SCHWARZ–PICK TYPE ESTIMATES OF PLURIHARMONIC MAPPINGS IN THE UNIT POLYDISK

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ABSTRACT. In this paper, we will give Schwarz–Pick type estimates of arbitrary order partial derivatives for bounded pluriharmonic mappings defined in the unit polydisk. Our main results are generalizations of results of Colonna for planar harmonic mappings in (*Indiana Univ. Math. J.* **38** (1989) 829–840).

1. Introduction and main results

Let $\mathbb{C}^n$ denote the complex Euclidean n-space. For $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, the conjugate of z, denoted by $\overline{z}$, is defined by $\overline{z} = (\overline{z}_1, \ldots, \overline{z}_n)$. For z and $w = (w_1, \ldots, w_n) \in \mathbb{C}^n$, the *inner product* on $\mathbb{C}^n$ and the *Euclidean norm* of z are given by $\langle z, w \rangle := \sum_{k=1}^n z_k \overline{w}_k$ and $\|z\| := \langle z, z \rangle^{1/2}$, respectively. For $a \in \mathbb{C}^n$, $\mathbb{B}^n(a, r) = \{z \in \mathbb{C}^n : \|z - a\| < r\}$ is the (open) ball of radius r with center a. Also, we let $\mathbb{B}^n(r) := \mathbb{B}^n(0, r)$ and use $\mathbb{B}^n$ to denote the unit ball $\mathbb{B}^n(1)$, and $\mathbb{D} = \mathbb{B}^1$. Let $\mathbb{D}^n = \mathbb{D} \times \cdots \times \mathbb{D}$ (n times) be the polydisc in $\mathbb{C}^n$ and $\mathbb{T}^n = \mathbb{T} \times \cdots \times \mathbb{T}$ (n times), where $\mathbb{T}$ is the unit circle in $\mathbb{C}^1$. A multi-index $k = (k_1, \ldots, k_n)$ consists of n nonnegative integers k_j, where $j \in \{1, \ldots, n\}$. The degree of a multi-index k is the sum $|k| = \sum_{j=1}^n k_j$. Given another multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$, let $k^\alpha = (k_1^{\alpha_1}, \ldots, k_n^{\alpha_n})$. For $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, let $\|z\| = (\sum_{j=1}^n |z_j|^2)^{1/2}$, $\|z\|_\infty = \max_{1 \le j \le n} |z_j|$ and $z^k = \Pi_{k=1}^n z_j^{k_j}$.

A continuous complex-valued function f defined on a domain $\Omega \subset \mathbb{C}^n$ is said to be *pluriharmonic* if for each fixed $z \in \Omega$ and $\theta \in \partial \mathbb{B}^n$, the function $f(z + \theta \zeta)$ is harmonic in $\{\zeta : \|\zeta\| < d_\Omega(z)\}$, where $d_\Omega(z)$ denotes the distance from z to the boundary $\partial \Omega$ of Ω (cf. [26]). If $\Omega \subset \mathbb{C}^n$ is a simply connected domain, then a function $f : \Omega \to \mathbb{C}$ is pluriharmonic if and only if f has a representation $f = h + \overline{g}$, where h and g are holomorphic in Ω (see [30]). Let $\mathcal{P}(\Omega, \mathbb{C}^N)$ be the class of all pluriharmonic mappings

$f = (f_1, \ldots, f_N)$ from a domain $\Omega \subset \mathbb{C}^n$ to $\mathbb{C}^N$, where N is a positive integer and f_j $(1 \leq j \leq N)$ are pluriharmonic mappings from Ω into $\mathbb{C}$. For a mapping $f \in \mathcal{P}(\Omega, \mathbb{C}^N)$, we use Df and $\overline{D}f$ to denote the two $N \times n$ matrices $(\partial f_j / \partial z_m)_{N \times n}$ and $(\partial f_j / \partial \overline{z}_m)_{N \times n}$, respectively. We refer to [7], [10], [11], [12], [19], [22] for more details on pluriharmonic mappings. In particular, if $n = 1$, then pluriharmonic mappings are planar harmonic mappings (cf. [14], [18]). Therefore, pluriharmonic mappings can be understood as the natural generalization of planar harmonic mappings to several complex variables.

We first recall the classical Schwarz–Pick lemma for analytic functions f of $\mathbb{D}$ into itself:

$$(1.1) \qquad \left| f'(z) \right| \leq \frac{1 - |f(z)|^2}{1 - |z|^2}, \quad z \in \mathbb{D}.$$

In 1920, Szász [29] extended the inequality (1.1) to the following estimate involving higher order derivatives:

$$(1.2) \qquad \left| f^{(2m+1)}(z) \right| \leq \frac{(2m+1)!}{(1 - |z|^2)^{2m+1}} \sum_{k=0}^{m} \binom{m}{k}^2 |z|^{2k},$$

where $m \in \{1, 2, \ldots\}$. In 1985, Ruscheweyh (cf. [3], [4], [27]) improved (1.2) to the following sharp estimate:

$$(1.3) \qquad \left| f^{(n)}(z) \right| \leq \frac{n!(1 - |f(z)|^2)}{(1 - |z|)^n (1 + |z|)}.$$

Recently, the inequality (1.3) was generalized into a variety of forms (see [1], [2], [4], [16], [17], [23], [31]).

In 1989, Colonna established an analogue of the Schwarz–Pick lemma for planar harmonic mappings, which is the following.

THEOREM A ([15, Theorems 3 and 4]). *Let f be a harmonic mapping of $\mathbb{D}$ into $\mathbb{D}$. Then for $z \in \mathbb{D}$,*

$$\left| \frac{\partial f(z)}{\partial z} \right| + \left| \frac{\partial f(z)}{\partial \overline{z}} \right| \leq \frac{4}{\pi} \frac{1}{1 - |z|^2}.$$

This estimate is sharp, and all the extremal functions are

$$f(z) = \frac{2\gamma}{\pi} \arg\left(\frac{1 + \psi(z)}{1 - \psi(z)} \right),$$

where $|\gamma| = 1$ and ψ is a conformal automorphism of $\mathbb{D}$.

We refer to [7], [5], [6], [8], [9], [10], [13], [20], [24], [28] for further discussion on this topic.

In this paper, we generalize Theorem A to higher dimensional case, and give the estimate for the partial derivatives of arbitrary order. One should

note that the higher dimensional case is very different from the one dimensional situation and, because we are dealing with partial derivatives of arbitrary order, the method of proof from [15] can not be used. By using the coefficient estimates and the Cauchy integral formula, we prove the following result.

THEOREM 1. *Let* $f \in \mathcal{P}(\mathbb{D}^n, \mathbb{D})$. *Then*

$$\left| \frac{\partial^\alpha f(z)}{\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}} \right| + \left| \frac{\partial^\alpha f(z)}{\partial \overline{z}_1^{\alpha_1} \cdots \partial \overline{z}_n^{\alpha_n}} \right| \leq \alpha! \frac{4}{\pi} \frac{(1 + \|z\|_\infty)^{|\alpha|-n}}{(1 - \|z\|_\infty^2)^{|\alpha|}},$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$ *is a multi-index with* $\alpha_j > 0, j \in \{1, \ldots, n\}$.

We remark that if $|\alpha| = n = 1$, then Theorem 1 coincides with Theorem A.

It is well known that there are no biholomorphic mappings between $\mathbb{D}^n$ and $\mathbb{B}^n$ (cf. [25], [26]). Hence, pluriharmonic mappings between $\mathbb{D}^n$ and $\mathbb{B}^n$ are particularly interesting in the theory of several complex variables. The following result is an analogue of [7, Theorem 4] for vector-valued pluriharmonic mappings defined in $\mathbb{B}^n$.

THEOREM 2. *If* $f \in \mathcal{P}(\mathbb{D}^n, \mathbb{B}^N)$, *then*

$$(1.4) \qquad \max_{\theta \in \mathbb{C}^n, \|\theta\|_\infty = 1} \left\| Df(z)\theta + \overline{D}f(z)\overline{\theta} \right\| \leq \frac{4}{\pi(1 - \|z\|_\infty^2)},$$

where θ *is regarded as a column vector.*

If $f \in \mathcal{P}(\mathbb{D}^n, \mathbb{B}^N)$ *with* $f(0) = 0$, *then*

$$(1.5) \qquad \|f(z)\| \leq \frac{4}{\pi} \arctan \|z\|_\infty.$$

We remark that if $n = N = 1$, then the estimates (1.4) and (1.5) coincide with Theorem A and [21, Lemma], respectively.

2. The proofs of the main results

LEMMA 1. *Let* m *be a positive integer and* γ *be a real constant. Then*

$$\int_0^{2\pi} \left| \cos(m\theta + \gamma) \right| d\theta = 4.$$

Proof. By elementary calculations, we have

$$\int_0^{2\pi} \left| \cos(m\theta + \gamma) \right| d\theta = \frac{1}{m} \int_\gamma^{2m\pi + \gamma} |\cos t| \, dt$$

$$= \frac{1}{m} \sum_{k=1}^{2m} \int_{(k-1)\pi + \gamma}^{k\pi + \gamma} |\cos t| \, dt$$

$$= \frac{1}{m} \sum_{k=1}^{2m} \int_0^\pi |\cos t| \, dt$$

$$= 2 \int_0^\pi |\cos t|\, dt$$
$$= 4.$$

The proof of the lemma is complete. $\qquad\square$

Proof of Theorem 1. Since $\mathbb{D}^n$ is a simply connected domain in $\mathbb{C}^n$, we see that f has a representation $f = h + \overline{g}$, where h and g are holomorphic in $\mathbb{D}^n$. Let $k = (k_1, \ldots, k_n)$ be a multi-index. Then f can be expressed as a power series as follows

$$f(z) = h(z) + \overline{g(z)} = \sum_k a_k z^k + \sum_k \overline{b_k} \overline{z}^k.$$

CLAIM 1. *For $|k| \geq 1$, $|a_k| + |b_k| \leq \frac{4}{\pi}$.*

Now we prove Claim 1. Let $z = (z_1, \ldots, z_n) = (r_1 e^{i\theta_1}, \ldots, r_n e^{i\theta_n}) \in \mathbb{D}^n$, where $0 \leq r_j < 1$ and $\theta_j \in \mathbb{R}$ for all $j \in \{1, \ldots, n\}$. Then for $|k| \geq 1$,

$$(2.1) \quad a_k r_1^{k_1} \cdots r_n^{k_n}$$
$$= \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} f\left(r_1^{k_1} e^{i\theta_1}, \ldots, r_n^{k_n} e^{i\theta_n}\right) e^{-i\sum_{j=1}^n k_j \theta_j}\, d\theta_1 \cdots d\theta_n$$

and

$$(2.2) \quad \overline{b_k} r_1^{k_1} \cdots r_n^{k_n}$$
$$= \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} f\left(r_1^{k_1} e^{i\theta_1}, \ldots, r_n^{k_n} e^{i\theta_n}\right) e^{i\sum_{j=1}^n k_j \theta_j}\, d\theta_1 \cdots d\theta_n.$$

By (2.1) and (2.2), we get

$$(2.3) \quad r_1^{k_1} \cdots r_n^{k_n} \left(|a_k| + |b_k|\right)$$
$$= \left| \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} \left(e^{-i\sum_{j=1}^n k_j \theta_j} e^{-i\arg a_k} \right. \right.$$
$$\left. \left. + e^{i\sum_{j=1}^n k_j \theta_j} e^{i\arg b_k} \right) f\left(r_1^{k_1} e^{i\theta_1}, \ldots, r_n^{k_n} e^{i\theta_n}\right) d\theta_1 \cdots d\theta_n \right|$$
$$\leq \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} \left| 1 + e^{(2\sum_{j=1}^n k_j \theta_j + \arg a_k + \arg b_k)i} \right|$$
$$\times \left| f\left(r_1^{k_1} e^{i\theta_1}, \ldots, r_n^{k_n} e^{i\theta_n}\right) \right| d\theta_1 \cdots d\theta_n$$
$$\leq \frac{2}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} \left| \cos\left[\sum_{j=1}^n k_j \theta_j + \frac{(\arg a_k + \arg b_k)}{2} \right] \right| d\theta_1 \cdots d\theta_n.$$

Since $|k| \geq 1$, without loss of generality, we assume that $k_1 \neq 0$. By using Lemma 1, we see that

$$(2.4) \qquad \int_0^{2\pi} \left| \cos\left[\sum_{j=1}^n k_j \theta_j + \frac{(\arg a_k + \arg b_k)}{2} \right] \right| d\theta_1 = 4.$$

Then (2.3) and (2.4) yield that

$$r_1^{k_1} \cdots r_n^{k_n} \left(|a_k| + |b_k| \right) \leq \frac{4}{\pi}.$$

For $j \in \{1, \ldots, n\}$, by letting $r_j \to 1^-$, we obtain the desired result.

For $j \in \{1, \ldots, n\}$ and $z = (z_1, \ldots, z_n) \in \mathbb{D}^n$, let

$$\phi(\zeta) = \big(\phi_1(\zeta_1), \ldots, \phi_n(\zeta_n) \big),$$

where $\zeta = (\zeta_1, \ldots, \zeta_n) \in \mathbb{D}^n$ and

$$\phi_j(\zeta_j) = \frac{z_j + \zeta_j}{1 + \overline{z_j}\zeta_j}.$$

Then $f \circ \phi$ can be written in the following form:

$$T(\zeta) = f\big(\phi(\zeta) \big) = H(\zeta) + \overline{G(\zeta)} = \sum_k c_k \zeta^k + \sum_k \overline{d_k} \overline{\zeta}^k,$$

where $H = h \circ \phi$ and $G = g \circ \phi$. By using the proof of Claim 1, we get

$$(2.5) \qquad |c_k| + |d_k| \leq \frac{4}{\pi}.$$

For $r \in (0, 1)$ and $z \in \mathbb{D}^n$ with $\|z\|_\infty < r$, by the Cauchy integral formula (cf. [25], [31]), we see that

$$(2.6) \qquad f(z) = \frac{1}{(2\pi i)^n} \int_{|\eta_1|=r} \cdots \int_{|\eta_n|=r} \frac{h(\eta_1, \ldots, \eta_n)}{\prod_{j=1}^n (\eta_j - z_j)} d\eta_1 \cdots d\eta_n$$
$$+ \frac{1}{(2\pi i)^n} \int_{|\eta_1|=r} \cdots \int_{|\eta_n|=r} \frac{g(\eta_1, \ldots, \eta_n)}{\prod_{j=1}^n (\eta_j - z_j)} d\eta_1 \cdots d\eta_n,$$

which implies that

$$(2.7) \qquad \frac{\partial^\alpha f(z)}{\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}} = \frac{\alpha!}{(2\pi i)^n} \int_{|\eta_1|=r} \cdots \int_{|\eta_n|=r} \frac{h(\eta_1, \ldots, \eta_n)}{\prod_{j=1}^n (\eta_j - z_j)^{\alpha_j + 1}} d\eta_1 \cdots d\eta_n$$

and

$$(2.8) \qquad \frac{\partial^\alpha f(z)}{\partial \overline{z}_1^{\alpha_1} \cdots \partial \overline{z}_n^{\alpha_n}}$$
$$= \frac{\alpha!}{(2\pi i)^n} \int_{|\eta_1|=r} \cdots \int_{|\eta_n|=r} \frac{g(\eta_1, \ldots, \eta_n)}{\prod_{j=1}^n (\eta_j - z_j)^{\alpha_j + 1}} d\eta_1 \cdots d\eta_n.$$

For $j \in \{1, \ldots, n\}$, by taking $\eta_j = \phi_j(\zeta_j) = \frac{z_j + \zeta_j}{1 + \overline{z}_j \zeta_j}$, we see from (2.7) and (2.8) that

$$
\frac{\partial^\alpha f(z)}{\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}}
$$

$$
= \frac{\alpha!}{(2\pi i)^n \prod_{j=1}^n (1 - |z_j|^2)^{\alpha_j}}
$$

$$
\times \int_{|\phi_1(\zeta_1)|=r} \cdots \int_{|\phi_n(\zeta_n)|=r} \frac{H(\zeta_1, \ldots, \zeta_n) \prod_{j=1}^n (1 + \overline{z}_j \zeta_j)^{\alpha_j - 1}}{\prod_{j=1}^n \zeta_j^{\alpha_j + 1}} \, d\zeta_1 \cdots d\zeta_n
$$

$$
= \frac{\alpha!}{\prod_{j=1}^n (1 - |z_j|^2)^{\alpha_j}}
$$

$$
\times \sum_{k_1=0}^{\alpha_1-1} \cdots \sum_{k_n=0}^{\alpha_n-1} \binom{\alpha_1 - 1}{k_1} \cdots \binom{\alpha_n - 1}{k_n} c_{\alpha_1 - k_1, \ldots, \alpha_n - k_n} \prod_{j=1}^n z_j^{k_j}
$$

and

$$
\frac{\partial^\alpha f(z)}{\partial \overline{z}_1^{\alpha_1} \cdots \partial \overline{z}_n^{\alpha_n}}
$$

$$
= \frac{\alpha!}{\overline{(2\pi i)^n} \prod_{j=1}^n (1 - |z_j|^2)^{\alpha_j}}
$$

$$
\times \overline{\int_{|\phi_1(\zeta_1)|=r} \cdots \int_{|\phi_n(\zeta_n)|=r} \frac{G(\zeta_1, \ldots, \zeta_n) \prod_{j=1}^n (1 + \overline{z}_j \zeta_j)^{\alpha_j - 1}}{\prod_{j=1}^n \zeta_j^{\alpha_j + 1}} \, d\zeta_1 \cdots d\zeta_n}
$$

$$
= \frac{\alpha!}{\prod_{j=1}^n (1 - |z_j|^2)^{\alpha_j}}
$$

$$
\times \sum_{k_1=0}^{\alpha_1-1} \cdots \sum_{k_n=0}^{\alpha_n-1} \binom{\alpha_1 - 1}{k_1} \cdots \binom{\alpha_n - 1}{k_n} \overline{d}_{\alpha_1 - k_1, \ldots, \alpha_n - k_n} \prod_{j=1}^n z_j^{k_j}.
$$

It follows from (2.5) that

$$
\left| \frac{\partial^\alpha f(z)}{\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}} \right| + \left| \frac{\partial^\alpha f(z)}{\partial \overline{z}_1^{\alpha_1} \cdots \partial \overline{z}_n^{\alpha_n}} \right|
$$

$$
\leq \frac{\alpha!}{\prod_{j=1}^n (1 - |z_j|^2)^{\alpha_j}} \sum_{k_1=0}^{\alpha_1-1} \cdots \sum_{k_n=0}^{\alpha_n-1} \binom{\alpha_1 - 1}{k_1} \cdots \binom{\alpha_n - 1}{k_n}
$$

$$
\times \left(|c_{\alpha_1 - k_1, \ldots, \alpha_n - k_n}| + |d_{\alpha_1 - k_1, \ldots, \alpha_n - k_n}| \right) \prod_{j=1}^n |z_j|^{k_j}
$$

$$
\leq \frac{4}{\pi} \frac{\alpha!}{\prod_{j=1}^n (1 - |z_j|^2)^{\alpha_j}} \sum_{k_1=0}^{\alpha_1-1} \cdots \sum_{k_n=0}^{\alpha_n-1} \binom{\alpha_1 - 1}{k_1} \cdots \binom{\alpha_n - 1}{k_n} \prod_{j=1}^n |z_j|^{k_j}
$$

$$\leq \frac{4}{\pi} \frac{\alpha!}{\prod_{j=1}^{n}(1-|z_j|^2)^{\alpha_j}} \prod_{j=1}^{n} (1+|z_j|)^{\alpha_j-1}$$

$$= \alpha! \frac{4}{\pi} \prod_{j=1}^{n} \frac{(1+|z_j|)^{\alpha_j-1}}{(1-|z_j|^2)^{\alpha_j}}$$

$$\leq \alpha! \frac{4}{\pi} \frac{(1+\|z\|_\infty)^{|\alpha|-n}}{(1-\|z\|_\infty^2)^{|\alpha|}}.$$

The proof of the theorem is complete. $\square$

LEMMA 2. *Let* $f = (f_1, \ldots, f_N) \in \mathcal{P}(\mathbb{D}^n, \mathbb{B}^N)$ *and* $\beta = (\beta_1, \ldots, \beta_n)$ *be a multi-index consisting of* n *nonnegative integers* β_v, *where* $v \in \{1, \ldots, n\}$. *Suppose that for* $z \in \mathbb{D}^n$, $f(z) = \sum_\beta a_\beta z^\beta + \sum_\beta \bar{b}_\beta \bar{z}^\beta$, *where for* $j \in \{1, \ldots, N\}$, $f_j = \sum_\beta a_{j,\beta} z^\beta + \sum_\beta \bar{b}_{j,\beta} \bar{z}^\beta$, $a_\beta = (a_{1,\beta}, \ldots, a_{N,\beta})$ *and* $b_\beta = (b_{1,\beta}, \ldots, b_{N,\beta})$. *Then*

(1) *for* $m \in \{1, 2, \ldots\}$ *and* $z \in \mathbb{D}^n$,

(2.9)
$$\left\| \sum_{|\beta|=m} a_\beta z^\beta + \sum_{|\beta|=m} a_\beta z^\beta \right\| \leq \frac{4}{\pi};$$

(2) *for* $z \in \mathbb{D}^n$,

(2.10)
$$\left\| f(0) \right\|^2 + \sum_{|\beta|=1}^{\infty} \left(\|a_\beta\|^2 + \|b_\beta\|^2 \right) \leq 1.$$

Proof. We first prove (2.9). Since

$$\sum_{|\beta|=m} a_\beta z^\beta = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta} z) e^{-im\theta} \, d\theta$$

and

$$\sum_{|\beta|=m} \bar{b}_\beta \bar{z}^\beta = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta} z) e^{im\theta} \, d\theta,$$

by Lemma 1, we see that

$$\left\| \sum_{|\beta|=m} a_\beta z^\beta + \sum_{|\beta|=m} a_\beta z^\beta \right\| = \left\| \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta} z)(e^{-im\theta} + e^{im\theta}) \, d\theta \right\|$$

$$\leq \frac{1}{2\pi} \int_0^{2\pi} \left\| f(e^{i\theta} z) \right\| \left| e^{-im\theta} + e^{im\theta} \right| \, d\theta$$

$$\leq \frac{1}{\pi} \int_0^{2\pi} |\cos m\theta| \, d\theta$$

$$= \frac{4}{\pi}.$$

Now we prove (2.10). For $j \in \{1, \ldots, n\}$, let $\xi_j \in \mathbb{D}$ and $\xi = (\xi_1, \ldots, \xi_n)$. Then

$$
\frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} \left\| f\left(\xi_1 e^{i\theta_1}, \ldots, \xi_n e^{i\theta_n}\right) \right\|^2 d\theta_1 \cdots d\theta_n
$$

$$
= \left\| f(0) \right\|^2 + \sum_{|\beta|=1}^{\infty} \left(\|a_\beta\|^2 + \|b_\beta\|^2 \right) \left| \xi^\beta \right|^2 \leq 1.
$$

By letting $\xi \to \partial \mathbb{D}^n$, we get the desired result. $\qquad \square$

Proof of Theorem 2. We first prove (1.4). For any fixed $z \in \mathbb{D}^n$, let ϕ_z be a holomorphic automorphism of $\mathbb{D}^n$ with $\phi_z(0) = z$. For $\varsigma \in \mathbb{D}^n$, let $F(\varsigma) = f(\phi_z(\varsigma))$. Then

$$
DF(\varsigma) = Df\left(\phi_z(\varsigma)\right) D\phi_z(\varsigma) \quad \text{and} \quad \overline{D}F(\varsigma) = \overline{D}f\left(\phi_z(\varsigma)\right) \overline{D\phi_z(\varsigma)}.
$$

Applying (2.9) to F, we get

$$
(2.11) \qquad \left\| DF(0)\varsigma + \overline{D}F(0)\overline{\varsigma} \right\| = \left\| Df(z)D\phi_z(0)\varsigma + \overline{D}f(z)\overline{D\phi_z(0)\varsigma} \right\| \leq \frac{4}{\pi},
$$

where ς is regarded as a column vector and

$$
D\phi_z(0) = \begin{pmatrix} 1 - |z_1|^2 & 0 & 0 & \cdots & 0 \\ 0 & 1 - |z_2|^2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 1 - |z_{n-1}|^2 & 0 \\ 0 & 0 & \cdots & 0 & 1 - |z_n|^2 \end{pmatrix}.
$$

By applying (2.11), and by letting $\varsigma \to \partial \mathbb{D}^n$, we have

$$
\left\| Df(z)\theta + \overline{D}f(z)\overline{\theta} \right\| \leq \frac{4}{\pi(1 - \min_{1 \leq k \leq n} |z_k|^2)} \leq \frac{4}{\pi(1 - \|z\|_\infty^2)},
$$

where $\theta \in \mathbb{C}^n$ and $\|\theta\|_\infty = 1$.

Now we prove (1.5). For any fixed $z' \in \mathbb{D}^n \backslash \{0\}$, let

$$
F(\zeta) = \left\langle f\left(\frac{\zeta z'}{\|z'\|_\infty}\right), \frac{f(z')}{|f(z')|} \right\rangle, \quad \zeta \in \mathbb{D}.
$$

By using [21, Lemma], we have

$$
\left| F\left(\|z'\|_\infty \right) \right| = \left\| f(z') \right\| \leq \frac{4}{\pi} \arctan \left\| z' \right\|_\infty.
$$

The proof of the theorem is complete. $\qquad \square$

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