

CONSTANT MEAN CURVATURE k -NOIDS IN HOMOGENEOUS MANIFOLDS

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ABSTRACT. For each $k \geq 2$, we construct a family of surfaces in $\Sigma(\kappa) \times \mathbb{R}$ with constant mean curvature $H \in [0, 1/2]$, where $\kappa + 4H^2 \leq 0$ and $\Sigma(\kappa)$ is a two-dimensional space form. The surfaces are invariant under $2\pi/k$ -rotations about a vertical fiber of $\Sigma(\kappa) \times \mathbb{R}$, have genus zero, and $2k$ ends. Each surface arises as the sister surface of a minimal graph in a homogeneous 3-manifold. The domain of the graph is non-convex. We use the sisters of a generalization of Jorge–Meeks- k -noids in homogeneous 3-manifolds as barriers in the conjugate Plateau construction.

1. Introduction

Recently, various mathematicians constructed minimal ([MR12], [Pyo11], [You10]) and constant mean curvature ([MT12]) surfaces in three dimensional homogeneous manifolds with four dimensional isometry group via minimal (vertical) graphs above convex domains Ω . The existence of minimal surfaces follows since in those manifolds the preimage $\pi^{-1}(\Omega)$ under the Riemannian fibration π is a mean convex domain. In this paper, we construct surfaces with constant mean curvature $H \in [0, 1/2]$ in $\Sigma(\kappa) \times \mathbb{R}$ with $\kappa + 4H^2 \leq 0$, which arise from minimal graphs, to some extent above non-convex domains. This work is based on the PhD thesis of the author, [Ple12a].

The paper begins with the setup of sister surfaces in homogeneous manifolds in Section 2. In Section 3, we define a number of reference surfaces, which we use as barriers. First of all, we summarize a classification of ruled minimal surfaces in homogeneous manifolds. It is followed by a subsection on graphs, where we prove the existence of a Scherk type minimal surface in

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homogeneous manifolds, which generalizes known surfaces. The section ends with the construction of k -noids with constant mean curvature $H \in [0, 1/2]$, we use the minimal sister as a barrier later on. In the final section, we construct a family of less symmetric $2k$ -noids. The main step is the construction of an appropriate mean convex domain in order to solve a Plateau problem, see Section 4.2.

2. Sister surfaces in homogeneous 3-manifolds

We construct constant mean curvature (cmc) surfaces in a simply connected homogeneous 3-manifold E with an at least 4-dimensional isometry group. Such a manifold is a Riemannian fibration $\pi : E \rightarrow \Sigma(\kappa)$ with geodesic fibers and constant bundle curvature τ , where $\Sigma(\kappa)$ is a two-dimensional space form. We write $E(\kappa, \tau)$ for these spaces, cp. [Dan07]. See [Thu97] and [Sco83] for a complete classification of homogeneous 3-manifolds and their geometry.

The isometry group of $E(\kappa, \tau)$ depends on the signs of κ and τ , and is equivalent to the isometry group of one of the following Riemannian manifolds:

Curv.	$\kappa < 0$	$\kappa = 0$	$\kappa > 0$
$\tau = 0$	$\mathbb{H}^2 \times \mathbb{R}$	\mathbb{R}^3	$\mathbb{S}^2 \times \mathbb{R}$
$\tau \neq 0$	$\widetilde{\text{SL}}_2(\mathbb{R})$	$\text{Nil}_3(\mathbb{R})$	(Berger-) \mathbb{S}^3

In $E(\kappa, \tau)$, there exists a unit Killing field ξ tangent to the fibres, called the vertical vector field. Translations along the fibres are isometries. We call surfaces and curves in $E(\kappa, \tau)$ vertical resp. horizontal if their tangent space is parallel resp. orthogonal to ξ at each point.

An interpretation of the bundle curvature τ is the vertical distance of a horizontal lift of a closed curve, which was proven in [Ple12b]. Let $d(p, q)$ denote the signed vertical distance for two points p, q in one fibre, which is positive if \overline{pq} is in the fibre-direction ξ .

LEMMA 1 (Vertical distances). *Let γ be a piecewise differentiable closed Jordan curve in the base manifold $\Sigma(\kappa)$ of a Riemannian fibration $E(\kappa, \tau)$ with geodesic fibres and constant bundle curvature τ . Assume that there exists a bounded domain Δ such that $\partial\Delta = \gamma$, then*

$$d(\tilde{\gamma}(0), \tilde{\gamma}(l)) = 2\tau \text{area}(\Delta),$$

where $\tilde{\gamma}$ is a horizontal lift. Observe that $\pi(\tilde{\gamma}(0)) = \pi(\tilde{\gamma}(l))$ and area denotes the oriented volume.

One way to construct cmc surfaces is the conjugate Plateau construction, see [Kar05] for an introduction in the case of space forms. This approach uses the Lawson correspondence between isometric surfaces in space forms [Law70]. Recently, Daniel generalized the correspondence to homogeneous 3-manifolds. Since the construction uses reflection about vertical and horizontal planes, we consider only a special case of Daniel's correspondence.

THEOREM 2 ([Dan07, Theorem 5.2]). *There exists an isometric correspondence between a simply-connected cmc H -surface \tilde{M} in $\Sigma(\kappa) \times \mathbb{R} = E(\kappa, 0)$ and a simply-connected minimal surface M in $E(\kappa + 4H^2, H)$. Their shape operators \tilde{S}, S are related by*

$$(1) \quad \tilde{S} = JS + Hid,$$

where J denotes the $\pi/2$ rotation on the tangent bundle of a surface. Moreover, the normal and tangential projections of the vertical vector fields $\tilde{\xi}$ and ξ are related by

$$(2) \quad \langle \tilde{\xi}, \tilde{\nu} \rangle = \langle \xi, \nu \rangle, \quad Jdf^{-1}(T) = d\tilde{f}^{-1}(\tilde{T}),$$

where f and \tilde{f} denote the parametrizations of M and \tilde{M} respectively, ν and $\tilde{\nu}$ their unit normals, and T, \tilde{T} the projections of the vertical vector fields on TM and $T\tilde{M}$.

We call the isometric surfaces M and \tilde{M} sister surfaces, or sisters in short. The correspondence between surfaces implies a correspondence of curvature and torsion of curves in related surfaces. Let $c = f \circ \gamma$ be a curve parametrized by arc length in a hypersurface $M = f(\Omega) \subset E$ with (surface) normal ν . The normal curvature k and the normal torsion t along c are defined by

$$k := \nu \cdot \nabla_{c'} c' = -\nabla_{c'} \nu \cdot c' = \langle S\gamma', \gamma' \rangle, \quad t := -\nabla_{c'} \nu \cdot Jc' = \langle S\gamma', J\gamma' \rangle.$$

We call related curves $\tilde{c} \subset \tilde{M}$ and $c \subset M$ sister curves. One computes directly that their normal curvatures and torsions are associated as follows:

$$(3) \quad \tilde{k} = -t + H \quad \text{and} \quad \tilde{t} = k.$$

The idea of the conjugate Plateau construction is to solve a Plateau problem for a polygon, which consists of horizontal and vertical geodesics. Equations (3) imply that the sister surface is bounded by a piecewise smooth curve contained in totally geodesic vertical/horizontal planes and the surface conormal is perpendicular to those planes. The planes are called mirror planes, see [MT12] where they construct minimal surfaces in $E(\kappa + 4H^2, H)$ with $\kappa + 4H^2 > 0$ and the sisters. A curve in which the surface meets a mirror plane orthogonally is called mirror curve. Under certain assumptions, Schwarz reflection about the horizontal and vertical mirror planes extends the surface smoothly without branch points.

We are interested in surfaces with cmc H in product spaces $\Sigma(\kappa) \times \mathbb{R}$ with $\kappa \leq 0$. The behaviour of the surfaces depends on the pair (H, κ) . Let us distinguish two cases:

- $\kappa = 0$: There is an isometric correspondence between surfaces with cmc H in \mathbb{R}^3 and minimal surfaces in 3-dimensional space forms with curvature H . Since the hyperbolic 3-manifold is not a Riemannian fibration this case is not covered by the relation above and we do not treat it here. For $H = 0$ we get two conjugate minimal surfaces in \mathbb{R}^3 , cp. [Kar89]. If $H > 0$, the

minimal surface is constructed in a Berger sphere. Since the base manifold of this fibration is compact, this case is different and we do not consider it.

- $\kappa < 0$: All product spaces have the same isometry group, so we consider $\mathbb{H}^2 \times \mathbb{R}$, hence $\kappa = -1$. There are different cases: For $H = 0$, the relation describes two minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$, for $H \in (0, 1/2)$ the surfaces arise from minimal surfaces in $E(4H^2 - 1, H)$, which has the same isometry group as $\widetilde{\text{PSL}}_2(\mathbb{R})$. The third class of surfaces has constant mean curvature $1/2$ and its sister surface is minimal in Nil_3 . We do not consider the case $H > 1/2$, since this would lead to a compact base manifold as above.

3. Reference surfaces

3.1. Ruled surfaces in homogeneous 3-manifolds. In [GK09], ruled minimal surfaces in homogeneous manifolds and their sisters are discussed systematically. Since we need some of them as barriers in our surface construction, we present a short outline of the work.

In \mathbb{R}^3 , a ruled surface is defined for an arc-length parametrized curve $c : I \rightarrow \mathbb{R}^3$ and a unit vector field $v : I \rightarrow \mathbb{R}^3$ along c with $v(t) \perp c'(t)$ as the mapping

$$f : \mathbb{R} \times I \rightarrow \mathbb{R}^3, \quad f(s, t) := c(t) + sv(t).$$

The curve c is called directrix; the rulings $\gamma(s) := f(s, t_0)$ are asymptote lines.

The helicoid $f(s, t) = (s \cos t, s \sin t, ht) \subset \mathbb{R}^3$, $h \in \mathbb{R} \cup \{\pm\infty\}$ is an example of a ruled surface. Its axis is a vertical geodesic $c(t) := (0, 0, ht)$ and it has horizontal geodesics as rulings $\gamma(s) := (s \cos t_0, s \sin t_0, ht_0)$. The parameter h is related to the constant rotation-speed. For $h = 0$ it is a horizontal plane and for $h = \pm\infty$ it is a vertical plane.

We claim it is minimal, because the helicoid is invariant under π -rotation about its rulings: Let $\tilde{\gamma}$ be a curve perpendicular to a ruling γ . The normal curvature $\kappa_{\text{norm}}(\tilde{\gamma})$ changes sign under rotation, therefore $\kappa_{\text{norm}}(\tilde{\gamma}) = 0$, that is, $\tilde{\gamma}$ is an asymptotic direction and perpendicular to γ . With the Euler-curvature-formula

$$g(Sv_\alpha, v_\alpha) = \kappa_1 \cos^2 \alpha + \kappa_2 \sin^2 \alpha$$

we get $\kappa_1 = -\kappa_2$, since $g(Sv_{\alpha_i}, v_{\alpha_i}) = 0$, for $i = 1, 2$ where $\alpha_1 = \alpha_2 - \pi/2$.

In $E(\kappa, \tau)$, each π -rotation about horizontal and vertical geodesics is an isometry. Therefore, surfaces which are invariant under π -rotations about those geodesics are minimal. Hence, we consider surfaces foliated by geodesics.

(1) Vertical planes

A vertical plane is defined as the preimage $\pi^{-1}(c)$ of a geodesic $c \subset \Sigma$. Vertical planes are minimal, since the horizontal lift of c is a geodesic. Therefore, the surface is foliated by geodesics. Moreover, a π -rotation about each geodesic leaves the plane invariant. In a product space we

have for example, $\{c\} \times \mathbb{R}$ for a geodesic $c \in \Sigma(\kappa)$. In $E(4, 1) = \mathbb{S}^3$ a vertical plane is a Clifford torus.

(2) Horizontal umbrellas

Horizontal umbrellas correspond to horizontal planes. They are defined by the exponential map of the horizontal tangent subspace of $T_p E(\kappa, \tau)$ in a point $p \in E(\kappa, \tau)$. Therefore, each umbrella consists of all horizontal radial geodesics starting at p . In $\Sigma(\kappa) \times \mathbb{R}$, a horizontal umbrella is totally geodesic, whereas for $\tau \neq 0$ the surface has non-horizontal tangent spaces except in p . Horizontal umbrellas are minimal. For $\kappa \leq 0$ or $\tau = 0$ they are sections. Each umbrella is of disc type for $\kappa \leq 0$ and a sphere for $\kappa > 0$, for example, a geodesic 2-sphere in \mathbb{S}^3 .

(3) Horizontal slices

We interpret a horizontal slice as a type of horizontal helicoid, where the axis is a horizontal geodesic c and the rulings are the horizontal geodesics, which are perpendicular to c . Horizontal slices are minimal. Topologically it is a disc for $\kappa \leq 0$, a torus if $\kappa > 0, \tau \neq 0$, and a sphere if $\kappa > 0, \tau = 0$.

(4) Vertical helicoids

Last but not least we consider vertical helicoids $M(s)$. $M(s)$ is family of minimal surfaces, where the axis is a fiber of $\pi : E \rightarrow \Sigma$ and the rulings are horizontal geodesics which rotate along the axis with constant speed s . As in \mathbb{R}^3 , we have special cases: The surface $M(\tau) \subset E(\kappa, \tau)$ is a vertical plane and $M(\pm\infty)$ are horizontal umbrellas.

3.2. Minimal surface equation for graphs. To derive a minimal surface equation, we consider $\mathbb{R} \times \mathbb{R}_{>0} \times \mathbb{R}$ with

$$(4) \quad ds^2 = \lambda^2(dx^2 + dy^2) + (2Hy\lambda^2 dx + dz)^2,$$

where $\lambda = \frac{1}{\sqrt{-\kappa - 4H^2y}}$. This is a model for $E(\kappa + 4H^2, H)$ with $\kappa + 4H^2 < 0$.

The Riemannian fibration $\pi : E(\kappa + 4H^2, H) \rightarrow \Sigma(\kappa + 4H^2)$ for these coordinates is given by the projection onto the first two coordinates. The vertical vector field is $\xi = \partial_z$. We consider graphs over a domain $\Omega \subset \Sigma(\kappa + 4H^2)$ with respect to this fibration.

Let M be a coordinate graph $z = u(x, y)$ in $E(\kappa + 4H^2, H)$ endowed with the metric (4). Then M is minimal if u is a solution of

$$(5) \quad 2w(u_{xx} + u_{yy}) - \left(\left(\frac{2H}{y(-4H^2 - \kappa)} + u_x \right) w_x + u_y w_y \right) = 0,$$

where $w = 1 + (u_x/\lambda + 2H\lambda y)^2 + (u_y/\lambda)^2$.

Let us change the coordinates $x = r \cos s, y = r \sin s, r > 0, 0 < s < \pi/2$ and assume that the solution $u(r, s)$ is constant for fixed s , i.e. $\partial_r u = 0$ and therefore let $'$ denote the derivate with respect to s . Equation (5) is then equivalent

to

$$2w(4H^2 + \kappa)u'' - w'(2H + (4H^2 + \kappa)u') = 0.$$

This is equivalent to

$$\frac{(2H + (4H^2 + \kappa)u')^2}{w} = c, \quad \text{for a constant } c \in \mathbb{R}.$$

Using $w = 1 - \frac{4H}{4H^2 + \kappa} - 4H \sin^2(s)u' - (4H^2 + \kappa) \sin^2(s)u'^2$ we get

$$u'(s) = \frac{1}{4H^2 + \kappa} \left(2H \pm \sqrt{\frac{(4H^2 \cos^2 s - (4H^2 + \kappa)) \frac{-c}{4H^2 + \kappa}}{1 - \frac{-c}{4H^2 + \kappa} \sin^2 s}} \right),$$

hence with $c = -(4H^2 + \kappa)$,

$$(6) \quad u(s) = \frac{1}{4H^2 + \kappa} \left(2Hs \pm \int_0^s \sqrt{\frac{4H^2 \cos^2 t - (4H^2 + \kappa)}{1 - \sin^2 t}} dt \right)$$

is a minimal section.

For $\kappa = -1$ and $H = 0$, this surface arises also in a family of screw motion surfaces in $\mathbb{H}^2 \times \mathbb{R}$ deduced in [Ear08], moreover in [CR10] it is used in the construction of a complete minimal graph to prove the existence of a harmonic diffeomorphism from \mathbb{C} to \mathbb{H}^2 .

In the universal cover of $\text{PSL}_2(\mathbb{R})$, the solution was derived in [You10] and used to prove a Jenkins–Serrin type theorem on compact domains.

In general, Equation (6) parametrizes a minimal graph of Scherk type in $E(\kappa + 4H^2, H)$, $\kappa + 4H^2 < 0$, where we consider polar coordinates in the upper half-plane model of $\Sigma(\kappa + 4H^2)$. For the right choice of signs, the surface has zero boundary data for $s = 0$ (positive x -axis) and tends to infinity for $s \rightarrow \pi/2$. Hence, M is a minimal graph above a domain which is bounded by a geodesic.

By isometries of $E(\kappa + 4H^2, H)$, we may define a minimal graph M for any geodesic $\gamma \subset \Sigma(\kappa + 4H^2)$, such that M tends to infinity on γ and has asymptotic values zero on a subset of $\partial\Sigma(\kappa + 4H^2)$.

3.3. Constant mean curvature k -noids in $\Sigma(\kappa) \times \mathbb{R}$. In [GK10], Große-Brauckmann and Kusner describe the conjugate Plateau construction in $E(\kappa, \tau)$. They outline the construction of a one-parameter family of surfaces in $\Sigma(\kappa) \times \mathbb{R}$ with constant mean curvature $H \geq 0$, which have k ends and dihedral symmetry. The parameter of the family a is related to the necksize of the surfaces. The idea is to solve a compact Plateau problem of disc type in $E(\kappa + 4H^2, H)$ and take a limit of Plateau solutions in order to solve an improper Plateau problem. The sister of the limit generates the cmc surface by reflections about horizontal and vertical planes. We use the limiting minimal disc $M = M(a, k)$ as a barrier in our construction of less symmetric surfaces, see Section 4.

We make use of the minimal graph from Section 3.2 to prove that a sequence of compact minimal discs has the minimal surface M as its limit. Each compact minimal disc arises as a Plateau solution M_r of a polygonal boundary Γ_r . In order to define Γ_r we consider a triangle Δ_r in the base manifold $\Sigma(\kappa + 4H^2)$, it is well-defined by a hinge of lengths a and r enclosing an angle π/k . We lift the hinge horizontally, add a vertical geodesic of length r^2 in fiber direction to the end of the edge of length a and lift the remaining edge of the geodesic triangle. By Lemma 1, the distance of the two endpoints is $r^2 - 2\tau \text{area}(\Delta_r)$, therefore the distance is always in fiber direction as sketched in Figure 1. We define Γ_r by adding the remaining vertical geodesic, it is contained in the boundary of a mean convex domain $\Omega_r := \pi^{-1}(\Delta_r)$. By [MY82], there exists an embedded minimal surface M_r with boundary Γ_r . Moreover, it is a unique section of the line bundle $\pi : E(\kappa + 4H^2, H) \rightarrow \Sigma(\kappa + 4H^2)$ projecting to Δ_r and extends without branch points, see [Ple12b].

The Jordan curves Γ_r converge to the boundary Γ of the desired minimal disc M , in the sense that $\Gamma_n \cap K_x = \Gamma \cap K_x$ for any compact neighborhood $K_x \subset E(\kappa + 4H^2, H)$ for $x \in \Gamma$ and $n > N_x$ large enough. Γ consists of two horizontal geodesics enclosing an angle π/k and one vertical geodesic. One of the horizontal geodesics has length a , the other two geodesics have infinite length.

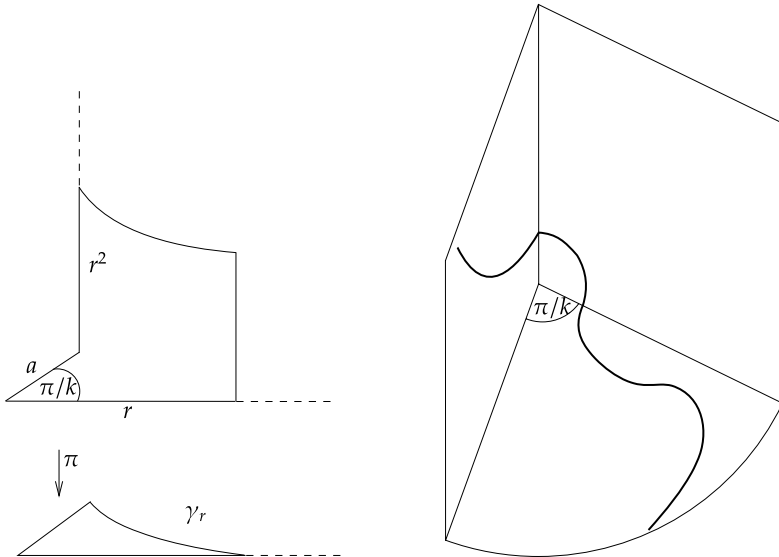


FIGURE 1. Left: The boundary Γ_r of the minimal disc M_r in $E(\kappa + 4H^2, H)$ and its projection, here $\kappa + 4H^2 < 0$. Right: The boundary of the desired cmc surface in $\Sigma(\kappa) \times \mathbb{R}$.

THEOREM 3. *Suppose $\kappa + 4H^2 \leq 0$, $a \in \mathbb{R}_{>0}$ and $k \geq 2$. Then there exists a minimal surface $M = M(a, k)$ with boundary Γ , such that M is a graph projecting to $\Delta := \lim_{r \rightarrow \infty} \Delta_r$, and it extends without branch points by Schwarz reflection about the edges of Γ .*

The case $\kappa = -1$, $H = 1/2$, that is, the existence of a minimal surface with those properties in Nil_3 was already proven in [Ple12b], the case $\kappa = -1$, $H = 0$ was shown in [Pyo11].

Proof of Theorem 3. To prove that the sequence of minimal sections, M_r converges to a minimal surface M , we have to show that a barrier exists. We distinguish three cases to show that the sequence is uniformly bounded on each Δ_n :

- $\kappa + 4H^2 < 0$: We claim the sequence is uniformly bounded by the Scherk-type minimal graph defined by Equation (6) in polar coordinates (r, s) on the first quadrant of the upper half-plane. For $\kappa + 4H^2 < 0$ the limit of Δ_r is a triangle in $\Sigma := \Sigma(\kappa + 4H^2)$ with one ideal vertex in $\partial\Sigma$, let γ_r denote the edge which closes the hinge, see Figure 1. We have seen that we find a Scherk-type minimal graph for any geodesic $\gamma \subset \Sigma$. For each $n \in \mathbb{N}$ consider the Scherk-type minimal surface S_n for the geodesic γ_n . Since M_n and S_n are both graphs, we can move S_n in vertical direction such that it is a barrier from above. By the maximum principle, the sequence M_r is uniformly bounded by S_n on Δ_n .
- $\kappa = 0$, $H = 0$: On each Δ_n the sequence of minimal graphs $M_r = (x, y, u_r(x, y)) \subset \mathbb{R}^3$ is bounded from above by a helicoid H_n with horizontal axis, that depends on a and π/k only.
- $\kappa + 4H^2 = 0$, $H \neq 0$: The manifold $E(\kappa + 4H^2, H)$ is isometric to Nil_3 . In [DH09] Daniel and Hauswirth proved the existence of a horizontal helicoid in Nil_3 , which is a barrier from above, see [Ple12b].

Hence for all pairs (κ, H) with $\kappa + 4H^2 \leq 0$ the sequence M_r of minimal graphs is uniformly bounded on each Δ_n . By the compactness theorem a subsequence converges uniformly on compact subsets to a minimal graph over Δ_n . A diagonal process yields a minimal graph $M(a, k)$ over Δ with boundary Γ .

To see that $M(a, k)$ extends without branch points by Schwarz reflection, we distinguish two cases. If p is not a vertex of Γ , it is no branch point by [GL73]. If p is a vertex, then the angle is of the form π/m with $m \geq 2$ and m copies of $M(a, k)$, obtained by successive rotations about the appropriate edges, have a barrier by construction. Moreover, the boundary is continuously differentiable in p , hence we are in the first case and p is no branch point. \square

The minimal graph $M(a, k)$ has a simply connected sister in a product space, which generates a complete cmc surface in $\Sigma(\kappa) \times \mathbb{R}$ by Schwarz reflection:

THEOREM 4. *For $H \in [0, 1/2]$ and $\kappa + 4H^2 \leq 0$ there exists a family of complete surfaces M_a in $\Sigma(\kappa) \times \mathbb{R}$ with constant mean curvature H , k ends, one horizontal and k vertical symmetry planes, $a > 0$.*

Proof. By Daniel’s correspondence (Theorem 2) the minimal graph $M(a, k) \subset E(\kappa + 4H^2, H)$ has a simply connected sister surface with cmc H in the product manifold $\Sigma(\kappa) \times \mathbb{R}$. Since $M(a, k)$ is a graph the sister surface is a (multi-)graph. Moreover since $M(a, k)$ is bounded by horizontal and vertical geodesics, the sister surface is bounded by mirror curves in vertical and horizontal planes. Hence by Schwarz reflection about those planes, we get a complete cmc surface \tilde{M}_a consisting of $4k$ fundamental pieces with the claimed symmetry planes. \square

4. Constant mean curvature $2k$ -noids

We construct a 2-parameter family of surfaces with cmc $H \in [0, 1/2]$ in $\Sigma(\kappa) \times \mathbb{R}$ with $2k$ ends and dihedral symmetry. Each surface has k vertical symmetry planes and one horizontal one, with $k \geq 2$. The construction is similar to the one of the k -noid from Section 3.3:

- (1) Define the boundary curve of the minimal surface in $E(\kappa + 4H^2, H)$, with $\kappa + 4H^2 \leq 0$ such that its sister cmc surface has the desired properties (Section 4.1).
- (2) Solve the Plateau problem for each truncated boundary curve (Proposition 6).
- (3) Show that the sequence of minimal discs has a limit (Theorem 8).
- (4) Consider the sister surface of the limit, it generates the desired $2k$ -noid (Theorem 9).

For $\kappa = -1$ and $H = 1/2$ the cmc surface corresponds to a minimal surface in $\text{Nil}_3(\mathbb{R}) = E(0, 1/2)$; if $0 < H < 1/2$ the MC H surface results from a minimal surface in $\widetilde{\text{PSL}}_2(\mathbb{R})$ and finally for $H = 0$ the surfaces are conjugate minimal surfaces in $\Sigma(\kappa) \times \mathbb{R}$, $\kappa \leq 0$.

4.1. Boundary construction. In $\Sigma(\kappa) \times \mathbb{R}$, the desired boundary is not connected. It consists of two components: The first component is a curve consisting of two mirror curves, each lying in a vertical plane. The two planes form an angle $\varphi = \pi/k$, $k \geq 2$. The second component is a mirror curve in a horizontal plane.

The mirror curves and their geodesic sisters are related: The horizontal mirror curve corresponds to a vertical geodesic and the mirror curves in vertical planes are related to horizontal geodesics enclosing an angle π/k . The relative position of the vertical and horizontal components determines the $2k$ -noid.

We denote the distance of the vertical geodesic to the vertex of the two horizontal geodesics with d , it is well-defined and realised by the length of

a horizontal geodesic γ . Its length is equal to the length of its projection $\pi(\gamma)$ to $\Sigma(\kappa + 4H^2)$, since it is a horizontal geodesic and the projection π is a Riemannian fibration. The same holds for the angle enclosed by γ and one of the horizontal rays, call it α . Since the sister surfaces are isometric, it is consistent to call the 2-parameter family of cmc surfaces which we will obtain $\tilde{M}_{d,\alpha}$, even though we cannot read off d and α directly.

To construct a minimal surface that is bounded by such a geodesic contour, we define Jordan curves Γ_n , $n > 0$ and consider their limit. To define Γ_n we consider a geodesic quadrilateral $\Delta_n := \Delta_n(d, \alpha)$ in the base $\Sigma(\kappa + 4H^2)$: The quadrilateral is well-defined by two edges of length n forming an angle π/k and intersecting in point \hat{p}_1 . Furthermore, its diagonal in \hat{p}_1 has length d and encloses an angle $\alpha \leq \varphi/2$ to one side. Let \hat{p} denote the endpoint of the diagonal. We consider the horizontal lift of $\partial\tilde{\Delta}_n$ starting in \hat{p} and going in positive direction. We label the endpoints with $\partial\tilde{\Delta}_n(0) = p_5$ and $\partial\tilde{\Delta}_n(l) = p_4$, by Lemma 1 the signed vertical distance is $d(p_5, p_4) = 2H \text{ area}(\Delta_n)$ and therefore, in positive ξ -direction. Now we translate the horizontal edge that ends in p_4 in positive ξ -direction by n and call the endpoint p_3 . Furthermore, we translate the horizontal edge that starts in p_5 in $-\xi$ -direction by n and call the endpoint p_6 . After the vertical translation, we have

$$d(p_5, p_4) = 2H \text{ area}(\Delta_n) + 2n.$$

Hence, p_5p_4 is in positive ξ -direction for all $n \geq 0$. We complete the Jordan curve Γ_n by adding two vertical edges of length n in p_3 and p_6 and label the intersections with the horizontal edges p_2 and p_7 , respectively. See Figure 2.

The polygon Γ_n has six right angles and one angle $\varphi = \pi/k$; the quadrilateral Δ_n is not necessarily convex for large n .

The sequence of Jordan curves Γ_n defines a limit $\Gamma := \Gamma_{d,\alpha}$, which is a non-connected curve and the boundary of the desired minimal disc. The convergence is in the sense that $\Gamma_n \cap K_x = \Gamma \cap K_x$ for any compact neighborhood $K_x \subset E(\kappa + 4H^2, H)$ for $x \in \Gamma_{d,\alpha}$ and $n > N_x$ large enough. This contour projects to the union of quadrilaterals

$$\Delta := \Delta(d, \alpha) = \bigcup_{n>0} \Delta_n.$$

Note that Δ is non-convex.

REMARK 1. With $\alpha = \varphi/2$, we get a contour $\Gamma_{d,\alpha}$, that is symmetric w.r.t. π -rotations about the horizontal diagonal of the lifted quadrilateral. Moreover, it is congruent to two copies of the contour $\Gamma_{(a,k)}$ with $(a, k) = (d, 2k)$ from Section 3.3, where one of the copies is rotated about the bounded horizontal geodesic of length a , cp. Figure 1. Accordingly, we continue the minimal surface $M(a, k)$ from Section 3.3 by rotating it about the same horizontal geodesic by π and get a smooth surface. We use this surface as a barrier in the proof of Proposition 6.

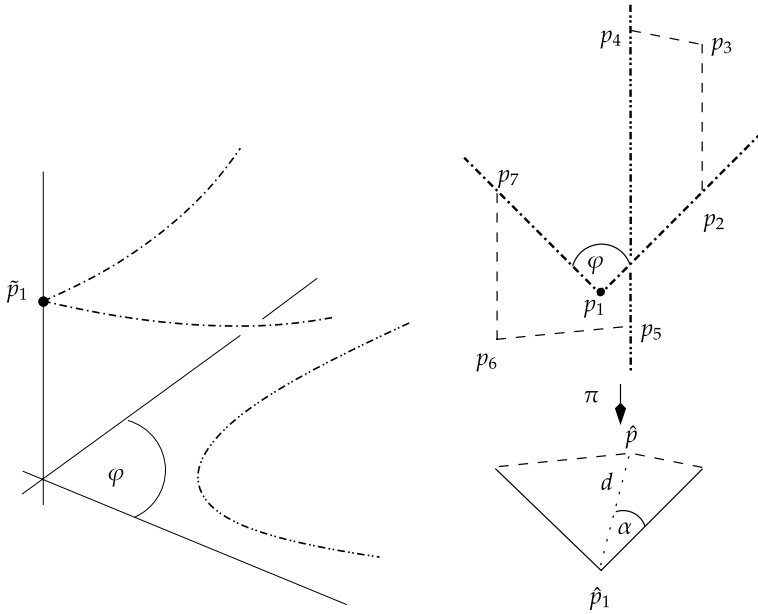


FIGURE 2. Left: The desired boundary of the cmc surface in $\Sigma(\kappa) \times \mathbb{R}$. Right: The corresponding boundary of the minimal sister surface in $E(\kappa + 4H^2, H)$. The single-dotted curve left corresponds to the single-dotted curve on the right.

4.2. Plateau solutions. The idea is to consider the Plateau solutions for Γ_n and to take their limit for $n \rightarrow \infty$. We show that there exists a domain Ω_n with mean convex boundary, such that $\Gamma_n \subset \partial\Omega_n$.

Definition 5. A Riemannian manifold N with boundary is mean convex if the boundary ∂N is piecewise smooth, each smooth subsurface of ∂N has non-negative mean curvature with respect to the inward normal, and there exists a Riemannian manifold N' such that N is isometric to a submanifold of N' and each smooth subsurface S of ∂N extends to a smooth embedded surface S' in N' such that $S' \cap N = S$. We call each surface S a barrier.

If such a domain Ω_n exists, the solution of the Plateau problem for Γ_n is an embedded minimal disc. In the last step (Theorem 8), it is shown that the sequence of minimal discs has a limiting minimal disc with boundary Γ .

PROPOSITION 6. *The special Jordan curve $\Gamma_n \subset E(\kappa + 4H^2, H)$ bounds a Plateau solution $M_n \subset E(\kappa + 4H^2, H)$ for large $n \in \mathbb{N}$, which extends without branch points by Schwarz reflection about the edges of Γ_n .*

Proof. We define the domain Ω_n with mean convex boundary as the intersection of five domains: two of them have horizontal umbrellas as boundaries, two have vertical planes as boundaries and the last domain has four fundamental pieces of the minimal surfaces from Theorem 3 in its boundary.

- (1) Take the halfspaces above the horizontal umbrella U_5 in p_5 and below the horizontal umbrella U_4 in p_4 . Below resp. above means in negative resp. positive ξ -direction. We call the intersection of the two halfspaces a *horizontal slab*. The umbrellas are with respect to the same fiber and therefore are parallel sections with vertical distance $d(U_5, U_4) = d(p_5, p_4)$. If $U_4 \cap \Gamma_n \setminus \{\overline{p_4 p_3}\} \neq \emptyset$ or $U_5 \cap \Gamma_n \setminus \{\overline{p_5 p_6}\} \neq \emptyset$, we redefine Γ_n by translating $p_{4/5}$ in $\pm \xi$ direction by a factor $c_{4/5}$. Since $U_{4/5}$ are sections, they are graphs above $\pi(\overline{p_1 p_2})$ and $\pi(\overline{p_1 p_7})$. Therefore we find constants $c_{4/5} > 0$ such that the new boundary curve, we call it again Γ_n does not intersect the umbrellas except for $\overline{p_4 p_3}$ and $\overline{p_5 p_6}$. The horizontal slab is a barrier for Γ_n , for all $n \in \mathbb{N}$.
- (2) Furthermore, consider the vertical halfspaces defined by the horizontal arcs $\overline{p_1 p_2}$ and $\overline{p_1 p_7}$, such that Γ_n lies inside.
- (3) The last domain is based on the minimal surface from Section 3.3: The idea is to consider a mean convex manifold sandwiched between two minimal surfaces as in Remark 1 with $(a, k) = (d, 2k)$, their vertical axes coincide with $\overline{p_4 p_5}$. We call one of the surfaces S_+ and the other S_- . Their position relative to Γ_n is given by a rotation angle δ and the vertical distances h_{\pm} from the horizontal umbrella U in p_1 . We define h_{\pm} and δ within the proof. We call the vertical distance to U *height*.

We claim, there exists $\delta, h_+, h_- > 0$ such that by rotation about $\overline{p_4 p_5}$ by angle δ followed by vertical translations by h_+ resp. h_- , the surfaces S_+ and S_- are barriers for Γ_n for large $n \in \mathbb{N}$. Compare Figure 3 for the projection of the three boundaries of M_n , S_+ and S_- to $\Sigma(\kappa + 4H^2)$.

We define the position of S_+ relative to Γ_n first: We choose an orientation such that the projection of its horizontal edge of length d coincides with the diagonal of the quadrilateral in the projection. Afterwards we translate the surface in ξ -direction by some small $h_+ > 0$. There exists $N \in \mathbb{N}$ such that for all $n \geq N$ the surface S_+ does not intersect Γ_n , because S_+ is graph above the projection of the horizontal edges of Γ_n . In the projection, the horizontal hinge of Γ_n encloses an angle $\delta := \varphi/2 - \alpha > 0$ with the horizontal hinge of ∂S_+ .

To define the position of S_- we rotate it about $\overline{p_4 p_5}$ such that in the projection the two edges of length d enclose an angle δ . The surface S_- is a graph and there exists $h_- > 0$ such that after translating the surface by h_- in $-\xi$ -direction it lies below a bounded component of $\overline{p_1 p_2}$, hence $S_- \cap \overline{p_1 p_2} = \emptyset$. The other edges of Γ_n are uncritical for n large enough.

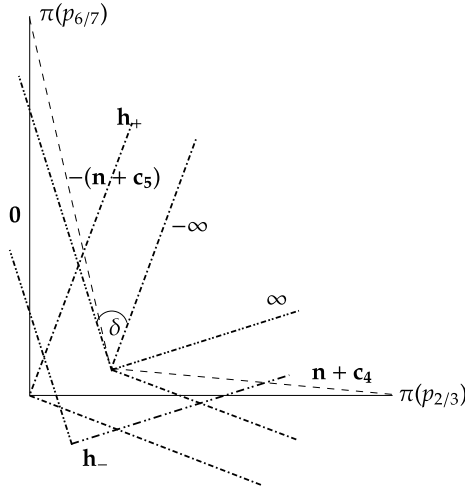


FIGURE 3. The projection of the three boundaries of M_n , S_+ (single-dotted) and S_- (double-dotted) to $\Sigma(\kappa + 4H^2)$. The heights of the horizontal geodesics are printed in bold letters. Here $k = 2$ and $\kappa + 4H^2 = 0$.

It remains to show that for every $p \in p_4p_5$ the opening angle $\psi(h)$ in height h of the tangent cone T_pC of S is less than π . This is clear for p at height $|h| > \max\{h_{\pm}\}$. The angle depends on h_+ and h_- and is bounded by $2(\pi - \varphi - \epsilon) + \delta$ where $\epsilon \geq 0$ denotes the defect depending on $\Sigma(\kappa + 4H^2)$. We consider the level curves of S_+ and S_- in height h . The level curves define angles $\beta_+(h)$ and $\beta_-(h)$ given in the projection by the angle of the projected conormal in height h of S_{\pm} and the edge of length d of the corresponding surface, see Figure 4. Therefore, we have

$$\psi(h) = \beta_+(h) + \beta_-(h) + \delta.$$

But h_+ was chosen independently of n , δ and h_- , moreover for $h_+ \rightarrow 0$ we have $\beta_+(h) \rightarrow 0$. Hence, we conclude

$$\psi(h) \rightarrow \beta_-(h) + \delta < \pi - \frac{\pi}{2k} - \epsilon + \delta < \pi - \alpha - \epsilon.$$

Hence, depending on α there exists $h_+ > 0$ such that $\psi(h) < \pi$ at all heights h .

We summarise: Γ_n lies in-between two copies of a cmc $2k$ -noid, that is, there exists $N \in \mathbb{N}$ such that $(S_+ \cup S_-) \cap \Gamma_n = p_4p_5$ for $n \geq N$.

We complete the last barrier by subsets of the two horizontal umbrellas at heights h_{\pm} given by the edges of S_+ and S_- ; it defines the boundary of the halfspace S .

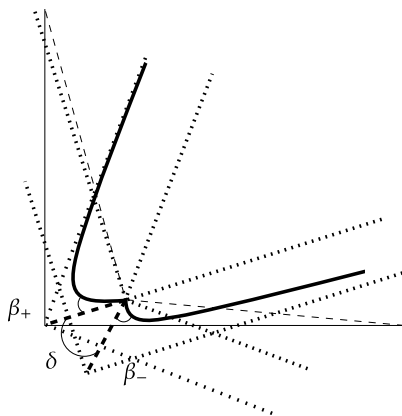


FIGURE 4. Level curves: The solid lines sketch the intersection of the horizontal umbrella U with M_n and the symmetric $2k$ -noids (bold). The dashed lines indicate the remaining boundaries. Here $k = 2$.

We define the mean convex domain Ω_n as the intersection of the five halfspaces. Since Γ_n lies in the boundary of a mean convex domain, the existence of an embedded minimal surface M_n of disc-type with boundary Γ_n follows from [HS88]. \square

REMARK 2. The definition of Ω_n would be more direct if we could define it as the intersection of halfspaces. A vertical plane/horizontal umbrella separates $E(\kappa + 4H^2, H)$ into two connected components, but two fundamental patches of the minimal surface from Section 3.3 do not separate $E(\kappa + 4H^2, H)$ in two connected components. This is because of the normal turning along the vertical geodesic. To get several connected components, we have to use $S_+ \cup S_-$, but their boundary would not be smooth anymore.

PROPOSITION 7. *The Plateau solution M_n is a graph over a simply connected domain Δ_n enclosed by $\partial\Delta_n := \pi(\Gamma_n)$ and unique among all Plateau solutions with the prescribed boundary values for each $n \in \mathbb{N}$.*

Proof. The proof consists of two steps. First, we show that the projection of M_n is an immersion. Second, we show that the projection is one-to-one.

(1) We show that M_n does not have any vertical tangent planes.

Suppose there exists a vertical plane V that is tangent to M_n at some $p \in M_n$. We consider the intersection $V \cap \overline{M_n}$: Since M_n and V are both minimal but not identical, their intersection $M_n \cap V$ is an union of analytic curves ending on $\partial M_n = \Gamma_n$. At p at least two of them meet. Since $M_n \cap V$ cannot contain a loop and the analytic curves have at least four endpoints on Γ_n .

Let us consider the intersection of the vertical plane with the mean convex domain $\Omega_n \cap V$. It might consist of more than one connected component. For the connected component V_p containing p , we know that $\Gamma_n \cap V_p$ has at most two connected components. Hence, we get a contradiction, that is, there exists no vertical tangent plane.

- (2) By construction, Γ_n is embedded and graph over $\partial\Delta_n$ except at three points. Since $M_n \subset \Omega_n$ it is clear, that M_n is locally a graph along $\overline{p_1p_2}$ and $\overline{p_1p_7}$ over a subset of Δ_n . The same is true along $\overline{p_3p_4}$ and $\overline{p_6p_7}$, since each edge defines a vertical half plane V_1 resp. V_2 as a barrier, such that $\overline{p_3p_4} \subset V_1$ resp. $\overline{p_6p_7} \subset V_2$, and $\partial V_1 = \partial V_2 = \pi^{-1}(\hat{p})$.

Remains to consider the vertical components of Γ_n , here the normal ν is horizontal. Assume there exists a point p where M_n is locally not a graph, that is, $d\nu_p = 0$. Therefore, the intersection of the vertical tangent plane in p and M_n is an union of at least three analytic curves, hence the intersection with Γ_n consists of three components. This a contradiction to the construction.

We have seen, in a neighbourhood of ∂M_n the surface is a graph over a region in Δ_n . The complement of this neighbourhood is compact, simply-connected and its boundary is itself a graph. Since the projection is an immersion, this implies M_n is a graph.

The uniqueness follows from the maximum principle. □

Now we can take the limit $n \rightarrow \infty$:

THEOREM 8. *There exists a minimal surface $M_\infty \subset E(\kappa + 4H^2, H)$, $\kappa + 4H^2 \leq 0$ which is a section over Δ and extends without branch points by Schwarz reflection across its edges.*

Proof. It is sufficient to prove that the sequence is uniformly bounded on each Δ_k . We modify the proof of Theorem 3: For each Δ_k we consider two fundamental pieces M_\pm of the minimal surface from Section 3.3, one with the end going to infinity and the other going to minus infinity. As before, we may orientate them such that the positive end of M_+ lies above the boundary component of M_k with value k and the negative end of M_K lies below the boundary component of M_k with value $-k$. We continue M_\pm with a horizontal umbrella such that each is a minimal section well-defined on Δ_k . By the maximum principle, there is no point of contact if we consider the sequence M_n for $n \geq k$. After diagonalization, we obtain a minimal surface M_∞ which is a section over Δ . As in the proof of Theorem 3, it extends by construction without branch points by Schwarz reflection across its edges. □

The minimal surface M_∞ is a fundamental piece of an interesting minimal surface itself. We reflect its sister surface to construct a cmc $2k$ -noid in $\Sigma(\kappa) \times \mathbb{R}$:

THEOREM 9. For $H \in [0, 1/2]$ and $k \geq 2$ there exists a two-parameter family

$$\{\tilde{M}_{d,\alpha} : d > 0, 0 < \alpha \leq \pi/(2k)\}$$

of constant mean curvature H surfaces in $\Sigma(\kappa) \times \mathbb{R}$, $\kappa \leq 0$, such that:

- $\tilde{M}_{d,\alpha}$ has k vertical mirror planes enclosing an π/k -angle,
- $\tilde{M}_{d,\alpha}$ has one horizontal mirror plane and
- for $\alpha = \pi/(2k)$ the surface $\tilde{M}_{d,\alpha}$ is symmetric and coincides with the surface from Section 3.3.

Proof. By Daniel's correspondence ([Dan07], see Theorem 2 above) the fundamental piece M_∞ has a sister surface \tilde{M}_∞ with constant mean curvature H in $\Sigma(\kappa) \times \mathbb{R}$. By construction, \tilde{M}_∞ has three curves in mirror planes: one in a horizontal and two in vertical planes; the two vertical mirror planes enclose an angle π/k . Schwarz reflection about those planes extends the surface to a complete MC H surface $\tilde{M}_{d,\alpha}$ with $2k$ ends. The MC H -surface $\tilde{M}_{d,\alpha}$ consists of $4k$ fundamental pieces \tilde{M}_∞ . \square

REFERENCES

- [CR10] P. Collin and H. Rosenberg, *Construction of harmonic diffeomorphisms and minimal graphs*, Ann. of Math. (2) **172** (2010), no. 3, 1879–1906. MR 2726102
- [Dan07] B. Daniel, *Isometric immersions into 3-dimensional homogeneous manifolds*, Comment. Math. Helv. **82** (2007), no. 1, 87–131. MR 2296059
- [DH09] B. Daniel and L. Hauswirth, *Half-space theorem, embedded minimal annuli and minimal graphs in the Heisenberg group*, Proc. Lond. Math. Soc. (3) **98** (2009), no. 2, 445–470. MR 2481955
- [Ear08] R. Sa Earp, *Parabolic and hyperbolic screw motion surfaces in $\mathbb{H}^2 \times \mathbb{R}$* , J. Aust. Math. Soc. **85** (2008), no. 1, 113–143. MR 2460869
- [GK09] K. Große-Brauckmann and R. B. Kusner, *Ruled minimal surfaces in homogeneous manifolds and their sisters*, preprint, 2009.
- [GK10] K. Große-Brauckmann and R. B. Kusner, *Conjugate Plateau constructions for homogeneous 3-manifolds*, preprint, 2010.
- [GL73] R. Gulliver and F. D. Lesley, *On boundary branch points of minimizing surfaces*, Arch. Ration. Mech. Anal. **52** (1973), 20–25. MR 0346641
- [HS88] J. Hass and P. Scott, *The existence of least area surfaces in 3-manifolds*, Trans. Amer. Math. Soc. **310** (1988), no. 1, 87–114. MR 0965747
- [Kar89] H. Karcher, *Construction of minimal surfaces*, Surveys in geometry, University of Tokyo, 1989.
- [Kar05] H. Karcher, *Introduction to conjugate Plateau constructions*, Global theory of minimal surfaces, Proceedings of the Clay Mathematics Institute 2001 summer school (Berkeley, CA, USA, June 25–July 27, 2001), American Mathematical Society (AMS), Providence, RI, 2005, pp. 137–161. MR 2167258
- [Law70] H. B. Lawson, *Complete minimal surfaces in \mathbb{S}^3* , Ann. of Math. (2) **2** (1970), no. 92, 335–374. MR 0270280
- [MR12] F. Morabito and M. M. Rodríguez, *Saddle towers and minimal k -noids in $\mathbb{H}^2 \times \mathbb{R}$* , J. Inst. Math. Jussieu **11** (2012), no. 2, 333–349. MR 2905307
- [MT12] J. M. Manzano and F. Torralbo, *New examples of constant mean curvature surfaces in $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$* , Michigan Math. J. **63** (2014), 701–723; available at arXiv:1104.1259. MR 3286667

- [MY82] W. H. Meeks III and S.-T. Yau, *The existence of embedded minimal surfaces and the problem of uniqueness*, Math. Z. **179** (1982), 151–168. MR 0645492
- [Ple12a] J. Plehnert, *Constant mean curvature surfaces in homogeneous manifolds*, Logos Verlag, Berlin, 2012.
- [Ple12b] J. Plehnert, *Surfaces with constant mean curvature $1/2$ and genus one in $\mathbb{H}^2 \times \mathbb{R}$* , preprint, 2012; available at arXiv:1212.2796.
- [Pyo11] J. Pyo, *New complete embedded minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$* , Ann. Global Anal. Geom. **40** (2011), no. 2, 167–176. MR 2811623
- [Sco83] P. Scott, *The geometries of 3-manifolds*, Bull. Lond. Math. Soc. **15** (1983), 401–487. MR 0705527
- [Thu97] W. P. Thurston, *Three-dimensional geometry and topology*, Princeton Mathematical Series, vol. 1, Princeton University Press, Princeton, NJ, 1997. MR 1435975
- [You10] R. Younes, *Minimal surfaces in $\widetilde{\text{PSL}}_2(\mathbb{R})$* , Illinois J. Math. **54** (2010), no. 2, 671–712. MR 2846478

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