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STANLEY DEPTH OF WEAKLY POLYMATROIDAL IDEALS AND SQUAREFREE MONOMIAL IDEALS

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ABSTRACT. Let I be a weakly polymatroidal ideal or a squarefree monomial ideal of a polynomial ring S. In this paper, we provide a lower bound for the Stanley depth of I and S/I. In particular, we prove that if I is a squarefree monomial ideal which is generated in a single degree, then $\mathrm{sdepth}(I) \geq n - \ell(I) + 1$ and $\mathrm{sdepth}(S/I) \geq n - \ell(I)$, where $\ell(I)$ denotes the analytic spread of I. This proves a conjecture of the author in a special case.

1. Introduction

Let $\mathbb{K}$ be a field and let $S = \mathbb{K}[x_1, \dots, x_n]$ be the polynomial ring in n variables over $\mathbb{K}$. Let M be a finitely generated $\mathbb{Z}^n$-graded S-module. Let $u \in M$ be a homogeneous element and $Z \subseteq \{x_1, \dots, x_n\}$. The $\mathbb{K}$-subspace $u\mathbb{K}[Z]$ generated by all elements uv with $v \in \mathbb{K}[Z]$ is called a *Stanley space* of dimension $|Z|$, if it is a free $\mathbb{K}[Z]$-module. Here, as usual, $|Z|$ denotes the number of elements of Z. A decomposition $\mathcal{D}$ of M as a finite direct sum of Stanley spaces is called a *Stanley decomposition* of M. The minimum dimension of a Stanley space in $\mathcal{D}$ is called the *Stanley depth* of $\mathcal{D}$ and is denoted by $\mathrm{sdepth}(\mathcal{D})$. The quantity

$$\mathrm{sdepth}(M) := \max\big\{\mathrm{sdepth}(\mathcal{D}) \mid \mathcal{D} \text{ is a Stanley decomposition of } M\big\}$$

is called the *Stanley depth* of M. Stanley [10] conjectured that

$$\mathrm{depth}(M) \leq \mathrm{sdepth}(M)$$

for every $\mathbb{Z}^n$-graded S-module M. For a reader friendly introduction to Stanley depth, we refer to [5].

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Let I be a monomial ideal of S with Rees algebra $\mathcal{R}(I)$ and let $\mathfrak{m} = (x_1, \ldots, x_n)$ be the graded maximal ideal of S. Then the $\mathbb{K}$-algebra $\mathcal{R}(I)/\mathfrak{m}\mathcal{R}(I)$ is called the *fibre ring* and its Krull dimension is called the *analytic spread* of I, denoted by $\ell(I)$. This invariant is a measure for the growth of the number of generators of the powers of I. Indeed, for $k \gg 0$, the Hilbert function $H(\mathcal{R}(I)/\mathfrak{m}\mathcal{R}(I), \mathbb{K}, k) = \dim_{\mathbb{K}}(I^k/\mathfrak{m}I^k)$, which counts the number of generators of the powers of I, is a polynomial function of degree $\ell(I) - 1$.

In this paper, we consider some linear algebraic approximations of the analytic spread of a monomial ideal. Indeed, assume that $v_1, \ldots, v_t$ are t vectors in $\mathbb{Q}^n$. Then they are said to be *linearly dependent* if there exist rational numbers $c_1, \ldots, c_t$, not all zero, for which

$$c_1v_1 + \cdots + c_tv_t = 0.$$

Similarly they are *affinely dependent*, if in addition the sum of the coefficients is zero:

$$\sum_{i=1}^{t} c_i = 0.$$

If $v_1, \ldots, v_t$ are not linearly dependent (resp. affinely dependent), then they are said to be *linearly independent* (resp. *affinely independent*). Now we associate two invariants to every monomial ideal I, which are called the rank and the affine rank of I. For every vector $\mathbf{a} = (a_1, \ldots, a_n)$ of non-negative integers, we denote the monomial $x_1^{a_1} \cdots x_n^{a_n}$ by $\mathbf{x}^{\mathbf{a}}$.

Definition 1.1. Let $I \subseteq S = \mathbb{K}[x_1, \ldots, x_n]$ be a monomial ideal and $G(I) = \{\mathbf{x}^{\mathbf{a_1}}, \ldots, \mathbf{x}^{\mathbf{a_m}}\}$ be the set of minimal monomial generators of I. The *rank* of I, denoted by $\operatorname{rank}(I)$ is the cardinality of the largest linearly independent subset of $\{\mathbf{a_1}, \ldots, \mathbf{a_m}\}$. Similarly the *affine rank* of I, denoted by $\operatorname{afrank}(I)$ is the cardinality of the largest affinely independent subset of $\{\mathbf{a_1}, \ldots, \mathbf{a_m}\}$.

It is clear from Definition 1.1 that for every monomial ideal I, the inequality $\operatorname{afrank}(I) \geq \operatorname{rank}(I)$ holds. It is known [2, Lemma 10.3.19] that if I is a monomial ideal which is generated in a single degree, then $\ell(I) = \operatorname{rank}(I)$. The following proposition shows that in this case we also have $\ell(I) = \operatorname{afrank}(I)$.

Proposition 1.2. *Let I be a monomial ideal, which is generated in a single degree. Then $\ell(I) = \operatorname{rank}(I) = \operatorname{afrank}(I)$.*

Proof. It is sufficient to prove the second equality. Assume that $\operatorname{afrank}(I) = t$. Therefore, there exist integers $1 \leq i_1 < \cdots < i_t \leq m$ such that the equalities

$$c_1\mathbf{a_{i_1}} + \cdots + c_t\mathbf{a_{i_t}} = 0$$

and

$$c_1 + \cdots + c_t = 0,$$

with $c_i \in \mathbb{Q}$, for every $1 \leq i \leq t$, imply that $c_1 = \cdots = c_t = 0$. Since I is generated in a single degree, say k, $\mathbf{a_{i_1}}, \ldots, \mathbf{a_{i_t}}$ are linearly independent over $\mathbb{Q}$. Indeed, assume that there exist rational numbers $d_1, \ldots, d_t$ such that

$$d_1\mathbf{a_{i_1}} + \cdots + d_t\mathbf{a_{i_t}} = 0.$$

Now for every $1 \leq j \leq t$, the sum of the components of $\mathbf{a_{i_j}}$ is equal to k and thus, the sum of the components of

$$d_1\mathbf{a_{i_1}} + \cdots + d_t\mathbf{a_{i_t}}$$

is equal to

$$d_1k + \cdots + d_tk$$

and this shows that

$$d_1 + \cdots + d_t = 0.$$

Therefore

$$d_1 = \cdots = d_t = 0.$$

Hence, $\mathbf{a_{i_1}}, \ldots, \mathbf{a_{i_t}}$ are linearly independent over $\mathbb{Q}$. Therefore, $\mathrm{rank}(I) \geq t$. Since we always have $\mathrm{afrank}(I) \geq \mathrm{rank}(I)$, it follows that $\mathrm{afrank}(I) = \mathrm{rank}(I)$. □

In [6], the authors prove that if $I \subset S$ is a weakly polymatroidal ideal I (see Definition 2.1), which is generated in a single degree, then $\mathrm{depth}(S/I) \geq n - \ell(I)$, $\mathrm{sdepth}(S/I) \geq n - \ell(I)$ and $\mathrm{sdepth}(I) \geq n - \ell(I) + 1$. In Section 2, we generalize this result by proving that for every weakly polymatroidal ideal I, the inequalities

$$\mathrm{sdepth}(I) \geq n - \mathrm{afrank}(I) + 1, \qquad \mathrm{sdepth}(S/I) \geq n - \mathrm{afrank}(I)$$

and

$$\mathrm{depth}(S/I) \geq n - \mathrm{afrank}(I)$$

hold (see Theorem 2.6).

In [8], the author conjectures that for every integrally closed monomial ideal, the inequalities $\mathrm{sdepth}(S/I) \geq n - \ell(I)$ and $\mathrm{sdepth}(I) \geq n - \ell(I) + 1$ hold (see Conjecture 3.1). In Section 3, we prove this conjecture for every squarefree monomial ideal which is generated in a single degree. In fact, we prove a stronger result. We show that for every squarefree monomial ideal I of the polynomial ring S, the inequalities

$$\mathrm{sdepth}(I) \geq n - \mathrm{rank}(I) + 1$$

and

$$\mathrm{sdepth}(S/I) \geq n - \mathrm{rank}(I)$$

hold (see Theorem 3.3).

2. Stanley depth of weakly polymatroidal ideals

Weakly polymatroidal ideals are generalization of polymatroidal ideals and they are defined as follows.

DEFINITION 2.1 ([4], Definition 1.1). A monomial ideal I of $S=\mathbb{K}[x_1,\dots,x_n]$ is called *weakly polymatroidal* if for every two monomials $u=x_1^{a_1}\cdots x_n^{a_n}$ and $v=x_1^{b_1}\cdots x_n^{b_n}$ in $G(I)$ such that $a_1=b_1,\dots,a_{t-1}=b_{t-1}$ and $a_t>b_t$ for some t, there exists $j>t$ such that $x_t(v/x_j)\in I$.

The aim of this section is to provide a lower bound for the depth and the Stanley depth of weakly polymatroidal ideals. As usual for every monomial u, the *support* of u, denoted by $\mathrm{Supp}(u)$, is the set of variables which divide u.

LEMMA 2.2. *Let I be a weakly polymatroidal ideal and let $G(I)=\{u_1,\dots,u_m\}$ be the set of minimal monomial generators of I. Assume that*

$$x_1\in\bigcup_{i=1}^{m}\mathrm{Supp}(u_i).$$

Then $(I:x_1)$ is a weakly polymatroidal ideal which is minimally generated by the set

$$\mathcal{G}=\left\{\frac{u_i}{x_1}\middle|u_i\in G(I)\ AND\ x_1\ divides\ u_i\right\}.$$

Proof. It is clear that the ideal generated by $\mathcal{G}$ is a weakly polymatroidal ideal. Thus, we prove that $(I:x_1)$ is generated by the set $\mathcal{G}$. Without loss of generality, we may assume that $u_1,\dots,u_t$ are divisible by x_1 and $u_{t+1},\dots,u_m$ are not divisible by x_1, where $1\le t\le m$. Let $v_i=u_i/x_1$ $(1\le i\le t)$. We should prove that $(I:x_1)$ is generated by $v_1,\dots,v_t$. Let $v\in(I:x_1)$ be a monomial. Then $x_1v\in I$ and so there exists $1\le i\le m$ in such a way that u_i divides x_1v. If $1\le i\le t$, then v is divisible by v_i and therefore, $v\in(v_1,\dots,v_t)$. Hence, we may assume that $i\ge t+1$. Now u_i is not divisible by x_1 and thus $u_i|v$. Since

$$x_1\in\bigcup_{i=1}^{m}\mathrm{Supp}(u_i),$$

Definition 2.1 implies that there exists $j\ge 2$ such that $x_1u_i/x_j\in I$. Hence, there exists $1\le s\le m$, such that u_s divides x_1u_i/x_j. If $t+1\le s\le m$, then u_s divides u_i/x_j and thus u_s properly divides u_i, which is a contradiction, because $G(I)$ is the set of minimal monomial generators of I. It follows that $1\le s\le t$. Therefore, v_s divides u_i/x_j and hence, it divides u_i. Since v is divisible by u_i, we conclude that v_s divides v. This shows that $v\in(v_1,\dots,v_t)$ and completes the proof of the lemma. □

The following lemma shows that the affine rank of a weakly polymatroidal ideal does not increase under the colon operation with respect to the variable x_1.

LEMMA 2.3. *Let I be a weakly polymatroidal ideal. Then* $\operatorname{afrank}((I:x_1)) \leq \operatorname{afrank}(I)$.

Proof. If $I=(I:x_1)$, then there is nothing to prove. So assume that $I \neq (I:x_1)$. Let $G(I)=\{u_1,\dots,u_m\}$ be the set of minimal monomial generators of I. Since $I \neq (I:x_1)$, it follows that

$$x_1 \in \bigcup_{i=1}^{m} \operatorname{Supp}(u_i).$$

Without loss of generality, we may assume that $u_1,\dots,u_t$ are divisible by x_1 and $u_{t+1},\dots,u_m$ are not divisible by x_1, where $1 \leq t \leq m$. Let $v_i=u_i/x_1$ $(1 \leq i \leq t)$. By Lemma 2.2, the set $\{v_1,\dots,v_t\}$ is the set of minimal monomial generators of $(I:x_1)$. For simplicity, we assume that $\mathbf{a_i}$ is the exponent vector of v_i $(1 \leq i \leq t)$. Suppose that $\operatorname{afrank}((I:x_1))=s$ and choose the monomials $v_{j_1},\dots,v_{j_s}$, such that the equalities

$$c_1\mathbf{a_{j_1}}+\cdots+c_s\mathbf{a_{j_s}}=0$$

and

$$c_1+\cdots+c_s=0,$$

with $c_i \in \mathbb{Q}$, for every $1 \leq i \leq s$, imply that $c_1=\cdots=c_s=0$. Note that for every $1 \leq i \leq t$, the exponent vector of u_i is equal to $\mathbf{a_i}+\mathbf{e_1}$, where $\mathbf{e_1}$ is the first vector in the standard basis of $\mathbb{Q}^n$. Now assume that there exist $d_1,\dots,d_s \in \mathbb{Q}$, such that $d_1+\cdots+d_s=0$ and

$$d_1(\mathbf{a_{j_1}}+\mathbf{e_1})+\cdots+d_s(\mathbf{a_{j_s}}+\mathbf{e_1})=0.$$

Therefore,

$$d_1\mathbf{a_{j_1}}+\cdots+d_s\mathbf{a_{j_s}}+(d_1+\cdots+d_s)\mathbf{e_1}=0.$$

Since $d_1+\cdots+d_s=0$, it follows that

$$d_1\mathbf{a_{j_1}}+\cdots+d_s\mathbf{a_{j_s}}=0.$$

By the choice of $v_{j_1},\dots,v_{j_s}$, we conclude that $d_1=\cdots=d_s=0$. Thus, $\operatorname{afrank}(I) \geq s$ and this proves our assertion. □

In the following lemma, we consider the behavior of the affine rank of an arbitrary monomial ideal under the elimination of x_1.

LEMMA 2.4. *Let I be a monomial ideal of $S=\mathbb{K}[x_1,\dots,x_n]$, such that*

$$x_1 \in \bigcup_{u \in G(I)} \operatorname{Supp}(u).$$

Let $S'=\mathbb{K}[x_2,\dots,x_n]$ be the polynomial ring obtained from S by deleting the variable x_1 and consider the ideal $I'=I \cap S'$. Then $\operatorname{afrank}(I')+1 \leq \operatorname{afrank}(I)$.

Proof. Let $G(I) = \{u_1, \dots, u_m\}$ be the set of minimal monomial generators of I. For simplicity we assume that $\mathbf{a_i}$ is the exponent vector of u_i $(1 \le i \le m)$. Without loss of generality, we may assume that $u_1, \dots, u_t$ are divisible by x_1 and $u_{t+1}, \dots, u_m$ are not divisible by x_1, where $1 \le t \le m$. Then the set $\{u_{t+1}, \dots, u_m\}$ is the set of minimal monomial generators of I'. Assume that $\mathrm{afrank}(I') = s$. Thus, there exist integers $t+1 \le j_1 < j_2 < \cdots < j_s \le m$, such that the equalities

$$c_1\mathbf{a_{j_1}} + \cdots + c_s\mathbf{a_{j_s}} = 0$$

and

$$c_1 + \cdots + c_s = 0,$$

with $c_i \in \mathbb{Q}$, for every $1 \le i \le s$, imply that $c_1 = \cdots = c_s = 0$. Now we consider the set $\{u_1, u_{j_1}, \dots, u_{j_s}\}$ and assume that there exist $d_0, d_1, \dots, d_s \in \mathbb{Q}$, such that $d_0 + d_1 + \cdots + d_s = 0$ and

$$d_0\mathbf{a_1} + d_1\mathbf{a_{j_1}} + \cdots + d_s\mathbf{a_{j_s}} = 0.$$

Looking at the first component of the vector $d_0\mathbf{a_1} + d_1\mathbf{a_{j_1}} + \cdots + d_s\mathbf{a_{j_s}}$, it follows that $d_0 = 0$ and hence, $d_1 + \cdots + d_s = 0$ and

$$d_1\mathbf{a_{j_1}} + \cdots + d_s\mathbf{a_{j_s}} = 0.$$

By the choice of integers $j_1, \dots, j_s$, we conclude that $d_1 = \cdots = d_s = 0$. Therefore, $\mathrm{afrank}(I) \ge s + 1 = \mathrm{afrank}(I') + 1$. □

REMARK 2.5. It is completely clear from the proof of the Lemma 2.4, that one can consider any arbitrary variable instead of x_1.

We are now ready to state and prove the main result of this section.

THEOREM 2.6. *Let I be a weakly polymatroidal ideal of $S = \mathbb{K}[x_1, \dots, x_n]$. Then we have the following assertions*:

(i) $\mathrm{sdepth}(I) \ge n - \mathrm{afrank}(I) + 1$ *and* $\mathrm{sdepth}(S/I) \ge n - \mathrm{afrank}(I)$,
(ii) $\mathrm{depth}(S/I) \ge n - \mathrm{afrank}(I)$.

Proof. We prove (i) and (ii) simultaneously by induction on n and

$$\sum_{u \in G(I)} \deg(u),$$

where $G(I)$ is the set of minimal monomial generators of I. If $n = 1$ or

$$\sum_{u \in G(I)} \deg(u) = 1,$$

then I is a principal ideal and so we have $\mathrm{afrank}(I) = 1$, $\mathrm{sdepth}(I) = n$, $\mathrm{depth}(S/I) = n - 1$ and by [7, Theorem 1.1], $\mathrm{sdepth}(S/I) = n - 1$. Therefore, in these cases, the inequalities in (i) and (ii) are trivial.

We now assume that $n \geq 2$ and

$$\sum_{u \in G(I)} \deg(u) \geq 2.$$

Let $S' = \mathbb{K}[x_2, \ldots, x_n]$ be the polynomial ring obtained from S by deleting the variable x_1 and consider the ideals $I' = I \cap S'$ and $I'' = (I : x_1)$. If

$$x_1 \notin \bigcup_{u \in G(I)} \mathrm{Supp}(u),$$

then the induction hypothesis on n implies that

$$\mathrm{depth}(S/I) = \mathrm{depth}\big(S'/I'\big) + 1 \geq (n-1) - \mathrm{afrank}\big(I'\big) + 1 = n - \mathrm{afrank}(I).$$

On the other hand, by [7, Theorem 1.1] and [3, Lemma 3.6], we conclude that $\mathrm{sdepth}(S/I) = \mathrm{sdepth}(S'/I') + 1$ and $\mathrm{sdepth}(I) = \mathrm{sdepth}(I') + 1$. Therefore, using the induction hypothesis on n we conclude that $\mathrm{sdepth}(I) \geq n - \mathrm{afrank}(I) + 1$ and $\mathrm{sdepth}(S/I) \geq n - \mathrm{afrank}(I)$. Therefore, we may assume that

$$x_1 \in \bigcup_{u \in G(I)} \mathrm{Supp}(u).$$

Now $I = I'S' \oplus x_1 I'' S$ and $S/I = (S'/I'S') \oplus x_1(S/I''S)$ (as vector spaces) and therefore by definition of the Stanley depth we have

$$\mathrm{sdepth}(I) \geq \min\big\{\mathrm{sdepth}_{S'}\big(I'S'\big), \mathrm{sdepth}_S\big(I''\big)\big\} \tag{1}$$

and

$$\mathrm{sdepth}(S/I) \geq \min\big\{\mathrm{sdepth}_{S'}\big(S'/I'S'\big), \mathrm{sdepth}_S\big(S/I''\big)\big\}. \tag{2}$$

On the other hand, by applying the depth lemma on the exact sequence

$$0 \longrightarrow S/(I : x_1) \longrightarrow S/I \longrightarrow S/(I, x_1) \longrightarrow 0$$

we conclude that

$$\mathrm{depth}(S/I) \geq \min\big\{\mathrm{depth}_{S'}\big(S'/I'S'\big), \mathrm{depth}_S\big(S/I''\big)\big\}. \tag{3}$$

Using Lemma 2.2, it follows that I'' is a weakly polymatroidal ideal and by Lemma 2.3 we conclude that $\mathrm{afrank}(I'') \leq \mathrm{afrank}(I)$. Hence, our induction hypothesis on

$$\sum_{u \in G(I)} \deg(u)$$

implies that

$$\mathrm{depth}_S\big(S/I''\big) \geq n - \mathrm{afrank}\big(I''\big) \geq n - \mathrm{afrank}(I),$$
$$\mathrm{sdepth}_S\big(S/I''\big) \geq n - \mathrm{afrank}\big(I''\big) \geq n - \mathrm{afrank}(I)$$

and

$$\mathrm{sdepth}_S\big(I''\big) \geq n - \mathrm{afrank}\big(I''\big) + 1 \geq n - \mathrm{afrank}(I) + 1.$$

On the other hand $I'S'$ is a weakly polymatroidal ideal and since

$$x_1 \in \bigcup_{u \in G(I)} \mathrm{Supp}(u),$$

using Lemma 2.4 we conclude that $\mathrm{afrank}(I'S') \leq \mathrm{afrank}(I) - 1$ and therefore by the induction hypothesis on n we conclude that

$$\begin{aligned}\mathrm{sdepth}_{S'}\big(I'S'\big) &\geq (n-1) - \mathrm{afrank}\big(I'S'\big) + 1 \geq (n-1) - \big(\mathrm{afrank}(I) - 1\big) + 1 \\ &= n - \mathrm{afrank}(I) + 1,\end{aligned}$$

and similarly $\mathrm{sdepth}_{S'}(S'/I'S') \geq n - \mathrm{afrank}(I)$ and $\mathrm{depth}_{S'}(S'/I'S') \geq n - \mathrm{afrank}(I)$. Now the assertions follow by inequalities (1), (2) and (3). □

Remark 2.7. Soleyman Jahan [9] proves that Stanley's conjecture holds true for I, when it has linear quotient. This shows that Theorem 2.6 is more interesting for the Stanley depth of S/I rather than I.

As an immediate consequence of Proposition 1.2 and Theorem 2.6, we conclude the following result which appeared in [6].

Corollary 2.8. *Let I be a weakly polymatroidal ideal of $S = \mathbb{K}[x_1, \dots, x_n]$ which is generated in a single degree. Then we have the following assertions*:

(i) $\mathrm{sdepth}(I) \geq n - \ell(I) + 1$ *and* $\mathrm{sdepth}(S/I) \geq n - \ell(I)$,

(ii) $\mathrm{depth}(S/I) \geq n - \ell(I)$.

Using Theorem 2.6, we provide an upper bound for the height of associated primes of a weakly polymatroidal ideal.

Corollary 2.9. *Let I be a weakly polymatroidal ideal of $S = \mathbb{K}[x_1, \dots, x_n]$. Then*

$$\max\big\{\mathrm{ht}(\mathfrak{p}) \mid \mathfrak{p} \in \mathrm{Ass}(S/I)\big\} \leq \mathrm{afrank}(I).$$

Proof. Let $\mathfrak{p} \in \mathrm{Ass}(S/I)$ be given. By [1, Proposition 1.2.13], we have $\mathrm{depth}(S/I) \leq n - \mathrm{ht}(\mathfrak{p})$, while by Theorem 2.6 we have $\mathrm{depth}(S/I) \geq n - \mathrm{afrank}(I)$. This implies that $\mathrm{ht}(\mathfrak{p}) \leq \mathrm{afrank}(I)$ for every $\mathfrak{p} \in \mathrm{Ass}(S/I)$ and completes the proof of the corollary. □

3. Stanley depth of squarefree monomial ideals

Let $I \subset S$ be an arbitrary ideal. An element $f \in S$ is *integral* over I, if there exists an equation

$$f^k + c_1 f^{k-1} + \cdots + c_{k-1} f + c_k = 0 \quad \text{with } c_i \in I^i.$$

The set of elements $\overline{I}$ in S which are integral over I is the *integral closure* of I. It is known that the integral closure of a monomial ideal $I \subset S$ is a monomial ideal generated by all monomials $u \in S$ for which there exists an integer k such that $u^k \in I^k$ (see [2, Theorem 1.4.2]).

In [8], the author proposed the following conjecture regarding the Stanley depth of integrally closed monomial ideals.

CONJECTURE 3.1. *Let $I \subset S$ be an integrally closed monomial ideal. Then* $\mathrm{sdepth}(S/I) \geq n - \ell(I)$ *and* $\mathrm{sdepth}(I) \geq n - \ell(I) + 1$.

In this section we prove that Conjecture 3.1 is true for every squarefree monomial ideal which is generated in a single degree. Indeed we show that for every squarefree monomial ideal I of the polynomial ring S, the inequalities $\mathrm{sdepth}(I) \geq n - \mathrm{rank}(I) + 1$ and $\mathrm{sdepth}(S/I) \geq n - \mathrm{rank}(I)$ hold (see Theorem 3.3).

First, we need the following lemma.

LEMMA 3.2. *Let I be a squarefree monomial ideal. Then for every $1 \leq j \leq n$ we have* $\mathrm{rank}((I : x_j)) \leq \mathrm{rank}(I)$.

Proof. Let $G(I) = \{u_1, \ldots, u_m\}$ be the set of minimal monomial generators of I. Without loss of generality, we may assume that $u_1, \ldots, u_t$ are divisible by x_j and $u_{t+1}, \ldots, u_m$ are not divisible by x_j, where $0 \leq t \leq m$. Put $v_i = u_i/x_j$, if $1 \leq i \leq t$ and $v_i = u_i$, if $t+1 \leq i \leq m$. For simplicity we assume that $\mathbf{a_i}$ is the exponent vector of u_i and $\mathbf{b_i}$ is the exponent vector of v_i $(1 \leq i \leq m)$. To prove the assertion one just note that for every $k \neq j$ and every $1 \leq i \leq m$, the kth component of $\mathbf{a_i}$ and $\mathbf{b_i}$ are the same and for $k = j$, the kth component of $\mathbf{b_i}$ is always zero. $\square$

We are now ready to state and prove the main result of this section.

THEOREM 3.3. *Let I be a squarefree monomial ideal of $S = \mathbb{K}[x_1, \ldots, x_n]$. Then* $\mathrm{sdepth}(I) \geq n - \mathrm{rank}(I) + 1$ *and* $\mathrm{sdepth}(S/I) \geq n - \mathrm{rank}(I)$.

Proof. Let $G(I)$ be the set of minimal monomial generators of I. We prove the assertions by induction on n. If $n = 1$, then I is a principal ideal and so we have $\mathrm{rank}(I) = 1$, $\mathrm{sdepth}(I) = n$ and by [7, Theorem 1.1], $\mathrm{sdepth}(S/I) = n - 1$. Therefore, in this case, there is nothing to prove.

We now assume that $n \geq 2$. Let $S' = \mathbb{K}[x_2, \ldots, x_n]$ be the polynomial ring obtained from S by deleting the variable x_1 and consider the ideals $I' = I \cap S'$ and $I'' = (I : x_1)$. If

$$x_1 \notin \bigcup_{u \in G(I)} \mathrm{Supp}(u),$$

then by [7, Theorem 1.1] and [3, Lemma 3.6], we conclude that $\mathrm{sdepth}(S/I) = \mathrm{sdepth}(S'/I') + 1$ and $\mathrm{sdepth}(I) = \mathrm{sdepth}(I') + 1$. Therefore, using our induction hypothesis, we conclude that $\mathrm{sdepth}(I) \geq n - \mathrm{rank}(I) + 1$ and $\mathrm{sdepth}(S/I) \geq n - \mathrm{rank}(I)$. Hence we may assume that

$$x_1 \in \bigcup_{u \in G(I)} \mathrm{Supp}(u).$$

Now $I = I'S' \oplus x_1 I''S$ and $S/I = (S'/I'S') \oplus x_1(S/I''S)$ and therefore by the definition of Stanley depth we have

$$\mathrm{sdepth}(I) \geq \min\{\mathrm{sdepth}_{S'}(I'S'), \mathrm{sdepth}_S(I'')\} \tag{1}$$

and

$$\text{sdepth}(S/I) \geq \min\{\text{sdepth}_{S'}(S'/I'S'), \text{sdepth}_S(S/I'')\}. \tag{2}$$

Note that the generators of I'' belong to S'. Therefore, our induction hypothesis implies that

$$\text{sdepth}_{S'}(S'/I'') \geq (n-1) - \text{rank}(I'')$$

and

$$\text{sdepth}_{S'}(I'') \geq (n-1) - \text{rank}(I'') + 1.$$

Using Lemma 3.2 together with [7, Theorem 1.1] and [3, Lemma 3.6], we conclude that

$$\text{sdepth}(S/I'') = \text{sdepth}_{S'}(S'/I'') + 1 \geq (n-1) - \text{rank}(I'') + 1 \geq n - \text{rank}(I)$$

and

$$\text{sdepth}_S(I'') = \text{sdepth}_{S'}(I'') + 1 \geq (n-1) - \text{rank}(I'') + 1 + 1 \geq n - \text{rank}(I) + 1.$$

On the other hand, since

$$x_1 \in \bigcup_{u \in G(I)} \text{Supp}(u),$$

it follows that $\text{rank}(I'S') \leq \text{rank}(I) - 1$ and therefore by our induction hypothesis we conclude that

$$\begin{aligned} \text{sdepth}_{S'}(I'S') &\geq (n-1) - \text{rank}(I'S') + 1 \geq (n-1) - (\text{rank}(I) - 1) + 1 \\ &= n - \text{rank}(I) + 1, \end{aligned}$$

and similarly $\text{sdepth}_{S'}(S'/I'S') \geq n - \text{rank}(I)$. Now the assertions follow by inequalities (1) and (2). □

As an immediate consequence of Proposition 1.2 and Theorem 3.3, we conclude that Conjecture 3.1 is true for every squarefree monomial ideal which is generated in a single degree.

Corollary 3.4. *Let I be a squarefree monomial ideal of $S = \mathbb{K}[x_1, \ldots, x_n]$ which is generated in a single degree. Then* $\text{sdepth}(I) \geq n - \ell(I) + 1$ *and* $\text{sdepth}(S/I) \geq n - \ell(I)$.

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