Illinois Journal of Mathematics
Volume 57, Number 3, Fall 2013, Pages 665–684
S 0019-2082

THE TOTAL ABSOLUTE TORSION OF OPEN CURVES IN E^3

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ABSTRACT. The total absolute torsion of smooth curves in E^3 is defined as the total integral of the absolute value of the torsion. This notion is extended to piecewise smooth curves. We study the infimum of the total absolute torsion in a certain set of curves, where the endpoints, the osculating planes at the endpoints and the length are all prescribed. We show how the infimum is calculated from the boundary data.

1. Introduction

Let Σ be a C^3 curve in the 3-dimensional Euclidean space E^3. For curves in E^3, two geometric quantities called curvature and torsion are defined. The total integral of curvature is called the total absolute curvature. The study of the total absolute curvature has a long history since the work by Fenchel ([5]). Seeing that most works had been done for closed curves, the authors studied the total absolute curvature of open (i.e., not closed) curves in [3] and [4]. The total integral of torsion is called the total torsion and the total integral of the absolute value of torsion is called the total absolute torsion. Note that the torsion may change its sign while the curvature of space curves is always nonnegative. Both the total torsion and the total absolute torsion have been studied for closed curves in E^3. An interesting property of the total torsion of a closed curve is that it becomes zero if the curve lies on the unit sphere ([7], [11], [12], [13]). For the total absolute torsion, see, for example, [6], [8], [10].

In the present paper, we study the total absolute torsion of open curves and determine the minimal possible value of the total absolute torsion in a certain family of open curves. The family of curves is described as follows.

Received February 23, 2013; received in final form December 3, 2013.

The second author was partially supported by Grant-in-Aid for Scientific Research (No. 23450098), JSPS.

2010 *Mathematics Subject Classification.* 53A04.

Let p, q be points in E^3, Π_p be an oriented plane through p, Π_q be an oriented plane through q and L be a positive constant not smaller than $|pq|$. We define $\mathcal{C}(p, \Pi_p, q, \Pi_q, L)$ as the set of all C^3 curves whose endpoints are p and q, osculating plane at p is Π_p, osculating plane at q is Π_q and length is L. The following theorem, which is the main result of the present paper, asserts that the infimum of the total absolute torsion in $\mathcal{C}(p, \Pi_p, q, \Pi_q, L)$ can be calculated using a piecewise linear curve with only two edges. In the statement, $\angle(\cdot, \cdot)$ denotes the angle between two oriented planes.

THEOREM 1.1. *For any p, q, Π_p, Π_q and L, there exist a point r, an oriented plane Π_1 containing the line segment pr, and an oriented plane Π_2 containing rq which have the following properties*:

(1) *The sum of the lengths of the line segments pr and rq is L.*
(2) *The sum of the angles $\angle(\Pi_p, \Pi_1) + \angle(\Pi_1, \Pi_2) + \angle(\Pi_2, \Pi_q)$ gives the infimum of the total absolute torsion in $\mathcal{C}(p, \Pi_p, q, \Pi_q, L)$.*

To prove this theorem, we first extend the notion of the total absolute torsion to curves which are C^3 only piecewise. This makes it possible to study the total absolute torsion of piecewise linear curves. The notion of the total absolute torsion for piecewise linear curves is generalized to what we call the total rotation of unit normal vector fields along piecewise linear curves. We will show that, for any piecewise linear curve with three edges, it is always possible to find a piecewise linear curve with two edges which has a unit normal vector field of smaller total rotation, preserving the boundary condition (Lemma 3.5). By an induction argument, we see that minimal value of the total rotation under the given boundary condition is attained by a unit normal vector field along a piecewise linear curve with 2 edges (Proposition 3.7). We will show that the minimization of the total rotation leads to the minimization of the total absolute torsion (Lemma 3.1). Finally, making use of an approximation of smooth curves by piecewise linear curves, we prove our main theorem. As a corollary, we give a proof of a theorem by Aratake ([1]).

The authors express their thanks to the referee for helpful comments.

2. Total absolute torsion

Let Σ be a C^3 curve in the 3-dimensional Euclidean space E^3. Let L be the length of Σ and $x(s)$ $(0 \leq s \leq L)$ be a parameterization of Σ by its arclength. Let

$$T(s) = \frac{dx}{ds}$$

and

$$\kappa(s) = \left|\frac{dT}{ds}\right| = \left|\frac{d^2x}{ds^2}\right|.$$

$\kappa(s)$ is called the curvature of Σ.

If $\kappa(s) \neq 0$ on Σ, a unit vector field

$$N(s) = \frac{dT/ds}{\kappa(s)}$$

is defined. Note that $\langle N(s), T(s)\rangle = 0$, where $\langle \cdot,\cdot\rangle$ is the inner product of E^3. Let $B(s) = T(s) \times N(s)$. Then $\{T(s), N(s), B(s)\}$ forms a positively oriented orthonormal frame field of E^3 defined along Σ. The torsion of Σ is, by definition,

$$\tau(s) = \left\langle \frac{dN}{ds}, B(s) \right\rangle.$$

The *total absolute torsion* is defined by

$$\mathrm{TAT}(\Sigma) = \int_0^L \left|\tau(s)\right| ds.$$

We regard $T(s)$ as a curve in S^2 and call it the *tangent indicatrix* of Σ, which will be denoted by T_Σ. $N(s)$ becomes a unit tangent vector of T_Σ and $B(s)$ becomes a unit normal vector of T_Σ. Since $|dT/ds| = \kappa(s)$, the oriented geodesic curvature K of T_Σ is given by

$$K(s) = \left\langle \frac{1}{\kappa(s)} \frac{dN}{ds}, B(s) \right\rangle = \frac{\tau(s)}{\kappa(s)}.$$

The integral of $|K|$ along T_Σ is called the total absolute curvature (as a curve in S^2) of T_Σ, which will be denoted by $\mathrm{TAC}(T_\Sigma)$. Then we have

$$\begin{aligned} \mathrm{TAC}(T_\Sigma) &= \int_0^L \left|K(s)\right| \left|\frac{dT}{ds}\right| ds \\ &= \int_0^L \left|K(s)\right| \kappa(s)\, ds \\ &= \int_0^L \left|\tau(s)\right| ds \\ &= \mathrm{TAT}(\Sigma). \end{aligned} \tag{2.1}$$

Since

$$\frac{dB}{ds} = -\tau(s) N(s), \tag{2.2}$$

we have

$$\mathrm{TAT}(\Sigma) = \int_0^L \left|\frac{dB}{ds}\right| ds. \tag{2.3}$$

This means that the total absolute torsion is equal to the length of $B(s)$ regarded as a curve in S^2. (2.1) and (2.3) give two different interpretations of the total absolute torsion.

We next consider the case when Σ has a point with vanishing curvature. If $k(s) = 0$ for $a \leq s \leq b$ (a and b may coincide), $\{T(s) : a \leq s \leq b\}$ shrinks to a point and T_Σ becomes a piecewise C^2 curve. The definition of the total absolute curvature is extended to piecewise C^2 curves in S^2 as the sum of the total integral of the curvature and the rotation angles of the tangent vector at nonsmooth points. In our case, if Σ is a C^3 curve whose curvature $\kappa(s)$ vanishes for $s \in [a_1, b_1] \cup [a_2, b_2] \cup \cdots \cup [a_r, b_r]$ ($a_1 \leq b_1 < a_2 \leq b_2 < \cdots < a_r \leq b_r$), then T_Σ becomes a piecewise C^2 curve in S^2 and its total absolute curvature is given by

$$\begin{aligned}\mathrm{TAC}(T_\Sigma) = \int_0^{a_1} \big|K(s)\big|\kappa(s)\,ds + \sum_{i=1}^{r-1} \int_{b_i}^{a_{i+1}} \big|K(s)\big|\kappa(s)\,ds \\ + \int_{b_r}^{L} \big|K(s)\big|\kappa(s)\,ds + \sum_{i=1}^{r} \varphi_i,\end{aligned}$$

where φ_i $(i = 1, \dots, r)$ be the angle between $\lim_{s\to a_i-0} N(s)$ and $\lim_{s\to b_i+0} N(s)$ $(0 \leq \varphi_i \leq \pi)$. Following (2.1), we define the total absolute torsion of Σ by

$$\mathrm{TAT}(\Sigma) = \mathrm{TAC}(T_\Sigma).$$

We make further extension of the definition of the total absolute torsion for curves which are C^3 only piecewise. Such a curve is expressed as

$$\Sigma = \bigcup_{i=1}^{n} \big\{x(s) \mid s_{i-1} < s < s_i\big\},$$

where $0 = s_0 < s_1 < \cdots < s_{n-1} < s_n = L$ and each $\{x(s) \mid s_i < s < s_{i+1}\}$ is a C^3 curve. We define the tangent indicatrix T_Σ of Σ as follows. Let $\Sigma_i = \{x(s) \mid s_{i-1} < s < s_i\}$ and let T_{Σ_i} be the tangent indicatrix of Σ_i. T_{Σ_i} is a continuous, piecewise C^2 curve in S^2 which has endpoints at $\lim_{s\to s_{i-1}+0} T(s)$ and $\lim_{s\to s_i-0} T(s)$. We define T_Σ as a curve which consists of $\bigcup_{i=1}^{n}\{T(s) \mid s_{i-1} < s < s_i\}$ and the geodesic arcs between $\lim_{s\to s_i-0} T(s)$ and $\lim_{s\to s_i+0} T(s)$ $(i = 1, \dots, n)$. Since T_Σ is a piecewise C^2 curve in S^2, the total absolute curvature of T_Σ is defined. Again, the total absolute torsion of Σ is defined by

$$\mathrm{TAT}(\Sigma) = \mathrm{TAC}(T_\Sigma).$$

As a special case, we consider the total absolute torsion of piecewise linear curves. A piecewise linear curve P is written as

$$P = p_0p_1 \cup p_1p_2 \cup \cdots \cup p_{n-1}p_n,$$

where $p_{i-1}p_i$ is the line segment which joins two points p_{i-1} and p_i. Let

$$T_i = \frac{\overrightarrow{p_{i-1}p_i}}{|\overrightarrow{p_{i-1}p_i}|}.$$

Then the tangent indicatrix T_P of P becomes

$$T_P = T_1T_2 \cup \cdots \cup T_{n-1}T_n,$$

where T_iT_{i+1} is the geodesic segment in S^2 which joins T_i and T_{i+1}. Let φ_i be the exterior angle of T_P at T_i. Then the total absolute torsion of P, or the total absolute curvature of T_P, is given by

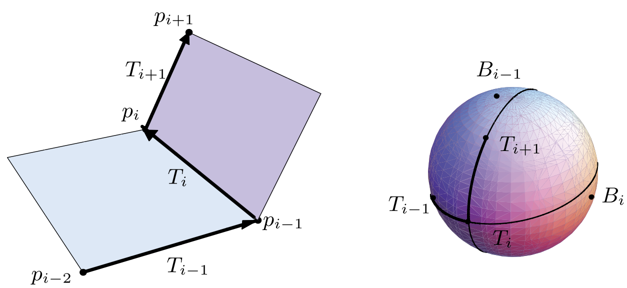
$$\mathrm{TAT}(P) = \mathrm{TAC}(T_P) = \sum_{i=1}^{n-1} \varphi_i.$$

The angle φ_i is equal to the angle between two oriented planes, one spanned by $\{\overrightarrow{p_{i-2}p_{i-1}}, \overrightarrow{p_{i-1}p_i}\}$ and one spanned by $\{\overrightarrow{p_{i-1}p_i}, \overrightarrow{p_ip_{i+1}}\}$. Hence φ_i is the angle (valued in $[0,\pi]$) between two vectors $T_{i-1} \times T_i$ and $T_i \times T_{i+1}$. If we set

$$B_i = \frac{T_i \times T_{i+1}}{|T_i \times T_{i+1}|},$$

then φ_i is the distance $d(B_{i-1}, B_i)$ between B_{i-1} and B_i as points in S^2. Thus, we have another expression of $\mathrm{TAT}(P)$ as

$$\mathrm{TAT}(P) = \sum_{i=1}^{n-1} d(B_{i-1}, B_i). \tag{2.4}$$



Our extension of the notion of the total absolute torsion to piecewise smooth curves is natural in the following sense. Let $\Sigma : x(s)$ be a C^3 curve. Let n be a positive integer and let

$$\Delta_n : 0 = s_0 < s_1 < \cdots < s_{n-1} < s_n = L$$

be a division of the interval $[0, L]$. Let $p_i = x(s_i)$ and let

$$P_n : p_0p_1 \cup p_1p_2 \cup \cdots \cup p_{n-2}p_{n-1} \cup p_{n-1}p_n.$$

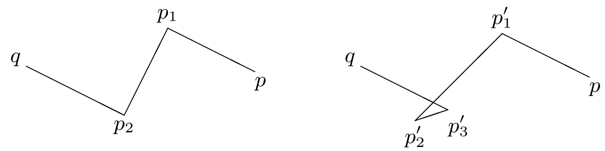
For any Σ, there exists a sequence of divisions $\{\Delta_n\}_{n=1}^{\infty}$ such that $\sum_{i=1}^{n-1} d(B_{i-1}, B_i)$ converges to $\int_0^L |\frac{dB}{ds}|\, ds$ (= the length of $B(s)$ as a curve in S^2) as $n \to \infty$, which implies

$$\lim_{n\to\infty} \mathrm{TAT}(P_n) = \mathrm{TAT}(\Sigma). \tag{2.5}$$

REMARK 2.1. The notion of the total absolute torsion for piecewise linear curves is also given by Banchoff ([2]) and McRae ([8]). Our definition coincides with them.

REMARK 2.2. We make several remarks on curves lying in a plane.

(1) If Σ is a curve with $\kappa \neq 0$ which lies in a plane, then $\mathrm{TAT}(\Sigma) = 0$. T_Σ becomes a subarc of a great circle in this case.
(2) For a curve in a plane, the oriented curvature $\tilde{\kappa}$ is defined. If $\tilde{\kappa}(s) > 0$ for $0 \leq s < s_0$, $\tilde{\kappa}(s_0) = 0$ and $\tilde{\kappa}(s) < 0$ for $s_0 < s \leq L$, then $\mathrm{TAT}(\Sigma) = \pi$.
(3) The total absolute torsion of a piecewise linear curve in a plane becomes zero if and only if all oriented angles from $p_{i-1}p_i$ to p_ip_{i+1} have the same orientation. The piecewise linear curve $p_0p_1 \cup p_1p_2 \cup p_2p_3$ shown in the picture below is an example of a planar curve with nonzero total absolute torsion $(= \pi)$. However, if one deforms the curve into a curve like $pp_1' \cup p_1'p_2' \cup p_2'p_3' \cup p_3'q$, one can construct a curve of vanishing total absolute torsion which has the same endpoints and the same length as the original curve.



Now we extend the notion of the total absolute torsion for piecewise linear curves to a little more general notion which we call the total rotation of unit normal fields along piecewise linear curves.

Let

$$P = p_0p_1 \cup p_1p_2 \cup \cdots \cup p_{n-2}p_{n-1} \cup p_{n-1}p_n$$

be a piecewise linear curve in E^3. Let ν_i be a unit vector perpendicular to $p_{i-1}p_i$. We denote the set of the unit vectors $\{\nu_i : i = 1, \ldots, n\}$ by $\bar{\nu}$. We call $\bar{\nu}$ a *unit normal vector field* along the piecewise linear curve P. We define the *total rotation* $\mathrm{TR}(P, \bar{\nu})$ by

$$\mathrm{TR}(P, \bar{\nu}) = \sum_{i=1}^{n-1} \angle(\nu_i, \nu_{i+1}).$$

If we regard unit vectors as points in the unit sphere S^2, we may rewrite the total rotation, using the distance $d(\cdot, \cdot)$ in S^2, as

$$\mathrm{TR}(P, \bar{\nu}) = \sum_{i=1}^{n-1} d(\nu_i, \nu_{i+1}).$$

Let

$$B_i = \frac{\overrightarrow{p_{i-1}p_i} \times \overrightarrow{p_ip_{i+1}}}{|\overrightarrow{p_{i-1}p_i} \times \overrightarrow{p_ip_{i+1}}|} \quad (i = 1, \ldots, n-1). \tag{2.6}$$

If we attach B_i to $p_{i-1}p_i$ for each $i = 1, \ldots, n-2$ and use B_{n-1} again and attach it to $p_{n-1}p_n$, then

$$\bar{B} = \{B_1, \ldots, B_{n-2}, B_{n-1}, B_{n-1}\} \tag{2.7}$$

defines a unit normal vector field along P. In terms of the total rotation, (2.4) can be rewritten as

$$\mathrm{TAT}(P) = \mathrm{TR}(P, \bar{B}). \tag{2.8}$$

3. Curves with fixed endpoints, end-osculating-planes, length

Let p, q be points in E^3. Let Π_p and Π_q be oriented planes which pass through p and q, respectively. Let L be a positive constant not smaller than $|pq|$. We define several classes of curves:

$\mathcal{C}(p,q)$: The set of all piecewise C^3 curves which has endpoints at p and q.
$\mathcal{C}(p,q,L)$: The set of elements of $\mathcal{C}(p,q)$ whose length is L.
$\mathcal{C}(p,\Pi_p,q,\Pi_q)$: The set of elements of $\mathcal{C}(p,q)$ such that T_Σ at the initial point is tangent to the oriented great circle corresponding to Π_p, and T_Σ at the terminal point is tangent to the oriented great circle corresponding to Π_q.
$\mathcal{C}(p,\Pi_p,q,\Pi_q,L) = \mathcal{C}(p,q,L) \cap \mathcal{C}(p,\Pi_p,q,\Pi_q)$.

Let n be a positive integer and let $\mathcal{P}_n$ be the set of all piecewise linear curves with n edges. For all $m < n$, we regard $\mathcal{P}_m$ as a subset of $\mathcal{P}_n$ by allowing angles between two edges to be zero. Let $\mathcal{P}_n(p,q) = \mathcal{P}_n \cap \mathcal{C}(p,q)$ and $\mathcal{P}_n(p,q,L) = \mathcal{P}_n \cap \mathcal{C}(p,q,L)$. Any element P of $\mathcal{P}_n(p,q)$ may be written as

$$P = pp_1 \cup p_1p_2 \cup \cdots \cup p_{n-2}p_{n-1} \cup p_{n-1}q.$$

P becomes an element of $\mathcal{C}(p,\Pi_p,q,\Pi_q)$ if and only if $p_1, p_2 \in \Pi_p$ and $\{\overrightarrow{pp_1}, \overrightarrow{p_1p_2}\}$ is positively oriented in Π_p and $p_{n-2}, p_{n-1} \in \Pi_q$ and $\{\overrightarrow{p_{n-2}p_{n-1}}, \overrightarrow{p_{n-1}q}\}$ is positively oriented in Π_q. If ν_i $(i = 1, \ldots, n)$ is a unit vector perpendicular to $p_{i-1}p_i$ (setting $p_0 = p$ and $p_n = q$), then

$$\bar{\nu} = \{\nu_1, \ldots, \nu_n\}$$

defines a unit normal vector field along P. Let $\mathcal{PN}(p,q,L)$ be the set of all $(P, \bar{\nu})$ such that $P \in \mathcal{P}(p,q,L)$ and $\bar{\nu}$ is a unit normal field along P. Let $\mathcal{PN}_n(p,q,L)$ be the set of all elements of $\mathcal{PN}(p,q,L)$ which have n edges. For our purpose, we define the *extended total rotation* $\widetilde{\mathrm{TR}}(P, \bar{\nu})$ by

$$\widetilde{\mathrm{TR}}(P, \bar{\nu}) = \angle(\nu_p, \nu_1) + \sum_{i=1}^{n-1} \angle(\nu_i, \nu_{i+1}) + \angle(\nu_n, \nu_q).$$

Note that if $\nu_1 = \nu_p$ and $\nu_n = \nu_q$, then $\widetilde{\mathrm{TR}}(P, \bar{\nu}) = \mathrm{TR}(P, \bar{\nu})$. As the following lemma shows, the problem of minimizing the total absolute torsion in $\mathcal{P}(p,\Pi_p,q,\Pi_q,L)$ is reduced to the problem of minimizing the extended total rotation in $\mathcal{PN}(p,q,L)$.

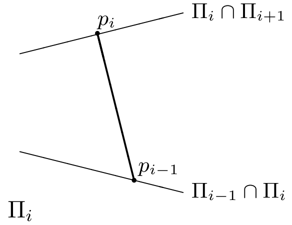
LEMMA 3.1. *We have*

$$\inf\{\mathrm{TAT}(P) \mid P \in \mathcal{P}(p, \Pi_p, q, \Pi_q, L)\} = \inf\{\widetilde{\mathrm{TR}}(P, \bar{\nu}) \mid (P, \bar{\nu}) \in \mathcal{PN}(p, q, L)\}.$$

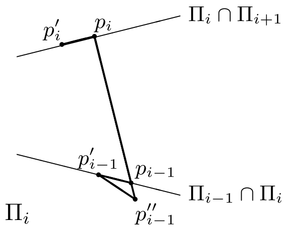
Proof. Let P be an element of $\mathcal{P}(p, \Pi_p, q, \Pi_q, L)$. If we define a unit normal vector field $\bar{B}$ along P by (2.6) and (2.7), then we have $\mathrm{TAT}(P) = \mathrm{TR}(P, \bar{B})$ as in (2.8). Since $B_1 = \nu_p$ and $B_{n-1} = \nu_q$, we have $\mathrm{TR}(P, \bar{B}) = \widetilde{\mathrm{TR}}(P, \bar{B})$. This implies that

$$\begin{aligned} \inf\{\mathrm{TAT}(P) \mid P \in \mathcal{P}(p, \Pi_p, q, \Pi_q, L)\} \\ \geq \inf\{\widetilde{\mathrm{TR}}(P, \bar{\nu}) \mid (P, \bar{\nu}) \in \mathcal{PN}(p, q, L)\}. \end{aligned} \tag{3.1}$$

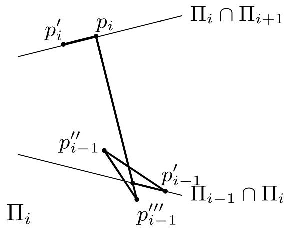
Conversely, let $(P, \bar{\nu})$ be an element of $\mathcal{PN}(p, q, L)$. If $(P, \bar{\nu}) \in \mathcal{PN}_n(p, q, L)$, we may write $P = pp_1 \cup p_1p_2 \cup \cdots \cup p_{n-2}p_{n-1} \cup p_{n-1}q$ and $\bar{\nu} = \{\nu_1, \nu_2, \ldots, \nu_{n-1}, \nu_n\}$. For each $i = 1, \ldots, n$, let Π_i be the oriented plane which contains the edge $p_{i-1}p_i$ and has ν_i as its oriented unit normal vector. The figure (a) shows the plane Π_i with the points p_{i-1} lying on the line $\Pi_{i-1} \cap \Pi_i$ and p_i lying on the line $\Pi_i \cap \Pi_{i+1}$. In the following, we will add some edges in the plane Π_i to make a detour of $p_{i-1}p_i$. The purpose of doing this is to construct a piecewise linear curve whose total absolute torsion is equal to $\widetilde{\mathrm{TR}}(P, \bar{\nu})$. The figures (b) and (c) both show how those detours of $p_{i-1}p_i$ are constructed in the plane Π_i. The difference between (b) and (c) comes from the difference of the direction of ν_{i-1}.



(a)



(b)



(c)

First, we take a point p_{i-1}' on the line $\Pi_{i-1} \cap \Pi_i$. p_{i-1}' is taken so that the direction of $\overrightarrow{p_{i-2}p_{i-1}} \times \overrightarrow{p_{i-1}p_{i-1}'}$ coincides with the direction of ν_{i-1}. The figures (b) and (c) show the difference of the direction of ν_{i-1}. Then we take

p''_{i-1} so that $\overrightarrow{p_{i-1}p'_{i-1}} \times \overrightarrow{p'_{i-1}p''_{i-1}}$ coincides with the direction of ν_i. In both figures (b) and (c), we assume that the normal vector ν_i of Π_i is heading toward the reader, but the argument will be similar even if ν_i heads into the other direction. Now we want a closed curve coming back to p_{i-1}. In (b), $p_{i-1}p'_{i-1} \cup p'_{i-1}p''_{i-1} \cup p''_{i-1}p_{i-1}$ is closed, while in (c) we need another point p'''_{i-1} to make a closed curve $p_{i-1}p'_{i-1} \cup p'_{i-1}p''_{i-1} \cup p''_{i-1}p'''_{i-1} \cup p'''_{i-1}p_{i-1}$. Finally, we take a point p'_i on the line $\Pi_i \cap \Pi_{i+1}$ so that $\overrightarrow{p_{i-1}p_i} \times \overrightarrow{p_ip'_i}$ coincides with the direction of ν_i, which gives at the same time the first step of our construction procedure in the next plane Π_{i+1}. We replace the edge $p_{i-1}p_i$ of the original curve P by $p_{i-1}p'_{i-1} \cup p'_{i-1}p''_{i-1} \cup p''_{i-1}p_i$ (the case (b)) or $p_{i-1}p'_{i-1} \cup p'_{i-1}p''_{i-1} \cup p''_{i-1}p'''_{i-1} \cup p'''_{i-1}p_i$ (the case (c)) for every i to construct a new curve P' which is an element of $\mathcal{P}_k(p, \Pi_p, q, \Pi_q, L)$ for some $k \le 4n$. By the way of construction, we have

$$\mathrm{TAT}\big(P'\big) = \widetilde{\mathrm{TR}}(P, \bar{\nu}). \tag{3.2}$$

Since we can make the length of the edges added as small as we want, the length of P' can be made smaller than $L + \varepsilon$ for any given $\varepsilon > 0$. (3.2) implies that

$$\begin{aligned} &\inf\big\{\mathrm{TAT}(P) \mid P \in \mathcal{P}_{4n}(p, \Pi_p, q, \Pi_q, L + \varepsilon)\big\} \\ &\qquad \le \inf\big\{\widetilde{\mathrm{TR}}(P, \bar{\nu}) \mid (P, \bar{\nu}) \in \mathcal{PN}_n(p, q, L)\big\} \end{aligned}$$

for any $\varepsilon > 0$. Note that $\mathcal{P}_k(p, \Pi_p, q, \Pi_q, L + \varepsilon)$ $(k \le 4n)$ is considered as a subset of $\mathcal{P}_{4n}(p, \Pi_p, q, \Pi_q, L + \varepsilon)$. Since $\inf\{\mathrm{TAT}(P) \mid P \in \mathcal{P}_{4n}(p, \Pi_p, q, \Pi_q, L)\}$ is continuous in L, we must have

$$\begin{aligned} &\inf\big\{\mathrm{TAT}(P) \mid P \in \mathcal{P}_{4n}(p, \Pi_p, q, \Pi_q, L)\big\} \\ &\qquad \le \inf\big\{\widetilde{\mathrm{TR}}(P, \bar{\nu}) \mid (P, \bar{\nu}) \in \mathcal{PN}_n(p, q, L)\big\} \end{aligned}$$

for every n. Thus we have

$$\begin{aligned} &\inf\big\{\mathrm{TAT}(P) \mid P \in \mathcal{P}(p, \Pi_p, q, \Pi_q, L)\big\} \\ &\qquad \le \inf\big\{\widetilde{\mathrm{TR}}(P, \bar{\nu}) \mid (P, \bar{\nu}) \in \mathcal{PN}(p, q, L)\big\}. \end{aligned} \tag{3.3}$$

Now the lemma follows from (3.1) and (3.3). □

From now on, we consider the problem of minimizing the extended total rotation in $\mathcal{PN}(p, q, L)$.

We start with the case when the curve has only one edge, i.e., the case when $L = |pq|$ and $P = pq$. In this case, our problem is simply the problem of minimizing $\angle(\nu_p, \nu) + \angle(\nu, \nu_q)$ for $\nu \perp pq$. The set of all unit vectors perpendicular to pq forms a great circle, which will be denoted as $(pq)^\perp$. Our problem is to find $\min\{d(\nu_p, \nu) + d(\nu, \nu_q) \mid \nu \in (pq)^\perp\}$. The answer is trivial if the geodesic segment between ν_p and ν_q intersects $(pq)^\perp$, but if not, the minimum is not so trivial, as we see in the following lemma.

LEMMA 3.2.

(1) *If the great circle* $(pq)^\perp$ *intersects the minimizing geodesic between* ν_p *and* ν_q *(or equivalently, if* $\langle \nu_p, \overrightarrow{pq}\rangle\langle \nu_q, \overrightarrow{pq}\rangle \le 0$*), then we have* $\min\{d(\nu_p,\nu) + d(\nu,\nu_q) \mid \nu \in (pq)^\perp\} = d(\nu_p,\nu_q)$.

(2) *If the great circle* $(pq)^\perp$ *does not intersect the minimizing geodesic between* ν_p *and* ν_q *(or equivalently, if* $\langle \nu_p, \overrightarrow{pq}\rangle\langle \nu_q, \overrightarrow{pq}\rangle > 0$*), then we have*

$$\begin{aligned}&\min\bigl\{d(\nu_p,\nu) + d(\nu,\nu_q) \mid \nu \in (pq)^\perp\bigr\}\\ &= \arccos\left(\langle \nu_p,\nu_q\rangle - 2\frac{\langle \nu_p, \overrightarrow{pq}\rangle\langle \nu_q, \overrightarrow{pq}\rangle}{|pq|^2}\right).\end{aligned}$$

Proof. If the minimizing geodesic between ν_p and ν_q intersects $(pq)^\perp$ at ν_0, then we have

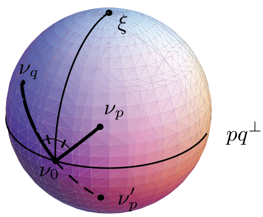
$$d(\nu_p,\nu_0) + d(\nu_0,\nu_q) = d(\nu_p,\nu_q),$$

which shows ν_0 attains the minimum.

Suppose that the great circle $(pq)^\perp$ does not intersect the minimizing geodesic between ν_p and ν_q. Set $\xi = \overrightarrow{pq}/|pq|$. If $\nu_0 \in (pq)^\perp$ minimizes $d(\nu_p,\nu) + d(\nu,\nu_q)$ for $\nu \in (pq)^\perp$, we must have

$$\angle \xi\nu_0\nu_p = \angle \xi\nu_0\nu_q = \theta \tag{3.4}$$

for some $\theta \in [0,\pi/2]$. (Here $\angle$ denotes the angle on S^2.) This follows from an ordinary reflection argument as the picture below shows. In this picture, ν_p' is the point on S^2 symmetric to ν_p with respect to the great circle $pq^\perp$. ν_0 is determined as the intersection of the great circle $pq^\perp$ and the great circle through ν_q and ν_p'.



Applying the Law of Cosine of the spherical trigonometry to the spherical triangle $\triangle\nu_0\nu_p\nu_q$, we have

$$\begin{aligned}\cos d(\nu_p,\nu_q) = {}&\cos d(\nu_0,\nu_p)\cos d(\nu_0,\nu_q)\\ &+ \sin d(\nu_0,\nu_p)\sin d(\nu_0,\nu_q)\cos 2\theta.\end{aligned} \tag{3.5}$$

Similarly, in $\triangle\nu_0\nu_p\xi$, we have

$$\begin{aligned}\cos d(\nu_p,\xi) &= \cos d(\nu_0,\nu_p)\cos d(\nu_0,\xi) + \sin d(\nu_0,\nu_p)\sin d(\nu_0,\xi)\cos\theta\\ &= \cos d(\nu_0,\nu_p)\cos\frac{\pi}{2} + \sin d(\nu_0,\nu_p)\sin\frac{\pi}{2}\cos\theta\\ &= \sin d(\nu_0,\nu_p)\cos\theta\end{aligned}$$

and in $\triangle \nu_0 \nu_q \xi$

$$\begin{aligned}\cos d(\nu_q,\xi) &= \cos d(\nu_0,\nu_q)\cos d(\nu_0,\xi) + \sin d(\nu_0,\nu_q)\sin d(\nu_0,\xi)\cos\theta \\ &= \sin d(\nu_0,\nu_q)\cos\theta.\end{aligned}$$

Now we have

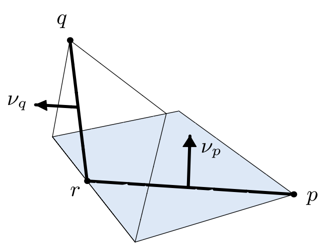
$$\begin{aligned}2\cos d(\nu_p,\xi)\cos d(\nu_q,\xi) &= 2\sin d(\nu_0,\nu_p)\sin d(\nu_0,\nu_q)\cos^2\theta \\ &= \sin d(\nu_0,\nu_p)\sin d(\nu_0,\nu_q)(\cos 2\theta+1) \\ &= \cos d(\nu_p,\nu_q) - \cos d(\nu_0,\nu_p)\cos d(\nu_0,\nu_q) \\ &\quad\ + \sin d(\nu_0,\nu_p)\sin d(\nu_0,\nu_q) \\ &= \cos d(\nu_p,\nu_q) - \cos\big(d(\nu_0,\nu_p)+d(\nu_0,\nu_q)\big),\end{aligned}$$

which gives

$$\begin{aligned}\cos\big(d(\nu_0,\nu_p)+d(\nu_0,\nu_q)\big) &= \cos d(\nu_p,\nu_q) - 2\cos d(\nu_p,\xi)\cos d(\nu_q,\xi) \\ &= \langle \nu_p,\nu_q\rangle - 2\langle \xi,\nu_p\rangle\langle \xi,\nu_q\rangle.\end{aligned}$$

Since $d(\nu_0,\nu_p)+d(\nu_0,\nu_q)\le\pi$ when ν_0 is the minimizer, we obtain the desired equation. $\square$

Now we deal with the case when the piecewise linear curve consists of two edges. For any $L\ge|pq|$, there exists an element of $\mathcal{P}_2(p,q,L)$. Let $L_0=\min\{|pr|+|rq| \mid r\in\Pi_p\cap\Pi_q\}$. L_0 is a positive constant determined by p, q, ν_p and ν_q. Let r_0 be the point in $\Pi_p\cap\Pi_q$ with $|pr_0|+|r_0q|=L_0$. Then $(pr_0\cup r_0q,\{\nu_p,\nu_q\})$ is an element of $\mathcal{PN}_2(p,q,L)$ whose total rotation is $d(\nu_p,\nu_q)$. For any positive constant $L\ge L_0$, there exists a point r in $\Pi_p\cap\Pi_q$ with $|pr|+|rq|=L$. Then $(pr\cup rq,\{\nu_p,\nu_q\})$ is an element of $\mathcal{PN}_2(p,q,L)$ whose total rotation is $d(\nu_p,\nu_q)$.



Thus we obtain the following lemma.

LEMMA 3.3. *Let L be any positive constant with $L\ge L_0$. Then we have*

$$\min\big\{\mathrm{TR}(P,\bar\nu) \mid (P,\bar\nu)\in\mathcal{PN}_2(p,q,L)\big\} = d(\nu_p,\nu_q).$$

Lemma 3.3 shows that the problem to be considered is essentially the case when $L_0>L\ge|pq|$.

Now we deal with the case when the piecewise linear curve consists of three edges. We note that for any positive constant $L > |pq|$ there exists an element $(pp_1 \cup p_1p_2 \cup p_2q, \{\nu_1, \nu_2, \nu_3\})$ of $\mathcal{PN}_3(p, q, L)$ which satisfies $\nu_1 = \nu_p$ and $\nu_3 = \nu_q$. The following lemma for curves with 3 edges is the key in this work.

Lemma 3.4. *Let* $(P, \bar{\nu}) = (pp_1 \cup p_1p_2 \cup p_2q, \{\nu_1, \nu_2, \nu_3\})$ *be an element of* $\mathcal{PN}_3(p, q, L)$. *Suppose that* $\nu_1 = \nu_p$ *and* $\nu_3 = \nu_q$. *Then, there exists an element of* $\mathcal{PN}_2(p, q, L)$, $(P', \bar{\nu}') = (pp_1' \cup p_1'q, \{\nu_1', \nu_2'\})$ *which satisfies the following conditions*:

(1) $\widetilde{\mathrm{TR}}(P', \bar{\nu}') \leq \mathrm{TR}(P, \bar{\nu})$.
(2) *Either* $\nu_1' = \nu_p$ *or* $\nu_2' = \nu_q$ *holds.*

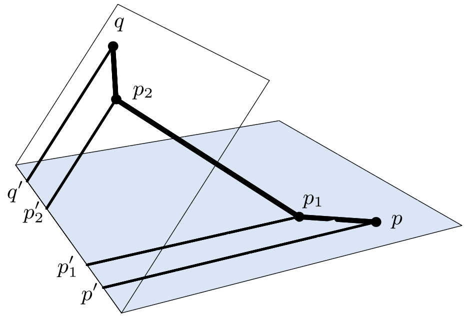
Proof. By Lemma 3.2, our problem becomes to find p_1 and p_2 which minimize

$$\frac{\langle \overrightarrow{p_1p_2}, \nu_p \rangle \langle \overrightarrow{p_1p_2}, \nu_q \rangle}{|p_1p_2|^2}$$

under the constraint

$$|pp_1| + |p_1p_2| + |p_2q| = L.$$

We will express the locations of p_1 and p_2 using 2 parameters for each and then express $\langle \overrightarrow{p_1p_2}, \nu_p \rangle \langle \overrightarrow{p_1p_2}, \nu_q \rangle / |p_1p_2|^2$ as a function of 4 variables. Let p', q', p_1', p_2' be the orthogonal projections of p, q, p_1, p_2 onto the line $\Pi_p \cap \Pi_q$, respectively. We may assume that p', p_1', p_2' and q' lie on $\Pi_p \cap \Pi_q$ in this order. Set $x_1 = |p_1p_1'|$, $x_2 = |p_2p_2'|$, $y_1 = |p'p_1'|$ and $y_2 = |p_2'q'|$. We also set $a = |pp'|$, $b = |qq'|$, $c = |p'q'|$ and $\alpha = d(\nu_p, \nu_q)$.



Now we can write $\langle \overrightarrow{p_1p_2}, \nu_p \rangle \langle \overrightarrow{p_1p_2}, \nu_q \rangle / |p_1p_2|^2$ as a function of x_1, y_1, x_2, y_2 as

$$\begin{aligned} f(x_1, y_1, x_2, y_2) &= \frac{\langle \overrightarrow{p_1p_2}, \nu_p \rangle \langle \overrightarrow{p_1p_2}, \nu_q \rangle}{|p_1p_2|^2} \\ &= \frac{x_1x_2 \sin^2 \alpha}{x_1^2 + x_2^2 - 2x_1x_2 \cos \alpha + (c - y_1 - y_2)^2}. \end{aligned}$$

We also define a function $g(x_1, y_1, x_2, y_2)$ by

$$\begin{aligned}g(x_1, y_1, x_2, y_2) &= |pp_1| + |p_1p_2| + |p_2q| \\ &= \sqrt{(x_1-a)^2+y_1^2} \\ &\quad + \sqrt{x_1^2+x_2^2-2x_1x_2\cos\alpha+(c-y_1-y_2)^2} \\ &\quad + \sqrt{(x_2-b)^2+y_2^2}.\end{aligned}$$

Suppose that (x_1, y_1, x_2, y_2) minimizes f under the constraint $g = L$ and both f and g are differentiable there. Then there exists a constant λ such that

$$\frac{\partial f}{\partial x_1} = \lambda \frac{\partial g}{\partial x_1}, \tag{3.6}$$

$$\frac{\partial f}{\partial y_1} = \lambda \frac{\partial g}{\partial y_1}, \tag{3.7}$$

$$\frac{\partial f}{\partial x_2} = \lambda \frac{\partial g}{\partial x_2}, \tag{3.8}$$

$$\frac{\partial f}{\partial y_2} = \lambda \frac{\partial g}{\partial y_2}. \tag{3.9}$$

Since

$$\frac{\partial f}{\partial y_1} = \frac{\partial f}{\partial y_2},$$

we must have either $\lambda = 0$ or

$$\frac{\partial g}{\partial y_1} = \frac{\partial g}{\partial y_2}. \tag{3.10}$$

(3.10) gives

$$\frac{y_1}{\sqrt{(x_1-a)^2+y_1^2}} = \frac{y_2}{\sqrt{(x_2-b)^2+y_2^2}}. \tag{3.11}$$

If $\lambda = 0$, (x_1, y_1, x_2, y_2) becomes a critical point of $f(x_1, y_1, x_2, y_2)$. Since $f(x_1, y_1, x_2, y_2)$ is maximal rather than minimal at any critical point, we see that $\lambda = 0$ is not a solution for our problem.

By (3.11), we may define a function θ $(-\pi/2 \le \theta \le \pi/2)$ by

$$\sin\theta = \frac{y_1}{\sqrt{(x_1-a)^2+y_1^2}} = \frac{y_2}{\sqrt{(x_2-b)^2+y_2^2}}.$$

Then we have

$$\cos\theta = \frac{|x_1-a|}{\sqrt{(x_1-a)^2+y_1^2}} = \frac{|x_2-a|}{\sqrt{(x_2-b)^2+y_2^2}}.$$

Now (3.6), (3.7), (3.8) are written as

$$\frac{x_2 \sin^2\alpha(-x_1^2+x_2^2+(c-y_1-y_2)^2)}{(x_1^2+x_2^2-2x_1x_2\cos\alpha+(c-y_1-y_2)^2)^2} = \lambda\left(\pm\cos\theta + \frac{x_1-x_2\cos\alpha}{\sqrt{x_1^2+x_2^2-2x_1x_2\cos\alpha+(c-y_1-y_2)^2}}\right), \tag{3.12}$$

$$\frac{2x_1x_2 \sin^2\alpha(c-y_1-y_2)}{(x_1^2+x_2^2-2x_1x_2\cos\alpha+(c-y_1-y_2)^2)^2} = \lambda\left(\sin\theta - \frac{c-y_1-y_2}{\sqrt{x_1^2+x_2^2-2x_1x_2\cos\alpha+(c-y_1-y_2)^2}}\right), \tag{3.13}$$

$$\frac{x_1 \sin^2\alpha(x_1^2-x_2^2+(c-y_1-y_2)^2)}{(x_1^2+x_2^2-2x_1x_2\cos\alpha+(c-y_1-y_2)^2)^2} = \lambda\left(\pm\cos\theta + \frac{x_2-x_1\cos\alpha}{\sqrt{x_1^2+x_2^2-2x_1x_2\cos\alpha+(c-y_1-y_2)^2}}\right). \tag{3.14}$$

By (3.12), (3.13), (3.14), we have

$$\begin{cases} (c-y_1-y_2)(x_1^2+x_2^2-2x_1x_2\cos\alpha+(c-y_1-y_2)^2)^{1/2} \\ \quad = \sin\theta(x_2^2-x_1^2+(c-y_1-y_2)^2) \pm 2\cos\theta x_1(c-y_1-y_2), \\ (c-y_1-y_2)(x_1^2+x_2^2-2x_1x_2\cos\alpha+(c-y_1-y_2)^2)^{1/2} \\ \quad = \sin\theta(x_1^2-x_2^2+(c-y_1-y_2)^2) \pm 2\cos\theta x_2(c-y_1-y_2). \end{cases} \tag{3.15}$$

The system (3.15) is equivalent to

$$\begin{cases} (x_1-x_2)(-\sin\theta(x_1+x_2)\pm\cos\theta(c-y_1-y_2))=0, \\ (c-y_1-y_2)(\pm\cos\theta(x_1+x_2)+\sin\theta(c-y_1-y_2) \\ \quad -(x_1^2+x_2^2-2x_1x_2\cos\alpha+(c-y_1-y_2)^2)^{1/2})=0. \end{cases} \tag{3.16}$$

(3.16) implies that the following 4 cases (3.17)–(3.20) are possible.

$$\begin{cases} x_1-x_2=0, \\ c-y_1-y_2=0, \end{cases} \tag{3.17}$$

$$\begin{cases} -\sin\theta(x_1+x_2)\pm\cos\theta(c-y_1-y_2)=0, \\ c-y_1-y_2=0, \end{cases} \tag{3.18}$$

$$\begin{cases} -\sin\theta(x_1+x_2)\pm\cos\theta(c-y_1-y_2)=0, \\ \pm\cos\theta(x_1+x_2)+\sin\theta(c-y_1-y_2) \\ \quad -(x_1^2+x_2^2-2x_1x_2\cos\alpha+(c-y_1-y_2)^2)^{1/2}=0, \end{cases} \tag{3.19}$$

$$\begin{cases} x_1-x_2=0, \\ \pm\cos\theta(x_1+x_2)+\sin\theta(c-y_1-y_2) \\ \quad -(x_1^2+x_2^2-2x_1x_2\cos\alpha+(c-y_1-y_2)^2)^{1/2}=0. \end{cases} \tag{3.20}$$

Suppose that (3.17) holds. Then (3.13) implies $\sin\theta = 0$, since $\lambda \neq 0$. Now (3.14) gives

$$\frac{x_1(1-\cos\alpha)}{\sqrt{2x_1^2 - 2x_1^2\cos\alpha}} = \pm 1. \tag{3.21}$$

This yields $\alpha = \pi$ and there is no solution in this case.

Suppose that (3.18) holds. Then we have $\sin\theta = 0$. By (3.12) and (3.14), we have

$$\frac{x_2 \sin^2\alpha(-x_1^2 + x_2^2)}{(x_1^2 + x_2^2 - 2x_1x_2\cos\alpha)^2} = \lambda\left(\pm 1 + \frac{x_1 - x_2\cos\alpha}{\sqrt{x_1^2 + x_2^2 - 2x_1x_2\cos\alpha}}\right) \tag{3.22}$$

and

$$\frac{x_1 \sin^2\alpha(x_1^2 - x_2^2)}{(x_1^2 + x_2^2 - 2x_1x_2\cos\alpha)^2} = \lambda\left(\pm 1 + \frac{x_2 - x_1\cos\alpha}{\sqrt{x_1^2 + x_2^2 - 2x_1x_2\cos\alpha}}\right). \tag{3.23}$$

It follows from (3.22) and (3.23) (where the signs for ± 1 coincide) that

$$\frac{\sin^2\alpha(x_1+x_2)^2(x_1-x_2)}{(x_1^2 + x_2^2 - 2x_1x_2\cos\alpha)^2} = \frac{\lambda(-1+\cos\alpha)(x_1-x_2)}{\sqrt{x_1^2 + x_2^2 - 2x_1x_2\cos\alpha}}. \tag{3.24}$$

Since we are assuming $0 < \alpha < \pi$, (3.24) implies $x_1 = x_2$, and we have the same conclusion as the first case (3.17).

If (3.19) holds, we have

$$(x_1+x_2)^2 + (c - y_1 - y_2)^2 = x_1^2 + x_2^2 - 2x_1x_2\cos\alpha + (c - y_1 - y_2)^2, \tag{3.25}$$

which does not have a solution under the assumption that $0 < \alpha < \pi$.

Not like other cases, (3.20) can possibly have a solution. This solution, however, does not minimize f under the constraint $g = L$, as we show in the following. Let $x_1 = x_2 = a$ and $y_1 = b_1$, $y_2 = b_2$ be a solution of (3.20). Then we have

$$\pm 2a\cos\theta + (c - b_1 - b_2)\sin\theta = \left(2a^2 - 2a^2\cos\alpha + (c - b_1 - b_2)^2\right)^{1/2}. \tag{3.26}$$

Let $p_1(t)$ be the point defined by

$$x_1 = a + t\cos\theta,$$
$$y_1 = b_1 - t\sin\theta$$

and $p_2(s)$ be the point defined by

$$x_2(s) = a - s\cos\theta,$$
$$y_2(s) = b_2 + s\sin\theta.$$

If $t > 0$, then $p_1(t)$ moves toward p along a straight line. If $|pp_1(t)| + |p_1(t)p_2(s)| + |p_2(s)q| = L$, we must have

$$\left|p_1(t)p_2(s)\right| + \left|p_2(s)p_2(0)\right| = \left|p_1(t)p_1(0)\right| + \left|p_1(0)p_2(0)\right|, \tag{3.27}$$

which gives

$$(1+\cos\alpha)\cos\theta(ts\cos\theta - at - as) = 0. \tag{3.28}$$

Thus, if $\alpha \neq \pi$ and $\cos\theta \neq 0$, then we have

$$s = \frac{at}{t\cos\theta + a}. \tag{3.29}$$

(3.29) allows us to express x_2 and y_2 in terms of t as

$$\begin{aligned} x_2(t) &= \frac{a^2}{t\cos\theta + a}, \\ y_2(t) &= b_2 + \frac{at\sin\theta}{t\cos\theta + a}. \end{aligned}$$

Now we can write f in terms of t as

$$\begin{aligned} f(t) &= f\big(x_1(t), y_1(t), x_2(t), y_2(t)\big) \\ &= \frac{x_1(t)x_2(t)\sin^2\alpha}{x_1(t)^2 + x_2(t)^2 - 2x_1(t)x_2(t)\cos\alpha + (c - y_1(t) - y_2(t))^2} \\ &= a^2\sin^2\alpha\left(t - \frac{at}{t\cos\theta + a} + 2a\cos\theta + (c - b_1 - b_2)\sin\theta\right)^{-2} \\ &= a^2\sin^2\alpha\left(\frac{1}{\cos\theta}(t\cos\theta + a) + \frac{1}{\cos\theta}\frac{a^2}{t\cos\theta + a} - \frac{2a}{\cos\theta}\right. \\ &\qquad \left. + 2a\cos\theta + (c - b_1 - b_2)\sin\theta\right)^{-2} \\ &\leq a^2\sin^2\alpha\left(\frac{2a}{\cos\theta} - \frac{2a}{\cos\theta} + 2a\cos\theta + (c - b_1 - b_2)\sin\theta\right)^{-2} \\ &= a^2\sin^2\alpha\big(2a\cos\theta + (c - b_1 - b_2)\sin\theta\big)^{-2} \\ &= f(0). \end{aligned}$$

Since the equality holds only when $t = 0$, we have $f(t) < f(0)$, which shows that the solution of (3.20) does not minimize f.

Now we conclude that, if $\{x_1, y_1, x_2, y_2\}$ minimizes f under the constraint that $g = L$, then either f or g is not differentiable there, with the only possible exception that $p_1 = p_2$. The possibility of $p_1 = p_2$ is eliminated by our assumption on the length of $pp_1 \cup p_1p_2 \cup p_2q$. Since f or g is not differentiable at the minimizing point, we have $p_1 = p$ or $p_2 = q$. □

Now we use Lemma 3.4 to deal with the case when the piecewise linear curve has 3 edges in general. ($\nu_1 = \nu_p$ and $\nu_3 = \nu_q$ are not assumed here.)

LEMMA 3.5. *Let $(P, \bar\nu)$ be an element of $\mathcal{PN}_3(p,q,L)$. Then there exists an element $(P', \bar\nu')$ of $\mathcal{PN}_2(p,q,L)$ such that $\widetilde{\mathrm{TR}}(P', \bar\nu') \leq \widetilde{\mathrm{TR}}(P, \bar\nu)$.*

Proof. We write $(P,\bar{\nu})$ as

$$P = pp_1 \cup p_1p_2 \cup p_2q, \qquad \bar{\nu} = \{\nu_1,\nu_2,\nu_3\}.$$

By Lemma 3.4, there exists an element $(P',\bar{\nu}') = (pp_1' \cup p_1'q, \{\nu_1',\nu_2'\})$ of $\mathcal{PN}_2(p,q,L)$ for which one of the following holds:

(1) $\nu_1' = \nu_1$ and $d(\nu_1,\nu_2') + d(\nu_2',\nu_3) \le d(\nu_1,\nu_2) + d(\nu_2,\nu_3)$.
(2) $\nu_2' = \nu_3$ and $d(\nu_1,\nu_1') + d(\nu_1',\nu_3) \le d(\nu_1,\nu_2) + d(\nu_2,\nu_3)$.

If (1) occurs, then we have

$$\begin{aligned}
\widetilde{\mathrm{TR}}(P,\bar{\nu}) &= d(\nu_p,\nu_1) + d(\nu_1,\nu_2) + d(\nu_2,\nu_3) + d(\nu_3,\nu_q)\\
&\ge d(\nu_p,\nu_1) + d\big(\nu_1,\nu_2'\big) + d\big(\nu_2',\nu_3\big) + d(\nu_3,\nu_q)\\
&= d\big(\nu_p,\nu_1'\big) + d\big(\nu_1',\nu_2'\big) + d\big(\nu_2',\nu_3\big) + d(\nu_3,\nu_q)\\
&\ge d\big(\nu_p,\nu_1'\big) + d\big(\nu_1',\nu_2'\big) + d\big(\nu_2',\nu_q\big)\\
&= \widetilde{\mathrm{TR}}\big(P',\bar{\nu}'\big).
\end{aligned}$$

Similarly, if (2) occurs, then we have

$$\begin{aligned}
\widetilde{\mathrm{TR}}(P,\bar{\nu}) &= d(\nu_p,\nu_1) + d(\nu_1,\nu_2) + d(\nu_2,\nu_3) + d(\nu_3,\nu_q)\\
&\ge d(\nu_p,\nu_1) + d\big(\nu_1,\nu_1'\big) + d\big(\nu_1',\nu_3\big) + d(\nu_3,\nu_q)\\
&= d(\nu_p,\nu_1) + d\big(\nu_1,\nu_1'\big) + d\big(\nu_1',\nu_2'\big) + d\big(\nu_2',\nu_q\big)\\
&\ge d\big(\nu_p,\nu_1'\big) + d\big(\nu_1',\nu_2'\big) + d\big(\nu_2',\nu_q\big)\\
&= \widetilde{\mathrm{TR}}\big(P',\bar{\nu}'\big).
\end{aligned}$$

□

We use Lemma 3.5 to prove the following proposition for piecewise linear curves with arbitrary number of edges.

LEMMA 3.6. *Let $(P,\bar{\nu})$ be an element of $\mathcal{PN}_n(p,q,L)$ with $n \ge 3$. Then there exists an element $(P',\bar{\nu}')$ of $\mathcal{PN}_{n-1}(p,q,L)$ such that $\widetilde{\mathrm{TR}}(P',\bar{\nu}') \le \widetilde{\mathrm{TR}}(P,\bar{\nu})$.*

Proof. We set

$$P = pp_1 \cup p_1p_2 \cup \cdots \cup p_{n-2}p_{n-1} \cup p_{n-1}q$$

and

$$\bar{\nu} = \{\nu_1,\nu_2,\ldots,\nu_{n-1},\nu_n\}.$$

If we apply Lemma 3.5 to the subarc $p_{n-3}p_{n-2} \cup p_{n-2}p_{n-1} \cup p_{n-1}q$ associated with $\{\nu_{n-2},\nu_{n-1},\nu_n\}$, regarding p_{n-3} as p, ν_{n-3} as ν_p in Lemma 3.5, we see that there exists a piecewise linear curve $p_{n-3}p_{n-2}' \cup p_{n-2}'q$ associated with a unit normal field $\{\nu_{n-2}',\nu_{n-1}'\}$ such that

$$\begin{aligned}
&d\big(\nu_{n-3},\nu_{n-2}'\big) + d\big(\nu_{n-2}',\nu_{n-1}'\big) + d\big(\nu_{n-1}',\nu_q\big)\\
&\quad \le d(\nu_{n-3},\nu_{n-2}) + d(\nu_{n-2},\nu_{n-1}) + d(\nu_{n-1},\nu_n) + d(\nu_n,\nu_q).
\end{aligned} \tag{3.30}$$

Now we set

$$P' = pp_1 \cup p_1p_2 \cup \cdots \cup p_{n-4}p_{n-3} \cup p_{n-3}p'_{n-2} \cup p'_{n-2}q$$

and

$$\bar{\nu}' = \{\nu_1, \nu_2, \ldots, \nu_{n-3}, \nu'_{n-2}, \nu'_{n-1}\}.$$

Then $(P', \bar{\nu}') \in \mathcal{PN}_{n-1}(p,q,L)$ and (3.30) implies

$$\widetilde{\mathrm{TR}}(P', \bar{\nu}') \le \widetilde{\mathrm{TR}}(P, \bar{\nu}). \qquad \square$$

By an inductive argument based on Lemma 3.6, we obtain the following.

PROPOSITION 3.7. *Let $(P, \bar{\nu})$ be an element of $\mathcal{PN}_n(p,q,L)$. Then there exists an element $(P', \bar{\nu}')$ of $\mathcal{PN}_2(p,q,L)$ such that $\widetilde{\mathrm{TR}}(P', \bar{\nu}') \le \widetilde{\mathrm{TR}}(P, \bar{\nu})$.*

Proposition 3.7, together with Lemma 3.1 and (2.8), gives the following theorem, which shows that our main theorem holds for piecewise linear curves.

THEOREM 3.8. *For any p, q, Π_p, Π_q and L, there exist a point r, an oriented plane Π_1 containing the line segment pr, and an oriented plane Π_2 containing rq which have the following properties*:

(1) *The sum of the lengths of the line segments pr and rq is L.*
(2) *The sum of the angles $\angle(\Pi_p, \Pi_1) + \angle(\Pi_1, \Pi_2) + \angle(\Pi_2, \Pi_q)$ gives the infumum of the total absolute torsion in $\mathcal{P}(p, \Pi_p, q, \Pi_q, L)$.*

Now we give a proof of our main theorem.

Proof of Theorem 1.1. Let Σ be a curve in $\mathcal{C}(p, \Pi_p, q, \Pi_q, L)$. For each integer n, we can construct a division

$$0 = s_0 < s_1 < \cdots < s_{n-1} < s_n = L$$

so that the total absolute torsion of the piecewise linear curve

$$P_n : px(s_1) \cup x(s_1)x(s_2) \cup \cdots \cup x(s_{n-2})x(s_{n-1}) \cup x(s_{n-1})q$$

converges to $\mathrm{TAT}(\Sigma)$. The length of each of these piecewise linear curves is not greater than L, but by attaching a small planar closed curve, it is easy to construct a piecewise linear curve whose length is L and total absolute torsion is equal to $\mathrm{TAT}(P_n)$. By Theorem 3.8, there exists

$$P = pr \cup rq$$

such that

$$\mathrm{TAT}(P) \le \mathrm{TAT}(P_n).$$

Since $\mathrm{TAT}(P_n)$ converges to $\mathrm{TAT}(\Sigma)$, we must have

$$\mathrm{TAT}(P) \le \mathrm{TAT}(\Sigma),$$

which completes the proof of Theorem 1.1. $\square$

Theorem 1.1 and Lemma 3.3 give the following corollary. Here, as in Lemma 3.3, $L_0 = \min\{|pr| + |rq| \mid r \in \Pi_p \cap \Pi_q\}$.

COROLLARY 3.9. *If* $L \geq L_0$, *then*

$$\inf\{\mathrm{TAT}(\Sigma) \mid \Sigma \in \mathcal{C}(p, \Pi_p, q, \Pi_q, L)\} = \angle(\Pi_p, \Pi_q).$$

REMARK 3.10. The problem we study in this paper may be regarded as a "torsion version" of our results in [4], in which the total absolute curvature of open curves are studied. The strategy we are taking here is somehow similar to the one taken in [4]. Corollary 3.9 shows, however, that there is a difference between them. The total torsion tends to be small when the length gets larger, while the total absolute curvature tends to be large.

The following is first proved by Aratake [1]. Thus our theorem may be regarded as a refinement of Aratake's theorem.

COROLLARY 3.11. *The infimum of the total absolute torsion in* $\mathcal{C}(p, q, L)$ *tends to zero as the length* L *tends to infinity.*

Proof. Let ε be any positive constant. We take an oriented plane Π_p through p and an oriented plane Π_q through q so that $\angle(\Pi_p, \Pi_q) < \varepsilon$. Again, we consider $L_0 = \min\{|pr| + |rq| \mid r \in \Pi_p \cap \Pi_q\}$. Since L_0 depends on p, q, Π_p and Π_q, we denote it as $L_0(p, q, \Pi_p, \Pi_q)$. We define $L_0(p, q, \varepsilon)$ by

$$L_0(p, q, \varepsilon) = \inf\{L_0(p, q, \Pi_p, \Pi_q) \mid \angle(\Pi_p, \Pi_q) < \varepsilon\}.$$

If $L \geq L_0(p, q, \varepsilon)$, we have $\inf\{\mathrm{TAT}(\Sigma) \mid \Sigma \in \mathcal{C}(p, q, L)\} < \varepsilon$. □

REMARK 3.12. In this paper, we extend the notion of the vector field B from smooth curves to piecewise linear curves, and interpret the total absolute torsion as the total rotation of the unit normal vector field along a piecewise linear curve. Lemma 3.5 makes it possible to reduce the number of edges and the total rotation, preserving the boundary condition. At the end, we obtain a curve with only two edges whose total rotation gives the minimal possible value of the total absolute torsion in $\mathcal{C}(p, \Pi_p, q, \Pi_q, L)$. An extension of the vector field N to piecewise linear curves is possible by defining N by the relation $N = B \times T$. If a piecewise linear curve approximates a smooth curve, then the total rotation of N along the piecewise linear curve approximates the integral $\int_\Sigma \sqrt{\kappa^2 + \tau^2}\, ds$. Lemma 3.5 again works to reduce the total rotation of N and the number of edges preserving the boundary condition. However, there is no lemma like Lemma 3.1 for N and the resulting piecewise linear curve with two edges does not give the infimum of $\int_\Sigma \sqrt{\kappa^2 + \tau^2}\, ds$ in $\mathcal{C}(p, \Pi_p, q, \Pi_q, L)$. The minimization of $\int_\Sigma \sqrt{\kappa^2 + \tau^2}\, ds$ may be another interesting problem.

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