x + x[a_{x,x^2}, x] = T(x)x + xT(x),$$

that is,

$$T\left(x^2\right) = T(x)x + xT(x), \tag{2.5}$$

for every $x \in M_{sa}$.

By Proposition 2.13 and Lemma 2.2, $T|_{M_{sa}} : M_{sa} \to M_{sa}$ is a real linear mapping. Now, we consider the linear extension $\widehat{T}$ of $T|_{M_{sa}}$ to M defined by

$$\widehat{T}(x_1 + ix_2) = T(x_1) + iT(x_2), \quad x_1, x_2 \in M_{sa}.$$

Taking into account the homogeneity of T, Proposition 2.13 and the identity (2.5) we obtain that $\widehat{T}$ is a Jordan derivation on M. By [5, Theorem 1] any Jordan derivation on a semi-prime algebra is a derivation. Since M is von Neumann algebra, $\widehat{T}$ is a derivation on M (see also [33] and [16]). Therefore, there exists an element $a \in M$ such that $\widehat{T}(x) = [a, x]$ for all $x \in M$. In particular, $T(x) = [a, x]$ for all $x \in M_{sa}$. Since $T(M_{sa}) \subseteq M_{sa}$, we can assume that $a^* = -a$, which completes the proof. □

LEMMA 2.16. *Let $T : M \to M$ be a (not necessarily linear nor continuous) 2-local triple derivation on a von Neumann algebra. If $T|_{M_{sa}} \equiv 0$, then $T \equiv 0$.*

Proof. Let $x \in M$ be an arbitrary element and let $x = x_1 + ix_2$, where $x_1, x_2 \in M_{sa}$. Since T is homogeneous, by passing to the element $(1 + \|x_2\|)^{-1}x$ if necessary, we can suppose that $\|x_2\| < 1$. In this case, the element $y = \mathbf{1} + x_2$ is positive and invertible. Take skew-Hermitian elements $a_{x,y}, b_{x,y} \in M$ such that

$$T(x) = [a_{x,y}, x] + b_{x,y} \circ x, \quad \text{and} \quad T(y) = [a_{x,y}, y] + b_{x,y} \circ y.$$

Since $T(y) = 0$, we get $[a_{x,y}, y] + b_{x,y} \circ y = 0$. By Lemma 2.4 we obtain that $[a_{x,y}, y] = 0$ and $ib_{x,y} \circ y = 0$. Taking into account that $ib_{x,y}$ is Hermitian, y is positive and invertible, Lemma 2.5 implies that $b_{x,y} = 0$.

We further note that $0 = [a_{x,y}, y] = [a_{x,y}, \ \mathbf{1} + x_2] = [a_{x,y}, x_2]$, that is, $[a_{x,y}, x_2] = 0$. Now, $T(x) = [a_{x,y}, x] + b_{x,y} \circ x = [a_{x,y}, x_1 + ix_2] = [a_{x,y}, x_1]$, i.e. $T(x) = [a_{x,y}, x_1]$. Therefore,

$$T(x)^* = [a_{x,y}, x_1]^* = \left[x_1, a_{x,y}^*\right] = [x_1, -a_{x,y}] = [a_{x,y}, x_1] = T(x).$$

So $T(x)^* = T(x)$. Now, replacing x by ix we obtain, from the homogeneity of T, that $T(x)^* = -T(x)$. Combining the last two identities, we obtain that $T(x) = 0$, which finishes the proof. □

Proof of Theorem 2.14. Let us define $\widehat{T} = T - \delta(\frac{1}{2}T(\mathbf{1}), \mathbf{1})$. Then $\widehat{T}$ is a 2-local triple derivation on M with $\widehat{T}(\mathbf{1}) = 0$ (cf. Lemma 2.1) and $\widehat{T}(x) = \widehat{T}(x)^*$, for every $x \in M_{sa}$ (cf. Lemma 2.2). By Lemma 2.15, there exists an element $a \in M$ such that $\widehat{T}(x) = [a, x]$ for all $x \in M_{sa}$. Consider the 2-local triple derivation $\widehat{T} - [a, \cdot]$. Since $(\widehat{T} - [a, \cdot])|_{M_{sa}} \equiv 0$, Lemma 2.16 implies that $\widehat{T} = [a, \cdot]$, and hence $T = [a, \cdot] + \delta(\frac{1}{2}T(\mathbf{1}), \mathbf{1})$, witnessing the desired statement. □

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