Illinois Journal of Mathematics
Volume 58, Number 1, Spring 2014, Pages 97–112
S 0019-2082

BAIRE CLASSES OF L_1-PREDUALS AND C^*-ALGEBRAS

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ABSTRACT. Let X be a separable real or complex L_1-predual such that its dual unit ball B_{X^*} has the set $\operatorname{ext} B_{X^*}$ of its extreme points of type F_σ. We identify intrinsic Baire classes of X with the spaces of odd or homogeneous Baire functions on $\operatorname{ext} B_{X^*}$. Further, we answer a question of S. A. Argyros, G. Godefroy and H. P. Rosenthal by showing that there exists a separable C^*-algebra X (the so-called CAR-algebra) for which the second intrinsic Baire class of X^{**} does not coincide with the second Baire class of X^{**}.

1. Introduction

A real (or complex) Banach space X is called an L_1*-predual* (sometimes a *Lindenstrauss space*) if its dual X^* is isometric to a real (or complex) space $L^1(X, \mathcal{S}, \mu)$ for a measure space $(X, \mathcal{S}, \mu)$. Real L_1-preduals were in depth investigated in papers [6], [9], [10], [3], [18], [25], [11], [26], [12], [4] or [5]. The complex variant of L_1-preduals was studied for example, in [14], [28], [20], [29], [8] or recently, in [27]. It has turned out that a real Banach space V is an L_1-predual if and only if its dual unit ball B_{V^*} satisfies a "simplex-like" condition (see [19]). A complex version of this "simplex-like" characterization was provided by Effros in [7]. It is mentioned in this paper that "we have reason to believe that this result will make theory of complex Lindenstrauss spaces as accessible as that for real spaces."

Received July 4, 2013; received in final form September 16, 2014.
The second author was supported in part by the grant GAČR P201/12/0290 and by The Foundation of Karel Janeček for Science and Research.
2010 *Mathematics Subject Classification.* Primary 46B25, 26A21. Secondary 46B04, 46A55, 46L05.

The goal of our paper is to support this belief by results on real and complex L_1-predual spaces and their Baire classes. The significance of Effros's characterization becomes apparent especially from the comparison of Sections 2 and 3 of the paper in hand.

Let $\mathbb{F}$ denote the field $\mathbb{R}$ or $\mathbb{C}$.

For a topological space K, let $\mathcal{B}(K,\mathbb{F})$ be the space of all Borel functions with values in $\mathbb{F}$ and $\mathcal{B}^b(K,\mathbb{F})$ be the space of all bounded Borel functions on K with values in $\mathbb{F}$. For a compact (Hausdorff) topological space K, let $\mathcal{C}(K,\mathbb{F})$ stand for the space of all continuous functions on K with values in $\mathbb{F}$. In case K is compact, we write $\mathcal{M}(K,\mathbb{F})$ for the space of Radon measures on K and $\mathcal{M}^1(K)$ for Radon probability measures on K.

Let $\mathcal{H}$ be a subset of $\mathcal{C}(K,\mathbb{F})$. Then we set $\mathcal{B}^0(\mathcal{H})=\mathcal{H}$ and, for $\alpha\in(0,\omega_1)$, let $\mathcal{B}^\alpha(\mathcal{H})$ consist of all pointwise limits of elements from $\bigcup_{\beta<\alpha}\mathcal{B}^\beta(\mathcal{H})$. Further, we denote by $\mathcal{B}^{\alpha,b}(\mathcal{H})$ the set of all bounded elements from $\mathcal{B}^\alpha(\mathcal{H})$. The symbol $\mathcal{B}^{\alpha,bb}(\mathcal{H})$ denotes the inductive families created by means of pointwise limits of bounded sequences of lower classes.

If we start the inductive procedure from the space of all continuous functions, we write simply $\mathcal{B}^\alpha(K,\mathbb{F})$ and $\mathcal{B}^{\alpha,b}(K,\mathbb{F})$ for the obtained spaces of Baire-α functions. Then we have $\mathcal{B}^{\alpha,b}(K,\mathbb{F})=\mathcal{B}^{\alpha,bb}(K,\mathbb{F})$. Let us remind that for a metrizable K holds $\mathcal{B}^b(K,\mathbb{F})=\bigcup_{\alpha<\omega_1}\mathcal{B}^{\alpha,b}(K,\mathbb{F})$. Having started with the space $\mathcal{A}(K,\mathbb{F})$ of all continuous affine functions on a compact convex set K in a locally convex space, we obtain spaces $\mathcal{A}^\alpha(K,\mathbb{F})$, $\mathcal{A}^{\alpha,b}(K,\mathbb{F})$ and $\mathcal{A}^{\alpha,bb}(K)$. As a consequence of the uniform boundedness principle we get $\mathcal{A}^{\alpha,bb}(K,\mathbb{F})=\mathcal{A}^{\alpha,b}(K,\mathbb{F})=\mathcal{A}^\alpha(K,\mathbb{F})$ (see, e.g., [24, Lemma 5.36]) and elements of this set we call *functions of affine class* α.

If X is a Banach space over $\mathbb{F}$ and B_{X^*} is its dual unit ball endowed with the weak* topology, X is isometrically embedded in $\mathcal{C}(B_{X^*},\mathbb{F})$ via the canonical embedding. We recall definitions of Baire classes of X^{**} from [2]. For $\alpha\in[0,\omega_1)$, we call $\mathcal{B}^\alpha(X)$ the *intrinsic α-Baire class of* X^{**}. Following [2, p. 1044], we denote the intrinsic αth Baire class as X^{**}_α. Let us remark, that our definition is formally slightly different from the one introduced in [2]. While in our case elements of X^{**}_α are restrictions of the uniquely determined elements from X^{**} to the closed unit ball B_{X^*}, the functions considered in [2] are precisely these extensions. This is substantiated by Lemma 2.2.

Still considering X as a subspace of $\mathcal{C}(B_{X^*})$, *the αth Baire class* of X^{**} is defined as

$$X^{**}_{\mathcal{B}_\alpha}=\left\{x^{**}\in X^{\perp\perp};x^{**}|_{B^*_X}\in\mathcal{B}^\alpha(B_{X^*},\mathbb{F})\right\}.$$

It can be verified that $x^{**}\in X^{**}_{\mathcal{B}_\alpha}$ if and only if $x^{**}|_{B_{X^*}}\in\mathcal{B}^\alpha(B_{X^*},\mathbb{F})$ and $x^{**}|_{B_{X^*}}$ satisfies the barycentric calculus, that is,

$$x^{**}\left(\int_{B_{X^*}}\mathrm{id}\,d\mu\right)=\int_{B_{X^*}}x^{**}\,d\mu$$

for every probability measure $\mu \in \mathcal{M}^1(B_{X^*})$. Where no confusion can arise, we do not distinguish between $X^{**}_{\mathcal{B}_\alpha}$ and $X^{**}_{\mathcal{B}_\alpha}|_{B_{X^*}}$.

Obviously, $X^{**}_\alpha \subset X^{**}_{\mathcal{B}_\alpha}$ but the converse need not hold by [36, Theorem] (for a detailed exposition on Baire classes of Banach spaces we refer the reader to [2, pp. 1043–1048]).

The first goal of our paper is the extension of the following result by Lindenstrauss and Wulbert proved in [21, Theorem 1]:

Let X be a real L_1-predual and T stand for the closure of extreme points $\operatorname{ext} B_{X^*}$ *of* B_{X^*}. *If* $T = \operatorname{ext} B_{X^*}$, *then* $X = \mathcal{C}_\Sigma(T,\mathbb{R})$, *where* $\Sigma(x^*) = -x^*$, $x^* \in T$ *and* $\mathcal{C}_\Sigma(T,\mathbb{R})$ *consists of real continuous functions on T satisfying* $f(x^*) = -f(-x^*)$.

We show in Theorem 2.10 that for a real L_1-predual X the space X^{**}_α can be identified with the space $\mathcal{B}^{\alpha,b}_\sigma(\operatorname{ext} B_{X^*},\mathbb{R})$ of all odd bounded Baire-α functions in case $\operatorname{ext} B_{X^*}$ is of type F_σ. An analogous result for complex L_1-predual is also valid.

The second goal of the paper is to extend to the complex setting the following result from [23] (see [23, Theorem 1.4]):

Let X be a real L_1-predual and let $x^{**} \in X^{**}$ satisfy $f = x^{**}|_{B_{X^*}} \in \mathcal{B}^\alpha(B_{X^*},\mathbb{R})$ for $\alpha \in [2,\omega_1)$. Then $f \in X^{**}_{\alpha+1}$ if $\alpha < \omega_0$ and $f \in X^{**}_\alpha$ if $\alpha \geq \omega_0$.

Our technique allows us to fulfill our intentions at least for separable complex L_1-preduals. This is achieved by a complex variant of Proposition 2.6 (see Section 3 or [22, Proposition 3.3.6]).

Finally, a question posed in [2, p. 1048] asks *whether for a separable C^*-algebra X holds $X^{**}_{\mathcal{B}_\alpha} = X^{**}_\alpha$*. We answer this question in the negative, more precisely we prove using [27] and [34] that there is a separable C^*-algebra X satisfying $X^{**}_{\mathcal{B}_2} \neq X^{**}_2$.

Throughout the paper, we work within separable Banach spaces, since our methods are based on the metrizability of their dual unit balls. The question of validity of the presented results for the case of nonseparable spaces is still open.

2. Real L_1-preduals

Let K be a compact convex set in a locally convex topological vector space. To a point $x \in K$, we can assign the set $\mathcal{M}^1_x(K)$ consisting of all probability measures on K satisfying $\int_K \operatorname{id} d\mu = x$ (equivalently, $\mu(h) = h(x)$ for any continuous affine function h on K). A function f on K is *strongly affine* if f is μ-measurable for each $\mu \in \mathcal{M}^1(K)$ and $f(x) = \mu(f)$ for any $x \in K$ and $\mu \in \mathcal{M}^1_x(K)$. Any strongly affine function is bounded (see, e.g., [24, Lemma 4.5]).

The usual dilation order $\prec$ on $\mathcal{M}^1(K)$ is defined as $\mu \prec \nu$ if and only if $\mu(f) \leq \nu(f)$ for any convex continuous function f on K. We write $\mathcal{M}^{\max}(K)$ for the set of all probability measures on K which are maximal with respect to $\prec$. A measure $\mu \in \mathcal{M}(K,\mathbb{F})$ is *boundary* if either $\mu = 0$ or the probability measure $\frac{|\mu|}{\|\mu\|}$ is maximal.

For a function $f \in \mathcal{C}(K,\mathbb{R})$, let

$$\widehat{f}(x) = \sup\bigl\{\mu(f); \mu \in \mathcal{M}_x^1(K)\bigr\}.$$

By the Choquet representation theorem, for any $x \in K$ there exists $\mu \in \mathcal{M}_x^1(K) \cap \mathcal{M}^{\max}(K)$ (see [17, p. 192, Corollary]). The set K is termed *simplex* if this measure is uniquely determined for each $x \in K$ (see [17, §20, Theorem 3]). In case K is metrizable, maximal measures are carried by the G_δ set $\operatorname{ext} K$ of extreme points of K (see [17, §20, Theorem 5]).

Let X be a real Banach space. Then $\sigma(x^*) = -x^*$, $x^* \in B_{X^*}$, is a natural affine homeomorphism of B_{X^*} onto itself. A set $B \subset B_{X^*}$ is *symmetric* if $\sigma(B) = B$. An example of a symmetric set is the set $\operatorname{ext} B_{X^*}$. For a function f defined on a symmetric set $B \subset B_{X^*}$ we define

$$(\operatorname{odd} f)\bigl(x^*\bigr) = \frac{1}{2}\bigl(f\bigl(x^*\bigr) - f\bigl(-x^*\bigr)\bigr), \quad x^* \in B.$$

A function f defined on a symmetric subset is odd if $\operatorname{odd} f = f$.

For $\mu \in \mathcal{M}(B_{X^*},\mathbb{R})$, let $\operatorname{odd}\mu \in \mathcal{M}(B_{X^*},\mathbb{R})$ be defined as

$$(\operatorname{odd}\mu)(f) = \mu(\operatorname{odd} f), \quad f \in \mathcal{C}(B_{X^*},\mathbb{R}).$$

The following characterization of L_1-preduals is due to Lazar (see [19, Theorem] or [17, §21, Theorem 7]):

Let X be a Banach space. Then X is an L_1-predual if and only if $\operatorname{odd}\mu = \operatorname{odd}\nu$ *for each $x^* \in B_{X^*}$ and $\mu,\nu \in \mathcal{M}_{x^*}^1(B_{X^*}) \cap \mathcal{M}^{\max}(B_{X^*})$.*

Let X be a real separable L_1-predual and f be a bounded Borel function f defined on a Borel subset of B_{X^*} containing $\operatorname{ext} B_{X^*}$. We define

$$Tf\bigl(x^*\bigr) = (\operatorname{odd}\mu)(f), \quad x^* \in B_{X^*}, \mu \in \mathcal{M}_{x^*}^1(B_{X^*}) \cap \mathcal{M}^{\max}(B_{X^*}). \tag{2.1}$$

Notice that Tf is well defined because of Lazar's characterization and because $\operatorname{odd}\mu$, as a boundary measure, is carried by the G_δ set $\operatorname{ext} B_{X^*}$.

The described mapping T is a natural generalization of the dilation mapping defined in the simplicial case for example, in [24, Definition 6.7].

PROPOSITION 2.1. *Let X be a real separable L_1-predual and T be defined as in* (2.1).

(a) *If $f \in \mathcal{C}(B_{X^*},\mathbb{R})$, then Tf is Baire*-1.
(b) *If $f \in \mathcal{B}^b(\operatorname{ext} B_{X^*},\mathbb{R})$, then Tf is an odd Borel strongly affine function on B_{X^*}.*

Proof. Since X is separable, B_{X^*} is a metrizable compact convex set (see [30, Theorems 3.15, 3.16]), and thus there exists a mapping $S: B_{X^*} \to \mathcal{M}^{\max}(B_{X^*})$ such that $Sx^* = \nu_{x^*} \in \mathcal{M}_{x^*}^1(B_{X^*})$ and the function $Sf: x^* \mapsto \nu_{x^*}(f)$ is a Baire-1 function on B_{X^*} for each continuous function f on B_{X^*} (see [35, Théoréme 1] or [24, Theorem 11.41]).

(a) Let $f \in \mathcal{C}(B_{X^*}, \mathbb{R})$ be given. Then $\operatorname{odd} f$ is a continuous function on B_{X^*} and, for a fixed $x^* \in B_{X^*}$, we have

$$Tf\big(x^*\big) = (\operatorname{odd} \nu_{x^*})(f) = \nu_{x^*}(\operatorname{odd} f) = S(\operatorname{odd} f)\big(x^*\big).$$

Thus, $Tf = S(\operatorname{odd} f)$ is a Baire-1 function on B_{X^*}.

(b) Let now

$$\mathcal{F} = \big\{f \in \mathcal{B}^b(B_{X^*}, \mathbb{R}); Tf \text{ is Borel}\big\}.$$

Then $\mathcal{F}$ is closed under the taking pointwise limits of bounded sequences by the Lebesgue dominated convergence theorem and contains $\mathcal{C}(B_{X^*}, \mathbb{R})$. Hence, it contains any bounded Borel function on B_{X^*}.

Let f be a bounded Borel function on $\operatorname{ext} B_{X^*}$. Since $\operatorname{ext} B_{X^*}$ is a Borel set, we can consider f to be a bounded Borel function on B_{X^*}. Hence $f \in \mathcal{F}$ and Tf is Borel.

Let us show that Tf is strongly affine, that is, that $\nu(Tf) = Tf(y^*)$ for each $y^* \in B_{X^*}$ and $\nu \in \mathcal{M}^1_{y^*}(B_{X^*})$. Given y^* and ν as above, let $\mu \in \mathcal{M}^1(B_{X^*})$ be defined as

$$\mu(g) = \int_{B_{X^*}} \nu_{x^*}(g)\, d\nu\big(x^*\big), \quad g \in \mathcal{C}(B_{X^*}, \mathbb{R}).$$

If g is a convex continuous function and $\widehat{g}$ is its upper envelope, due to Mokobodzki's maximality test (e.g., [17, §20, Theorem 2]), we have $\nu_{x^*}(g) = \nu_{x^*}(\widehat{g})$, $x^* \in B_{X^*}$, and thus

$$\mu(\widehat{g}) = \int_{B_{X^*}} \nu_{x^*}(\widehat{g})\, d\nu\big(x^*\big) = \int_{B_{X^*}} \nu_{x^*}(g)\, d\nu\big(x^*\big) = \mu(g).$$

Hence, μ is maximal. Further, for an affine continuous function h on B_{X^*} we have

$$\mu(h) = \int_{B_{X^*}} \nu_{x^*}(h)\, d\nu\big(x^*\big) = \int_{B_{X^*}} h\big(x^*\big)\, d\nu\big(x^*\big) = h\big(y^*\big),$$

and thus $\mu \in \mathcal{M}^1_{y^*}(B_{X^*})$. Hence, $Tf(y^*) = (\operatorname{odd} \mu)(f)$ and it follows that

$$\begin{aligned}\nu(Tf) &= \int_{B_{X^*}} Tf\big(x^*\big)\, d\nu\big(x^*\big) = \int_{B_{X^*}} \nu_{x^*}(\operatorname{odd} f)\, d\nu\big(x^*\big)\\ &= \mu(\operatorname{odd} f) = (\operatorname{odd} \mu)(f) = Tf\big(y^*\big).\end{aligned}$$

Hence, $\nu(Tf) = Tf(y^*)$ and Tf is strongly affine.

Finally, we show that Tf is odd. Since Tf is affine, it is enough to show that $Tf(0) = 0$. Let x^* be an extreme point of B_{X^*}. Then the combination $\mu = \frac{1}{2}(\varepsilon_{x^*} + \varepsilon_{-x^*})$ of the Dirac measures ε_{x^*}, ε_{-x^*} is contained in $\mathcal{M}^1_0(B_{X^*}) \cap \mathcal{M}^{\max}(B_{X^*})$. Because $\operatorname{odd} f$ is an odd function,

$$Tf(0) = (\operatorname{odd} \mu)(f) = \mu(\operatorname{odd} f) = \frac{1}{2}\Big((\operatorname{odd} f)\big(x^*\big) + (\operatorname{odd} f)\big(-x^*\big)\Big) = 0.$$

Hence, Tf is odd. □

Lemma 2.2. *Let X be a real Banach space and let f be an odd strongly affine function on the closed unit ball B_{X^*}. Then f is a restriction of a uniquely determined element of X^{**}.*

Proof. This simple observation is based on the fact that a strongly affine function f on B_{X^*} is bounded (e.g., [24, Lemma 4.5]). Thus, the uniquely defined linear extension of f is an element of X^{**}. □

Proposition 2.3. *Let X be a real separable L_1-predual and $f \in \mathcal{B}^b(\operatorname{ext} B_{X^*}, \mathbb{R})$. If h is an odd strongly affine function on B_{X^*} extending f, then $h = Tf$.*

Proof. The function f, being extended by an odd function h, is odd as well.

Let $y^* \in B_{X^*}$ be given. We choose a maximal measure $\mu \in \mathcal{M}^1_{y^*}(B_{X^*})$ and compute

$$Tf(y^*) = (\operatorname{odd}\mu)(f) = \mu(\operatorname{odd} f) = \mu(f) = \int_{\operatorname{ext} B_{X^*}} h(x^*)\, d\mu(x^*) = h(y^*).$$

This concludes the proof. □

Proposition 2.4. *Let X be a real separable L_1-predual and assume that $f \in \mathcal{B}^{\alpha,b}(\operatorname{ext} B_{X^*}, \mathbb{R})$.*

(a) *If $\alpha \in [1, \omega_0)$, then $Tf \in X^{**}_{\alpha+1}$.*

(b) *If $\alpha \in [\omega_0, \omega_1)$, then $Tf \in X^{**}_{\alpha}$.*

(c) *If $\alpha \in [1, \omega_1)$ and $\operatorname{ext} B_{X^*}$ is of type F_σ, then $Tf \in X^{**}_{\alpha}$.*

Proof. (a) If $\alpha = 1$, f can be extended to a bounded Baire-1 function on B_{X^*} ([1, Corollary I.4.4] and [16, §35, VI, Theorem]). Let (f_n) be a bounded sequence in $\mathcal{C}(B_{X^*})$ converging to this extension on B_{X^*}. For a given $x^* \in B_{X^*}$, let $\mu \in \mathcal{M}^1_{x^*}(B_{X^*}) \cap \mathcal{M}^{\max}(B_{X^*})$ be chosen. Then we have

$$Tf_n(x^*) = (\operatorname{odd}\mu)(f_n) \to (\operatorname{odd}\mu)(f) = Tf(x^*).$$

Since $Tf_n = \operatorname{odd} f_n$ on $\operatorname{ext} B_{X^*}$, each Tf_n is a continuous function on $\operatorname{ext} B_{X^*}$. By Proposition 2.1 and [23, Theorem 5.2], each Tf_n is an odd Baire-1 strongly affine function on B_{X^*}. By the Mokobodzki theorem ([2, Theorem II.1.2(a)]), $Tf_n \in X^{**}_1$. Hence $Tf \in X^{**}_2$.

The rest of the proof follows by induction.

(b) Let $\alpha = \omega_0$ and $f \in \mathcal{B}^{\alpha,b}(\operatorname{ext} B_{X^*}, \mathbb{R})$. Let (f_n) be a bounded sequence of functions from $\mathcal{B}^{\alpha_n,b}(\operatorname{ext} B_{X^*}, \mathbb{R})$, where $\alpha_n < \alpha$, converging to f. Then $Tf_n \to Tf$ and $Tf_n \in X^{**}_{\alpha_n+1}$ by (a). Hence $Tf \in X^{**}_{\alpha}$. For higher Baire classes the proof follows by transfinite induction.

(c) Let $\operatorname{ext} B_{X^*}$ is of type F_σ. If $f \in \mathcal{B}^{1,b}(\operatorname{ext} B_{X^*}, \mathbb{R})$, then also $\operatorname{odd} f \in \mathcal{B}^{1,b}(\operatorname{ext} B_{X^*}, \mathbb{R})$. Because a restriction of Tf to $\operatorname{ext} B_{X^*}$ is equal to $\operatorname{odd} f$, it is of Baire class 1. By Proposition 2.1 and [23, Theorem 1.3], Tf is a strongly affine function in $\mathcal{B}^{1,b}(B_{X^*}, \mathbb{R})$, and it is in X^{**}_1 by the Mokobodzki theorem (see [2, Theorem II.1.2(a)]).

For functions of higher Baire classes we proceed by transfinite induction. $\square$

We follow with a result based on [18, Theorem 4.4] which could serve as a motivation for Proposition 2.6.

PROPOSITION 2.5. *Let X be a real separable L_1-predual with* $\operatorname{ext} B_{X^*} \cup \{0\}$ *closed. If $\alpha \in [0, \omega_1)$ and f is an odd strongly affine function on B_{X^*} such that $f|_{\operatorname{ext} B_{X^*} \cup \{0\}}$ is a function of Baire class α, then $f \in X^{**}_\alpha$.*

Proof. Let $\alpha = 0$ and let f be as in the hypothesis. By Theorem 2.3, then $f = Tf$. Due to [18, Theorem 4.4] there exists an odd continuous affine function g such that $g = f$ on $\operatorname{ext} B_{X^*}$. Employing Theorem 2.3 again we get $g = Tf$, hence $f = g$ and therefore $f \in X$.

If $\alpha > 0$, let $F = \operatorname{ext} B_{X^*} \cup \{0\}$. Since X is separable, the set $\{0\}$ is of type G_δ. Then $\operatorname{ext} B_{X^*} = F \setminus \{0\}$ is of type F_σ. Applying Proposition 2.4(c) to a Baire-α function $f|_{\operatorname{ext} B_{X^*}}$ we get that Tf is of Baire class-α. Since $f = Tf$ due to Proposition 2.3, the proof is finished. $\square$

PROPOSITION 2.6. *Let X be a real separable L_1-predual, f an odd strongly affine function on B_{X^*} such that $f|_{\operatorname{ext} B_{X^*}}$ is of Baire class α on* $\operatorname{ext} B_{X^*}$.

(a) *If $\alpha \in [0, \omega_0)$, then $f \in X^{**}_{\alpha+1}$.*
(b) *If $\alpha \in [\omega_0, \omega_1)$, then $f \in X^{**}_\alpha$.*
(c) *If $\alpha \in [1, \omega_1)$ and* $\operatorname{ext} B_{X^*}$ *is of type F_σ, then $f \in X^{**}_\alpha$.*

Proof. (a) Let $\alpha \in [0, \omega_0)$ and f be an odd strongly affine function on B_{X^*} such that $f|_{\operatorname{ext} B_{X^*}}$ is of Baire class α. If $\alpha = 0$, that is, f is continuous on $\operatorname{ext} B_{X^*}$, then f is Baire-1 on B_{X^*} by [23, Theorem 5.2]. As an odd strongly affine Baire-1 function, f is in X^{**}_1 by [2, Theorem II.1.2(a)]. If $\alpha \in [1, \omega_0)$, $f = Tf$ due to Proposition 2.3. By Proposition 2.4(a), $f \in X^{**}_{\alpha+1}$. This finishes the proof of (a).

(b) If $\alpha \in [\omega_0, \omega_1)$, f is an odd strongly affine function and $f|_{\operatorname{ext} B_{X^*}}$ is of Baire class α, then $f = Tf$ by Proposition 2.3. It follows from Proposition 2.4(b) that $f \in X^{**}_\alpha$.

(c) It suffices to use Propositions 2.3 and 2.4(c). $\square$

THEOREM 2.7. *Let X be a real separable L_1-predual and let f be an odd function in $\mathcal{B}^{\alpha,b}(\operatorname{ext} B_{X^*}, \mathbb{R})$.*

(a) *If $\alpha \in [0, \omega_1)$, then there exists a function h such that $h = f$ on* $\operatorname{ext} B_{X^*}$ *and $h \in X^{**}_{\alpha+1}$ in case $\alpha \in [0, \omega_0)$ and $h \in X^{**}_\alpha$ in case $\alpha \in [\omega_0, \omega_1)$.*
(b) *If* $\operatorname{ext} B_{X^*}$ *is of type F_σ, then for any $\alpha \in [1, \omega_1)$ and an odd function $f \in \mathcal{B}^{\alpha,b}(\operatorname{ext} B_{X^*}, \mathbb{R})$ there exists a function $h \in X^{**}_\alpha$ such that $h = f$ on* $\operatorname{ext} B_{X^*}$.

Proof. (a) Let f be an odd bounded Borel function on $\operatorname{ext} B_{X^*}$. Thus by Proposition 2.1, the function Tf is an odd Borel strongly affine function on B_{X^*} satisfying

$$Tf\big(x^*\big) = (\operatorname{odd}\varepsilon_{x^*})(f) = \varepsilon_{x^*}(\operatorname{odd} f) = f\big(x^*\big), \quad x^* \in \operatorname{ext} B_{X^*}.$$

By Proposition 2.6(a), (b), the function $h = Tf$ is in $X^{**}_{\alpha+1}$ in case $\alpha \in [0, \omega_0)$ and $h \in X^{**}_\alpha$ in case $\alpha \in [\omega_0, \omega_1)$.

(b) We argue as above, only we use Proposition 2.6(c) instead. □

THEOREM 2.8. *Let X be a real separable L_1-predual. If the set $\operatorname{ext} B_{X^*}$ is not of type F_σ, then there exists an odd function $f \in \mathcal{B}^{1,b}(\operatorname{ext} B_{X^*}, \mathbb{R})$ that is not extensible to a function from X^{**}_1.*

Proof. Assume that $\operatorname{ext} B_{X^*}$ is not of type F_σ. Since it is a G_δ subset of a compact metrizable space, by the Hurewicz theorem (see [15, Theorem 21.18]) there exists a closed set $A \subset B_{X^*}$ satisfying

$$\overline{A \cap \operatorname{ext} B_{X^*}} = \overline{A \setminus \operatorname{ext} B_{X^*}} = A$$

with $A \setminus \operatorname{ext} B_{X^*}$ countable. Let $\{x^*_n; n \in \mathbb{N}\}$ be an enumeration of $A \setminus \operatorname{ext} B_{X^*}$. For each $n \in \mathbb{N}$ we select a maximal measure $\mu_n \in \mathcal{M}^1_{x^*_n}(B_{X^*})$ and using the regularity of Radon measures we find a compact set $K_n \subset \operatorname{ext} B_{X^*}$ such that $\mu_n(K_n) > 1 - \frac{1}{n}$. Without loss of generality, we may assume that $K_n = -K_n$. The set $\bigcup_n K_n$ is of type F_σ and $A \cap \operatorname{ext} B_{X^*} \setminus \bigcup K_n$ cannot be F_σ-separated from $A \setminus \operatorname{ext} B_{X^*}$ (i.e., there does not exist any F_σ set $F \in B_{X^*}$ such that $A \cap \operatorname{ext} B_{X^*} \setminus \bigcup K_n \subset F$ and $F \cap A \setminus \operatorname{ext} B_{X^*} = \emptyset$), otherwise $A \cap \operatorname{ext} B_{X^*}$ would be an F_σ set which is impossible. An application of [15, Theorem 21.22] then provides a closed set $B \subset A \setminus \bigcup_n K_n$ such that

$$\overline{B \cap \operatorname{ext} B_{X^*}} = \overline{B \setminus \operatorname{ext} B_{X^*}} = B.$$

Let $b^* \in B$ be distinct from 0 and V be its closed neighborhood satisfying $V \cap -V = \emptyset$. Set $C = B \cap V$. Then

$$C \cap (-C) \subset V \cap (-V) = \emptyset.$$

Let

$$f\big(x^*\big) = \frac{1}{2}\big(\chi_C\big(x^*\big) - \chi_{-C}\big(x^*\big)\big), \quad x^* \in B_{X^*}.$$

Then f is a bounded odd Baire-1 function on B_{X^*}, and thus its restriction to $\operatorname{ext} B_{X^*}$ is also a bounded odd Baire-1 function on $\operatorname{ext} B_{X^*}$. We show that there is no odd Baire-1 strongly affine extension of $f|_{\operatorname{ext} B_{X^*}}$ to B_{X^*}.

Let h be such an extension. Then $h = Tf$ by Proposition 2.3. Let $n \in \mathbb{N}$ be such that

$$x^*_n \in C \setminus \operatorname{ext} B_{X^*} \subset A \setminus \operatorname{ext} B_{X^*} = \big\{x^*_k; k \in \mathbb{N}\big\}.$$

Since $K_n = -K_n$ and $C \cap K_n = \emptyset$, $(C \cup -C) \cap K_n = \emptyset$. Thus, $\mu_n(C \cup -C) < \frac{1}{n}$ by the choice of the set K_n. Then we get

$$\begin{aligned}\big|Tf\big(x_n^*\big)\big| &= \big|(\operatorname{odd}\mu_n)(f)\big| = \big|\mu_n(f)\big| \\ &\le \frac{1}{2}\big(\mu_n(C) + \mu_n(-C)\big) \le \frac{1}{n}.\end{aligned}$$

On the other hand, if $x^* \in C \cap \operatorname{ext} B_{X^*}$, then $x^* \notin -C$ as $C \cap -C = \emptyset$. Hence, it follows that

$$\big|Tf\big(x^*\big)\big| = \big|(\operatorname{odd}\varepsilon_{x^*})(f)\big| = \big|\varepsilon_{x^*}(f)\big| = \left|\frac{1}{2}(1-0)\right| = \frac{1}{2}.$$

Since both $C \cap \operatorname{ext} B_{X^*}$ and $C \setminus \operatorname{ext} B_{X^*}$ are dense in C, $h = Tf$ has no point of continuity on C. In particular, h is not a Baire-1 function on B_{X^*} by [15, Theorem 24.14], which concludes the proof. □

By a rephrasing a part of the previous results, we get an analogue of [18, Theorem 4.4].

Corollary 2.9. *Let X be a separable real Banach space. Then the following statements are equivalent.*

(i) *A space X is a real L_1-predual and* $\operatorname{ext} B_{X^*}$ *is an F_σ set.*
(ii) *Every odd function $f \in \mathcal{B}^{1,b}(\operatorname{ext} B_{X^*}, \mathbb{R})$ can be extended to a function in X_1^{**}.*

Proof. (i) $\Longrightarrow$ (ii). Due to Theorem 2.7(b).

(ii) $\Longrightarrow$ (i). Assume $x^* \in X^*$ and let $\mu, \nu \in \mathcal{M}^1_{x^*}(B_{X^*}) \cap \mathcal{M}^{\max}(B_{X^*})$. For any $f \in \mathcal{C}(B_{X^*}, \mathbb{R})$ then there exists by (ii) a function $h \in X_1^{**}$ extending $\operatorname{odd} f|_{\operatorname{ext} B_{X^*}}$. Maximal measures are carried by $\operatorname{ext} B_{X^*}$ and h is a strongly affine function, hence

$$(\operatorname{odd}\mu)(f) = \mu(\operatorname{odd} f) = \mu(h) = h\big(x^*\big) = \nu(h) = \nu(\operatorname{odd} f) = (\operatorname{odd}\nu)(f).$$

Thus $\operatorname{odd}\mu = \operatorname{odd}\nu$ and using Lazar's characterization of the real Lindenstrauss spaces (see [19, Theorem] or [17, §21, Theorem 7]) we get that X is an L_1-predual.

Finally, due to Theorem 2.8, the set $\operatorname{ext} B_{X^*}$ is of type F_σ. □

For a symmetric set B and $\alpha \in [0, \omega_1)$, we denote a space of all bounded odd Baire-α function on B by $\mathcal{B}^{\alpha,b}_\sigma(B, \mathbb{R})$.

The following result extends [21, Theorem 1] of Lindenstrauss and Wulbert.

Theorem 2.10. *Let X be a real separable L_1-predual such that* $\operatorname{ext} B_{X^*}$ *is an F_σ set. Then for any $\alpha \in [1, \omega_1)$, the space X_α^{**} is isometric to the space $\mathcal{B}^{\alpha,b}_\sigma(\operatorname{ext} B_{X^*}, \mathbb{R})$.*

Proof. A function $f \in X_\alpha^{**}$ is bounded, Baire-α and strongly affine. The restriction mapping $r : X_\alpha^{**} \to \mathcal{B}_\sigma^{\alpha,b}(\operatorname{ext} B_{X^*}, \mathbb{R})$ is therefore an isometric isomorphism due to Theorem 2.7(b) and the minimum principle exposed by [24, Theorem 3.86]. □

Further, one can be tempted to investigate whether a topological quality of $\operatorname{ext} B_{X^*}$ can characterize possibility of extending Baire functions of higher classes. Theorem 2.11 below shows that this is not the case.

Let K be a compact convex set in a locally convex space and set $X = \mathcal{A}(K, \mathbb{R})$. Then we can make the natural identifications

$$\begin{aligned} B_{X^*} &= \operatorname{conv}(K \cup -K), \\ \operatorname{ext} B_{X^*} &= \operatorname{ext} K \cup -\operatorname{ext} K \end{aligned} \tag{2.2}$$

using an affine homeomorphism $\varphi\colon \operatorname{conv}(K \cup -K) \to B_{X^*}$ defined by the formula $\varphi(\lambda k_1 - (1-\lambda)k_2)(h) = \lambda h(k_1) - (1-\lambda)h(k_2)$, $\lambda \in [0,1]$, $k_1, k_2 \in K$ and $h \in X$.

Further, we need to establish a mapping I from the space $\mathcal{A}^\alpha(K, \mathbb{R})$ to a space of all affine functions on B_{X^*} by setting

$$If(s) = \mu(f), \quad \text{where } \mu \in B_{\mathcal{M}(K,\mathbb{R})} \text{ is any measure extending } s \in B_{X^*}.$$

For more detailed information concerning the mapping I consult for example, [34, Theorem 2.5] or [24, Chapter 5.6].

Theorem 2.11. *There exist real separable L_1-preduals X, Y with the following properties.*

(a) *The set* $\operatorname{ext} B_{X^*}$ *is homeomorphic to* $\operatorname{ext} B_{Y^*}$ *and they are of type G_δ.*
(b) *For any $\alpha \in [2, \omega_1)$ and any function $f \in \mathcal{B}_\sigma^{\alpha,b}(\operatorname{ext} B_{Y^*}, \mathbb{R})$ there exists a function $h \in Y_\alpha^{**}$ such that $h = f$ on* $\operatorname{ext} B_{Y^*}$.
(c) *There exists a function $f \in \mathcal{B}_\sigma^{2,b}(\operatorname{ext} B_{X^*}, \mathbb{R})$ not extensible to an element of X_2^{**}.*

Proof. By [33, Theorem 1.1], there exists a couple of metrizable simplices K, L with the following properties:

- The set $\operatorname{ext} K$ is homeomorphic to $\operatorname{ext} L$.
- The sets $\operatorname{ext} K$ and $\operatorname{ext} L$ are of type G_δ.
- For $\alpha \in [2, \omega_1)$, any bounded Baire-α function on $\operatorname{ext} L$ can be extended to a function of affine class α on L.
- There exists a bounded function g on $\operatorname{ext} K$ of Baire-2 class that is not extensible to a function on K of affine class 2.

We set $X = \mathcal{A}(K, \mathbb{R})$ and $Y = \mathcal{A}(L, \mathbb{R})$. Then X and Y are separable L_1-preduals (see [17, §19, Theorem 2]).

(a) The assertion follows from the identification (2.2) and the metrizability of B_{X^*} (see [1, Corollary I.4.4]).

(b) We claim that, for any $\alpha\in[2,\omega_1)$, every function $f\in\mathcal{B}^{\alpha,b}_\sigma(\operatorname{ext}B_{Y^*},\mathbb{R})$ can be extended to a function $h\in Y^{**}_\alpha$.

Indeed, let $f\in\mathcal{B}^{\alpha,b}_\sigma(\operatorname{ext}B_{Y^*},\mathbb{R})$. Using the identification (2.2), we may assert that $f|_{\operatorname{ext}L}\in\mathcal{B}^{\alpha,b}(\operatorname{ext}L,\mathbb{R})$ and set $g=f|_{\operatorname{ext}L}$. Due to the hypotheses there exists a function $\tilde g\in\mathcal{A}^\alpha(L,\mathbb{R})$ extending g. Then $I\tilde g|_L=\tilde g$ and applying [34, Theorem 2.5(f)] (see also [24, Theorem 5.40(f)]) we get that $I\tilde g\in Y^{**}_\alpha$. Hence, $I\tilde g=f$ on $\operatorname{ext}B_{Y^*}$ and we may define $h=I\tilde g$ as the desired function.

(c) Let $g\in\mathcal{B}^{2,b}(\operatorname{ext}K,\mathbb{R})$ be a function not extensible to a function from $\mathcal{A}^2(K,\mathbb{R})$. The function g can be nevertheless naturally extended to an odd function $\tilde g$ defined on $\operatorname{ext}K\cup-\operatorname{ext}K$. Due to the identification (2.2), we may see $\tilde g$ as a function from $\mathcal{B}^{2,b}_\sigma(\operatorname{ext}B_{X^*},\mathbb{R})$. We claim that the function $\tilde g$ cannot be extended to an element of X^{**}_2.

Suppose the contrary and let $\tilde f\in X^{**}_2$ be an extension of $\tilde g$. Due to [34, Theorem 2.5(f)] (see also [24, Theorem 5.40(f)]) there exists $f\in\mathcal{A}^2(K,\mathbb{R})$ such that $If=\tilde f$. The definition of I immediately provides that $f=g$ on $\operatorname{ext}K$ which gives us a contradiction with the properties of g. $\square$

3. Complex L_1-preduals

The validity of the most of the results of Section 2 can be essentially extended to the complex setting. The principal technical inconvenience consists in the impossibility of using the notion of odd functions in the complex setting. The role of odd functions play homogeneous functions here.

The following notions are due to Effros (see [7]). Let $\mathbb{T}$ stand for the unit circle endowed with the unit Haar measure $d\alpha$. Let X be a complex Banach space. A set $B\subset B_{X^*}$ is called *homogeneous* if $\alpha B=B$ for each $\alpha\in\mathbb{T}$. An example of a homogeneous set is $\operatorname{ext}B_{X^*}$. A function f on a homogeneous set $B\subset B_{X^*}$ is called *homogeneous* (see, e.g., [7, p. 53], [17, p. 240]) if

$$f(\alpha x^*)=\alpha f(x^*),\quad (\alpha,x^*)\in\mathbb{T}\times B.$$

If f is a Borel function defined on a homogeneous Borel set $B\subset B_{X^*}$, we set

$$(\operatorname{hom}f)(x^*)=\int_{\mathbb{T}}\alpha^{-1}f(\alpha x^*)\,d\alpha,\quad x^*\in B.$$

Then the function $\operatorname{hom}f$ is homogeneous on B and it is easy to see that it is continuous in case $f\in\mathcal{C}^b(B,\mathbb{C})$. By the Lebesgue dominated convergence theorem, $\operatorname{hom}f$ is well defined for each bounded Baire function on B and $\operatorname{hom}f$ is Baire-α whenever $f\in\mathcal{B}^{\alpha,b}(B,\mathbb{C})$. A function f is homogeneous if and only if $\operatorname{hom}f=f$.

The mapping hom provides a mapping on $\mathcal{M}(B_{X^*},\mathbb{C})$ defined as

$$(\operatorname{hom}\mu)(f)=\mu(\operatorname{hom}f),\quad f\in\mathcal{C}(B_{X^*},\mathbb{C}),\mu\in\mathcal{M}(B_{X^*},\mathbb{C}).$$

For $x^*\in B_{X^*}$, let $\mathcal{M}^1_{x^*}(B_{X^*})$ be defined as in Section 2. Similarly, symbols $\prec$ and $\mathcal{M}^{\max}(B_{X^*})$ are defined as above.

If X is a complex Banach space, then we have the following analogue of [19, Theorem] due to Effros:

The Banach space X is an complex L_1-predual if and only if, for any $x^* \in B_{X^*}$ and measures $\mu, \nu \in \mathcal{M}^1_{x^*}(B_{X^*}) \cap \mathcal{M}^{\max}(B_{X^*})$, it holds $\hom \mu = \hom \nu$ (see [7, Theorem 4.3] or [17, §23, Theorem 5]).

This theorem permits to define a mapping T analogously as in the real case (see Section 2). Namely, for a separable complex L_1-predual X and a bounded Borel function f defined at least on $\operatorname{ext} B_{X^*}$ we set

$$Tf\big(x^*\big) = (\hom \mu)(f), \quad \mu \in \mathcal{M}^1_{x^*}(B_{X^*}) \cap \mathcal{M}^{\max}(B_{X^*}). \tag{3.1}$$

Since $\hom \mu$ is a boundary measure if μ is maximal (see [7, Lemma 4.2] or [17, §23, Lemma 10]), the mapping T is well defined.

Employing a newly defined mapping T and a notion of homogeneous function instead of odd function we claim that Propositions 2.1, 2.3, 2.6, Theorems 2.7, 2.8, 2.10 and Corollary 2.9 are valid also in complex setting. Putting aside some additional technical obstacles the proofs are similar to the real case. For the proofs in full details, we refer the reader to [22, Chapter 3.3].

4. C^*-algebras

The main result of this section answers a question from [2, p. 1048].

In order to prove it, we need to recall a notion of a *function space* which is a linear subspace of $\mathcal{C}(K,\mathbb{F})$ containing constants and separating points of K. If $\mathcal{H} \subset \mathcal{C}(K,\mathbb{F})$ is a function space, we write $\mathcal{H}^{\perp\perp}$ for the set of all bounded Borel functions on K satisfying $\mu(f) = 0$ for each $\mu \in \mathcal{H}^{\perp}$.

Proposition 4.1. *Let K be a metrizable compact space and $f \in \mathcal{B}^b(K,\mathbb{C})$. Then the function $F : B_{\mathcal{M}(K,\mathbb{C})} \to \mathbb{C}$ defined as $F(\mu) = \mu(f)$, $\mu \in B_{\mathcal{M}(K,\mathbb{C})}$, is strongly affine on $B_{\mathcal{M}(K,\mathbb{C})}$.*

Proof. If $f \in \mathcal{C}(K,\mathbb{C})$, F is strongly affine on $B_{\mathcal{M}(K,\mathbb{C})}$ by the definition. If (f_n) is a bounded sequence of Borel functions pointwise converging to f such that the relevant functions F_n are strongly affine on $B_{\mathcal{M}(K,\mathbb{C})}$, (F_n) converges pointwise to F by the Lebesgue dominated convergence theorem. Since F_n are strongly affine, F is strongly affine as well again due to the Lebesgue dominated convergence theorem.

Hence, the family of all Borel functions f, for which F is strongly affine, is closed under the taking pointwise limits of bounded sequences and contains continuous functions. Hence, it contains any bounded Borel function. ☐

Next, we recall a result which is essentially from [32].

Proposition 4.2. *Let $\pi : K \to L$ be a continuous affine surjection of a compact convex set K onto a compact convex set L. Let $g : L \to \mathbb{C}$ be a bounded function. Then g is strongly affine on L if and only if $g \circ \pi$ is strongly affine on K.*

Proof. We notice that a function $g: L \to \mathbb{C}$ is strongly affine if and only if both $\operatorname{Re} g$ and $\operatorname{Im} g$ are strongly affine. Then use [32, Proposition 3.2] (see also [24, Proposition 5.29]). □

PROPOSITION 4.3. *Let K be a metrizable compact space, $\mathcal{A} \subset \mathcal{C}(K,\mathbb{C})$ be a function space and let $\pi : B_{\mathcal{M}(K,\mathbb{C})} \to B_{\mathcal{A}^*}$ be the restriction mapping. If $f \in \mathcal{B}^{\alpha,b}(K,\mathbb{C}) \cap \mathcal{A}^{\perp\perp}$, then the function $F : B_{\mathcal{A}^*} \to \mathbb{C}$ defined as*

$$F(a^*) = \mu(f), \quad \mu \in B_{\mathcal{M}(K,\mathbb{C})}, \pi(\mu) = a^*,$$

is a well defined homogeneous strongly affine function on $B_{\mathcal{A}^*}$ of Baire class α.

Proof. Let $f \in \mathcal{B}^{\alpha,b}(K,\mathbb{C}) \cap \mathcal{A}^{\perp\perp}$. First, we notice that F is well defined. Indeed, if $a^* \in B_{\mathcal{A}^*}$, let $\mu \in B_{\mathcal{M}(K,\mathbb{C})}$ be extending a^*. If $\nu \in B_{\mathcal{M}(K,\mathbb{C})}$ is another extension, then $\mu - \nu \in \mathcal{A}^{\perp}$, and thus $\mu(f) = \nu(f)$.

Let $\alpha \in \mathbb{T}$, $a^* \in B_{\mathcal{A}^*}$ and $\mu \in B_{\mathcal{M}(K,\mathbb{C})}$ such that $\pi(\mu) = a^*$. Then $\pi(\alpha\mu) = \alpha a^*$ and $F(\alpha a^*) = \alpha F(a^*)$, thus F is homogeneous. For the verification of the strong affinity of F, we use Proposition 4.2. Let $G : B_{\mathcal{M}(K,\mathbb{C})} \to \mathbb{C}$ be defined as $G(\mu) = \mu(f)$, $\mu \in B_{\mathcal{M}(K,\mathbb{C})}$. Then

$$G = F \circ \pi.$$

Since π is a continuous affine surjection of the compact convex set $B_{\mathcal{M}(K,\mathbb{C})}$ onto the compact convex set $B_{\mathcal{A}^*}$, the strong affinity of F follows from Propositions 4.1 and 4.2. If f is of Baire class α, G is of class α as well by the Lebesgue dominated convergence theorem. Hence, F is of class α by [31] (see also [24, Theorem 5.16]). □

THEOREM 4.4. *Let X be the CAR-algebra (see [27, p. 104]). Then $X^{**}_{\mathcal{B}_2} \neq X^{**}_2$.*

Proof. A rather intricate construction in [34, Section 5] provides a function space $\mathcal{H} \subset \mathcal{C}(K,\mathbb{R})$ such that,

- $\mathcal{H}$ is closed in $\mathcal{C}(K,\mathbb{R})$, and thus in $\mathcal{C}(K,\mathbb{C})$ (see [34, p. 1674]),
- K is metrizable (see [34, p. 1673]), and thus $\mathcal{H}$ is separable,
- $\mathcal{H}$ is a so-called simplicial function space (see [24, Section 6.1]), and thus a real L_1-predual (see [34, Lemma 6.1(a)] and [24, Theorem 6.25]), and,
- by [34, Lemmas 6.5, 6.6],

$$\mathcal{B}^{2,bb}(\mathcal{H}) \subsetneq \mathcal{B}^{2,b}(K,\mathbb{R}) \cap \mathcal{H}^{\perp\perp}.$$

Let

$$\mathcal{A} = \{g \in \mathcal{C}(K,\mathbb{C}); \operatorname{Re} g, \operatorname{Im} g \in \mathcal{H}\}.$$

Then $\mathcal{A}$ is selfadjoint and $\operatorname{Re}\mathcal{A} = \mathcal{H}$ is a real L_1-predual. Thus, $\mathcal{A}$ is a complex L_1-predual by [13, Theorem 2] (see also [17, §23, Theorem 6]). We claim that $\mathcal{A}^{**}_2 \subsetneq \mathcal{A}^{**}_{\mathcal{B}_2}$. Indeed, pick

$$f \in \left(\mathcal{B}^{2,b}(K,\mathbb{R}) \cap \mathcal{H}^{\perp\perp}\right) \setminus \mathcal{B}^{2,bb}(\mathcal{H}).$$

Since $f \in \mathcal{H}^{\perp\perp}$, clearly $f \in \mathcal{A}^{\perp\perp}$ as well. Due to Proposition 4.3, we are able to define $F : B_{\mathcal{A}^*} \to \mathbb{C}$ as

$$F\big(a^*\big) = \mu(f), \quad \mu \in B_{\mathcal{M}(K,\mathbb{C})}, \pi(\mu) = a^*,$$

such that F is a homogeneous strongly affine function on $B_{\mathcal{A}^*}$ of Baire class 2.

On the other hand, $F \notin \mathcal{A}_2^{**}$. Indeed, assume that $F \in \mathcal{A}_2^{**}$. Let

$$S = \big\{\phi(k); k \in K\big\} \subset B_{\mathcal{A}^*},$$

where $\phi(k)(a) = a(k)$, $a \in \mathcal{A}$. Then $\phi : K \to S$ is a homeomorphic embedding and $f = F \circ \phi$. Since $F \in \mathcal{A}_2^{**}$, also $f = F \circ \phi \in \mathcal{B}^{2,bb}(\mathcal{A})$. So let

$$\{a_{nk}; n, k \in \mathbb{N}\}$$

be a family in $\mathcal{A}$ such that

$$f = \lim_{n\to\infty} \lim_{k\to\infty} a_{nk},$$

where $(\lim_{k\to\infty} a_{nk})_{n\in\mathbb{N}}$ is a bounded sequence as well as every sequence $(a_{nk})_{k\in\mathbb{N}}$ for any given $n \in \mathbb{N}$. Since f is real,

$$f = \operatorname{Re} f = \lim_{n\to\infty} \lim_{k\to\infty} \operatorname{Re} a_{nk},$$

and thus $f \in \mathcal{B}^{2,bb}(\mathcal{H})$, which is not the case. Thus $F \notin \mathcal{A}_2^{**}$.

Now we use [27, Theorem] asserting that $\mathcal{A}$ is a 1-complemented subspace of the separable C^*-algebra X. We claim that $X_{\mathcal{B}_2}^{**} \neq X_2^{**}$. Indeed, recall that $F \in \mathcal{A}_{\mathcal{B}_2}^{**} \setminus \mathcal{A}_2^{**}$. Let $P : X \to \mathcal{A}$ be a projection of norm 1 and $\pi : B_{X^*} \to B_{\mathcal{A}^*}$ be the restriction mapping. Then

$$\big(\pi \circ P^*\big)\big(a^*\big) = a^*, \quad a^* \in B_{\mathcal{A}^*}.$$

Let

$$G = F \circ \pi.$$

By Proposition 4.2, $G \in X_{\mathcal{B}_2}^{**}$. Suppose $G \in X_2^{**}$ and let $(x_{nk})_{n,k\in\mathbb{N}}$ witness that $G \in X_2^{**}$. Then $(Px_{nk})_{n,k\in\mathbb{N}}$ witness that $F \in \mathcal{A}_2^{**}$, because

$$\begin{aligned} F\big(a^*\big) &= F\big(\pi\big(P^* a^*\big)\big) = G\big(P^* a^*\big) = \lim_{n\to\infty} \lim_{k\to\infty} x_{nk}\big(P^* a^*\big) \\ &= \lim_{n\to\infty} \lim_{k\to\infty} Px_{nk}\big(a^*\big), \quad a^* \in B_{\mathcal{A}^*}. \end{aligned}$$

But this contradicts our choice of F. $\square$

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