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RIEMANNIAN ALMOST *CR* MANIFOLDS WITH TORSION

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ABSTRACT. We characterize and study Riemannian almost *CR* manifolds admitting characteristic connections, that is, metric connections with totally skew-symmetric torsion parallelizing the almost *CR* structure. Natural constructions are provided of new nontrivial examples. We study the influence of the curvature of the metric on the underlying almost *CR* structure. A global classification is obtained under flatness assumption of a characteristic connection, provided that the fundamental 2-form of the structure is closed (quasi Sasakian condition).

1. Introduction

The study of Riemannian geometry on (almost) complex and (almost) contact manifolds is a primary topic in differential geometry. In particular, Kähler metrics and Sasakian metrics are widely studied in the literature and their most important features are well understood. The study of the interplay of curvature properties of the metric and the underlying structure is a fascinating subject, including for instance the classification and structure theory of Hermitian symmetric spaces and Sasakian φ-symmetric spaces.

On the other hand, both almost complex structures and almost contact structures are instances of the more general concept of *almost CR structure*. Recall that an almost CR manifold of type (n,k) is a triple (M,HM,J), where M is a real smooth manifold of dimension $2n+k$, $HM \subset TM$ is a subbundle of rank $2n$ of the tangent bundle and $J: HM \to HM$ is a partial almost complex structure, that is, a vector bundle endomorphism such that $J^2 = -\mathrm{Id}$. The integers n and k are respectively, the CR dimension and the CR codimension of M. When (HM,J) satisfies a standard condition of integrability one speaks of a *CR structure* (see Section 2). In particular, we recall

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that Sasakian manifolds are strongly pseudoconvex *CR* manifolds of hypersurface type, that is, $k = 1$ (see, e.g., [9]). For the notion of pseudoconvexity and pseudoconcavity in *CR* geometry, we refer the reader to [27]; see also Section 2.

Given an almost *CR* manifold, the most natural Riemannian metrics to study on M are those *compatible* with the underlying structure, that is

$$g(JX, JY) = g(X, Y)$$

for all sections X, Y of HM; we refer to (HM, J, g) as a *Riemannian almost CR structure*. In the present paper, we single out a special class of compatible Riemannian metrics on almost *CR* manifolds. They are defined by the requirement that M should admit a metric linear connection parallelizing the almost *CR* structure, and whose torsion tensor is totally skew-symmetric (with respect to the metric). Given such a metric g, any connection on M having the properties stated above will be called *characteristic*, while (M, HM, J, g) will be referred to as a *Riemannian almost CR manifold with torsion*. The main motivation in the choice of this kind of metrics is in a recent development of the theory of metric G-structures by Friedrich and Ivanov, see [15]; their study reveals that for a subgroup G of $O(n)$, the circumstance that a G-structure on a manifold admits an adapted connection with (nonvanishing) totally skew-symmetric torsion leads to remarkable features of the structure itself, often of interest in the theoretical and mathematical physics literature. For a comprehensive discussion of this subject, see the survey article [2] of Agricola.

In [15], the authors obtain two characterizations of almost Hermitian and almost contact metric structures admitting a characteristic connection, also proving the uniqueness (see Theorems 10.1 and 8.2 in [15]). The almost Hermitian structures satisfying this condition are exactly those belonging to the class $\mathcal{G}_1$ in the Gray–Hervella classification. For Kähler manifolds, the Levi–Civita connection is the unique characteristic one, while for Hermitian manifolds, it is the Bismut connection. For Sasakian structures, the characteristic connection is the one introduced first by Okumura in [28].

We generalize and unify the above results providing necessary and sufficient conditions for a Riemannian almost *CR* manifold (M, HM, J, g) to admit a characteristic connection (Theorem 3.3); such a connection is unique only in the case where the *CR* codimension is less than 3. We also furnish an explicit description of the torsion of any characteristic connection.

Examples of Riemannian almost *CR* manifolds with torsion having *CR* codimension higher than 1 come from the theory of homogeneous naturally reductive spaces, standard *CR* manifolds according to Tanaka's theory [36], f-structures with complemented frames [8], 3-Sasakian manifolds, complex contact manifolds [9], and from the geometry of *CR* submanifolds according to Bejancu [6]. See Section 4 for a detailed description of these examples.

Our first objective is to construct a wide class of new examples, showing that this kind of structures actually arise in natural constructions. For instance, starting from a generalized flag manifold $N = G/H$, we prove that each G-invariant Kähler structure (J, g), J being the canonical complex structure on N, admits a canonical lift to a G-invariant Riemannian *CR* structure with torsion on every homogeneous space $M = G/K$, with K an arbitrary closed subgroup of H. For instance, this construction is available on Stiefel manifolds.

In particular, every compact semisimple Lie group admits a family of natural left-invariant Riemannian *CR* structures with torsion; these structures have some interesting features resembling Sasakian ones; namely, the fundamental 2-form Φ, defined as customary by

$$\Phi(X,Y) := g(X, JPY),$$

is closed; here $P : TM \to HM$ denotes the orthogonal projection. Moreover, the underlying *CR* structure is strongly pseudoconvex, in fact the metric g coincides on HM with the Levi form in a suitable direction (Corollary 6.2).

A Riemannian almost *CR* manifold with torsion whose fundamental 2-form is closed will be called a *quasi Sasakian CR manifold*. This terminology is consistent with the notion of quasi Sasakian manifolds in almost contact metric geometry (cf. [7]).

Actually, we prove that a homogeneous space $M = G/H$ of a compact semisimple Lie group carries a G-invariant quasi Sasakian *CR* structure if and only if it carries a G-invariant closed 2-form; the structure is *regular* in the sense that the distribution $HM^{\perp}$ is integrable and the corresponding foliation $\mathfrak{F}$ is regular; the space of leaves $M/\mathfrak{F}$ is a flag manifold and (HM, J, g) projects onto an invariant Kähler structure (Proposition 3.12 and Theorem 6.8). We remark that the above considerations applied to the orthogonal group, yield that the bundle of orthonormal frames $O(M)$ of any Riemannian manifold (M, g) is a natural way a Riemannian *CR* manifold with torsion.

In Section 5, we treat the general problem of lifting Riemannian almost *CR* structures with torsion from the base to the total space of a principal fiber bundle $Q \to M$, endowed with a principal connection. The necessary and sufficient condition is formulated in terms of the curvature of the connection (see Theorem 5.1); it is automatically satisfied for flat bundles. It holds true also for the frame bundle $L(M)$ of any Riemannian almost *CR* manifold M with *parallel* torsion, such as nearly Kähler and Sasakian manifolds.

Another objective of the paper is to investigate the influence of the curvature on the underlying almost *CR* structure. It is well known that for Sasakian manifolds all the *mixed sectional curvatures* $K(X, \xi)$ of 2-planes spanned by a holomorphic vector $X \in HM$ and $\xi \in HM^{\perp}$ are equal to 1. The situation changes radically in our more general setting. For instance, when $k = 2$ and the almost CR structure is at least partially integrable, if all the mixed sectional curvatures do not vanish at one point x_o, then (M, HM, J) must be pseudoconcave at x_o. Moreover, n must be even. More generally, for $k \geq 2$, assuming in addition that $HM^{\perp}$ is an integrable distribution, then all Levi forms at x_o turn out to be nondegenerate, so that (M, HM, J) is pseudoconcave and we obtain the sharp inequality

$$k \leq 2b + 1,$$

where 2^b is the greatest power of 2 which divides n. Section 7 contains other results concerning the interplay between Levi forms and mixed sectional curvatures, yielding some obstructions to the integrability of $HM^{\perp}$ or of HM (Levi-flatness).

We also prove that a Riemannian space form (M, g) cannot carry a non-Levi-flat compatible almost CR structure with CR codimension $k \geq 2$, turning it into an a Riemannian almost CR manifold with parallel torsion (see Theorems 7.1 and 7.17). The parallelizable 7-dimensional sphere with the round metric provides an example showing that the last condition on torsion cannot be dropped.

Finally, the complete, simply connected, irreducible quasi Sasakian CR manifolds admitting a *flat* characteristic connection are classified in Theorem 8.2. In accordance with a classical result of Cartan–Schouten, these must be compact simple Lie groups; precisely, we show they are those arising as the universal covering of the connected component G of the isometry group of an irreducible Hermitian symmetric space of compact type G/H. The quasi Sasakian CR structure coincides with the one obtained, according to the above cited Corollary 6.2, from the Kähler–Einstein structure on G/H. It should be observed again that, in particular, the underlying CR structure is strongly pseudoconvex. This is remarkable, since for any quasi Sasakian manifold of dimension at least 5, flatness of the characteristic connection forces the Levi degeneracy of HM (cf. Proposition 8.1).

The above classification is of global nature; on the other hand, we prove existence results of local nature, stating that given a Hermitian locally symmetric space, respectively a Sasakian locally φ-symmetric space N, if the sectional curvature at a fixed point is nonnegative, respectively ≥ -3, there exists a quasi Sasakian CR manifold admitting a flat characteristic connection and fibering onto N.

2. Preliminaries

An *almost CR structure of type* (n, k) on a differentiable manifold M of dimension $2n + k$, $k \geq 0$, is a pair (HM, J), where HM is a rank $2n$ vector subbundle of the tangent bundle TM, and $J : HM \to HM$ is a smooth fiber preserving bundle isomorphism, such that $J^2 = -\mathrm{Id}$. The triple (M, HM, J)

will be called an *almost CR manifold of type* (n,k); n and k are the *CR dimension* and the *CR codimension* of M, respectively. The almost *CR* structure (HM,J) will be called *partially integrable* if

$$[X,Y]-[JX,JY]\in\Gamma HM \quad \forall X,Y\in\Gamma HM. \tag{2.1}$$

If, in addition, the formal integrability condition

$$N_J(X,Y):=[JX,JY]-[X,Y]-J\big([JX,Y]+[X,JY]\big)=0 \tag{2.2}$$

is satisfied for every $X,Y\in\Gamma HM$, (M,HM,J) is called a *CR manifold.* Here and in the following we use the notation ΓE for the module of smooth sections of a vector bundle E; we also set $\mathfrak{X}(M):=\Gamma TM$.

We recall that a *CR map* between two almost *CR* manifolds (M,HM,J) and (M',HM',J') is a smooth map $f:M\to M'$ whose tangent map at each point $x\in M$ satisfies $f_*(H_xM)\subset H_{f(x)}M'$ and $J'_{f(x)}\circ f_*=f_*\circ J_x$. We say that M and M' are *CR diffeomorphic* provided there exits a *CR* map $f: M\to M'$ which is also a diffeomorphism.

If D is any differentiable distribution on a manifold M, the *Levi–Tanaka form* of D is the $\mathcal{C}^\infty(M)$-bilinear map $L:\Gamma D\times\Gamma D\to\Gamma(TM/D)$ defined by

$$L(X,Y):=\pi[X,Y],$$

where $\pi:TM\to TM/D$ is the projection onto the quotient bundle (see [32]). The vanishing of L is equivalent to the integrability of the distribution D in the sense of Frobenius; in this case D is called *Levi flat.* We shall denote by $N(L_{x_o})$ the kernel of the determination L_{x_o} of the Levi–Tanaka form at the point x_o. If $N(L_{x_o})=0$, D is said to be *Levi nondegenerate* at x_o. In all that follows, we shall adopt the notation $N(L)$ for the kernel of a skew-symmetric bilinear map $L:V\times V\to W$, where V,W are vector spaces.

Denoting by $D^o\subset T^*M$ the annihilator of D and given a global section $\eta\in\Gamma D^o$, the corresponding (scalar) *Levi form* is defined by

$$L_\eta(X,Y):=-2d\eta(X,Y)=\eta\big([X,Y]\big), \quad \forall X,Y\in\Gamma D. \tag{2.3}$$

If furthermore M is endowed with a Riemannian metric g, both the bundles TM/D and D^o will be identified with $D^\perp$, and π with the orthogonal projection with respect to g. In this context, if $\xi\in\Gamma D^\perp$ and η is the dual 1-form, we shall denote the Levi form L_η by L_ξ, so that

$$L_\xi(X,Y)=g\big([X,Y],\xi\big).$$

A kind 2 distribution is defined by the requirement that

$$\mathfrak{X}(M)=\Gamma D+[\Gamma D,\Gamma D], \tag{2.4}$$

which is equivalent to require that at each point $x\in M$,

$$\big\{\xi\in D_x^\perp : L_\xi=0\big\}=\{0\}.$$

When the above condition holds at one point x_o, we also say that D has kind 2 at x_o.

For a partially integrable almost CR structure (HM, J), for any $\eta \in \Gamma H^o M$ we can also introduce a Hermitian symmetric $\mathcal{C}^\infty(M)$-bilinear map $\mathcal{L}_\eta : \Gamma HM \times \Gamma HM \to \mathcal{C}^\infty(M)$, also called *Levi form*, defined as follows:

$$\mathcal{L}_\eta(X, Y) := \eta\big([X, JY]\big).$$

A partially integrable almost CR manifold is called *pseudoconvex* at a point $x \in M$ if $\mathcal{L}_\eta$ is positive definite for some $\eta \in H^o_x M$. It is called *q-pseudoconcave* at x, if for every $\eta \in H^o_x M$, $\mathcal{L}_\eta$ has at least q negative eigenvalues (see, e.g., [27]). Finally, M is called *strongly pseudoconvex* if there exists a global section η of $H^o M$ such that $\mathcal{L}_\eta$ is everywhere positive definite.

Next, we recall some basic information concerning metric connections with torsion. Let (M, g) be a Riemannian manifold; a metric connection ∇ with torsion T is said to have (totally) skew-symmetric torsion if the $(0, 3)$-tensor field T defined by

$$T(X, Y, Z) = g\big(T(X, Y), Z\big)$$

is a 3-form. In this case, the relation between ∇ and the Levi–Civita connection ∇^g is

$$\nabla_X Y = \nabla^g_X Y + \frac{1}{2} T(X, Y). \tag{2.5}$$

Denoting by R the curvature tensor of ∇, we define the $(0, 4)$ curvature tensor by

$$R(X, Y, Z, V) := g\big(R(Z, V)Y, X\big),$$

which satisfies

$$R(X, Y, Z, V) = -R(Y, X, Z, V) = -R(X, Y, V, Z),$$

and the Bianchi identity:

$$\begin{aligned}\mathfrak{S}_{XYZ} R(V, X, Y, Z) = 4dT(X, Y, Z, V) - \sigma_T(X, Y, Z, V) \\ + (\nabla_V T)(X, Y, Z),\end{aligned} \tag{2.6}$$

where σ_T is the 4-form given by

$$\begin{aligned}\sigma_T(X, Y, Z, V) = g\big(T(X, Y), T(Z, V)\big) + g\big(T(Y, Z), T(X, V)\big) \\ + g\big(T(Z, X), T(Y, V)\big).\end{aligned} \tag{2.7}$$

Moreover, the exterior derivative of T satisfies

$$\begin{aligned}4dT(X, Y, Z, V) = \mathfrak{S}_{XYZ}\big[(\nabla_X T)(Y, Z, V) - (\nabla_V T)(X, Y, Z)\big] \\ + 2\sigma_T(X, Y, Z, V).\end{aligned} \tag{2.8}$$

If $\nabla T = 0$, then one can show that

$$R(X, Y, Z, V) = R(Z, V, X, Y). \tag{2.9}$$

Finally, the Riemannian curvature and R are related by:

$$R^g(V,Z,X,Y) = R(V,Z,X,Y) - \frac{1}{2}(\nabla_X T)(Y,Z,V) + \frac{1}{2}(\nabla_Y T)(X,Z,V) - \frac{1}{4}g\big(T(X,Y),T(Z,V)\big) - \frac{1}{4}\sigma_T(X,Y,Z,V). \tag{2.10}$$

See for more details [2] and references therein.

3. Characteristic connections on Riemannian almost *CR* manifolds

Let (M,HM,J,g) be a Riemannian almost CR manifold of type (n,k), that is (HM,J) is an almost CR structure of type (n,k) and g is a compatible Riemannian metric on M, that is,

$$g(JX,JY) = g(X,Y) \quad \forall X,Y \in \Gamma HM.$$

Let

$$P : TM \to HM$$

be the orthogonal projection and define an operator $\Gamma : \Gamma HM \times \Gamma HM \to \Gamma HM$ by

$$\Gamma_X Y := P\big(\nabla^g_X Y\big).$$

We also introduce the tensor $N : \Gamma HM \times \Gamma HM \to \Gamma HM$ defined by

$$N(X,Y) := P\big([JX,JY] - [X,Y]\big) - JP\big([JX,Y] + [X,JY]\big). \tag{3.1}$$

Notice that if the almost CR structure is partially integrable, the tensor N coincides with N_J. It can be verified that

$$N(X,Y) = (\Gamma_{JX} J)Y - (\Gamma_{JY} J)X + (\Gamma_X J)JY - (\Gamma_Y J)JX, \tag{3.2}$$

where, for any $X \in \Gamma HM$, $\Gamma_X J : \Gamma HM \to \Gamma HM$ is the $C^\infty(M)$-linear operator defined by

$$(\Gamma_X J)Y := \Gamma_X(JY) - J(\Gamma_X Y).$$

This operator is skew-symmetric with respect to g and anticommutes with J. We shall denote by the same symbol N the $(0,3)$-tensor defined by

$$N(X,Y,Z) := g\big(N(X,Y),Z\big) \quad \forall X,Y \in \Gamma HM.$$

PROPOSITION 3.1. *The following conditions are equivalent*:

(a) *the* $(0,3)$*-tensor* N *is skew-symmetric*;
(b) *for every* $X \in \Gamma HM$, $(\Gamma_X J)X = (\Gamma_{JX} J)JX$;
(c) *for every* $Y,Z \in \Gamma HM$, $(\Gamma_Y J)Z + (\Gamma_Z J)Y = (\Gamma_{JY} J)JZ + (\Gamma_{JZ} J)JY$.

Proof. Using (3.2) and the fact that the operators $\Gamma_X J$ are skew-symmetric and anticommute with J, a straightforward computation shows that

$$N(X,Y,JZ) - N(Y,Z,JX) + N(Z,X,JY)$$
$$= 2g\big((\Gamma_{JX}J)JY,Z\big) - 2g\big((\Gamma_X J)Y,Z\big).$$

Applying this formula for $Z = JX$, we have

$$N(X,Y,X) + N(Y,JX,JX) - N(JX,X,JY) \tag{3.3}$$
$$= 2g\big((\Gamma_X J)Y,JX\big) - 2g\big((\Gamma_{JX}J)JY,JX\big).$$

Now, the tensor N satisfies $N(Y,JX) = -JN(Y,X)$ so that

$$N(Y,JX,JX) = -g\big(JN(Y,X),JX\big) = N(X,Y,X).$$

Being also $N(X,JX) = 0$, (3.3) yields

$$N(X,Y,X) = g\big((\Gamma_{JX}J)JX,JY\big) - g\big((\Gamma_X J)X,JY\big)$$

for every $X,Y \in \Gamma HM$, which implies the equivalence of (a) and (b). The equivalence of (b) and (c) is immediate. □

DEFINITION 3.2. Let (M,HM,J,g) be a Riemannian almost *CR* manifold. We say that (M,HM,J,g) is a *Riemannian almost CR manifold with torsion* if there exists a metric connection on M with totally skew-symmetric torsion which parallelizes the structure (HM,J). Such a connection will be called *characteristic*.

In order to provide necessary and sufficient conditions for the existence of a characteristic connection, we introduce for every $\xi \in \Gamma HM^\perp$, the bundle endomorphism $\theta_\xi : HM \to HM$ defined by

$$\theta_\xi(X) := P[\xi,JX] - JP[\xi,X], \quad X \in \Gamma HM.$$

We shall also denote by $L : \Gamma HM \times \Gamma HM \to \Gamma HM^\perp$ and $L' : \Gamma HM^\perp \times \Gamma HM^\perp \to \Gamma HM$ the Levi–Tanaka forms of HM and $HM^\perp$, respectively. Then we state the following theorem.

THEOREM 3.3. *Let (M,HM,J,g) be a Riemannian almost CR manifold. Then M admits a characteristic connection if and only if the following conditions are satisfied*:

(1) *the tensor N is skew-symmetric*,
(2) $g(\theta_\xi(X),Y) = g([JX,Y]+[X,JY],\xi)$,
(3) $(\mathcal{L}_\xi g)(X,Y) = 0$,
(4) $(\mathcal{L}_X g)(\xi,\xi') = 0$,

for every $X,Y \in \Gamma HM$, and $\xi,\xi' \in \Gamma HM^\perp$. Furthermore, the torsion of each characteristic connection satisfies:

$$T(X,Y,Z) = N(X,Y,Z) - \mathfrak{S}_{XYZ}\, g\big((\Gamma_{JX}J)Y,Z\big), \tag{3.4}$$

$$T(X,Y,\xi) = -g\big([X,Y],\xi\big) = -L_\xi(X,Y), \tag{3.5}$$

$$T(X,\xi,\xi') = -g([\xi,\xi'],X) = -L'_X(\xi,\xi') \tag{3.6}$$

for every $X,Y,Z\in\Gamma HM$, *and* $\xi,\xi'\in\Gamma HM^\perp$.

REMARK 3.4. We point out the circumstance that a Riemannnian almost *CR* manifold with torsion of *CR* codimension $k<3$ admits a unique characteristic connection. For $k\geq 3$ the characteristic connections are in a one-to-one correspondence with the smooth sections of $\Lambda^3(HM^\perp)$. In fact, for each $A\in\Gamma\Lambda^3(HM^\perp)$, the corresponding characteristic connection is the one whose torsion satisfies

$$T(\xi,\xi',\xi'') = A(\xi,\xi',\xi'') \tag{3.7}$$

for every $\xi,\xi',\xi''\in\Gamma HM^\perp$.

Proof of Theorem 3.3. Assume that M carries a characteristic connection ∇, given as in (2.5). If $D:\mathfrak{X}(M)\times\Gamma HM\to\Gamma HM$ is the induced connection on HM, being $DJ=0$ one gets

$$2g((\Gamma_X J)Y,Z) + T(X,JY,Z) + T(X,Y,JZ) = 0 \tag{3.8}$$

for every $X,Y,Z\in\Gamma HM$. Using (3.2) and (3.8), we get

$$\begin{aligned} N(X,Y,Z) = {} & T(X,Y,Z) - T(JX,JY,Z) \\ & - T(JX,Y,JZ) - T(X,JY,JZ), \end{aligned} \tag{3.9}$$

which implies that N is skew-symmetric, thus proving (1). Before proving (2)–(4), we observe that the torsion satisfies Formulas (3.5) and (3.6) since ∇ parallelizes HM. We also prove (3.4); applying (3.8), we have

$$\mathfrak{S}_{XYZ}\, g((\Gamma_X J)Y,Z) = -\mathfrak{S}_{XYZ}\, T(X,Y,JZ). \tag{3.10}$$

Now, since the operator $\Gamma_{JX}J$ anticommutes with J, we have

$$g((\Gamma_{JX}J)Y,Z) = -g((\Gamma_{JX}J)JY,JZ),$$

and applying (3.9) and (3.10), we obtain (3.4).

In order to prove (2), using the parallelism of (HM,J), we have

$$\begin{aligned} g(\theta_\xi(X),Y) &= g([\xi,JX],Y) + g([\xi,X],JY) \\ &= g(\nabla_\xi(JX) - \nabla_{JX}\xi - T(\xi,JX),Y) \\ &\quad + g(\nabla_\xi X - \nabla_X\xi - T(\xi,X),JY) \\ &= -T(JX,Y,\xi) - T(X,JY,\xi), \end{aligned}$$

and applying (3.5), we get (2). Using $\nabla g=0$, we also have

$$\begin{aligned} (\mathcal{L}_\xi g)(X,Y) &= \xi(g(X,Y)) - g(\nabla_\xi X - \nabla_X\xi - T(\xi,X),Y) \\ &\quad - g(X,\nabla_\xi Y - \nabla_Y\xi - T(\xi,Y)) \\ &= T(\xi,X,Y) + T(\xi,Y,X) = 0. \end{aligned}$$

Analogously, we get (4).

As for the converse, let us suppose that (1)–(4) hold. Let ∇ be the linear connection given by (2.5), where T as a $(0,3)$-tensor is determined by (3.4)-(3.7) and

$$T(\xi,X,Y) = -T(X,\xi,Y) = T(X,Y,\xi),$$
$$T\big(\xi,\xi',X\big) = -T\big(\xi,X,\xi'\big) = T\big(X,\xi,\xi'\big).$$

In order to check that T is a 3-form, a simple computation using (3.2) gives

$$T(X,Y,Z) = g\big((\Gamma_X J)Y - (\Gamma_Y J)X, JZ\big) - g\big((\Gamma_{JZ} J)X, Y\big), \tag{3.11}$$

for every $X,Y,Z \in \Gamma HM$, which implies that $T(X,X,Z) = 0$. Moreover,

$$T(X,Y,X) = -g\big((\Gamma_X J)JX + (\Gamma_{JX} J)X, Y\big) - g\big((\Gamma_Y J)X, JX\big).$$

Being N skew-symmetric, from (c) of Proposition 3.1, $(\Gamma_X J)JX + (\Gamma_{JX} J)X = 0$. On the other hand, $g((\Gamma_Y J)X, JX) = 0$ since the operators $\Gamma_Y J$ and J are skew-symmetric and anticommute. Thus, $T(X,Y,X) = 0$.

Now, we prove that ∇ parallelizes HM. Indeed, for every $X,Y \in \Gamma HM$ and $\xi \in \Gamma HM^\perp$, we have

$$\begin{aligned} g(\nabla_X Y,\xi) &= g\big(\nabla^g_X Y,\xi\big) - \frac{1}{2} g\big([X,Y],\xi\big) \\ &= \frac{1}{2} g\big(\nabla^g_X Y + \nabla^g_Y X,\xi\big) \\ &= -\frac{1}{2} g\big(Y, \nabla^g_\xi X - [\xi,X]\big) - \frac{1}{2} g\big(X, \nabla^g_\xi Y - [\xi,Y]\big) \\ &= -\frac{1}{2} (\mathcal{L}_\xi g)(X,Y) \end{aligned}$$

which vanishes because of (3). Analogously, by assumption (4), for every $X \in \Gamma HM$ and $\xi,\xi' \in \Gamma HM^\perp$,

$$g\big(\nabla_\xi X,\xi'\big) = \frac{1}{2} (\mathcal{L}_X g)\big(\xi,\xi'\big) = 0.$$

Finally, denoting by $D : \mathfrak{X}(M) \times \Gamma HM \to \Gamma HM$ the induced connection on HM, we prove that $DJ = 0$. For every $X,Y,Z \in \Gamma HM$, we have

$$\begin{aligned} g\big((D_X J)Y, Z\big) &= g\big(\nabla_X (JY), Z\big) + g(\nabla_X Y, JZ) \\ &= g\big((\Gamma_X J)Y, Z\big) + \frac{1}{2}\big(T(X,JY,Z) + T(X,Y,JZ)\big), \end{aligned}$$

which vanishes, since applying (3.11) and (c) of Proposition 3.1, we have

$$\begin{aligned} T(X,JY,Z) + T(X,Y,JZ) &= g\big((\Gamma_X J)JY - (\Gamma_{JY} J)X, JZ\big) \\ &\quad - g\big((\Gamma_X J)Y - (\Gamma_Y J)X, Z\big) \\ &\quad - g\big((\Gamma_{JZ} J)X, JY\big) + g\big((\Gamma_Z J)X, Y\big) \\ &= -2g\big((\Gamma_X J)Y, Z\big). \end{aligned}$$

Now, notice that, since ∇ parallelizes HM, for every $X,Y\in\Gamma HM$ and $\xi\in\Gamma HM^{\perp}$,

$$T(X,\xi,Y)=-g(\nabla_{\xi}X,Y)-g\big([X,\xi],Y\big).$$

Hence, using also (3.5), we have

$$g(\nabla_{\xi}X,Y)=-g\big([X,Y],\xi\big)+g\big([\xi,X],Y\big). \tag{3.12}$$

Therefore, for every $\xi\in\Gamma HM^{\perp}$ and $X,Y\in\Gamma HM$,

$$\begin{aligned} g\big((D_{\xi}J)X,Y\big)&=g\big(\nabla_{\xi}(JX),Y\big)+g(\nabla_{\xi}X,JY)\\ &=-g\big([JX,Y]+[X,JY],\xi\big)+g\big(\theta_{\xi}(X),Y\big)\end{aligned}$$

which vanishes because of (2). □

REMARK 3.5. If R is the curvature tensor of a characteristic connection ∇, then

$$R(JX,JY,Z,V)=R(X,Y,Z,V)$$

for every $X,Y,Z,V\in\Gamma HM$. If furthermore $\nabla T=0$, then by (2.9), we have

$$R(X,Y,JZ,JV)=R(X,Y,Z,V). \tag{3.13}$$

REMARK 3.6. For a Riemannian almost *CR* manifold with torsion, the integrability of the distributions HM and $HM^{\perp}$ can be characterized by means of the torsion of any characteristic connection. Indeed, from (3.5) and (3.6), it follows that

- (i) HM is integrable iff $T(X,Y,\xi)=0$, for any $X,Y\in\Gamma HM$ and $\xi\in\Gamma HM^{\perp}$;
- (ii) $HM^{\perp}$ is integrable iff $T(X,\xi,\xi')=0$ for any $X\in\Gamma HM$ and $\xi,\xi'\in\Gamma HM^{\perp}$;
- (iii) HM and $HM^{\perp}$ are both integrable iff $T(X,\xi)=0$, for any $X\in\Gamma HM$ and $\xi\in\Gamma HM^{\perp}$.

Moreover, the almost *CR* structure (HM,J) is partially integrable if and only if $\theta_{\xi}=0$ for every $\xi\in\Gamma HM^{\perp}$. If furthermore $N=0$, the structure is integrable.

PROPOSITION 3.7. *Let (M,HM,J,g) be a Riemannian almost CR manifold with torsion. Assume that $HM^{\perp}$ is integrable, then*

- (i) *the foliation $\mathfrak{F}$ determined by $HM^{\perp}$ is Riemannian with totally geodesic leaves*;
- (ii) *if $\mathfrak{F}$ is regular, the space of leaves $M/\mathfrak{F}$ is an almost Hermitian manifold of type $\mathcal{G}_1$, provided that J is partially integrable. Furthermore, $M/\mathfrak{F}$ is Hermitian if and only if (HM,J) is integrable.*

In the case (ii), *we say that M is* regular.

Proof. From (2.5) and (3.6) it follows that the leaves of $\mathfrak{F}$ are totally geodesic, while condition (3) in Theorem 3.3 guarantees that the foliation is Riemannian. Assuming $\mathfrak{F}$ regular, g is projectable and induces a Riemannian metric on $M/\mathfrak{F}$, making the natural projection $\pi: M \to M/\mathfrak{F}$ a Riemannian submersion. Now, if J is partially integrable, being $\theta_\xi = 0$ for every $\xi \in \Gamma HM^\perp$, the (1,1)-tensor field φ defined by $\varphi Z = JPZ$ is projectable (cf. e.g., (P.1) in [4]) and induces an almost complex structure J on $M/\mathfrak{F}$, making π a *CR* map. Finally, it is straightforward to see that for every $X, Y, Z \in \Gamma HM$ which are projectable vector fields with respect to π, we have

$$N(X,Y,Z) \circ \pi = N(\pi_* X, \pi_* Y, \pi_* Z),$$

where the tensor N in the right-hand side is the Nijenhuis tensor of $M/\mathfrak{F}$. This justifies the last claims. □

Next we describe some special classes of Riemannian almost *CR* manifolds which may be thought as generalizations of Kähler and (quasi) Sasakian manifolds. In order to do this, we introduce the *fundamental* 2*-form* Φ defined by

$$\Phi(U,V) := g(PU, JPV)$$

for every $U, V \in \mathfrak{X}(M)$.

Proposition 3.8. *Let (M, HM, J, g) be a Riemannian almost CR manifold with torsion. Then, the following conditions are equivalent*:

(i) $\Gamma_X J = 0$ *for every* $X \in \Gamma HM$,
(ii) $T(X,Y,Z) = 0$ *for every* $X, Y, Z \in \Gamma HM$,
(iii) $d\Phi(X,Y,Z) = 0$ *for every* $X, Y, Z \in \Gamma HM$.

If any of the above conditions is satisfied, then $N = 0$.

Proof. The equivalence of (i) and (ii) is an immediate consequence of (3.11) and (3.8). Now a simple computation shows that

$$3d\Phi(X,Y,Z) = -\mathfrak{S}_{XYZ}\, g\big((\Gamma_X J)Y, Z\big)$$

for every $X, Y, Z \in \Gamma HM$, and thus (i) implies (iii). Conversely, supposing (iii), from (3.4) we have that T and N coincide on HM. Hence, from (3.10), we have

$$N(X,Y,JZ) + N(Y,Z,JX) + N(Z,X,JY) = 0$$

for every $X, Y, Z \in \Gamma HM$. Since N is skew-symmetric and $N(JX,Y) = -JN(X,Y)$, we deduce that $N(X,Y,JZ) = 0$. Therefore (ii), or equivalently (i), holds. □

A Riemannian almost *CR* manifold with torsion satisfying any of the equivalent conditions in the previous result will be called *of Kähler type*. This terminology is in accordance with [14], where it has been adopted in the more general context of (generalized) pseudohermitian geometry.

PROPOSITION 3.9. *Let (M,HM,J,g) be a Riemannian almost CR manifold with torsion. Then the fundamental 2-form is closed if and only if the almost CR structure (HM,J) is partially integrable, M is of Kähler type and $HM^{\perp}$ is an integrable distribution. If these conditions are satisfied the almost CR structure is integrable.*

Proof. The partial integrability of the almost *CR* structure is equivalent to the vanishing of $d\Phi(\xi,X,Y)$ for every $\xi\in\Gamma HM^{\perp}$ and $X,Y\in\Gamma HM$. Indeed, applying (3) of Theorem 3.3, we have

$$\begin{aligned}3d\Phi(\xi,X,Y)&=\xi\big(g(X,JY)\big)-g\big([\xi,X],JY\big)-g\big([Y,\xi],JX\big)\\&=g\big(X,[\xi,JY]\big)-g\big(X,JP[\xi,Y]\big)\\&=g\big(\theta_{\xi}(Y),X\big).\end{aligned}$$

On the other hand, the distribution $HM^{\perp}$ is integrable iff $d\Phi(\xi,\xi',X)=0$ for every $\xi,\xi'\in\Gamma HM^{\perp}$ and $X\in\Gamma HM$, being

$$3d\Phi\big(\xi,\xi',X\big)=-g\big(\big[\xi,\xi'\big],JX\big).$$

Finally, taking into account Proposition 3.8 the proof is easily completed. □

REMARK 3.10. In the case of an almost contact metric manifold (M,φ,ξ,η,g), necessary and sufficient conditions for the existence of a characteristic connection have been already determined in [15, Theorem 8.2], namely such a connection exists if and only if the tensor $N_{\varphi}:=[\varphi,\varphi]+2d\eta\otimes\xi$ is totally skew-symmetric and ξ is Killing. Here $[\varphi,\varphi]$ denotes the Nijenhuis tensor of φ. Consequently, if these conditions are satisfied, it can be easily seen that the requirement $d\Phi=0$ is equivalent to the circumstance that the structure is quasi Sasakian in the sense of Blair [7] (see also [15, Theorem 8.4]).

According to this remark, we shall adopt the following terminology:

DEFINITION 3.11. A Riemannian almost *CR* manifold with torsion, whose fundamental 2-form Φ is closed, will be called a *quasi Sasakian CR manifold.*

PROPOSITION 3.12. *Every homogeneous quasi Sasakian CR manifold $M=G/K$, where G is a compact semisimple Lie group is regular. The space of leaves $M/\mathfrak{F}$ is a generalized flag manifold G/H and the projected structure is Kähler and G-invariant.*

Proof. Consider the orthogonal decomposition $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{n}$ of $\mathfrak{g}$ with respect to the Killing form B, where $\mathfrak{k}=\mathrm{Lie}(K)$. Since the fundamental 2-form Φ of M is G-invariant, it determines an element $\bar{\Phi}\in\Lambda^2(\mathfrak{n})$, invariant under $\mathrm{Ad}(K)$. We extend $\bar{\Phi}$ to an element of $\Lambda^2(\mathfrak{g})$ in a trivial way. Since Φ is closed, a standard argument based on the assumption that G is semisimple shows that $\bar{\Phi}$ is given by

$$\bar{\Phi}(X,Y)=B\big([Z_o,X],Y\big)\quad\forall X,Y\in\mathfrak{g},\tag{3.14}$$

where $Z_o \in \mathfrak{g}$ is uniquely determined (cf. [11]). Since $\bar{\Phi}$ is $\mathrm{Ad}(K)$-invariant, we have that K is contained in the centralizer H of Z_o in G. Now G/H is a generalized flag manifold; denote by $\pi : M \to G/H$ the natural projection. Observe that, denoting by $\mathfrak{h}$ the Lie algebra of H, at the point $o \in M$ corresponding to the coset K, we have

$$H_o M^{\perp} = N(\bar{\Phi}) \cap \mathfrak{n} = \mathfrak{h} \cap \mathfrak{n}$$

which implies that $H_o M^{\perp} = \mathrm{Ker}(\pi_{*o})$. By G-invariance, it follows that $\mathfrak{F}$ is regular (cf. [37]) and that $M/\mathfrak{F} \cong G/H$. Finally, we know that the projected structure is Kähler according to Proposition 3.7. □

4. Examples

EXAMPLE 4.1. Let $M = G/H$ be a homogeneous space endowed with a G-invariant Riemannian almost CR structure (HM, J, g). If M is a naturally reductive space with respect to g (see e.g. [20, Ch. X]), then the canonical G-invariant connection determined by a reductive decomposition of the Lie algebra $\mathfrak{g}$ of G is a characteristic connection, since it parallelizes (HM, J) according to [20, Proposition 2.7, Ch. X]. A remarkable class is given by compact standard homogeneous CR manifolds associated to semisimple Levi–Tanaka algebras (see, for more information, [27], [24], [25]).

EXAMPLE 4.2. Let $\mathfrak{m} = \mathfrak{m}_{-1} \oplus \mathfrak{m}_{-2}$ be a pseudocomplex fundamental graded Lie algebra of kind 2, that is $\mathfrak{m}$ is $\mathbb{Z}$-graded and generated by $\mathfrak{m}_{-1}$, and $\mathfrak{m}_{-1}$ is endowed with a complex structure J such that

$$[JX, JY] = [X, Y] \quad \forall X, Y \in \mathfrak{m}_{-1}.$$

For more details, see [27]. The simply connected Lie group M with Lie algebra $\mathfrak{m}$ carries a canonical left invariant CR structure (HM, J) of type (n, k), where $2n = \dim \mathfrak{m}_{-1}$, $k = \dim \mathfrak{m}_{-2}$, HM is the left invariant distribution of kind 2 such that $H_e M = \mathfrak{m}_{-1}$, and $J_e = J$. Notice that M is CR diffeomorphic to an affine CR quadric in $\mathbb{C}^{n+k}$ [36]. Fix an inner product $\langle\ ,\ \rangle$ on $\mathfrak{m}$ which is Hermitian on $\mathfrak{m}_{-1}$ with respect to J and such that $\mathfrak{m}_{-1}$ and $\mathfrak{m}_{-2}$ are orthogonal. Then taking the left invariant Riemannian metric g on M determined by $\langle\ ,\ \rangle$, (M, HM, J, g) is a Riemannian CR manifold with torsion. Indeed the tensor N vanishes and conditions (2), (3), (4) in Theorem 3.3 can be readily verified using left invariant vector fields.

EXAMPLE 4.3. We discuss now $\mathcal{K}$-manifolds defined by Blair in [8]. Consider a manifold M of dimension $2n + k$ endowed with a normal f-structure φ of rank $2n$. This means that there exist vector fields ξ_i, $i = 1, \dots, k$, with dual forms η_i such that

$$\varphi\xi_i = 0, \qquad \eta_i \circ \varphi = 0, \qquad \varphi^2 = -I + \sum \eta_i \otimes \xi_i,$$
$$N_{\varphi} := [\varphi, \varphi] + 2 \sum d\eta_i \otimes \xi_i = 0.$$

The vanishing of N_φ ensures that M is a CR manifold of type (n,k) with structure (HM,J), where $HM=\operatorname{Im}\varphi$ and $J=\varphi|_{HM}$. Such a manifold is called a $\mathcal{K}$-manifold if it admits a Riemannian metric g such that

$$g(X,Y)=g(\varphi X,\varphi Y)-\sum\eta_i(X)\eta_i(Y),$$

and the fundamental 2-form Φ, defined by $\Phi(X,Y)=g(X,\varphi Y)$, is closed. Then M is a quasi Sasakian CR manifold in the sense of Definition 3.11. Indeed, being the structure CR integrable, condition (1) in Theorem 3.3 is satisfied; each operator θ_{ξ_i} vanishes since it is known that $\mathcal{L}_{\xi_i}\varphi=0$, thus yielding condition (2). The vector fields ξ_i are Killing (see [8]) and this gives (3). Finally, from $N_\varphi(\xi_i,X)=0$ it follows that $[\xi_i,X]\in\Gamma HM$ for every $X\in\Gamma HM$, implying $(\mathcal{L}_X g)(\xi_i,\xi_j)=0$ and thus (4) holds.

Example 4.4. A 3-Sasakian manifold is a $(4n+3)$-dimensional manifold M endowed with three Sasakian structures $(\varphi_\alpha,\xi_\alpha,\eta_\alpha,g)$, with the same compatible metric g, satisfying the following relations, for any even permutation (α,β,γ) of $\{1,2,3\}$:

$$\begin{aligned}\varphi_\gamma&=\varphi_\alpha\varphi_\beta-\eta_\beta\otimes\xi_\alpha=-\varphi_\beta\varphi_\alpha+\eta_\alpha\otimes\xi_\beta,\\ \xi_\gamma&=\varphi_\alpha\xi_\beta=-\varphi_\beta\xi_\alpha,\qquad \eta_\gamma=\eta_\alpha\circ\varphi_\beta=-\eta_\beta\circ\varphi_\alpha.\end{aligned}\tag{4.1}$$

It is known that the Reeb vector fields ξ_1,ξ_2,ξ_3 are orthonormal with respect to the metric g. We shall denote by $\mathcal{V}$ the distribution $\langle\xi_1,\xi_2,\xi_3\rangle$, which is invariant with respect to each φ_α.

Let HM be the $4n$-dimensional distribution given by

$$HM=\bigcap_{\alpha=1}^{3}\operatorname{Ker}(\eta_\alpha)$$

which is orthogonal to $\mathcal{V}$. Denote by J_α, $\alpha=1,2,3$, the endomorphism of HM induced by φ_α, which satisfies $J_\alpha^2=-\mathrm{Id}$; the quaternionic identities $J_\alpha J_\beta=J_\gamma=-J_\beta J_\alpha$ hold for every even permutation (α,β,γ) of $\{1,2,3\}$. Given three real constants a_1,a_2,a_3 such that $a_1^2+a_2^2+a_3^2=1$, we consider in the following the almost CR structure (HM,J), where $J:=a_1J_1+a_2J_2+a_3J_3$.

The Riemannian almost CR structure (HM,J,g) does not admit any characteristic connection since one can verify that conditions (1), (3), (4) in Theorem 3.3 are satisfied, but not condition (2); however, this problem is rectified by considering instead of g the modified metric g' defined as follows:

$$g'(X,Y):=g(X,Y),\qquad g'(X,\xi)=0,\qquad g'(\xi,\xi'):=\frac{1}{2}g(\xi,\xi')\tag{4.2}$$

for every $X,Y\in\Gamma HM$, $\xi,\xi'\in\Gamma\mathcal{V}$.

First, it is known that $\Gamma_X J_\alpha=0$ for each $\alpha=1,2,3$ and $X\in\Gamma HM$, cf. [14, Prop. 8.1]. Hence, (1) holds. As regards (2), first we observe that the

Levi–Tanaka form relative to HM is given by

$$(4.3)\quad L(X,Y)=\sum_{\delta=1}^{3}\eta_\delta[X,Y]\xi_\delta=-2\sum_{\delta=1}^{3}d\eta_\delta(X,Y)\xi_\delta=-2\sum_{\delta=1}^{3}\Phi_\delta(X,Y)\xi_\delta,$$

where Φ_δ is the fundamental 2-form of the Sasakian structure $(\varphi_\delta,\xi_\delta,\eta_\delta,g)$ defined by $\Phi_\delta(X,Y)=g(X,\varphi_\delta Y)$. Now, it is also known that $\Phi_\alpha(J_\alpha X,J_\alpha Y)=\Phi_\alpha(X,Y)$ and $\Phi_\alpha(J_\beta X,J_\beta Y)=-\Phi_\alpha(X,Y)$ for $\beta\neq\alpha$, and thus

$$\begin{aligned} g'\big([JX,Y]+[X,JY],\xi_\alpha\big) &= -\Phi_\alpha(JX,Y)-\Phi_\alpha(X,JY)\\ &= -2a_\beta g(X,J_\gamma Y)+2a_\gamma g(X,J_\beta Y)\end{aligned}$$

with (α,β,γ) an even permutation of $\{1,2,3\}$. On the other hand, using $\mathcal{L}_{\xi_\alpha}\varphi_\alpha=0$ and $(\mathcal{L}_{\xi_\beta}\varphi_\alpha)X=-2\varphi_\gamma X$ (see [12]), and taking into account that each ξ_α is an infinitesimal automorphism of HM, we see that

$$\theta_{\xi_\alpha}(X)=2a_\beta\varphi_\gamma X-2a_\gamma\varphi_\beta X$$

which implies condition (2). Finally, (3) holds since each ξ_α is a Killing vector field with respect to g; further it is immediately verified that $(\mathcal{L}_X g')(\xi_\alpha,\xi_\beta)=0$, which implies (4).

Example 4.5. Let M be a 3-Sasakian manifold of dimension $4n+3$ with structure $(\varphi_\alpha,\xi_\alpha,\eta_\alpha,g)$. For an even permutation (α,β,γ) of $\{1,2,3\}$, consider the *CR* structure (HM,J_α) of type $(1,4n+1)$, defined by $HM=\langle\xi_\beta,\xi_\gamma\rangle$ and $J_\alpha(\xi_\beta)=\xi_\gamma$. We shall verify that (HM,J_α,g) is a Riemannian *CR* structure with torsion.

Conditions (1) and (4) in Theorem 3.3 are trivially satisfied. As regards condition (2), we have $\theta_{\xi_\alpha}=0$ as consequence of the basic relation $[\xi_\alpha,\xi_\beta]=2\xi_\gamma$; furthermore, for every vector field ζ orthogonal to $\mathcal{V}=\langle\xi_1,\xi_2,\xi_3\rangle$, θ_ζ also vanishes since $[\zeta,\xi_i]$ is orthogonal to $\mathcal{V}$. Finally, condition (3) holds since ξ_α is Killing and $(\mathcal{L}_\zeta g)(\xi_i,\xi_j)=0$.

Concerning this example we also claim that (HM,J_α,g) *does not admit any characteristic connection with parallel torsion.* Indeed, we shall prove that any characteristic connection ∇ satisfies

$$(4.4)\qquad (\nabla_{\xi_\beta}T)\big(\xi_\gamma,\zeta,\zeta'\big)=4g\big(\zeta',\varphi_\alpha\zeta\big)$$

for every vector fields ζ, ζ' orthogonal to $\mathcal{V}$. Applying (3.6) the torsion T of any characteristic connection ∇ satisfies

$$(4.5)\qquad T\big(\xi_\gamma,\zeta,\zeta'\big)=-\eta_\gamma\big([\zeta,\zeta']\big)=2g\big(\zeta,\varphi_\gamma\zeta'\big),$$

$$(4.6)\qquad T(\xi_\gamma,\xi_\alpha,\zeta)=-g\big([\xi_\alpha,\zeta],\xi_\gamma\big)=0.$$

Notice that by (2.5), we get

$$g(\nabla_{\xi_\beta}\xi_\gamma,\xi_\beta)=g\big(\nabla^g_{\xi_\beta}\xi_\gamma,\xi_\beta\big)=-g\big(\xi_\gamma,\nabla^g_{\xi_\beta}\xi_\beta\big)=0,$$

which implies $\nabla_{\xi_\beta}\xi_\gamma = 0$, since HM is ∇-parallel. Since $\nabla_{\xi_\beta}\zeta \in \Gamma HM^\perp$, by (4.5) and (4.6) we have

$$T\big(\xi_\gamma, \nabla_{\xi_\beta}\zeta, \zeta'\big) = 2g\big(\nabla_{\xi_\beta}\zeta, \varphi_\gamma\zeta'\big).$$

Therefore,

$$\begin{aligned}(\nabla_{\xi_\beta}T)\big(\xi_\gamma, \zeta, \zeta'\big) &= 2\xi_\beta\big(g\big(\zeta, \varphi_\gamma\zeta'\big)\big) - 2g\big(\nabla_{\xi_\beta}\zeta, \varphi_\gamma\zeta'\big) + 2g\big(\nabla_{\xi_\beta}\zeta', \varphi_\gamma\zeta\big)\\ &= 2g\big(\zeta, (\nabla_{\xi_\beta}\varphi_\gamma)\zeta'\big).\end{aligned}$$

On the other hand, by (2.5)

$$g\big((\nabla_{\xi_\beta}\varphi_\gamma)\zeta', \zeta\big) = g\big(\big(\nabla^g_{\xi_\beta}\varphi_\gamma\big)\zeta', \zeta\big) + \frac{1}{2}\big(T\big(\xi_\beta, \varphi_\gamma\zeta', \zeta\big) + T\big(\xi_\beta, \zeta', \varphi_\gamma\zeta\big)\big).$$

Recalling that $(\varphi_\alpha, \xi_\alpha, \eta_\alpha, g)$ is a Sasakian structure, the first term on the right-hand side vanishes, and applying (4.5) we get

$$g\big((\nabla_{\xi_\beta}\varphi_\gamma)\zeta', \zeta\big) = -2g\big(\zeta', \varphi_\gamma\varphi_\beta\zeta\big) = 2g\big(\zeta', \varphi_\alpha\zeta\big)$$

and this completes the proof of (4.4).

Example 4.6. A *complex contact manifold* [9] is a complex manifold M of odd complex dimension $2n+1$, endowed with an open covering $\{\mathcal{O}\}$ of coordinate neighborhoods such that:

(1) On each $\mathcal{O}$ there is a holomorphic 1-form θ such that $\theta \wedge (d\theta)^n \neq 0$ everywhere on $\mathcal{O}$.
(2) If $\mathcal{O} \cap \mathcal{O}' \neq \emptyset$, there is a nonvanishing holomorphic function f such that $\theta' = f\theta$.

The local complex contact forms determine a global nonintegrable subbundle HM, defined by the equation $\theta = 0$, which has complex dimension $2n$ and is called the *complex contact subbundle* or the *horizontal subbundle.*

Denoting by J the complex structure on M, if g is a Hermitian metric, M is called a *complex almost contact metric manifold* if the following conditions hold:

(1) In each $\mathcal{O}$ there exist real 1-forms u and $v = u \circ J$ with dual vector fields U and $V = -JU$, and $(1,1)$-tensor fields G and $H = GJ$ such that

$$\begin{aligned}&G^2 = H^2 = -I + u \otimes U + v \otimes V,\\ &GJ = -JG, \qquad GU = 0, \qquad g(X, GY) = -g(GX, Y).\end{aligned}$$

(2) On the overlaps $\mathcal{O} \cap \mathcal{O}' \neq \emptyset$, the above tensors transform as

$$\begin{aligned}u' &= au - bv, \qquad v' = bu + av,\\ G' &= aG - bH, \qquad H' = bG + aH,\end{aligned}$$

for some functions a and b with $a^2 + b^2 = 1$.

A complex contact manifold always admits a complex almost contact metric structure for which the local contact form θ is $u-iv$ to within a nonvanishing complex valued function multiple. Such a structure is in fact a *complex contact metric structure*: the tensor fields G and H are related to du and dv by

$$du(X,Y)=g(X,GY)+(\sigma\wedge v)(X,Y), \tag{4.7}$$

$$dv(X,Y)=g(X,HY)-(\sigma\wedge u)(X,Y), \tag{4.8}$$

where σ is the 1-form given by $\sigma(X)=g(\nabla^g_X U,V)$. The tangent bundle splits as the orthogonal sum

$$TM=HM\oplus\mathcal{V},$$

where $\mathcal{V}$ is locally spanned by the vector fields U and V, and in the literature is usually assumed to be integrable; actually, the integral surfaces of $\mathcal{V}$ are totally geodesic submanifolds.

Now, the complex structure of M induces an almost CR structure (HM,J) of type $(2n,2)$, since HM is J-invariant. Notice that, being also $\mathcal{V}$ J-invariant, the orthogonal projection $P:TM\to HM$ commutes with J.

We claim that (HM,J) is not partially integrable. Indeed, using (4.7) and (4.8), the Levi–Tanaka form of HM is given by

$$L(X,Y)=u\big([X,Y]\big)U+v\big([X,Y]\big)V=-2\big(g(X,GY)U+g(X,HY)V\big) \tag{4.9}$$

which implies

$$L(X,Y)=-L(JX,JY). \tag{4.10}$$

This proves our claim because of the nonintegrability of the distribution HM. Nevertheless, the tensor N identically vanishes, since a simple computation using (4.9) and (4.10) shows that

$$N(X,Y)=[J,J](X,Y)=0$$

for every $X,Y\in\Gamma HM$.

Now, we restrict our attention to *normal* complex contact metric manifolds [21], in which case the covariant derivative of J is given by

$$\begin{aligned} g\big((\nabla^g_X J)Y,Z\big)&=u(X)\big(d\sigma(Z,GY)-2g(HY,Z)\big)\\ &\quad+v(X)\big(d\sigma(Z,HY)+2g(GY,Z)\big). \end{aligned} \tag{4.11}$$

The Levi–Civita connection also verifies

$$\nabla^g_X U=-GX+\sigma(X)V,\qquad \nabla^g_X V=-HX-\sigma(X)U, \tag{4.12}$$

for every $X\in\mathfrak{X}(M)$, yielding

$$\begin{aligned}&(\mathcal{L}_U g)(X,Y)=(\mathcal{L}_V g)(X,Y)=0,\\ &(\mathcal{L}_X g)(U,V)=(\mathcal{L}_X g)(U,U)=(\mathcal{L}_X g)(V,V)=0\end{aligned}$$

for every $X,Y\in\Gamma HM$.

Now, the Riemannian almost *CR* structure (HM, J, g) does not admit any characteristic connection. However, considering instead of g the modified metric g' defined by

$$g'(X,Y) := g(X,Y), \qquad g'(X,\xi) = 0, \qquad g'\big(\xi,\xi'\big) := \frac{1}{2} g\big(\xi,\xi'\big) \tag{4.13}$$

for every $X, Y \in \Gamma HM$, $\xi, \xi' \in \Gamma\mathcal{V}$, we shall see that conditions (1)–(4) in Theorem 3.3 hold if and only if g is Kähler.

Notice that

$$\big(\mathcal{L}_\xi g'\big)(X,Y) = (\mathcal{L}_\xi g)(X,Y) = 0, \qquad \big(\mathcal{L}_X g'\big)\big(\xi,\xi'\big) = \frac{1}{2}(\mathcal{L}_X g)\big(\xi,\xi'\big) = 0$$

for every $\xi, \xi' \in \Gamma\mathcal{V}$ and $X, Y \in \Gamma HM$. Since $N = 0$, it remains to verify that g is Kähler if and only if condition (2) in Theorem 3.3 holds for g', i.e.

$$g'\big(\theta_\xi(X), Y\big) = 2g'\big(L(X,JY),\xi\big) \tag{4.14}$$

for every $X, Y \in \Gamma HM$ and $\xi \in \Gamma\mathcal{V}$. From (4.9) we have

$$2g'\big(L(X,JY),U\big) = -2g(X,HY).$$

On the other hand,

$$\begin{aligned} g'\big(\theta_U(X), Y\big) &= g\big((\mathcal{L}_U J)X, Y\big) \\ &= g\big(\big(\nabla^g_U J\big)X - \nabla^g_{JX}U + J\big(\nabla^g_X U\big), Y\big) = d\sigma(Y, GX), \end{aligned}$$

where we used (4.11) and (4.12). Hence (4.14) holds for $\xi = U$ iff

$$d\sigma(Y, GX) = -2g(X, HY).$$

Analogously, (4.14) holds for $\xi = V$ iff

$$d\sigma(Y, HX) = 2g(X, GY).$$

The above two equation are both equivalent to require

$$d\sigma(X,Y) = -2g(X,JY)$$

for every $X, Y \in \Gamma HM$. From (4.11), it is readily seen that this is equivalent to $\nabla^g J = 0$.

As a remarkable case, we point out that the odd dimensional complex projective space $\mathbb{C}P^{2n+1}$ carries a normal complex contact metric structure whose Hermitian metric is Kähler, namely g is the Fubini–Study metric (see [21], [9]).

Example 4.7. Consider a Kähler manifold (N, J, g) of complex dimension n and a *CR*-submanifold $M \subset N$ of real dimension $2n - s$, $s \geq 1$. According to the definition in [6, p. 20], TM admits an orthogonal decomposition $TM = HM \oplus HM^\perp$, where HM is the maximal J-invariant subbundle of TM and $J(HM^\perp) \subset TM^\perp$. Recall that M carries a canonical f-structure $\varphi : TM \to TM$, where for every vector X tangent to M, φX is the tangential component of JX.

Taking into account the induced Riemannian metric g on M, we claim that (HM,J,g) is a Riemannian *CR* structure with torsion if and only if M is normal in the sense of [6, Ch. 3]. Indeed, according to [6, Theorem 3.2, Ch. 3], a necessary and sufficient condition for M to be normal is that for each local orthonormal frame $\xi_1,\dots,\xi_k$ of $HM^\perp$ and for each $X,Y\in\Gamma HM$,

$$g\big(\nabla^g_X\xi_i,Y\big)+g\big(X,\nabla^g_Y\xi_i\big)=0. \tag{4.15}$$

In fact this condition is equivalent to (3) of Theorem 3.3. Hence, it is enough to verify that if M is normal (2) and (4) of Theorem 3.3 hold. First, we prove that $\theta_{\xi_i}=0$, $i=1,\dots,k$. Denoting by $\bar\nabla$ the Levi–Civita connection of N and by A_i the Weingarten operator relative to $J\xi_i$, we have:

$$\begin{aligned}
g\big(\theta_{\xi_i}(X),Y\big)&=g(\bar\nabla_{\xi_i}JX,Y)-g(\bar\nabla_{JX}\xi_i,Y)+g(\bar\nabla_{\xi_i}X,JY)-g(\bar\nabla_X\xi_i,JY)\\
&=-g\big(\nabla^g_{JX}\xi_i,Y\big)-g\big(\nabla^g_X\xi_i,JY\big)\\
&=g\big(\nabla^g_Y\xi_i,JX\big)-g\big(\nabla^g_X\xi_i,JY\big)\\
&=g(A_iY,X)-g(A_iX,Y)=0,
\end{aligned}$$

where we used $\bar\nabla J=0$ and (4.15). As regards condition (4), notice that it is equivalent to

$$g\big(\nabla^g_{\xi_i}X,\xi_j\big)+g\big(\nabla^g_{\xi_j}X,\xi_i\big)=0.$$

Now,

$$g\big(\nabla^g_{\xi_i}X,\xi_j\big)=g(\bar\nabla_{\xi_i}JX,J\xi_j)=g(A_j\xi_i,\varphi X)=0,$$

where the last equality holds since it is known that, being M normal, each operator A_i commutes with φ.

EXAMPLE 4.8. We consider a parallelizable manifold M of dimension $2n+k$ with a global frame $\{E_i\}$. We assume that M is endowed with a Riemannian metric g *consistent* with $\{E_i\}$, that is,

$$g(E_i,E_j)=\delta_{ij},\qquad g\big([E_i,E_j],E_k\big)+g\big(E_j,[E_i,E_k]\big)=0$$

(cf. [38]). Consider the almost *CR* structure (HM,J), where HM is the subbundle of TM spanned by $E_1,\dots,E_{2n}$ and $J:HM\to HM$ is defined by

$$JE_i=E_{i+n},\qquad JE_{i+n}=-E_i,\quad i=1,\dots,n.$$

Then the flat connection ∇ associated with $\{E_i\}$ is a characteristic connection for the structure (HM,J,g); recall that ∇ is the unique linear connection parallelizing each E_i.

EXAMPLE 4.9. Let (M_1,HM_1,J_1,g_1) and (M_2,HM_2,J_2,g_2) be two Riemannian almost *CR* manifolds with torsion of type (n_1,k_1) and (n_2,k_2) respectively; then the product $M=M_1\times M_2$ is in a natural way a Riemannian almost *CR* manifold with torsion of type (n_1+n_2,k_1+k_2).

5. Riemannian almost *CR* structures on principal bundles

This section is devoted to a description of some Riemannian almost *CR* structures with torsion on the total space of a principal bundle, arising by lifting analogous structures from the base space.

Let $\pi : Q \to M$ be a principal bundle with structure group G and let $\omega : TQ \to \mathfrak{g}$ be the connection form of a given connection on Q, where $\mathfrak{g}$ is the Lie algebra of G. Denote by $\mathcal{V}$ the vertical subbundle of TQ and by $\mathcal{H}$ the horizontal subbundle determined by ω. We also denote by $\pi_* : \mathcal{H} \to TM$ the natural bundle isomorphism and by X^* the horizontal lift of a vector field X on M.

We shall call a *Jensen type metric* on Q any Riemannian metric g given by

$$(5.1)\quad g\big(X^*, Y^*\big) = h(X,Y)\circ\pi, \qquad g\big(X^*, A^*\big) = 0, \qquad g\big(A^*, B^*\big) = \langle A, B\rangle,$$

where h is a fixed Riemannian metric on M, and $\langle\ ,\ \rangle$ is an inner product on $\mathfrak{g}$ (see [17]); here $X, Y \in \mathfrak{X}(M)$ and A^*, B^* are the fundamental vertical vector fields corresponding to $A, B \in \mathfrak{g}$.

Now assume that (HM, J) is an almost *CR* structure on M. We define an almost *CR* structure (HQ, J) on Q by

$$(5.2)\qquad HQ := \pi_*^{-1}(HM), \qquad JX^* := (JX)^*$$

for every $X \in \Gamma HM$. This structure will be called the *canonical lift* of (HM, J) with respect to the connection ω.

We notice that if h is a Riemannian metric on M compatible with (HM, J), the Jensen type metric (5.1) is compatible with (HQ, J) and we have that

$$HQ^\perp = \mathcal{W} \oplus \mathcal{V},$$

where $\mathcal{W} := \pi_*^{-1}(HM^\perp)$. We shall denote by $\bar{N}$ and N the tensors associated to the structures (HM, J, h) and (HQ, J, g) according to (3.1).

Let Ω be the curvature form of ω; we prove the following theorem.

THEOREM 5.1. (Q, HQ, J, g) *is a Riemannian almost CR manifold with torsion if and only if the following conditions are satisfied*:

(1) (M, HM, J, h) *is a Riemannian almost CR manifold with torsion*,
(2) $\Omega(JX, JY) = \Omega(X, Y)$ *for any* $X, Y \in \Gamma HQ$,
(3) $\Omega(X, \xi) = 0$ *for any* $X \in \Gamma HQ$ *and* $\xi \in \Gamma\mathcal{W}$.

Furthermore, if the above conditions hold then:

(i) (HQ, J) *is partially integrable iff* (HM, J) *is*;
(ii) $\bar{N} = 0$ *iff* $N = 0$;
(iii) Q *is of Kähler type iff* M *is*;
(iv) $HQ^\perp$ *is integrable iff* $HM^\perp$ *is*;
(v) Q *is quasi Sasakian iff* M *is.*

Proof. We shall use Theorem 3.3. First, for any $X,Y,Z \in \Gamma HM$ we have

$$\begin{aligned}
&\bar{N}\big(X^*,Y^*,Z^*\big)\\
&\quad = g\big(\big[JX^*,JY^*\big]-\big[X^*,Y^*\big],Z^*\big)+g\big(\big[JX^*,Y^*\big]+\big[X^*,JY^*\big],JZ^*\big)\\
&\quad = g\big([JX,JY]^*-[X,Y]^*,Z^*\big)+g\big([JX,Y]^*+[X,JY]^*,(JZ)^*\big)\\
&\quad = h\big([JX,JY]-[X,Y],Z\big)\circ\pi+g\big([JX,Y]+[X,JY],JZ\big)\circ\pi\\
&\quad = N(X,Y,Z)\circ\pi.
\end{aligned}$$

Hence, $\bar{N}$ is skew-symmetric if and only if N is. Analogously, for any $\xi \in \Gamma HM^{\perp}$ and $X,Y \in \Gamma HM$, we have

$$g\big(\theta_{\xi^*}\big(X^*\big),Y^*\big)=h\big(\theta_\xi(X),Y\big)\circ\pi,$$

$$g\big(\big[JX^*,Y^*\big]+\big[X^*,JY^*\big],\xi^*\big)=h\big([JX,Y]+[X,JY],\xi\big)\circ\pi.$$

On the other hand, remarking that $\theta_{A^*}(X^*)=0$ for any $A \in \mathfrak{g}$ and $X \in \Gamma HM$, condition (2) in Theorem 3.3 is satisfied by the metric g if and only if it is satisfied by h and in addition

$$g\big(\big[JX^*,Y^*\big]+\big[X^*,JY^*\big],A^*\big)=0$$

for every $X,Y \in \Gamma HM$ and $A \in \mathfrak{g}$. The last condition is equivalent to

$$\omega\big(\big[JX^*,JY^*\big]-\big[X^*,Y^*\big]\big)=0,$$

that is

$$\Omega\big(JX^*,JY^*\big)=\Omega\big(X^*,Y^*\big).$$

Now, notice that for every $X,Y,Z \in \mathfrak{X}(M)$,

$$(\mathcal{L}_{X^*}g)\big(Y^*,Z^*\big)=(\mathcal{L}_X h)(Y,Z)\circ\pi.$$

Therefore, condition (3) in Theorem 3.3 is equivalent for the metrics g and h since $(\mathcal{L}_{A^*}g)(X^*,Y^*)=0$ for every $A \in \mathfrak{g}$ and $X,Y \in \Gamma HM$. Finally,

$$(\mathcal{L}_{X^*}g)\big(A^*,B^*\big)=0, \qquad (\mathcal{L}_{X^*}g)\big(\xi^*,A^*\big)=-g\big(\big[X^*,\xi^*\big],A^*\big)$$

for every $X \in \Gamma HM$, $A,B \in \mathfrak{g}$ and $\xi \in \Gamma HM^{\perp}$. Hence, the Riemannian metric g satisfies condition (4) if and only if h satisfies it and $\omega([X^*,\xi^*])=0$ or equivalently

$$\Omega\big(X^*,\xi^*\big)=0.$$

The proof of the last claims is straightforward. □

COROLLARY 5.2. *Let (M,HM,J,h) be a Riemannian almost CR manifold. Let ∇ be a linear connection on M and $L(M)$ the bundle of linear frames. Endow $L(M)$ with the canonical lift of (HM,J) determined by ∇ and fix a Jensen type metric g. Then, $L(M)$ admits a characteristic connection if and only if the following conditions hold*:

(1) *M admits a characteristic connection,*
(2) *$R(JX,JY)=R(X,Y)$ for any $X,Y \in \Gamma HM$,*

(3) *$R(X,\xi)=0$ for any $X\in\Gamma HM$ and $\xi\in\Gamma HM^{\perp}$,*
where R is the curvature tensor of ∇.

Corollary 5.3. *Let (M,HM,J,h) be a Riemannian almost CR manifold with torsion. If M admits a characteristic connection with parallel torsion, then $L(M)$ admits a Riemannian almost CR structure with torsion.*

Proof. If ∇ is a characteristic connection satisfying $\nabla T=0$, then the above corollary applies since conditions (2) and (3) hold. Indeed, (2) is consequence of (3.13); moreover, by (2.9) and being HM ∇-parallel, we have:

$$g\big(R(X,\xi)Y,Z\big)=g\big(R(Y,Z)X,\xi\big)=0,$$
$$g\big(R(X,\xi)\xi',\xi''\big)=g\big(R(\xi',\xi'')X,\xi\big)=0$$

for every $X,Y,Z\in\Gamma HM$ and $\xi,\xi',\xi''\in\Gamma HM^{\perp}$, thus proving (3). □

Remark 5.4. We point out that the characteristic connection of a nearly Kähler manifold or a Sasakian manifold does have parallel torsion (see [2]). The canonical invariant connection of a naturally reductive homogeneous Riemannian *CR* manifold also has this property.

Another criterion providing Riemannian almost *CR* structures with torsion on the total space of a principal bundle is given by the following result.

Proposition 5.5. *Let $\pi:Q\to M$ be a G-principal bundle. Assume G is endowed with a left invariant Riemannian almost CR structure with torsion $(HG,J,\langle\ ,\ \rangle)$. Then Q inherits a Riemannian almost CR structure with torsion (HQ,J,g) where g is any Jensen type metric (5.1) on Q, and (HQ,J) is defined by*

$$H_uQ=\lambda(H_eG),\qquad JA^*=(JA)^*,\quad A\in H_eG.$$

Here, for each $u\in Q$, $\lambda:\mathfrak{g}\to V_u$ is the canonical isomorphism $A\mapsto A^*_u$.

Proof. One checks for (HQ,J,g) the validity of conditions (1)–(4) in Theorem 3.3 with the same technique of the proof of Theorem 5.1. □

6. Some homogeneous models

Let G be a compact semisimple Lie group. Let $N=G/H$ be a (generalized) flag manifold of G. Recall that H is the centralizer of a torus $T\subset G$. We shall assume for semplicity that G acts almost effectively on N.

It is known that N admits a canonical G-invariant complex structure J, see [10] or [11]. We shall consider a G-invariant metric g_o Hermitian with respect to J. Among the possible choices of g_o, we recall that up to scaling, there exists a unique G-invariant compatible Kähler–Einstein metric. There is a canonical reductive decomposition of the Lie algebra $\mathfrak{g}$ of G:

$$\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{m}, \tag{6.1}$$

where $\mathfrak{m}$ is the orthogonal complement of $\mathfrak{h} := \mathrm{Lie}(H)$ with respect to the Killing form B of $\mathfrak{g}$. We shall identify $\mathfrak{m}$ with the tangent space of N at the point corresponding to the coset H in a canonical way.

Let $K \subset H$ be a closed subgroup. Since $\mathrm{Ad}(K)\mathfrak{m} \subset \mathfrak{m}$, and $J : \mathfrak{m} \to \mathfrak{m}$ is $\mathrm{Ad}(K)$-invariant, the homogeneous space $M = G/K$ admits a G-invariant almost CR structure (HM, J) such that $H_oM = \mathfrak{m}$, where $o \in M$ is the point corresponding to the coset K. Next, we introduce a G-invariant metric on M as follows; let $\mathfrak{p} \subset \mathfrak{h}$ be the orthogonal complement of the Lie algebra $\mathfrak{k}$ of K with respect to the Killing form. We consider the G-invariant metric g on M determined by the orthogonal direct sum inner product of $-B$ on $\mathfrak{p}$ and of g_o on $\mathfrak{m}$. Clearly, (HM, J, g) is a Riemannian almost CR sructure; we shall call it the *canonical lift* to G/K of the Hermitian structure (J, g_o) of the flag manifold N.

THEOREM 6.1. *$M = G/K$ is a homogeneous Riemannian CR manifold with torsion. The distribution $HM^{\perp}$ is integrable, the associated foliation $\mathfrak{F}$ is regular and the space of leaves $M/\mathfrak{F}$ is N. If g_o is Kähler, then M is quasi Sasakian.*

Proof. We prove the first claim. Consider the principal K-bundle $p : G \to M$ and the principal H-bundle $q : G \to N$. Let $\omega : TG \to \mathfrak{h}$ be the left invariant connection form on G whose horizontal space at $e \in G$ is $\mathfrak{m}$, and let $\omega' : TG \to \mathfrak{k}$ be the left invariant connection form on G whose horizontal space at e is $\mathfrak{p} \oplus \mathfrak{m}$. We observe that the canonical lifts of (HN, J) with respect to ω and of (HM, J) with respect to ω' coincide; we shall denote this lift by (HG, J). Moreover, the Jensen type metrics

$$p^*g - B(\omega', \omega'), \qquad q^*g_o - B(\omega, \omega)$$

also coincide; we shall denote this metric by $\hat{g}$. By virtue of Theorem 5.1, we are reduced to prove that $(HG, J, \hat{g})$ admits a characteristic connection.

In order to apply Theorem 5.1 to the bundle $q : G \to N$, since (N, J, g_o) admits a characteristic connection, it suffices to prove that the curvature form Ω of ω satisfies

$$\Omega(JX, JY) = \Omega(X, Y) \tag{6.2}$$

for every $X, Y \in \Gamma HG$. Observe that the structure $(HG, J, \hat{g})$ is left invariant by construction and $H_eG = \mathfrak{m}$. Hence, it suffices to prove (6.2) for left invariant vector fields $X, Y \in \mathfrak{m}$. According to Theorem 11.1 in [19, p. 103], we have to show that

$$[JX, JY]_{\mathfrak{h}} = [X, Y]_{\mathfrak{h}}. \tag{6.3}$$

Now let ∇ be the canonical G-invariant linear connection on N determined by the decomposition (6.1). Its curvature tensor satisfies

$$R(JX, JY) = R(X, Y) \tag{6.4}$$

for every $X, Y \in \mathfrak{X}(N)$, by Remark 3.5, because $\nabla J = 0$, ∇ has parallel torsion and the normal G-invariant metric on N (i.e., the naturally reductive metric induced by $-B$) is compatible with J (cf. e.g., [11, (3.52)]).

On the other hand, at the point $o = q(e)$, R_o is given by

$$R_o(X,Y) = -\mathrm{ad}\big([X,Y]_{\mathfrak{h}}\big)$$

for every $X, Y \in \mathfrak{m} \cong T_oN$, where $\mathrm{ad} : \mathfrak{h} \to \mathrm{End}(\mathfrak{m})$ is the adjoint representation. Observe that this representation is faithful since we assumed that G acts almost effectively. Hence, (6.4) implies (6.3). Thus, we have showed that the $(HG, J, \hat{g})$ is a Riemannian almost CR structure with torsion.

The integrability of $HM^{\perp}$ and of the almost CR structure (HM, J) follows applying (i), (ii) and (iv) in Theorem 5.1 to both the bundles $q : G \to N$ and $p : G \to M$. Moreover, if g_o is Kähler, taking into account (v), we deduce that M is quasi Sasakian. Finally, $\mathfrak{F}$ is regular in accordance with Proposition 3.12; it is straightforward to verify that the induced Kähler structure on $M/\mathfrak{F} \cong N$, is just (J, g_o). □

Corollary 6.2. *Let G be a compact semisimple Lie group and let $H \subset G$ be the centralizer of a torus $T \subset G$. Choose a G-invariant Kähler metric g_o with respect to the canonical complex structure on the flag manifold G/H. Then the canonical lift to G of (J, g_o) is a left invariant, quasi Sasakian strongly pseudoconvex CR structure.*

Proof. The above theorem applies taking $K = \{e\}$ and provides the quasi Sasakian CR structure on G. It remains to prove that (HG, J) is strongly pseudoconvex. Indeed it is known that there exists a vector ξ belonging to the center of $\mathfrak{h}$ such that

$$g_o(X, JY) = B\big(\xi, [X,Y]\big)$$

for every $X, Y \in \mathfrak{m}$ (see, e.g., [11, pp. 613–614]). This means that the Levi form of the corresponding left invariant section of $HG^{\perp}$ satisfies

$$g(X,Y) = \mathcal{L}_{\xi}(X,Y),$$

proving the assertion. □

Combining this corollary and Proposition 5.5, we get the following corollary.

Corollary 6.3. *The total space of any principal fiber bundle with compact semisimple structure group carries a Riemannian CR structure with torsion, strongly pseudoconvex and of Kähler type.*

Another application of Theorem 6.1 is the following corollary.

Corollary 6.4. *Every homogeneous manifold G/K where G is compact semisimple and K has nondiscrete center, carries a G-invariant quasi Sasakian CR structure.*

EXAMPLE 6.5. By applying Theorem 6.1, we see that the (oriented) Stiefel manifolds $V_{k,n}(\mathbb{K})$, $k \geq 2$, admit several homogeneous quasi Sasakian *CR* structures.

In the case $\mathbb{K} = \mathbb{R}$ the manifold $V_{n,k}(\mathbb{R}) = \mathrm{SO}(n)/\mathrm{SO}(n-k)$, can be endowed with a rich family of structures, projecting onto one of the following flag manifolds:

$$N = \mathrm{SO}(n)/U(n_1) \times \cdots \times U(n_p) \times U(1)^m \times \mathrm{SO}(r),$$

where

$$2\Big(\sum n_i + m\Big) + r = n, \qquad n - k \leq r \leq n - 2.$$

If $\mathbb{K} = \mathbb{C}$, for $V_{n,k}(\mathbb{C}) = \mathrm{SU}(n)/\mathrm{SU}(n-k)$, one can choose as base manifold

$$N = \mathrm{SU}(n)/S\big(U(n_1) \times \cdots \times U(n_p)\big),$$

where

$$n_1 \geq n_2 \geq \cdots \geq n_p \geq 1, \qquad n_p \geq n - k, \qquad \sum n_i = n.$$

Finally, in the case $\mathbb{K} = \mathbb{H}$, possible choices for N are

$$N = \mathrm{Sp}(n)/U(n_1) \times \cdots \times U(n_p) \times U(1)^m \times \mathrm{Sp}(r),$$

where

$$\Big(\sum n_i + m\Big) + r = n, \qquad n_1 \geq n_2 \geq \cdots \geq n_p > 1, \qquad n - k \leq r \leq n - 1.$$

We remark that, by construction, the metric of the Riemannian *CR* structure thus obtained on $V_{n,k}(\mathbb{K})$ in general is not the normal one.

EXAMPLE 6.6. Another family of examples is provided by the homogeneous spaces $\mathrm{SO}(4n)/\mathrm{Sp}(n)$; in this case they fiber onto the Hermitian symmetric space $\mathrm{SO}(4n)/U(2n)$.

EXAMPLE 6.7. According to Corollary 6.3, we see that for every Riemannian manifold (M, g), the orthonormal frame bundle $O(M)$ can be endowed with several Riemannian strongly pseudoconvex *CR* structures with torsion $(HO(M), J, \tilde{g})$, of Kähler type. Here $\tilde{g}$ is a suitable Jensen type metric

$$\tilde{g} = \pi^* g + \langle \omega, \omega \rangle,$$

where ω is the Levi–Civita connection. Alternatively, $\tilde{g}$ can be chosen as the diagonal lift of g, that is, taking $\langle\ ,\ \rangle$ equal to the opposite of the Killing form B (cf. [22]); however the resulting structure on $O(M)$ is not of Kähler type.

If $n = 2m$ is even and M is orientable, one can also consider the bundle $O_+(M)$ of positive orthonomal frames; in this case the fibration of $\mathrm{SO}(2m)$ onto the Hermitian symmetric space $\mathrm{SO}(2m)/U(m)$ yields a left invariant Kähler type structure on $\mathrm{SO}(2m)$ whose underlying metric is $-B$, so that $O_+(M)$ carries a Kähler type structure with $\tilde{g}$ equal to the diagonal lift.

THEOREM 6.8. *Let $M = G/K$ be a homogeneous manifold, where G is a compact semisimple Lie group. Then the following conditions are equivalent*:

(a) *M admits a G-invariant quasi Sasakian CR structure of CR dimension n.*
(b) *M admits a G-invariant closed 2-form Φ of rank $2n$.*

If (b) *holds*, *Φ is the fundamental 2-form associated to some G-invariant quasi Sasakian CR structure.*

Proof. (a) $\Rightarrow$ (b) is clear since the fundamental 2-form must be G-invariant.

(b) $\Rightarrow$ (a) Consider the orthogonal decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{n}$ of $\mathfrak{g}$ with respect to the Killing form B, where $\mathfrak{k} = \mathrm{Lie}(K)$. Fix a G-invariant closed 2-form Φ of rank $2n$; following the same argument as in the proof of Proposition 3.12, the natural extension $\bar{\Phi} \in \Lambda^2(\mathfrak{g})$ of Φ is given by

$$\bar{\Phi}(X,Y) = B\big([Z_o, X], Y\big) \quad \forall X, Y \in \mathfrak{g}, \tag{6.5}$$

where $Z_o \in \mathfrak{g}$ and K is contained in the centralizer H of Z_o in G. Being $N(\bar{\Phi}) = \mathfrak{h}$, the flag manifold G/H has real dimension $2n$; we consider the Kirillov–Kostant–Souriau symplectic 2-form determined by Z_o, which is the Kähler form of a metric g compatible with the canonical complex structure J (see [10, p. 220]). Hence, according to Theorem 6.1, M inherits a G-invariant quasi Sasakian CR structure of CR dimension n projecting onto (J,g). ☐

By means of a construction similar to that of Theorem 6.1, one can also prove the following result allowing to construct several homogeneous Riemannian almost CR manifolds starting from naturally reductive homogeneous almost Hermitian manifolds.

Proposition 6.9. *Let $N = G/H$ be a naturally reductive homogeneous almost Hermitian manifold with structure (J, g_o). Let $K \subset H$ be a closed subgroup. Then $M = G/K$ admits a regular G-invariant Riemannian almost CR structure with torsion (HM, J, g), projecting onto (J, g_o).*

7. Riemannian curvature and Levi forms

Let (M, HM, J, g) be a Riemannian almost CR manifold. A sectional curvature $K^g(X,\xi)$ of a 2-plane spanned by unit vectors $X \in H_xM$ and $\xi \in H_xM^\perp$, $x \in M$, will be called a *mixed sectional curvature.*

Theorem 7.1. *Let (M, HM, J, g) be a Riemannian almost CR manifold with torsion*, *of type (n,k). Then all the mixed sectional curvatures are non-negative. If they are all vanishing*, *HM is Levi flat and M is locally the Riemannian product of a $2n$-dimensional almost Hermitian manifold of type $\mathcal{G}_1$ and a k-dimensional Riemannian manifold.*

Proof. Notice that for a characteristic connection ∇ on M, the sectional curvatures $K(X,\xi)$, $X \in \Gamma HM$ and $\xi \in \Gamma HM^\perp$, are all vanishing. Hence, applying (2.10), for any unit vector fields X and ξ, the mixed sectional curvatures are given by

$$K^g(X,\xi) = \frac{1}{4} g\big(T(X,\xi), T(X,\xi)\big). \tag{7.1}$$

If these sectional curvatures are all vanishing, (iii) in Remark 3.6 implies the Levi flatness, and by (i) of Proposition 3.7, the manifold is locally a Riemannian product. □

REMARK 7.2. The above theorem provides an alternative proof of Blair's Theorem 1.6 in [8] concerning $\mathcal{C}$-manifolds, that is, $\mathcal{K}$-manifolds with closed characteristic 1-forms η_i, and stating that such manifolds are locally the Riemannian product of a Kähler manifold and an Abelian Lie group. We refer to Example 4.3 for the notations.

EXAMPLE 7.3. We use (7.1) to show that every 3-Sasakian manifold M, endowed with the Riemannian metric g' discussed in Example 4.4, has constant mixed sectional curvature $c = \frac{1}{2}$. Indeed, by (4.3) and (3.5) one can easily compute

$$T(X,\xi) = \sum_{i=1}^{3} \eta_i(\xi)\varphi_i X,$$

which gives $K^{g'}(X,\xi) = \frac{1}{2}$.

PROPOSITION 7.4. *Let (M, HM, J, g) be a Riemannian almost CR manifold with torsion, of type (n,k). Assume that all the mixed sectional curvatures $K^g(X,\xi)$ at a point $x_o \in M$ are nonnull. Then*

(a) *Either L_{x_o} or L'_{x_o} is nondegenerate.*
(b) *If $k \geq 2n$, then $HM^\perp$ has kind 2 at x_o.*
(c) *If $k \leq 2n$, then HM has kind 2 at x_o.*

Proof. (a) This is a direct consequence of (7.1), (3.5) and (3.6).

(b) Suppose $k \geq 2n$ and assume by contradiction that there exists $X \in H_{x_o}M$, $X \neq 0$, such that $L'_X = 0$. This allows us to consider the linear map

$$\xi \in H_{x_o}M^\perp \mapsto T_{x_o}(\xi, X) \in H_{x_o}M,$$

where T is the torsion of a fixed characteristic connection. By virtue of (7.1) and the hypothesis on the curvatures, this map is injective. On the other hand, being $T(\xi, X, X) = 0$, it is not onto, yielding a contradiction.

The proof of (c) is analogous. □

PROPOSITION 7.5. *Let (M, HM, J, g) be a Riemannian almost CR manifold with torsion, of type (n,k). Let $x_o \in M$ and fix $\xi \in H_{x_o}M^\perp$, $\xi \neq 0$. If L_ξ is nondegenerate, then all the mixed sectional curvatures $K^g(X,\xi)$ are nonnull. Conversely, if all the mixed sectional curvatures $K^g(X,\xi)$ are nonnull, then*

$$\mathrm{rk}(L_\xi) \geq 2n - \mathrm{rk}(\psi_\xi), \tag{7.2}$$

where

$$\psi_\xi := L'_{x_o}(\xi, \cdot) : H_{x_o}M^\perp \to H_{x_o}M.$$

In particular, if $k = 2$ or $\xi \in N(L'_{x_o})$ then L_ξ is nondegenerate.

Proof. By virtue of (3.5), for every $X,Y \in H_{x_o}M$ we have

$$L_\xi(X,Y) = g\big(T_{x_o}(X,\xi),Y\big) \tag{7.3}$$

and the first claim follows taking into account (7.1). As for the converse, we argue by contradiction assuming

$$\mathrm{rk}(L_\xi) + \mathrm{rk}(\psi_\xi) < 2n,$$

which can be rewritten as

$$\dim N(L_\xi) + \dim\big(\mathrm{Im}(\psi_\xi)^\perp\big) > 2n.$$

Hence, there exists $X \in N(L_\xi) \cap \mathrm{Im}(\psi_\xi)^\perp$, $X \neq 0$, which satisfies $T_{x_o}(X,\xi) = 0$ by (7.3) and (3.6).

Finally, if $k = 2$, notice that $\mathrm{rk}(\psi_\xi) < 2$, and (7.2) ensures that L_ξ is nondegerate. ☐

A similar argument yields also the following proposition.

PROPOSITION 7.6. *Let (M,HM,J,g) be a Riemannian almost CR manifold with torsion, of type (n,k). Let $x_o \in M$ and fix $X \in H_{x_o}M$, $X \neq 0$. If L'_X is nondegenerate, then all the mixed sectional curvatures $K^g(X,\xi)$ are nonnull. Conversely, if all the mixed sectional curvatures $K^g(X,\xi)$ are nonnull, then*

$$\mathrm{rk}\big(L'_X\big) \geq k - \mathrm{rk}(\psi_X), \tag{7.4}$$

where

$$\psi_X := L_{x_o}(X,\cdot) : H_{x_o}M \to H_{x_o}M^\perp.$$

In particular, L'_X is nondegenerate provided that $X \in N(L_{x_o})$ or $n = 1$ and k is even.

THEOREM 7.7. *Let (M,HM,J,g) be a Riemannian almost CR manifold with torsion, of type (n,k). Assume that all the mixed sectional curvatures $K^g(X,\xi)$ are nonnull at a point $x_o \in M$.*

(a) *If $k = 2$, then n is even.*

(b) *If $n = 1$ and k is even, then k is a multiple of 4.*

Proof. Both claims are consequences of Propositions 7.5 and 7.6, and Theorem 3 in [5], stating that the maximum dimension of a linear space of real skew-symmetric matrices of order q is $\rho(q) - 1$, where ρ denotes the Radon–Hurwitz function. We recall that ρ is defined by

$$\rho(q) = 2^c + 8d,$$

where q is factorized as $q = (2a+1)2^{c+4d}$, $c \in \{0,1,2,3\}$. ☐

Using the same argument based on Propositions 7.5 and 7.6, one can also deduce the following theorem.

THEOREM 7.8. *Let (M,HM,J,g) be a Riemannian almost CR manifold with torsion, of type (n,k).*

(a) *If* $\dim N(L'_{x_o}) \geq 2$ *and all the mixed sectional curvatures* $K^g(X,\xi)$, *with* $\xi \in N(L'_{x_o})$, *are nonnull, then* n *is even and*

$$\dim N\big(L'_{x_o}\big) \leq \rho(2n) - 1.$$

(b) *If* $\dim N(L_{x_o}) \geq 2$ *and all the mixed sectional curvatures* $K^g(X,\xi)$, *with* $X \in N(L_{x_o})$, *are nonnull, then* k *is even and*

$$\dim N(L_{x_o}) \leq \rho(k) - 1.$$

COROLLARY 7.9. *Let* (M, HM, J, g) *be a Riemannian almost CR manifold with torsion, of type* (n,k). *Assume that all the mixed sectional curvatures* $K^g(X,\xi)$ *at a point* $x_o \in M$ *are nonnull.*

(a) *If* $HM^\perp$ *is integrable, then* n *is even and*

$$k \leq \rho(2n) - 1.$$

(b) *If* HM *is integrable, then* k *is even and*

$$2n \leq \rho(k) - 1.$$

REMARK 7.10. Notice that for a 7-dimensional 3-Sasakian manifold the inequality in (a) is in fact an equality (see Example 7.3).

THEOREM 7.11. *Let* (M, HM, J, g) *be a Riemannian partially integrable almost CR manifold with torsion, of type* (n,k). *Assume that* $HM^\perp$ *is integrable and all the mixed sectional curvatures* $K^g(X,\xi)$ *at a point* $x_o \in M$ *are nonnull. Then, setting* $n = (2a+1)2^b$,

$$k \leq 2b + 1. \tag{7.5}$$

If $k \geq 2$, *all the Levi forms* $\mathcal{L}_\xi$ *at* x_o, $\xi \neq 0$, *have signature* $(\frac{n}{2}, \frac{n}{2})$. *Hence,* (M, HM, J) *is* $\frac{n}{2}$*-pseudoconcave at* x_o.

Proof. Applying again Proposition 7.5, all the Levi forms $\mathcal{L}_\xi$ are nondegenerate. Now the maximum dimension of a linear space of $n \times n$ Hermitian matrices is $2b+1$ according to [1, Theorem 1]. It is known that the nonzero matrices of such a space have signature $(\frac{n}{2}, \frac{n}{2})$. Indeed, this is an application of the canonical simultaneous reduced form for a pair of Hermitian matrices (see [16, Theorem 4.5.19]). □

With the same argument one proves the following theorem.

THEOREM 7.12. *Let* (M, HM, J, g) *be a Riemannian partially integrable almost CR manifold with torsion, of type* $(n,2)$. *Assume that all the mixed sectional curvatures* $K^g(X,\xi)$ *at a point* $x_o \in M$ *are nonnull. Then,* (M, HM, J) *is* $\frac{n}{2}$*-pseudoconcave at* x_o.

REMARK 7.13. We provide a class of examples showing that inequality (7.5) is sharp. Consider a vector space V of $n \times n$ Hermitian matrices such that every nonzero $A \in V$ is nonsingular and V has the maximum dimension $2b+1$ (for an explicit example of such a space see, e.g., [13]). Then one can construct a pseudocomplex fundamental graded Lie algebra $\mathfrak{m} = \mathfrak{m}_{-1} \oplus \mathfrak{m}_{-2}$ of kind 2, where $\mathfrak{m}_{-1} = \mathbb{C}^n$ and $\mathfrak{m}_{-2} = V^*$, whose nontrivial bracket is defined by

$$[X,Y](A) = \Im\big({}^t\bar{X}AY\big), \quad X,Y \in \mathbb{C}^n, A \in V.$$

As in Example 4.2, we can endow the simply connected Lie group M corresponding to $\mathfrak{m}$ with a left invariant Riemannian CR structure with torsion, of type $(n, 2b+1)$. Clearly the distribution $HM^\perp$ is integrable; by construction, the space of the Levi forms $\mathcal{L}_\xi$, $\xi \in H_eM^\perp$, coincides with V and thus, by Proposition 7.5, all the mixed sectional curvatures are nonvanishing.

REMARK 7.14. Theorem 7.11 implies Blair's result stating the nonexistence of $\mathcal{S}$-manifolds of codimension $k \geq 2$ of constant curvature [8, Corollary 1.9]. Recall that $\mathcal{S}$-manifolds are $\mathcal{K}$-manifolds whose structure satisfies $d\eta_i = \Phi$ for every $i = 1, \dots, k$. In particular, these manifolds are strongly pseudoconvex.

As an application of Theorem 7.11, we prove the following result about CR submanifolds of $\mathbb{C}^n$ (cf. Example 4.7):

THEOREM 7.15. *Let $M \subset \mathbb{C}^n$ be a compact, orientable, normal CR submanifold of codimension $s \geq 1$ and CR dimension $n-s$. If all the mixed sectional curvatures of M are nonnull, then M is the sphere S^{2n-1}.*

Proof. We shall prove that M must be an hypersurface, that is, $s = 1$. Then the claim follows from Okumura's classification of normal almost contact hypersurfaces in Euclidean spaces [29]. Indeed, since M is compact, it is known that there exists a point $x_o \in M$ such that (M, HM, J) is pseudoconvex at x_o. Namely, there exists a normal vector ζ at a point x_o whose Weingarten operator A_ζ is positive definite; take $\eta \in T^*_{x_o}M$ defined by

$$\eta(X) := \langle X, \bar{J}\zeta \rangle.$$

Here $\bar{J}$ is the complex structure of $\mathbb{C}^n$. Then η belongs to $H^o_{x_o}M$ and the corresponding Levi form is

$$2\mathcal{L}_\eta(X,X) = \langle A_\zeta X, X \rangle + \big\langle A_\zeta(JX), JX \big\rangle,$$

which is positive definite. On the other hand, being $HM^\perp$ integrable (cf. [6]), Theorem 7.11 is applicable, forcing $k = s = 1$. ☐

We conclude this section studying the constant curvature case.

Lemma 7.16. *Let* (M, HM, J, g) *be a* 3*-dimensional orientable Riemannian almost CR manifold with torsion. If* M *has constant sectional curvature* $c > 0$*, then* g *is homothetic to a Sasakian metric.*

Proof. Since M is orientable, there exists a global section ξ of $HM^{\perp}$, with $g(\xi, \xi) = 1$. Denote by η the dual form of ξ and by φ the $(1,1)$-tensor field extending J in such a way that $\varphi\xi = 0$. Hence, (φ, ξ, η, g) is an almost contact metric structure. We shall prove that actually this is a quasi Sasakian structure; thus, a result of Olszak applies [26, Theorem 6.2]. First, by Proposition 3.8, being HM of rank 2, (HM, J, g) is of Kähler type. Hence, Proposition 3.9 ensures that Φ is closed. Recalling Remark 3.10, the structure is quasi Sasakian. □

Theorem 7.17. *Let* (M, HM, J, g) *be a Riemannian almost CR manifold with torsion, of type* (n, k)*,* $k \geq 2$*. Assume that* M *has constant sectional curvature* c *and* M *admits a characteristic connection with parallel torsion. If* M *is complete and simply connected, then* $c = 0$*.*

Proof. According to the Ambrose–Singer theorem [35] and to a Kirichenko's result (cf. [18]), since the Riemannian curvature is parallel with respect to the characteristic connection, there exists a Lie group G acting transitively on M and preserving the structure (HM, J, g). Arguing by contradiction, we suppose $c > 0$ and up to scaling we may assume as well $c = 1$, so that M is isometric to the sphere S^{2n+k}. Now, we make use of the classification of homogeneous almost *CR* structures on spheres obtained by Krüger in [23]; since $k \geq 2$, M must be a sphere S^{4m+3}, with (HM, J) invariant under the action of $\mathrm{Sp}(m+1)$, $m \geq 1$. Moreover, we have two possibilities:

(a) $k = 3$,
(b) $n = 1$.

We shall examine the two cases:

(a) Fixed a representation of M as the homogeneous space $\mathrm{Sp}(m+1)/\mathrm{Sp}(m)$, there exists a single equivalence class of $\mathrm{Sp}(m+1)$-invariant almost *CR* structures. Moreover, $\mathrm{Sp}(m+1)/\mathrm{Sp}(m)$ has a canonical $\mathrm{Sp}(m+1)$-invariant 3-Sasakian structure $(\varphi_\alpha, \xi_\alpha, \eta_\alpha, g_o)$ (see [9]). Hence, there exists a $\mathrm{Sp}(m+1)$-equivariant *CR* diffeomorphism $f : (M, H'M, J_\alpha) \to (M, HM, J)$, where $(H'M, J_\alpha)$ denotes the almost *CR* structure induced by φ_α in the manner described in Example 4.4. Now, $(H'M, J_\alpha, f^*g)$ is also a Riemannian almost *CR* structure with torsion. On the other hand, f^*g is of constant curvature 1 and, by equivariance of f, it is $\mathrm{Sp}(m+1)$-invariant. Therefore, we must have $f^*g = g_o$, since g_o is the unique $\mathrm{Sp}(m+1)$-invariant metric of constant curvature 1 on $\mathrm{Sp}(m+1)/\mathrm{Sp}(m)$ (cf. [39, §2]). This is impossible, since we already know that $(H'M, J_\alpha, g_o)$ does not admit any characteristic connection (see again Example 4.4).

(b) In this case, there exists a family of equivalence classes of $\mathrm{Sp}(m+1)$-invariant CR structures of type $(1,k)$ parametrized by $(0,1]$. Each structure is obtained in a natural way by choosing a homogeneous CR hypersurface structure on each fiber S^3 of the Hopf fibration $S^{4m+3} \to \mathbb{H}P^m$. The parameter value 1 corresponds to the standard structure inherited by S^3 from $\mathbb{C}^2$; we shall show that (HM, J) is actually equivalent to this structure. Indeed, the Riemannian metric g induced on S^3 turns S^3 into a Riemannian CR manifold with torsion; moreover, applying Lemma 7.16, each fiber, being totally geodesic, is a Sasaki space form of constant curvature 1. Therefore, each fiber is CR equivalent to S^3 with the standard CR structure [34]. We have in this case a $\mathrm{Sp}(m+1)$-equivariant CR diffeomorphism $f:(M, H'M, J_\alpha) \to (M, HM, J)$, where $(H'M, J_\alpha)$ denotes the CR structure on the 3-Sasakian homogeneous space $\mathrm{Sp}(m+1)/\mathrm{Sp}(m)$ discussed in Example 4.5. Arguing as above, $f^*g = g_o$ and this is not possible since we know that $(H'M, J_\alpha, g_o)$ does not admit any characteristic connection with parallel torsion. □

Remark 7.18. The assumption $\nabla T = 0$ in the above theorem is essential. Indeed, it is well known that the 7-dimensional sphere carries a parallelization $\{E_i\}_{i=1,\dots,7}$ with respect to which the standard metric g_o is consistent (cf. e.g., [3]), and such that $\bar{\nabla}\bar{T} \neq 0$ where $\bar{\nabla}$ is the corresponding flat connection. Fix i,j,k,l such that $\bar{\nabla}\bar{T}(E_i,E_j,E_k,E_l) \neq 0$. On the other hand, according to Example 4.8, we can construct an almost CR structure of codimension $k=3$ such that $E_i,E_j,E_k,E_l \in \Gamma HS^7$ and (HS^7, J, g_o) is a Riemannian almost CR structure with torsion. Now, we observe that if ∇ is any charactestic connection for (S^7, HS^7, J, g_o), then $\nabla T \neq 0$. Indeed, we have

$$(\nabla T)(E_i,E_j,E_k,E_l) = (\bar{\nabla}\bar{T})(E_i,E_j,E_k,E_l)$$

since according to (3.4) and (3.5), $T(X,Y) = \bar{T}(X,Y)$ and $\nabla_X Y = \bar{\nabla}_X Y$ for every $X, Y \in \Gamma HS^7$.

8. Flat characteristic connections

In this section, we investigate the existence of quasi Sasakian CR manifolds admitting flat characteristic connections. As a first remark, regarding the case of CR codimension 1, we prove the following proposition.

Proposition 8.1. *Let $(M, \varphi, \xi, \eta, g)$ be a quasi Sasakian manifold. If the characteristic connection is flat then at each point the Levi form L_η has rank* 0 *or* 2. *In particular, the Okumura connection of any Sasakian manifold of dimension at least* 5 *is not flat.*

Proof. By (ii) in Proposition 3.8, it follows that $(\nabla_X T)(Y,Z,V) = 0$ for every $X,Y,Z,V \in \Gamma HM$; therefore, using (2.8) and (2.6), under the assumption $R=0$, we see that $dT(X,Y,Z,V) = 0$. Now, recall that the torsion of the characteristic connection for a quasi Sasakian manifold is given by $T = 6\eta \wedge d\eta$ (see [15] or (3.5)). Hence, $\eta \wedge (d\eta)^2 = 0$. □

For CR codimensions at least 2, the existence of flat characteristic connections does not necessarily lead to the Levi degeneracy of the underlying CR structure, in fact to strong pseudoconvexity, as shown by the following results.

THEOREM 8.2. *Let $N=G/H$ be an irreducible Hermitian symmetric space, where G is a compact simple Lie group. Let (J,g_o) be the G-invariant Kähler structure on N, where g_o is the normal metric. Then the canonical lift (HG,J,g) to G of (J,g_o) is a left invariant quasi Sasakian strongly pseudoconvex CR structure projecting onto N, admitting a flat characteristic connection.*

Furthermore, any simply connected, complete and irreducible quasi Sasakian CR manifold admitting a flat characteristic connection arises by this construction, up to scaling the metric.

Proof. The first claim is a consequence of Corollary 6.2; in this case g is a bi-invariant metric on G; of course the $(-)$-connection is a characteristic connection on G.

Now, let (M,HM,J,g) be a simply connected, complete and irreducible quasi Sasakian CR manifold, of type (n,k), admitting a flat characteristic connection. Following [3], M admits a global orthonormal frame $\{e_h,\xi_j\}$, $h=1,\dots,2n$, $j=1,\dots,k$, consisting of Killing vector fields, where $e_{n+i}=Je_i$, $i=1,\dots,n$, constructed by parallel transport from an adapted basis of the tangent space at a fixed point. Now, there are two cases:

(a) $\sigma_T=0$,
(b) $\sigma_T\neq 0$.

We exhamine first (a). In this case, it is known that M is a compact simple Lie group G with Lie algebra $\mathfrak{g}$ spanned by e_h,ξ_j [3, p. 484]. Clearly, the structure (HM,J,g) is left invariant. Now, this Lie algebra decomposes as $\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{m}$, where $\mathfrak{h}=span\{\xi_j\}$ and $\mathfrak{m}=span\{e_h\}$. We claim that this is a symmetric decomposition. Indeed, being $HM^\perp$ integrable, $\mathfrak{h}$ is a subalgebra; moreover, since the vector fields of the orthonormal frame are parallel, we have

$$g\big([\xi_i,e_h],\xi_j\big)=-T(\xi_i,e_h,\xi_j)=0,$$

so that $[\mathfrak{h},\mathfrak{m}]\subset\mathfrak{m}$. Finally, according to Proposition 3.8, $T(e_i,e_j,e_h)=0$ yielding $[\mathfrak{m},\mathfrak{m}]\subset\mathfrak{h}$. Let H be the analytic Lie subgroup of G corresponding to $\mathfrak{h}$; the symmetric space $N:=G/H$ is in a canonical way an irreducible Hermitian symmetric space of compact type (cf. Prop. 7.4 in [20, p. 250] and Prop. 9.3 in [20, p. 260]). The result follows immediately.

Finally, we prove that the case (b) must be excluded. Indeed, when $\sigma_T\neq 0$, M is isometric to the sphere S^7. Using again Proposition 3.8, we have $T(e_i,e_j,Z)=0$ for every $Z\in\Gamma HM$, yielding $[e_i,e_j]\in\Gamma HM^\perp$. On the other hand, by the *CR* integrability, we have

$$[e_i, e_j] - [Je_i, Je_j] = 0.$$

Hence,

$$\big[[e_i, e_j], e_h\big] = \big[[Je_i, Je_j], e_h\big];$$

but, $R^g(e_i, e_j)e_h = -\frac{1}{4}[[e_i, e_j], e_h]$ (see [3, Prop. 2.3]), and thus

$$R^g(e_i, e_j)e_h = R^g(Je_i, Je_j)e_h.$$

This is impossible since g has constant curvature, provided $n > 1$. For the case $n = 1$, one can apply (a) of Theorem 7.8. □

We can weaken the global assumptions on N in the above theorem, and describe a construction providing again a quasi Sasakian *CR* manifold which fibers onto N and admits a flat characteristic connection.

THEOREM 8.3. *Let (N, J, h) be a Hermitian locally symmetric manifold of complex dimension n. If N has nonnegative sectional curvature at a point x, then there exists a quasi Sasakian CR manifold M admitting a flat characteristic connection, fibering onto N, and with strongly pseudoconvex CR structure provided that the local de Rham decomposition of N at x contains no flat factor.*

Proof. Denote by $U(N)$ the $U(n)$-structure of N and let $M := P(u) \subset U(N)$ be the holonomy bundle of the Levi–Civita connection through an adapted frame $u \in U(N)$ at $x \in N$. Let (HM, J) be the canonical lift of the *CR* structure of N with respect to the Levi–Civita connection. Theorem 5.1 ensures that every Jensen type metric g on M defined as in (5.1) makes M a quasi Sasakian *CR* manifold. Let $\Psi(x)$ be the linear holonomy group at x. We show that one can choose an inner product $\langle\, ,\, \rangle$ on the Lie algebra $\mathfrak{h} := \mathrm{Lie}(\Psi(x))$ in such a way that the corresponding g admits a flat characteristic connection.

In order to construct such an inner product on $\mathfrak{h}$, consider the Cartan–Nomizu Lie algebra

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m},$$

where $\mathfrak{m} := T_xN$, whose Lie bracket is given by

$$[X, Y] = -R_x(X, Y), \qquad [A, B] = AB - BA, \qquad [A, X] = AX$$

for every $X, Y \in \mathfrak{m}$ and $A, B \in \mathfrak{h}$. The Killing form $B_{\mathfrak{g}}$ satisfies

$$B_{\mathfrak{g}}(A, B) = B_{\mathfrak{h}}(A, B) + \mathrm{tr}(AB)$$

which implies that $B_{\mathfrak{g}}$ is negative definite on $\mathfrak{h}$, since the restricted holonomy group $\Psi^0(x)$ is compact. Now fix a decomposition

$$T_xN = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_s \tag{8.1}$$

into mutually orthogonal, invariant and irreducible subspaces with respect to the action of $\Psi^0(x)$. Then $B_{\mathfrak{g}}$ restricts to a $\Psi^0(x)$-invariant bilinear form on each subspace $\mathfrak{m}_i$; therefore there exists a real number λ_i such that

$$B_{\mathfrak{g}}(X,Y) = \lambda_i X \cdot Y$$

for every $X, Y \in \mathfrak{m}_i$ (see [19, Appendix 5]), where we use the symbol $\cdot$ to denote h_x. Observe that $\lambda_i = 0$ if and only if $R_x(X,Y) = 0$ for every $X, Y \in \mathfrak{m}_i$. Set $I = \{i \in \{1,\dots,s\} : \lambda_i \neq 0\}$. Actually we have $\lambda_i < 0$ for each $i \in I$; this follows from the curvature assumption, being

$$R_x(X,Y,X,Y) = \frac{1}{\lambda_i} B_{\mathfrak{g}}\big([X,Y],[X,Y]\big)$$

for every orthonormal set $\{X, Y\}$ in $\mathfrak{m}_i$. Now, according to the local de Rham decomposition theorem, applied to (8.1) (cf. e.g., [30], pp. 227–228), we can decompose the Lie algebra $\mathfrak{h}$ as

$$\mathfrak{h} = \mathfrak{h}_1 \oplus \cdots \oplus \mathfrak{h}_s,$$

where $\mathfrak{h}_i := \{A \in \mathfrak{h} | A|_{\mathfrak{m}_j} = 0 \ \forall j \neq i\}$ is an ideal of $\mathfrak{h}$. Of course, $\mathfrak{h}_i = \{0\}$ if $i \notin I$. Accordingly, we define an inner product on $\mathfrak{h}$ by setting

$$\langle \, , \rangle := \sum_{i \in I} \lambda_i B_{\mathfrak{g}}|_{\mathfrak{h}_i \times \mathfrak{h}_i}.$$

Now, let $\{e_1, \dots, e_{2n}\}$ be the standard basis of $\mathbb{R}^{2n}$ and $\{A_1, \dots, A_r\}$ a basis of $\mathfrak{h}$, orthonormal with respect to $\langle \, , \rangle$. Then $\{B(e_i), A_j^*\}$ is an absolute parallelism on M, where $B(e_i)$ is the standard horizontal vector field corresponding to e_i [19, p. 137]. According to Example 2.9, it suffices to show that the Jensen metric g corresponding to $\langle \, , \rangle$ is consistent with the absolute parallelism, which amounts to

$$g\big(\big[B(e_i),B(e_j)\big],B(e_k)\big) + g\big(\big[B(e_i),B(e_k)\big],B(e_j)\big) = 0, \tag{8.2}$$

$$g\big(\big[B(e_i),B(e_j)\big],A_k^*\big) + g\big(\big[B(e_i),A_k^*\big],B(e_j)\big) = 0, \tag{8.3}$$

$$g\big(\big[B(e_i),A_j^*\big],A_k^*\big) + g\big(\big[B(e_i),A_k^*\big],A_j^*\big) = 0, \tag{8.4}$$

$$g\big(\big[A_i^*,A_j^*\big],A_k^*\big) + g\big(\big[A_i^*,A_k^*\big],A_j^*\big) = 0. \tag{8.5}$$

First, we remark that using Proposition 5.5 in [19, p. 137], $\Omega(B(e_i), B(e_j))$ is a constant function on M and $[B(e_i), B(e_j)] = -2\Omega(B(e_i), B(e_j))^*$, which implies (8.2). On the other hand, we also have $[B(e_i), A_k^*] = -B(A_k e_i)$ giving (8.4). Observe that (8.5) holds true since the inner product $\langle \, , \rangle$ is ad($\mathfrak{h}$)-invariant.

Formula (8.3) is equivalent to

$$-2\big\langle \Omega\big(B(e_i),B(e_j)\big),A_k\big\rangle = A_k(e_i)\cdot e_j \tag{8.6}$$

which can be rewritten as

$$\big\langle u^{-1} \circ R_x(ue_i,ue_j) \circ u, A_k\big\rangle = -A_k(e_i)\cdot e_j, \tag{8.7}$$

where $\cdot$ here denotes the standard inner product on $\mathbb{R}^{2n}$. After identifying $\mathbb{R}^{2n}$ with T_xN by means of the linear isometry $u:\mathbb{R}^{2n}\to T_xN$, this condition can be rewritten

$$\big\langle R_x(X,Y),A\big\rangle = -A(X)\cdot Y \tag{8.8}$$

for every $X,Y\in T_xN$ and $A\in\mathfrak{h}$; here we use again the symbol $\cdot$ to denote h_x. In order to prove this formula, take $A\in\mathfrak{h}_i$ and $X,Y\in\mathfrak{m}_i$. Then

$$\big\langle R_x(X,Y),A\big\rangle = -\frac{1}{\lambda_i}B_{\mathfrak{g}}\big([X,Y],A\big) = -\frac{1}{\lambda_i}B_{\mathfrak{g}}(AX,Y) = -A(X)\cdot Y.$$

This completes the proof of the first statement.

As concerns the last assertion, if the local de Rham decomposition of N has no flat factor, then the Ricci tensor of N is nondegenerate. Hence, according to [20, Theorem 8.3, p. 173], we have that $J_x\in\mathfrak{h}$. We claim that the Levi form in the direction of the vector field J_x^* is positive definite. Indeed we compute

$$\begin{aligned}\mathcal{L}_{J_x^*}\big(B(e_i),B(e_j)\big) &= L_{J_x^*}\big(B(e_i),JB(e_j)\big) = L_{J_x^*}\big(B(e_i),B(J_oe_j)\big)\\ &= g\big(\big[B(e_i),B(J_oe_j)\big],J_x^*\big) = \big\langle\omega\big[B(e_i),B(J_oe_j)\big],\omega\big(J_x^*\big)\big\rangle\\ &= -2\big\langle\Omega\big(B(e_i),B(e_j)\big),J_o\big\rangle = \delta_{ij},\end{aligned}$$

where J_o is the standard complex structure on $\mathbb{R}^{2n}$ and the last equality is justified by (8.6). ☐

THEOREM 8.4. *Let (N,φ,ξ,η,g) be a locally φ-symmetric Sasakian manifold of dimension $2n+1$. Suppose that at a point $x\in N$ the sectional curvatures of 2-planes orthogonal to ξ_x are ≥ -3. Then there exists a quasi Sasakian CR manifold M admitting a flat characteristic connection and fibering onto N.*

Proof. Let $U(N)$ be the $U(n)\times 1$-structure of N and let $M:=P(u)\subset U(N)$ be the holonomy bundle of the Tanaka–Webster connection through an adapted frame $u\in U(N)$ at x. By virtue of the curvature properties of the Tanaka–Webster connection $\tilde\nabla$ (see for details [31]), Theorem 5.1 applies. Hence M admits a quasi Sasakian *CR* structure $(HM,J,\bar g)$, $\bar g$ being a Jensen metric defined by means of an inner product $\langle\ ,\ \rangle$ on the Lie algebra $\mathfrak{h}:=\mathrm{Lie}(\Phi^0(x))$, where $\Phi^0(x)$ is the restricted holonomy group at x of the Tanaka–Webster connection. Arguing as in the proof of Theorem 8.3, we shall determine a suitable inner product in such a way that M admits a flat characteristic connection.

Let U be an open neighborhood of x such that the induced Sasakian structure on U is regular and let $p:U\to\bar N$ be the corresponding fibering over the Hermitian locally symmetric space $\bar N=U/\xi$. The Riemannian curvature tensor R of $\bar N$ and the curvature tensor $\tilde R$ of $\tilde\nabla$ on U are related by

$$\tilde R\big(X^*,Y^*\big)Z^* = \big(R(X,Y)Z\big)^*$$

for every $X, Y, Z \in \mathfrak{X}(\bar{N})$; here X^* denotes the horizontal lift of X with respect to p (see [33]). On the other hand (see again [33]),

$$\tilde{R}(X^*, Y^*)Y^* = R(X^*, Y^*)Y^* + 3g(X^*, \varphi Y^*)\varphi Y^*,$$

so that $\bar{N}$ has nonnegative sectional curvature at $p(x)$, by our assumptions.

Now, the holonomy algebra $\mathfrak{h}$ is isomorphic to the Riemannian holonomy algebra $\mathfrak{h}_{\bar{N}}$ of $\bar{N}$ at $p(x)$ through the map

$$A \in \mathfrak{h} \longmapsto A : \langle \xi_x \rangle^\perp \to \langle \xi_x \rangle^\perp,$$

taking into account that $\langle \xi_x \rangle^\perp$ is canonically identified with $T_{p(x)}\bar{N}$.

By the proof of Theorem 8.3, there exists an $\mathrm{ad}(\mathfrak{h}_{\bar{N}})$-invariant inner product $\langle\ ,\ \rangle$ on $\mathfrak{h}_{\bar{N}}$ satisfying

$$\big\langle R_{p(x)}(X, Y), A \big\rangle = -A(X) \cdot Y \tag{8.9}$$

for every $X, Y \in T_{p(x)}\bar{N}$ and $A \in \mathfrak{h}_{\bar{N}}$. This inner product induces in a natural way an $\mathrm{ad}(\mathfrak{h})$-invariant inner product on $\mathfrak{h}$.

We consider now an absolute parallelism $\{B(e_i), A_j^*\}$ on M, $\{e_0, \dots, e_{2n}\}$ being the standard basis of $\mathbb{R}^{2n+1}$ and $\{A_1, \dots, A_r\}$ being an orthonormal basis of $\mathfrak{h}$. Recall that for a locally φ-symmetric Sasakian manifold $\tilde{\nabla}$ is invariant by parallelism (see again [33]). Therefore, the Jensen metric $\bar{g}$ is consistent with the absolute parallelism since

$$\big\langle u^{-1} \circ \tilde{R}_x(ue_i, ue_j) \circ u, A_k \big\rangle = -A_k(e_i) \cdot e_j$$

for every $i, j = 0, \dots, 2n$. In fact, the above formula holds when $i = 0$ being $u(e_0) = \xi_x$, and for $i, j \geq 1$ it holds by virtue of (8.9). □

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