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ON CONFORMAL FIELDS OF A RANDERS METRIC WITH ISOTROPIC S-CURVATURE

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Abstract. Randers metrics are natural and important Finsler metrics. This paper shows a non-existence theorem for non-homothetic conformal fields on a non-Riemannian Randers manifold with isotropic S-curvature.

1. Introduction

Conformal fields on a Riemann–Finsler manifold are vector fields induced by local 1-parameter group of conformal diffeomorphisms. They contain all Killing fields and homothetic fields.

Let F be an n-dimensional Randers metric and (h,V) its navigation data [2], [5]. It is well known that V is conformal with respect to h if and only if F is of isotropic S-curvature [13], [18]. Recall that the S-curvature is one of most important non-Riemannian quantities in Finsler geometry [3]. An n-dimensional Finsler metric F on a manifold M is said to be of *isotropic S-curvature* if $\mathbf{S}(x,y)=(n+1)cF(x,y)$ for some scalar function $c(x)$ on M. Very recently, Shen and Xia determine completely conformal fields of a Randers metric with isotropic S-curvature and scalar flag curvature [5, Theorem 4.3.5], [16].

A *Randers metric* F on a manifold M is a Finsler metric in the form $F=\alpha+\beta$, where $\alpha=\sqrt{a_{ij}y^iy^j}$ is a Riemannian metric and $\beta=b_iy^i$ is a 1-form with $\|\beta\|_x<1$ for any point x. Randers metrics were first studied by physicist, G. Randers, in 1941 [14] from the standard point of general relativity. Recently, a signification progress has been made in studying Randers metrics [5], [12]. In particular, the classification of Randers metrics with constant flag curvature has been completed by D. Bao, C. Robles and Z. Shen

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[2]. It is worth mentioning the flag curvature is an important quantity in Finsler geometry because it lies in the second variation formula of arc length and takes the place of the sectional curvature in the Riemannian case.

In this paper, we study conformal fields on a Randers manifold of isotropic S-curvature. We show that there exists no non-homothetic conformal field on such a Randers manifold. Precisely, we prove the following theorem.

THEOREM 1.1. *Let $F=\alpha+\beta$ be a Randers metric of isotropic S-curvature. Suppose that V is a conformal field of F with dilation $c(x)$ and $\beta\neq 0$. Then V is homothetic, that is, $c(x)=$ constant.*

It is worth mentioning the recent result [7], [8] by Matveev–Rademacher–Troyanov–Zeghib [7], [8] that if V is a complete conformal vector field on a non-Riemannian Finsler manifold (M,F) such that V is not a Killing field for some conformal deformation $e^{\sigma(x)}F$, then (M,F) is globally conformally equivalent to a Minkowski space and V is homothetic with respect to the Minkowski metric. If F has isotropic S-curvature, then (M,F) is trivial conformally equivalent to the Minkowski space and V is homothetic with respect to F. Here we weaken Matveev–Rademacher–Troyanov–Zeghib's conditions on the conformal field and impose the Randers type on the Finsler metric instead.

We will prove Theorem 1.1 in Section 4. The condition of isotropic S-curvature in Theorem 1.1 can not be dropped (see Example 4.5 below).

For interesting results of Randers metrics of isotropic S-curvature, we refer the reader to [3], [4], [5], [17], [18].

2. Preliminaries

In this section, we recall the basic definitions and notations. Let M be a manifold. A function $F=F(x,y)$ on TM is called a *Finsler metric* on M if it has the following properties:

(a) $F(x,y)$ is C^∞ on $TM_0:=TM\setminus\{0\}$;
(b) $F_x(y):=F(x,y)$ is a Minkowski norm on T_xM for any $x\in M$.

The pair (M,F) is called a *Finsler manifold* or a *Finsler space.*

Let $\alpha=\sqrt{a_{ij}(x)y^iy^j}$ be a Riemann metric on a manifold M and $\beta=b_i(x)y^i$ be a 1-form on M. Assume that

$$\|\beta_x\|_\alpha:=\sup_{y\in T_xM}\frac{\beta(x,y)}{\alpha(x,y)}<1.$$

Then $F:=\alpha+\beta$ is a Finsler metric, which called the *Randers metric.*

Let F be a Finsler metric. Define the (*mean*) *distortion* $\tau: SM\to R$ by [15]

$$\tau\big(x,[y]\big):=\ln\frac{\sqrt{\det(g_{ij}(y))}}{\sigma(x)},$$

where SM is the projective sphere bundle of M, obtained from TM by identifying non-zero vectors which differ from each other by a positive multiplicative factor and

$$\sigma(x) = \frac{\mathrm{Vol}(\mathrm{B}^n)}{\mathrm{Vol}\{(y^i) \in \mathbb{R}^n | F(x, y^i \frac{\partial}{\partial x^i}) < 1\}},$$

where B^n denotes the unit ball in $\mathbb{R}^n$ and Vol denotes the Euclidean measure on $\mathbb{R}^n$. To measure the rate of changes of the distortion along geodesics, we define

$$\mathbf{S}(x, y) := \frac{d}{dt}\big[\tau\big(\dot{c}(t)\big)\big]_{t=0},$$

where $c(t)$ is the geodesic with $\dot{c}(0) = y$. We call the scalar function $\mathbf{S}$ the *S-curvature*. S-curvature is an important non-Riemannian quantity. It plays an important role in the geometry of Finsler manifold and it interacts with the Riemann curvature in a delicate way.

Every vector $y \in T_xM \setminus \{0\}$ uniquely determines a covector $p \in T_x^*M \setminus \{0\}$ by

$$p(u) := \frac{1}{2}\frac{d}{dt}\big(F^2(x, y + tu)\big)\bigg|_{t=0}, \quad u \in T_xM.$$

The resulting map $L_x^F : y \in T_xM \to p \in T_x^*M$ is called the *Legendre transformation* at x.

3. Conformal vector fields

In this section, we are going to give some properties of conformal fields on a Finsler manifold.

Two Finsler metrics F and F_1 on an open subset $\mathcal{U} \subseteq M$ are called *conformally equivalent* if $F = e^{2\sigma}F_1$ for a scalar function σ on $\mathcal{U}$. We say that a differentiable mapping $f : (M_1, F_1) \to (M_2, F_2)$ is *conformal*, if the pullback of the metric F_2 is conformally equivalent to F_1.

A vector field on a Finsler manifold (M, F) is called to be *conformal* if its local flow acts by conformal local diffeomorphisms.

Note that a local diffeomorphism $\varphi : \mathcal{U} \to \varphi(\mathcal{U})$ can be (locally) lifted to the maps $\check{\varphi} : T\mathcal{U} \to T\varphi(\mathcal{U})$ where

$$\check{\varphi}(x, y) := \big(\varphi(x), \varphi_\star(y)\big), \quad y \in T_x\mathcal{U}. \tag{3.1}$$

Let V be a conformal field on (M, F) and ϕ_t its local flow. Then we have

$$\big(\check{\phi}_t^\star F\big)(x, y) = e^{2\sigma_t(x)}F(x, y). \tag{3.2}$$

The function $c(x) := \frac{d\sigma_t(x)}{dt}|_{t=0}$ is called the *dilation* of V. A vector field V is called to be *homothetic with dilation c* if $c(x) = \text{constant}$ [9], [11], [16].

By the relationship of vector fields and local flows, (3.1) induces a natural way to lift a vector field U on M to a vector field X_U on TM. In natural coordinates, we have

$$X_U = u^i \frac{\partial}{\partial x^i} + y^j \frac{\partial u^i}{\partial x^j} \frac{\partial}{\partial y^i}, \tag{3.3}$$

where $U = u^i \frac{\partial}{\partial x^i}$ [9], [10]. We have the following lemma.

LEMMA 3.1. *V is a conformal field of F with dilation $c(x)$ if and only if $X_V(F) = 2c(x)F$.*

Proof. First, suppose that V is conformal with dilation $c(x)$. Differentiating (3.2) with respect to t at $t=0$ yields

$$\begin{aligned} 2c(x)F &= 2\frac{d\sigma_t(x)}{dt}\bigg|_{t=0} F \\ &= \frac{d}{dt}\big(e^{2\sigma_t(x)}F\big)\bigg|_{t=0} \\ &= \frac{d}{dt}\big(\breve{\phi}_t^* F\big)\bigg|_{t=0} \\ &= \frac{d}{dt}(F \circ \breve{\phi}_t)\bigg|_{t=0} \\ &= \frac{d\breve{\phi}_t}{dt}\bigg|_{t=0} F = X_V(F). \end{aligned}$$

Conversely, suppose that $X_V(F) = 2c(x)F$. Put

$$f_t = \breve{\phi}_t^* F. \tag{3.4}$$

Then we have $f_0 = F$ and

$$\frac{df_t}{dt}\bigg|_{t=0} = \frac{d}{dt}\big(\breve{\phi}_t^* F\big)\bigg|_{t=0} = X_V(F) = 2c(x)F. \tag{3.5}$$

Recall that $\breve{\phi}_t^*$ is a local flow on the tangent bundle. It follows that

$$\breve{\phi}_s^* f_t = \breve{\phi}_s^* \circ \breve{\phi}_t^* F = \breve{\phi}_{s+t}^* F = f_{s+t}. \tag{3.6}$$

Differentiating (3.6) with respect to t at $t=0$ yields

$$\begin{aligned} \breve{\phi}_s^*(2c(x)F) &= \breve{\phi}_s^*\left(\frac{df_t}{dt}\bigg|_{t=0}\right) \\ &= \frac{d}{dt}\breve{\phi}_s^* f_t\bigg|_{t=0} \\ &= \frac{d}{dt} f_{s+t}\bigg|_{t=0} \\ &= \frac{d}{dt} f_{s+u}\bigg|_{u=0} = \frac{d}{dt} f_t\bigg|_{t=s}, \end{aligned} \tag{3.7}$$

where we have used (3.5). By (3.4), we have

$$\breve{\phi}_s^*\big(2c(x)F\big) = 2(c\circ\phi_s)f_s. \tag{3.8}$$

Plugging this into (3.7) yields $2(c\circ\phi_s)f_s = \frac{d}{dt}f_t|_{t=s}$. For any fixed (x,y), we consider the following ordinary differential equation

$$\begin{cases} \frac{d}{dt}f_t(x,y)|_{t=s} = 2c(\phi_s(x))f_s(x,y), \\ f_0(x,y) = F(x,y). \end{cases} \tag{3.9}$$

The unique solution of (3.9) is given by

$$f_s(x,y) = e^{2\sigma_s(x)}F(x,y),$$

where $\sigma_s(x) := \int_0^s c(\phi_t(x))\,dt$. Together with (3.4) we have $\breve{\phi}_s^*F = e^{2\sigma_s(x)}F$. Furthermore

$$\left.\frac{d\sigma_s(x)}{dt}\right|_{s=0} = c\big(\phi_s(x)\big)\big|_{s=0} = c(x).$$

It follows that V is conformal with dilation $c(x)$. □

EXAMPLE 3.2. Let $a\in\mathbb{R}^N$ be a constant vector with $|a|<1$. Define $F=\alpha+\beta$ by

$$F(x,y) = e^{\sigma(x)}\big(|y| + \langle a,y\rangle\big). \tag{3.10}$$

Then F is a Randers metric. Let V denote a vector field on $\mathbb{R}^N$ defined by

$$V_x = -xQ - b \quad \text{at } x\in\mathbb{R}^N, \tag{3.11}$$

where Q is skew-symmetric and satisfies that

$$Qa^T = 0 \tag{3.12}$$

and $b\in\mathbb{R}^N$ is a constant vector. By using (3.3) and (3.11), we have

$$X_V = V - y^j q_{ji}\frac{\partial}{\partial y^i}, \tag{3.13}$$

where $Q=(q_{ij})$. Since Q is anti-symmetric, we have

$$\sum_{i,j} y^i y^j q_{ij} = 0. \tag{3.14}$$

By simple calculations, we obtain

$$\frac{\partial|y|}{\partial y^i} = \frac{y^i}{|y|}, \qquad \frac{\partial\langle a,y\rangle}{\partial y^i} = a^i, \tag{3.15}$$

where $a=(a^1,\dots,a^n)$. Together with (3.10), (3.12) and (3.14) we get

$$-y^j q_{ji}\frac{\partial F}{\partial y^i} = -e^{\sigma(x)}\left(\frac{y^iy^jq_{ji}}{|y|} + a^iy^jq_{ji}\right) = 0. \tag{3.16}$$

It follows that

$$\begin{aligned} X_V(F) &= V(F) - y^j q_{ji}\frac{\partial F}{\partial y^i} \\ &= V^i \frac{\partial}{\partial x^i}\left[e^{\sigma(x)}\left(|y| + \langle a, y\rangle\right)\right] = \left(V^i \frac{\partial \sigma}{\partial x^i}\right)F, \end{aligned}$$

where we have made use of (3.10) and (3.13). Thus V is conformal with dilation $\frac{1}{2}\langle V, \nabla\sigma\rangle$.

Let us take a look at the special case: when $\sigma(x) = |x|^2$,

$$\frac{1}{2}\langle V, \nabla\sigma\rangle = \sum_{i,j}\left(-x^j q_{ji} - b^i\right)x^i = -\langle b, x\rangle.$$

It follows that V is a non-homothetic conformal field when $b \neq 0$.

LEMMA 3.3. *Let W and V be two conformal fields of F with dilation $\lambda(x)$ and $\mu(x)$ respectively. Then $[W, V]$ is a conformal field of F with dilation $W(\mu) - V(\lambda)$.*

Proof. By Lemma 3.1, we have

$$X_W(F) = 2\lambda(x)F, \qquad X_V(F) = 2\mu(x)F. \tag{3.17}$$

It follows that

$$\mu(x)X_W(F) - \lambda(x)X_V(F) = (2\lambda\mu)F - (2\mu\lambda)F = 0. \tag{3.18}$$

From (3.3), we obtain $X_V(\lambda) = V(\lambda)$ and $X_W(\mu) = W(\mu)$. Together with (3.17) and (3.18) we have

$$\begin{aligned} X_{[W,V]}(F) &= [X_W, X_V]F \\ &= X_W(X_V F) - X_V(X_W F) \\ &= 2X_W(\mu F) - 2X_V(\lambda F) \\ &= 2X_W(\mu)F + 2\mu X_W(F) - 2X_V(\lambda)F - 2\lambda X_V(F) \\ &= 2[W(\mu) - V(\lambda)]F. \end{aligned}$$

By Lemma 3.1 again, one obtains that $[W, V]$ is conformal with dilation $W(\mu) - V(\lambda)$. □

LEMMA 3.4. *Let W be a non-zero conformal fields of F and $c(x)$ a scalar function on an n-dimensional manifold M. Assume that $c(x)W$ is conformal with respect to F. Then $c(x)$ is a constant.*

Proof. From Lemma 3.1, for conformal fields W and $c(x)W$ we have $X_W(F) = \sigma(x)F$ and $X_{cW}(F) = \tau(x)F$ where $\sigma(x)$ and $\tau(x)$ are scalar functions on M. In natural coordinates, we have

$$W^i \frac{\partial F}{\partial x^i} + y^j \frac{\partial W^i}{\partial x^j}\frac{\partial F}{\partial y^i} = \sigma F \tag{3.19}$$

and

$$cW^i\frac{\partial F}{\partial x^i}+y^j\frac{\partial(cW^i)}{\partial x^j}\frac{\partial F}{\partial y^i}=\tau F, \tag{3.20}$$

where we have used (3.3). (3.20)-$c(x)\times$(3.19) yields

$$\left(y^j\frac{\partial c}{\partial x^j}\right)\left(W^i\frac{\partial F}{\partial y^i}\right)=(\tau-c\sigma)F, \tag{3.21}$$

where we have used the fact $\frac{\partial(cW^i)}{\partial x^j}=\frac{\partial c}{\partial x^j}W^i+c\frac{\partial W^i}{\partial x^j}$. Suppose there exist $x_0\in M$ such that $(\tau-c\sigma)(x_0)\neq 0$. Taking $y\in T_{x_0}M\setminus\{0\}$, where $y=y^j\frac{\partial}{\partial x^j}$ satisfies that

$$y^j\frac{\partial c}{\partial x^j}=0.$$

Note that $F(x_0,y)>0$. Hence, $(\tau-c\sigma)(x_0)=0$. This contradicts to our assumption. It follows that $\tau-c\sigma\equiv 0$. Plugging this into (3.21) yields

$$\left(y^j\frac{\partial c}{\partial x^j}\right)\omega(W)\equiv 0, \tag{3.22}$$

where $\omega:=\frac{\partial F}{\partial y^i}\,dx^i$ is the Hilbert form of F. Defined $f:\mathbb{R}^n\to\mathbb{R}$ by

$$f\left(y^1,\ldots,y^n\right)=\omega_{(x,[y^j\frac{\partial}{\partial x^j}])}(W).$$

Then $f(W^1,\ldots,W^n)=F(x,W)>0$. It follows that $f^{-1}(0)$ is an $(n-1)$-dimensional submanifold of $\mathbb{R}^n$ where we have used the theorem on original of regular value. Suppose that

$$\left(\frac{\partial c}{\partial x^1},\ldots,\frac{\partial c}{\partial x^n}\right)(x)\neq 0.$$

Then

$$\Omega:=\left\{\left(y^1,\ldots,y^n\right)\in\mathbb{R}^n\middle|y^j\frac{\partial c}{\partial x^j}=0\right\}$$

is an $(n-1)$-dimensional linear subspace of $\mathbb{R}^n$. It is easy to see that

$$\Omega\cup f^{-1}(0)\neq\mathbb{R}^n.$$

This contradicts to (3.22). We obtain that $\frac{\partial c}{\partial x^j}=0, j=1,\ldots,n$. Hence, c is a constant. $\square$

4. Conformal fields of a Randers metric

In this section, we are going to prove our main theorem and show that the condition of isotropic S-curvature cannot be omitted in Theorem 1.1.

LEMMA 4.1. *Let $F=\sqrt{h_{ij}(x)y^iy^j}$ be a Riemannian metric. Then its Cartan metric H is given by*

$$H(x,p)=\sqrt{h^{ij}(x)p_ip_j}, \tag{4.1}$$

where

$$p = p_i\, dx^i, \qquad \left(h^{ij}\right) := (h_{ij})^{-1}.$$

Proof. For a Riemannian metric, its Legendre transformation (at x) L_x^F is just the musical isomorphism $\flat$, that is,

$$L_x^F = \flat. \tag{4.2}$$

Moreover, F and H satisfy the following [6]

$$H(x,p) = F\big(x, \big(L_x^F\big)^{-1} p\big). \tag{4.3}$$

Now the formula (4.1) can be obtained from (4.2) and (4.3) immediately. □

It is known that a Randers metric $F = \alpha + \beta$ is a solution of Zermelo's negation problem on a Riemannian manifold (M, h) under the influence of a vector field W, which is expressed in terms of h and W by

$$F = \frac{\sqrt{\lambda h^2 + W_0^2}}{\lambda} - \frac{W_0}{\lambda}, \tag{4.4}$$

where $W_0 := W_j(x) y^j$ and $\lambda := 1 - \|W_x\|_h^2$ [2].

Proposition 4.2. *Let V be a vector field on a manifold M. Let $F = \alpha + \beta$ be a Randers metric on M, which is expressed in terms of h and W by* (4.4). *Then the following assertions are equivalent*:

(i) *V is a conformal field of F with dilation $c(x)$;*
(ii) *V is a conformal field of h with dilation $c(x)$ and*

$$[W, V] = 2cW. \tag{4.5}$$

Proof. Denote the Cartan metric of h by H. Lemma 4.1 tells us that (4.1) holds. By Lemma 6.1 in [11], we have

$$H(x,p) = \tilde{H}(x,p) + p(W), \tag{4.6}$$

where $\tilde{H}$ is the Cartan metric of F, that is, [6]

$$\tilde{H}(x,p) := \max_{y \in T_xM \setminus \{0\}} \frac{p(y)}{F(x,y)}. \tag{4.7}$$

First, suppose that V is a conformal field of F with dilation $c(x)$. By using (3.2) and (4.6), we have

$$\tilde{H}\big(\phi_t(x), \big(\phi_t^*\big)^{-1}(p)\big) = e^{-2\sigma_t(x)} \tilde{H}(x,p),$$

where $c(x) = \frac{d\sigma_t(x)}{dt}|_{t=0}$ and ϕ_t is the local flow of V. It follows that

$$X_V^*(\tilde{H}) = -2c(x)\tilde{H}, \tag{4.8}$$

where

$$X_V^* = V^i \frac{\partial}{\partial x^i} - p_j \frac{\partial V^j}{\partial x^i} \frac{\partial}{\partial p_i} \tag{4.9}$$

is the vector field on the cotangent bundle induced by ϕ_t. Plugging (4.6) into (4.8) yields

$$X_V^*(H) - X_V^*\big(p(W)\big) = X_V^*\big(H - p(W)\big) = -2c(x)\big(H - p(W)\big) = -2c(x)H + 2c(x)p(W). \tag{4.10}$$

From (4.1) and (4.9), we have $X_V^*(H)$ and $-2c(x)H$ are irrational with respect to p_j. It follows that both rational and irrational parts of (4.10) should vanish, that is,

$$X_V^*(H) = -2c(x)H \tag{4.11}$$

and

$$X_V^*\big(p(W)\big) = -2c(x)p(W). \tag{4.12}$$

(4.11) implies that V is a conformal field of H with dilation $c(x)$, and hence its local flow satisfies the following:

$$H\Big(\phi_t(x), \big(\phi_t^*\big)^{-1}(p)\Big) = e^{-2\sigma_t(x)} H(x, p), \tag{4.13}$$

where $c(x) = \frac{d\sigma_t(x)}{dt}|_{t=0}$. From the standard duality, we have

$$h(x, y) := \max_{p \in T_x^* M \setminus \{0\}} \frac{p(y)}{H(x, p)}.$$

Together with (4.13) we obtain that V is a conformal field of h with dilation $c(x)$.

By a simple calculation, we have

$$p(W) = p_j W^j \tag{4.14}$$

from which together with (4.9) we obtain

$$X_V^*\big(p(W)\big) = p_i V^j \frac{\partial W^i}{\partial x^j} - p_i W^j \frac{\partial V^i}{\partial x^j}. \tag{4.15}$$

Plugging (4.14) and (4.15) into (4.12) yields

$$p_i \left(V^j \frac{\partial W^i}{\partial x^j} - W^j \frac{\partial V^i}{\partial x^j} \right) = -2c p_i W^i. \tag{4.16}$$

Take $p = dx^i$ then

$$V^j \frac{\partial W^i}{\partial x^j} - W^j \frac{\partial V^i}{\partial x^j} = -2cW^i. \tag{4.17}$$

By a straightforward computation, one obtains

$$[W, V] = -\left(V^j \frac{\partial W^i}{\partial x^j} - W^j \frac{\partial V^i}{\partial x^j} \right) \frac{\partial}{\partial x^i}.$$

Together with (4.17) we get (4.5).

Conversely, suppose that assertion (ii) holds. We have

$$X_V(h) = 2c(x)h. \tag{4.18}$$

Then it is easy to see from (4.5) and (4.18) that V is a conformal field of F with dilation $c(x)$. □

Proof of Theorem 1.1. Let (h, W) denote the navigation data of F (see (4.4)). Since F is of isotropic S-curvature, W is a conformal field of h [13], [18].

Now we assume that V is a conformal field of F with dilation $c(x)$. It follows from Proposition 4.2 that V is a conformal field of h with dilation $c(x)$ and (4.5) holds. By Lemma 3.3, $[W, V]$ is a conformal field of h and so by (4.5), $c(x)W$ is conformal with respect to h. Since $\beta \neq 0$, we have W is a non-zero vector field on the open subset

$$A := \{x \in M | b(x) \neq 0\},$$

where $b = \|\beta\|_\alpha$. Together with Lemma 3.4, we conclude that

$$\nabla c = 0 \tag{4.19}$$

on A. By the continuity, (4.19) holds on the whole connected manifold M. It follows that $c(x) =$ constant on M. □

Corollary 4.3. *Let F be a non-Riemannian Randers metric of isotropic S-curvature. Then there exists no non-homothetic conformal field of F.*

Together with Theorem 3.2.2 in [5] and Theorem 1.1 in [17], we have the following:

Corollary 4.4. *Let F be a non-Riemannian Randers metric and V a conformal field of F. Suppose that F satisfies one of the following conditions*:

(i) *F has isotropic mean Berwald curvature*;
(ii) *F has almost vanishing H-curvature*;
(iii) *F has weakly isotropic S-curvature.*

Then V is a homothetic field of F.

The condition of isotropic S-curvature in Theorem 1.1 can not be dropped.

Example 4.5. Let us continue to investigate Example 3.2. Z. Shen's result tells us all Berwald metrics have zero S-curvature [15]. On the other hand, (local) Minkowski metrics are of Berwald type. It follows that Minkowski metric $|y| + \langle a, y\rangle$ has vanishing S-curvature. Combining with S-curvature changing relation under a conformal deformation [1], we conclude that a generic non-trivial conformal deformation, for instance, taking $F(x, y) := e^{|x|}(|y| + \langle a, y\rangle)$, will no longer have isotropic S-curvature.

From the conclusion of Example 3.2, there is many non-homothetic conformal field of F. Thus the condition of isotropic S-curvature in Theorem 1.1 cannot be omitted.

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