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THE CARPENTER AND SCHUR–HORN PROBLEMS FOR MASAS IN FINITE FACTORS

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Abstract. Two classical theorems in matrix theory, due to Schur and Horn, relate the eigenvalues of a self-adjoint matrix to the diagonal entries. These have recently been given a formulation in the setting of operator algebras as the Schur–Horn problem, where matrix algebras and diagonals are replaced respectively, by finite factors and maximal Abelian self-adjoint subalgebras (masas). There is a special case of the problem, called the carpenter problem, which can be stated as follows: for a masa A in a finite factor M with conditional expectation $\mathbb{E}_A$, can each $x \in A$ with $0 \le x \le 1$ be expressed as $\mathbb{E}_A(p)$ for a projection $p \in M$?

In this paper, we investigate these problems for various masas. We give positive solutions for the generator and radial masas in free group factors, and we also solve affirmatively a weaker form of the Schur–Horm problem for the Cartan masa in the hyperfinite factor.

1. Introduction

Two classical theorems due to Schur [17] and Horn [9], which relate the diagonal entries of an $n \times n$ self-adjoint matrix to its eigenvalues, have recently been reformulated in the setting of type II_1 factors M with normalized trace τ [3]. A special case of the problem, termed the carpenter problem in [10], [11], asks whether each element x in a masa $A \subseteq M$ satisfying $0 \le x \le 1$ can be expressed as $\mathbb{E}_A(p)$ for some projection $p \in M$. This entails $\tau(x) = \tau(p)$,

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so the analogous problem in complex matrix algebras places a constraint on the value of $\tau(x)$. Subject to this, Horn's theorem gives a positive solution for matrices.

The II_1 factor analogue of the diagonal subalgebra in the $n \times n$ matrices is a maximal Abelian (self-adjoint) subalgebra, called a masa, $A \subseteq M$. We let $\mathbb{E}_A$ denote the trace-preserving conditional expectation of M onto A. The carpenter problem in a II_1 factor is, given $x \in A$ with $0 \le x \le 1$, to find a projection $p \in M$ so that $\mathbb{E}_A(p) = x$; this problem remains open.

The Schur–Horn problem for a masa $A \subseteq M$ may be stated as follows: for a suitable notion of spectral majorization of $x \in A$ by $z \in M$ (described in Section 5), does there exist an element $y \in M$ having the same spectral distribution as z so that $x = \mathbb{E}_A(y)$? In this paper, we address these two questions for specific choices of masas. We give positive solutions to both the carpenter problem and the Schur–Horn problem when A is either a generator masa or the radial masa in a free group factor. We also investigate the Cartan masa in the hyperfinite factor, and obtain a version of the Schur–Horn theorem which is slightly weaker than the one above.

The paper is organized as follows. In Section 2, we present a technical result giving a sufficient condition for positive solutions of the carpenter problem (Lemma 2.1), and all of our subsequent results are based on this. The main results on masas in free group factors are contained in Section 3, while Section 4 is concerned with the carpenter problem for the Cartan masa in the hyperfinite factor. Here our results are less definitive, although we do present classes of elements in A for which a positive solution can be given. In a different direction, we also solve the carpenter problem for all elements of the Cartan masa A, but modulo an automorphism of A.

In the final section, we consider the Schur–Horn problem. We first consider a minor reformulation of Arveson and Kadison's version of the problem and show that it is equivalent to theirs. Then we give a positive solution for the generator masa and the radial masa in free group factors. We also investigate the Cartan masa, proving a weaker version of the Schur–Horn problem as mentioned above.

There has been considerable recent interest in these problems, and we have drawn heavily on the ideas and results presented in [1], [2], [3], [10], [11].

2. An existence method

In the first lemma below, we will describe a sufficient condition for solving the carpenter problem positively, and in subsequent sections we will apply it in various situations.

We fix a finite von Neumann algebra M with a normal normalized trace τ and a masa $A \subseteq M$. We denote the unique trace preserving conditional expectation of M onto A by $\mathbb{E}_A$. For each $x \in A$ satisfying $0 \le x \le 1$, we introduce the w^*-compact convex subset $\Gamma_x \subseteq M$, defined by

$$\Gamma_x = \{y \in M : 0 \le y \le 1, \mathbb{E}_A(y) = x\}. \tag{2.1}$$

This set is nonempty since it contains x, and any projection $p \in \Gamma_x$ is a solution of the carpenter problem for the element $x \in A$. Any such projection is automatically an extreme point of Γ_x, and so it suffices to consider the extreme points of Γ_x. These are abundant, by the Krein–Milman theorem.

For each nonzero projection $e \in M$, define a bounded map $\Phi_e : eMe \to A$ by

$$\Phi_e(exe) = \mathbb{E}_A(exe), \quad x \in M. \tag{2.2}$$

LEMMA 2.1. *Let A be a masa in a finite von Neumann algebra M, and suppose that Φ_e is not injective for each nonzero projection $e \in M$. Given $x \in A$ satisfying $0 \le x \le 1$, there exists a projection $p \in M$ such that $\mathbb{E}_A(p) = x$.*

Proof. Fix an arbitrary $x \in A$ satisfying $0 \le x \le 1$. Under the stated hypotheses, we will show that every extreme point of Γ_x is a projection and the result then follows. To obtain a contradiction, let y be an extreme point of Γ_x which is not a projection. For a sufficiently small choice of $\varepsilon > 0$, the spectral projection e of y for the interval $(\varepsilon, 1-\varepsilon)$ is nonzero. Since Φ_e is not injective we may choose a nonzero element $z \in eMe$ so that $\Phi_e(z) = 0$. By considering real and imaginary parts we may take z to be self-adjoint, and by scaling we may assume that $\|z\| \le \varepsilon$. Note that $\mathbb{E}_A(y \pm z) = \mathbb{E}_A(y) = x$. Since $\varepsilon e \le ye \le (1-\varepsilon)e$, it follows that $0 \le y \pm z \le 1$, and so $y \pm z \in \Gamma_x$ with $y = ((y+z)+(y-z))/2$. This contradicts the assumption that y is an extreme point, showing that every extreme point is a projection. □

The following proposition illustrates the use of Lemma 2.1.

PROPOSITION 2.2. *If M is a II_1 factor that is nonseparable as a von Neumann algebra and if $A \subseteq M$ is a masa that is separable as a von Neumann algebra, then given any $x \in A$ with $0 \le x \le 1$, there is a projection $p \in M$ such that $\mathbb{E}_A(p) = x$.*

Proof. By cardinality considerations, Φ_e must have nontrivial kernel for each nonzero projection $e \in M$. Indeed, since Φ_e is normal, its Banach space dual map restricted to the predual A_* of A yields a continuous linear map $(\Phi_e)_* : A_* \to M_*$, given by

$$\langle (\Phi_e)_*(\psi), x \rangle = \langle \psi, \Phi_e(x) \rangle, \tag{2.3}$$

where $\psi \in A_*$, $x \in M$ and $\langle \cdot, \cdot \rangle$ denotes pairings of von Neumann algebras and their preduals. But $(\Phi_e)_*(A_*)$ is a separable subspace of the nonseparable space M_*, so there must be $x \in M$ such that $\langle (\Phi_e)_*(\psi), x \rangle = 0$ for all $\psi \in A_*$, while $x \ne 0$. This, together with (2.3) implies $\Phi_e(x) = 0$.

Now the desired conclusion follows from Lemma 2.1. □

For example, we can now show that the carpenter problem has a positive solution for any masa in a free group factor with an uncountable number of generators.

THEOREM 2.3. *Let S be an uncountable set and let $\mathbb{F}_S$ be the free group on a set of generators indexed by S. If A is a masa in $L(\mathbb{F}_S)$ and $x \in A$ satisfies $0 \leq x \leq 1$, then there exists a projection $p \in L(\mathbb{F}_S)$ such that $\mathbb{E}_A(p) = x$.*

Proof. From [15], any masa A in $L(\mathbb{F}_S)$ is separable as a von Neumann algebra, while $L(\mathbb{F}_S)$ itself is not. Now Proposition 2.2 applies. $\square$

REMARK 2.4. (i) The maps Φ_e introduced above are normal and so have preduals. It is an easy calculation to see that $(\Phi_e)_* : L^1(A) \to L^1(eMe)$ is given by $(\Phi_e)_*(a) = eae$, $a \in A$, and extended by continuity to $L^1(A)$. It then follows that noninjectivity of Φ_e is equivalent to the failure of eAe to be $\|\cdot\|_1$-dense in eMe, a potentially useful reformulation.

(ii) In the case of type II_1 factors, we have no example of a nonzero projection e for which Φ_e is injective. However, this can occur for type I factors. Take A to be the diagonal masa in $B(H)$ and let $e \in A$ be a rank one projection. Then $eB(H)e = eAe$ and Φ_e is injective in this case.

(iii) If $e \in M$ is a projection such that $e\{e, A\}''e \neq eMe$, then the map Φ_e is not injective. To see this, let $N = \{e, A\}''$ and observe that the condition $eNe \neq eMe$ gives a nonzero element $exe \in eMe$ so that $\mathbb{E}_{eNe}(exe) = 0$. Then

$$\mathbb{E}_A(exe) = \mathbb{E}_A\big(\mathbb{E}_N(exe)\big) = \mathbb{E}_A\big(\mathbb{E}_{eNe}(exe)\big) = 0 \tag{2.4}$$

and Φ_e is not injective.

The third part of this remark leads to a connection with another open problem, the question of whether separable von Neumann algebras must be singly generated.

LEMMA 2.5. *Let M be a type II_1 factor and let A be a separable masa. If there exists a nonzero projection $e \in M$ such that Φ_e is injective, then M is singly generated.*

Proof. Let $N = \{A, e\}''$ and let z be the central support of e in N. Then z is the identity element for the w^*-closed ideal $\overline{NeN}^{w^*}$ in N. By Remark 2.4(iii), the injectivity of Φ_e implies that $eNe = eMe$. For any $m \in M$,

$$zmz \in \overline{NeNmNeN}^{w^*} \subseteq \overline{NeMeN}^{w^*} = \overline{NeNeN}^{w^*} = Nz, \tag{2.5}$$

showing that $zMz \subseteq zNz$. The reverse containment is obvious and so $zMz = Nz$. Since $z \in A$, this gives $zMz = \{Az, e\}''$, so the separability of A implies that zMz is generated by two self-adjoint elements x_1 and x_2. By adding a multiple of z and scaling, we may assume that $0 \leq x_1 \leq z$.

Since M is a finite factor, we can find projections $z_2, \dots, z_n \in M$ which are all equivalent to subprojections of z and such that $z + \sum_{i=2}^n z_i = 1$. Then choose partial isometries $v_2,\dots,v_n \in M$ so that $v_i^* v_i = z_i$ and $v_i v_i^* \le z$ for $2 \le i \le n$, and define

$$y_1 = x_1 + 2z_2 + \cdots + nz_n, \qquad y_2 = x_2 + v_2 + v_2^* + \cdots + v_n + v_n^*. \tag{2.6}$$

By construction, $z_2,\dots,z_n$ are spectral projections of y_1 and so lie in $\{y_1,y_2\}''$, showing that this algebra also contains x_1. Since $y_2 z_i = v_i$, $2 \le i \le n$, we see that $\{y_1,y_2\}''$ also contains $v_2,\dots,v_n$ and e, so in particular $zMz \subseteq \{y_1,y_2\}''$. Now $v_i^* z v_i = z_i$, and so $z_i M z_j \subseteq \{y_1,y_2\}''$, showing that $M = \{y_1,y_2\}''$. Thus, M is singly generated by $y_1 + iy_2$. □

It is currently unknown whether all separable type II_1 factors must be singly generated. Lemmas 2.1 and 2.5 show that any example of a separable II_1 factor that fails to be singly generated would have a positive solution to the carpenter problem for any masa A.

We conclude this section by presenting a class of masas for which the carpenter problem has a positive solution. We will need a preliminary lemma which gives a norm density result.

Lemma 2.6. *Let M be a separable type II_1 factor and let A be a masa in M.*

(i) *If $r \in \mathbb{Q} \cap [0,1]$, then there exists a projection $p \in M$ so that $\mathbb{E}_A(p) = r1$.*
(ii) *Given $\varepsilon > 0$ and $x \in A$ satisfying $0 \le x \le 1$, there exists a projection $p \in M$ such that $\|x - \mathbb{E}_A(p)\| < \varepsilon$.*

Proof. (i) The cases $r = 0$ and $r = 1$ are trivial so we may assume that $r = k/n$ where $1 \le k \le n-1$ for integers k,n. In A, choose n orthogonal projections $e_{11},\dots,e_{nn}$ of trace $1/n$ and choose a matrix algebra $\mathbb{M}_n \subseteq M$ with diagonal $\mathbb{D}_n$ so that the e_{ii}'s are the minimal diagonal projections. Since the e_{ii}'s lie in $\mathbb{D}_n$, the two conditional expectations $\mathbb{E}_A$ and $\mathbb{E}_{\mathbb{D}_n}$ agree on $\mathbb{M}_n$. From [9], there is a projection $p \in \mathbb{M}_n \subseteq M$ so that $\mathbb{E}_{\mathbb{D}_n}(p) = (k/n)I_n$, and so $\mathbb{E}_A(p) = (k/n)1 \in A$.

(ii) Now consider a fixed but arbitrary $x \in A$ satisfying $0 \le x \le 1$ and let $\varepsilon > 0$ be given. Since A is separable we may identify A with $L^\infty[0,1]$ and then we may choose projections e_k, $1 \le k \le n$, summing to 1, corresponding to disjoint measurable subsets of $[0,1]$, and constants $\lambda_k \in [0,1]$ so that

$$\left\| x - \sum_{k=1}^{n} \lambda_k e_k \right\|_\infty < \varepsilon. \tag{2.7}$$

A further approximation allows us to assume that each λ_k is rational in $[0,1]$. Applying (i) to the containment $Ae_k \subseteq e_k M e_k$, we find projections $p_k \le e_k$, $1 \le k \le n$, so that $\mathbb{E}_{Ae_k}(p_k) = \lambda_k e_k$. If we define a projection by $p = \sum_{k=1}^n p_k$, then $\mathbb{E}_A(p) = \sum_{k=1}^n \lambda_k e_k$ and $\|x - \mathbb{E}_A(p)\| < \varepsilon$ as required. □

THEOREM 2.7. *Let A be a masa in a type* II_1 *factor M and let ω be a free ultrafilter on* $\mathbb{N}$. *Then the carpenter problem has a positive solution for the masa $A^\omega \subseteq M^\omega$.*

Proof. Let $x \in A^\omega$ satisfy $0 \le x \le 1$ and choose a representative $(x_1, x_2, \ldots)$ for x where $x_n \in A$ and $0 \le x_n \le 1 - 1/n$. By Lemma 2.6, there exist elements $y_n \in A$, $0 \le y_n \le 1$, and projections $p_n \in M$ such that $\|x_n - y_n\| < \frac{1}{n}$ and $\mathbb{E}_A(p_n) = y_n$. Then $(y_1, y_2, \ldots)$ is also a representative for x, $p = (p_1, p_2, \ldots)$ is a projection in M^ω, and it follows that $\mathbb{E}_{A^\omega}(p) = (\mathbb{E}_A(p_1), \mathbb{E}_A(p_2), \ldots) = x$. □

3. Free group factors

In this section, we consider the carpenter problem in free group factors. Let $\mathbb{F}_n$ denote the free group on n generators $\{g_1, \ldots, g_n\}$, $2 \le n < \infty$. There are types of masas in the free group factor $L(\mathbb{F}_n)$ that have been much studied. Each g_i generates a masa A_i, called a generator masa. The second type is the radial or Laplacian masa, whose generator is the self-adjoint element $\sum_{i=1}^{n}(g_i + g_i^{-1})$. We consider first the generator masa.

THEOREM 3.1. *Let $g_1, \ldots, g_n$ be the generators for $\mathbb{F}_n$, $2 \le n \le \infty$, and let A_i be the ith generator masa, where i is fixed. Given $x \in A_i$, $0 \le x \le 1$, there exists a projection $p \in L(\mathbb{F}_n)$ such that $\mathbb{E}_{A_i}(p) = x$.*

Proof. We first consider the case $n = 2$, and without loss of generality we take $i = 1$. Let S_0 be an uncountable set and let $S = \{1, 2\} \cup S_0$. Then the free group factor $L(\mathbb{F}_S)$ with generators g_1, g_2, and g_s for $s \in S_0$ contains A_1 as a masa. By Theorem 2.3, there is a projection $q \in L(\mathbb{F}_S)$ such that $\mathbb{E}_{A_1}(q) = x$. The underlying Hilbert space $L^2(L(\mathbb{F}_S))$ has an orthonormal basis of group elements and the Fourier series for q can only have countably many nonzero terms. Thus there is a countable subset $T \subseteq S$, whose elements we list as $t_1, t_2, \ldots$ with $t_1 = 1$, so that $q \in L(\mathbb{F}_T)$. Define an embedding $\phi : L(\mathbb{F}_T) \to L(\mathbb{F}_2)$ on generators by $\phi(g_{t_i}) = g_2^{i-1} g_1 g_2^{1-i}$, $i \ge 1$. Then ϕ is the identity on A_1, and $\mathbb{E}_{A_1}(\phi(q)) = x$. The desired projection is then $p = \phi(q)$.

For the general case, choose an integer $j \neq i$. Then $A_i \subseteq L(\{g_i, g_j\}) \cong L(\mathbb{F}_2) \subseteq L(\mathbb{F}_n)$, and the result follows from above since the desired projection can be chosen from $L(\mathbb{F}_2)$. □

For the notion of freeness that is used below, see [19] or [20].

COROLLARY 3.2. *In a type* II_1 *factor M with tracial state τ, if $A \subseteq M$ is a masa and if $s \in M$ is a symmetry with $\tau(s) = 0$ and such that A and $\{s\}$ are free with respect to τ, then for every $x \in A$ satisfying $0 \le x \le 1$, there exists a projection $p \in M$ so that $\mathbb{E}_A(p) = x$.*

Proof. Since A and sAs are free and together generate a copy of $L(\mathbb{F}_2)$, this follows from Theorem 3.1. □

REMARK 3.3. Now it is clear that if a masa $A \subseteq L(\mathbb{F}_n)$ is supported on at most $n-1$ generators, then the carpenter problem for A has a positive solution.

We now consider the radial masa B in $L(\mathbb{F}_n)$ for $2 \leq n < \infty$.

THEOREM 3.4. *Let B be the radial masa in $L(\mathbb{F}_n)$ for a fixed n in the range $2 \leq n < \infty$. Given $x \in B$, $0 \leq x \leq 1$, there exists a projection $p \in L(\mathbb{F}_n)$ so that $\mathbb{E}_B(p) = x$.*

Proof. Let $g_1, \dots, g_n$ be the generators of $\mathbb{F}_n$ and let A_i be the ith generator masa. For each i, let h_i be $g_i + g_i^{-1}$ and let $L_i \subseteq A_i$ be the Abelian von Neumann algebra generated by h_i. We denote by L the von Neumann algebra generated by $\{L_i : 1 \leq i \leq n\}$ which can be regarded as the free product $L_1 * L_2 * \cdots * L_n$.

If we identify A_i with $L^\infty[-1,1]$, then L_i is the subalgebra of even functions. Let $v_i \in A_i$ be the self-adjoint unitary corresponding to the odd function $1 - 2\chi_{[0,1]}$. For each $f \in L_i$, fv_i is an odd function and so has trace 0. We now wish to show that the algebras v_1Lv_1, $v_2Lv_2, \dots, v_nLv_n$ are free.

Recall that the centered elements of a type II_1 factor N are $\overset{\circ}{N} = \{y \in N : \tau(y) = 0\}$. In order to show freeness, of the algebras v_iLv_i, it suffices to show that the trace vanishes on finite products of the form

$$v_{i_1}y_1v_{i_1}v_{i_2}y_2v_{i_2}\cdots v_{i_k}y_kv_{i_k}, \tag{3.1}$$

where each $y_i \in \overset{\circ}{L}$ and $i_j \neq i_{j+1}$ for $1 \leq j \leq k-1$. Products of the form $z_{r_1}z_{r_2}\cdots z_{r_s}$ with $z_{r_i} \in \overset{\circ}{L}_{r_i}$, $r_i \neq r_{i=1}$, span a weakly dense subspace of L so we may assume that each y_i has this form. Consider

$$v_{i_1}y_1v_{i_1} = v_{i_1}z_{r_1}\cdots z_{r_s}v_{i_1}. \tag{3.2}$$

A cancellation is only possible if $z_{r_1} \in L_{i_1} \subseteq A_{i_1}$ or $z_{r_s} \in L_{i_1} \subseteq A_{i_1}$. In the first case, $v_{i_1}z_{r_1}$ is an element of A_{i_1} and so corresponds to an odd function on $[-1,1]$. Thus $v_{i_1}z_{r_1} \in \overset{\circ}{A}_{i_1}$, and similarly, if $z_{r_s} \in L_{i_1}$, then $z_{r_s}v_{i_1} \in \overset{\circ}{A}_{i_1}$. Analyzing in the same way the behavior when each v_{i_j} is adjacent to a y_j leads to the conclusion that the element of (3.1) has trace 0. Thus the algebras $v_1Lv_1, \dots, v_nLv_n$ are free, implying that $v_1Bv_1, \dots, v_nBv_n$ are free. Thus B and $v_1v_2Bv_2v_1$ are free subalgebras of $L(\mathbb{F}_n)$ and Corollary 3.2 finishes the proof. ☐

4. Crossed products and tensor products

One of the most important masas is the Cartan masa A in the hyperfinite II_1 factor R. From [5], it is unique up to isomorphisms of R. While the carpenter problem is open in this case, significant progress has been made in

[1], [2]. In this section, we display classes of elements in A for which a positive solution can be given.

There are many ways of constructing the hyperfinite factor R. One that we will employ below is to let $\mathbb{Z}$ act on $L^\infty(\mathbb{T})$ by irrational rotation, whereupon the crossed product $L^\infty(\mathbb{T}) \rtimes \mathbb{Z}$ is isomorphic to R. In keeping with our earlier techniques, we will enlarge the crossed product and exploit the nonseparability of the resulting algebra.

As a vector space over the field of rationals $\mathbb{Q}$, the real field $\mathbb{R}$ has an uncountable Hamel basis $\{\theta_\alpha : \alpha \in S\}$, where S is an uncountable index set. For integers $n_1, \ldots, n_{k+1}$, the equation $\sum_{i=1}^k n_i\theta_{\alpha_i} = n_{k+1}$ can only be satisfied by taking all the n_i's to be 0. Then the group G, defined to be the set of all finite sums $\{n_1\theta_{\alpha_1} + \cdots + n_k\theta_{\alpha_k} : n_i \in \mathbb{Z}, \alpha_i \in S\}$ under addition, can be expressed as $\sum_{\alpha\in S} G_\alpha$, where $G_\alpha = \{n\theta_\alpha : n \in \mathbb{Z}\} \cong \mathbb{Z}$. The group G acts on $L^\infty(\mathbb{T})$ by irrational rotation, and the crossed product $L^\infty(\mathbb{T}) \rtimes G$ is a type II_1 factor and so has a faithful trace.

Theorem 4.1. *Let R be the separable hyperfinite* II_1 *factor and let A be the Cartan masa in R. Given $x \in A$, $0 \le x \le 1$, there exists a trace preserving automorphism ϕ of A and a projection $p \in R$ such that $\mathbb{E}_A(p) = \phi(x)$.*

Proof. Fix $x \in A$, $0 \le x \le 1$, and let $\phi_1 : A \to L^\infty(\mathbb{T})$ be an isomorphism that takes the trace on A to integration by Lebesgue measure on $\mathbb{T}$. The algebra $L^\infty(\mathbb{T})$ is a separable masa in the nonseparable factor $L^\infty(\mathbb{T}) \rtimes G$, and so by Proposition 2.2 there is a projection $q \in L^\infty(\mathbb{T}) \rtimes G$ so that $\mathbb{E}_{\phi_1(A)}(q) = \phi_1(x)$. Elements of $L^\infty(\mathbb{T}) \rtimes G$ have Fourier series $\sum_{g\in G} a_g g$ for $a_g \in L^\infty(\mathbb{T})$, only a countable number of whose terms are nonzero. Thus, there is a countable subgroup H of G so that $q \in L^\infty(\mathbb{T}) \rtimes H \subseteq L^\infty(\mathbb{T}) \rtimes G$. Now (after enlarging H if needed to contain an irrational element) $L^\infty(\mathbb{T}) \rtimes H$ is a copy of R, and so the uniqueness of Cartan subalgebras in R gives an isomorphism $\phi_2 : L^\infty(\mathbb{T}) \rtimes H \to R$ so that $\phi_2(L^\infty(\mathbb{T})) = A$, and note that ϕ_2 is trace preserving. Then

$$\mathbb{E}_A\big(\phi_2(q)\big) = \phi_2\big(\mathbb{E}_{L^\infty(\mathbb{T})}(q)\big) = \phi_2\phi_1(x). \tag{4.1}$$

Set $p = \phi_2(q)$ and $\phi = \phi_2\phi_1$ to conclude that $\mathbb{E}_A(p) = \phi(x)$. □

Remark 4.2. (i) If x is taken to be $\lambda 1$ for any $\lambda \in [0,1]$, then there exists a projection $p \in R$ such that $\mathbb{E}_A(p) = \phi(\lambda 1) = \lambda 1$.

(ii) In Theorem 4.1, an identical proof gives a more general result: given $\{x_i\}_{i=1}^\infty \in A$, $0 \le x_i \le 1$, there exists an automorphism ϕ of A and projections $p_i \in R$ so that $\mathbb{E}_A(p_i) = \phi(x_i)$, $i \ge 1$.

If A is the Cartan masa in R then, by uniqueness, the inclusions $A \subseteq R$ and $A \overline{\otimes} A \subseteq R \overline{\otimes} R$ are equivalent. In the latter formulation we now obtain a large class of elements for which we can solve the carpenter problem.

THEOREM 4.3. *Let A be the Cartan masa in the hyperfinite factor R and let $x \in A$, $0 \le x \le 1$. Then there exists a projection $p \in R \overline{\otimes} R$ so that $\mathbb{E}_{A \overline{\otimes} A}(p) = x \otimes 1$.*

Proof. Let S be an uncountable set, and for each $\alpha \in S$ let A_α be a copy of A inside R_α, a copy of R. Form $M = R \overline{\otimes} (\overline{\bigotimes}_{\alpha\in S} R_\alpha)$, and denote the Abelian subalgebra $A \overline{\otimes} (\overline{\bigotimes}_{\alpha\in S} A_\alpha)$ by N. We identify R with $R \otimes 1 \subseteq M$.

Let $e \in M$ be a nonzero projection. Then there exists a countable subset $T \subseteq S$ such that e lies in $R \overline{\otimes} (\overline{\bigotimes}_{\alpha\in T} R_\alpha)$. Then

$$eMe = e\left(R \overline{\otimes} \left(\overline{\bigotimes_{\alpha\in T}} R_\alpha\right)\right) e \overline{\otimes} \left(\overline{\bigotimes_{\alpha\in S\setminus T}} R_\alpha\right) \tag{4.2}$$

and

$$e\{N, e\}'' e = e\left(\left\{ A \overline{\otimes} \left(\overline{\bigotimes_{\alpha\in T}} A_\alpha\right), e\right\}''\right) e \overline{\otimes} \left(\overline{\bigotimes_{\alpha\in S\setminus T}} A_\alpha\right). \tag{4.3}$$

Thus $e\{N, e\}'' e \ne eMe$, and we can find a nonzero element $z \in eMe$ such that $\mathbb{E}_{e\{N,e\}''e}(z) = 0$. It follows that $\mathbb{E}_N(z) = 0$, so Lemma 2.1 applies to give a projection $q \in M$ such that $\mathbb{E}_N(q) = x \otimes 1$. Now q is supported by $R \overline{\otimes} (\overline{\bigotimes}_{\alpha\in S_2} R_\alpha)$ for a countable subset S_2 of S. Then there is an isomorphism $\phi : R \to \overline{\bigotimes}_{\alpha\in S_2} R_\alpha$ which maps A to $\overline{\bigotimes}_{\alpha\in S_2} A_\alpha$, and $\theta = 1 \otimes \phi : R \overline{\otimes} R \to R \otimes (\overline{\bigotimes}_{\alpha\in S_2} R_\alpha)$ is also an isomorphism. If we define a projection $p \in R \overline{\otimes} R$ by $p = \theta^{-1}(q)$, then it follows that $\mathbb{E}_{A \overline{\otimes} A}(p) = x \otimes 1$ as required. $\square$

5. The Schur–Horn theorem

Let A be a self-adjoint $n \times n$ matrix, let $\alpha_1 \ge \alpha_2 \ge \cdots \ge \alpha_n$ be a decreasing rearrangement of the diagonal entries and let $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$ be a decreasing ordering of the eigenvalues. A classical theorem of Schur [17] states that

$$\sum_{i=1}^{k} \alpha_i \le \sum_{i=1}^{k} \lambda_i, \quad 1 \le k \le n, \tag{5.1}$$

with equality when $k = n$. These inequalities can be used to define a partial ordering on general n-tuples of real numbers by $\alpha \preceq \lambda$ if (5.1) holds for the decreasing rearrangements of the entries, with equality when $k = n$. A converse to Schur's theorem was proved by Horn in [9]: if two n-tuples α and λ satisfy $\alpha \preceq \lambda$ then there is a self-adjoint matrix A so that the diagonal is α and the eigenvalues are the entries of λ. Collectively, these two results are known as the Schur–Horn theorem. If we denote by $\mathbb{E}_{\mathbb{D}_n}$ the conditional expectation of $\mathbb{M}_n$ onto the diagonal $\mathbb{D}_n$, then there is an equivalent reformulation of the Schur–Horn theorem as follows (see [10], [11]). If α and λ are n-tuples of real numbers and D_α is the diagonal matrix with entries from α, then $\alpha \preceq \lambda$ if and only if there exists a unitary matrix $U \in \mathbb{M}_n$ so that

$$\mathbb{E}_{\mathbb{D}_n}\big(UD_\lambda U^*\big) = D_\alpha. \tag{5.2}$$

When λ has entries that are 0 or 1, then D_λ is a projection, and (5.2) reduces to a solution of the carpenter problem for the masa $\mathbb{D}_n$. An appropriate formulation of the Schur–Horn theorem for type II_1 factors M with a normalized trace τ was given by Arveson and Kadison in [3] (see also the work of Hiai [7], [8]), as we now describe. For each self-adjoint $a \in M$, the distribution of a is the unique Borel probability measure m_a on $\mathbb{R}$ so that

$$\int_{\mathbb{R}} t^n \, dm_a(t) = \tau\big(a^n\big), \quad n = 0, 1, 2, \ldots. \tag{5.3}$$

To each Borel subset B of $\mathbb{R}$, there corresponds a spectral projection e_B of a, and it follows from (5.3) that $m_a(B) = \tau(e_B)$. Moreover, m_a is supported on the spectrum $\sigma(a)$ of a, and is called the spectral distribution of a. Following [3, Definition 6.2], we say that a compactly supported probability measure n on $\mathbb{R}$ dominates a similar probability measure m on $\mathbb{R}$ if

$$\begin{gathered} \int_{\mathbb{R}} t \, dm(t) = \int_{\mathbb{R}} t \, dn(t) \quad \text{and} \\ \int_t^\infty m\big((s,\infty)\big)\, ds \le \int_t^\infty n\big((s,\infty)\big)\, ds, \quad t \in \mathbb{R}, \end{gathered} \tag{5.4}$$

where $\int \cdots ds$ indicates integration with respect to Lebesgue measure. We deviate slightly from [3] which uses closed intervals $[s,\infty)$, but this makes no difference since $m((s,\infty))$ and $m([s,\infty))$ can only be unequal on a countable set of s-values. The relation $\preceq$ on the set $M_{\mathrm{s.a.}}$ of self-adjoint elements in M is defined by $a \preceq b$ if and only if $m_a \preceq m_b$.

This relation $a \preceq b$ can be rewritten to resemble more closely the classical condition (5.1) for matrices. Instead of the eigenvalue sequence of a matrix, for $a = a^* \in M_{\mathrm{s.a.}}$, we have the eigenvalue function, which is the real-valued, monotone nonincreasing, right-continuous function

$$\mu_t(a) = \inf\big\{ s \in \mathbb{R} : m_a\big((s,\infty)\big) \le t \big\}$$

of $t \in [0,1)$. This is the unique real-valued, nonincreasing, right-continuous function so that we have $a = \int_0^1 \mu_t(a)\, dE(t)$ for some projection-valued measure E on $[0,1)$ such that $\tau(E([0,t))) = t$ (the actual measure E is obtained by reparameterizing e_a). This eigenvalue function $\mu_t(a)$ is analogous to the decreasing eigenvalue sequence, and we have, for example,

$$\tau\big(a^k\big) = \int_0^1 \big(\mu_t(a)\big)^k \, dt \quad (k \ge 1).$$

It is, after a change of variable, the function defined by Murray and von Neumann [13, Lemma 15.2.1] and used in various forms by several authors (e.g., [12], [14], [4]). In terms of eigenvalue functions, the relation $a \preceq b$ is characterized by the inequalities

$$\int_0^t \mu_s(a)\, ds \le \int_0^t \mu_s(b)\, ds$$

for all $0 \le t \le 1$ with equality at $t = 1$. This follows by combining Theorem 2.1 of [1] (which comes from [7]) with Proposition 6.1 of [3].

The analog of Schur's theorem was established in [3, Theorem 7.2].

THEOREM 5.1. *If A is a masa in a type* II_1 *factor M, then $\mathbb{E}_A(x) \preceq x$ for all self-adjoint elements $x \in M$.*

Let $\mathcal{O}(x)$ denote the norm closure of the unitary orbit of a self-adjoint $x \in M$. Then $y \in \mathcal{O}(x)$ if and only if x and y have the same spectral data, i.e., $\mu_t(x) = \mu_t(y)$ for all $t \in [0,1)$ or, equivalently $m_x = m_y$. This was shown by Kamei [12], and also in [3].

The analog of Horn's theorem is then the following problem. If A is a masa in a type II_1 factor M and $x \in M_{\text{s.a.}}$ and $y \in A_{\text{s.a.}}$ satisfy $y \preceq x$, does y lie in $\mathbb{E}_A(\mathcal{O}(x))$? In attempting to answer this question, it is unchanged by adding multiples of the identity to x and y, and so it suffices to assume that $x, y \ge 0$. For $x \in M^+$, the eigenvalue function's values $\mu_t(x)$ are actually the generalized s-numbers of [13] (see the account in [6]), which are defined for $x \in M$ and $t \ge 0$ by

$$\mu_t(x) = \inf\bigl\{\|xe\| : e \in \mathcal{P}(M), \tau(e) \ge 1 - t\bigr\}, \tag{5.5}$$

where $\mathcal{P}(M)$ denotes the set of projections in M. This was established in [6, Proposition 2.2]. In particular, we have $\mu_0(x) = \|x\|$.

For $x \in M^+$ we will need the distribution function $\lambda_t(x)$, defined in [6, Definition 1.3] to be

$$\lambda_t(x) = m_x(t, \infty) = \tau\bigl(e_{(t,\infty)}(x)\bigr), \tag{5.6}$$

so that we have

$$\mu_t(x) = \inf\bigl\{s \ge 0 : \lambda_s(x) \le t\bigr\}. \tag{5.7}$$

Finally, we will need the Ky Fan norms

$$\|x\|_{(t)} = \int_0^t \mu_s(x)\, ds, \quad 0 \le t \le 1. \tag{5.8}$$

It is not obvious that these are norms for $t > 0$, but this fact is established in [6, Theorem 4.4(ii)].

Since $x \in \mathcal{O}(y)$ means that the spectral data of x and y agree, we have the following lemma.

LEMMA 5.2. *Let M be a type* II_1 *factor. Then the following are equivalent for elements $x, y \in M^+$:*

(i) $x \in \mathcal{O}(y)$.

(ii) $\|x\|_{(t)} = \|y\|_{(t)}, 0 \le t \le 1$.

We now use this to investigate the Schur–Horn theorem.

Let A be a masa in a type II_1 factor M, let $x \in A^+$, $0 \le x \le 1$, and $z \in M^+$ be elements such that $x \preceq z$. Then define

$$\Delta_{x,z} = \{y \in M : 0 \le y \le 1, y \preceq z, \mathbb{E}_A(y) = x\}, \tag{5.9}$$

which is nonempty since it contains x. From the Ky Fan norm characterization above, $\Delta_{x,z}$ is convex and it is w^*-compact from [1, Corollary 3.5]. By the Krein–Milman theorem, $\Delta_{x,z}$ has extreme points. Recall the definition of Φ_e from (2.2).

THEOREM 5.3. *Let A be a masa in a type II_1 factor M and suppose that Φ_e is noninjective for each nonzero projection $e \in M$. Let $x \in A$, $0 \le x \le 1$ and suppose that $x \preceq z$ for some element $z \in M^+$. Then every extreme point of $\Delta_{x,z}$ lies in $\mathcal{O}(z)$.*

Proof. Fix an extreme point b of $\Delta_{x,z}$. Then $\|b\|_{(t)} \le \|z\|_{(t)}$ for $0 \le t \le 1$ since $b \preceq z$. To derive a contradiction, suppose that there exists t so that $\|b\|_{(t)} < \|z\|_{(t)}$, for otherwise Lemma 5.2 gives the result. Since $\|b\|_{(0)} = \|z\|_{(0)} = 0$, and $\|b\|_{(1)} = \|z\|_{(1)} = \tau(b)$, this value of t lies in $(0,1)$. The function $\|z\|_{(t)} - \|b\|_{(t)}$ is continuous on $[0,1]$ and so attains its maximum value on a closed nonempty subset $\Lambda \subseteq [0,1]$. Let t_0 be the least value in Λ, and let $t_1 \in \Lambda$ be the largest value for which $[t_0, t_1] \subseteq \Lambda$. Since $0, 1 \notin \Lambda$, we have $0 < t_0 \le t_1 < 1$, and it is possible to have $t_0 = t_1$. By continuity, there exist $\delta_0 > 0$ and $\varepsilon < \min\{t_0, 1 - t_1\}$ so that

$$\|z\|_{(t)} - \|b\|_{(t)} > \delta_0 \quad \big(t \in [t_0 - \varepsilon, t_1 + \varepsilon]\big). \tag{5.10}$$

On $(t_0 - \varepsilon, t_0)$, the inequality $\mu_t(z) \le \mu_t(b)$ cannot hold everywhere because we would then have $\|z\|_{(t_0-\varepsilon)} - \|b\|_{(t_0-\varepsilon)} \ge \|z\|_{(t_0)} - \|b\|_{(b_0)}$, implying $t_0 - \varepsilon \in \Lambda$ and contradicting the minimal choice of t_0. Thus, there exist $\delta_1 > 0$ and $\varepsilon_0 \in (0, \varepsilon)$ so that

$$\mu_{t_0 - \varepsilon_0}(z) \ge \mu_{t_0 - \varepsilon_0}(b) + \delta_1. \tag{5.11}$$

Similarly, if we had $\mu_t(z) \ge \mu_t(b)$ for all $t \in (t_1, t_1 + \varepsilon)$ then it would follow that $[t_0, t_1 + \varepsilon] \subseteq \Lambda$, contradicting the maximal choice of t_1. Thus, there exists $\varepsilon_1 \in (0, \varepsilon)$ so that $\mu_{t_1+\varepsilon_1}(z) \le \mu_{t_1+\varepsilon_1}(b) - \delta_2$ for some $\delta_2 > 0$. Clearly, the values of δ_0, δ_1 and δ_2 can be replaced by their minimum value which we denote by $\delta > 0$. Thus, we have the inequalities

$$\|z\|_{(t)} - \|b\|_{(t)} > \delta, \quad t \in [t_0 - \varepsilon, t_1 + \varepsilon] \tag{5.12}$$

and

$$\mu_{t_0-\varepsilon_0}(z) \ge \mu_{t_0-\varepsilon_0}(b) + \delta, \qquad \mu_{t_1+\varepsilon_1}(z) \le \mu_{t_1+\varepsilon_1}(b) - \delta. \tag{5.13}$$

Now consider the interval $(t_0 - \varepsilon_0, t_1 + \varepsilon_1)$ to which we will associate a nonzero spectral projection e of b. There are two cases to consider. Suppose first that $\mu_s(b)$ takes at least three distinct values on this interval. Then there are points $\alpha_1 < \alpha_2 < \alpha_3 \in (t_0 - \varepsilon_0, t_1 + \varepsilon_1)$ so that $\mu_{\alpha_1}(b) > \mu_{\alpha_2}(b) > \mu_{\alpha_3}(b)$. Then the open interval $I = (\mu_{t_1+\varepsilon_1}(b), \mu_{t_0-\varepsilon_0}(b))$ contains a value $\mu_{\alpha_2}(b)$ in the spectrum $\sigma(b)$ of b. Second, if $\mu_s(b)$ takes at most two distinct values on $(t_0-\varepsilon_0, t_1+\varepsilon_1)$ then there exists an interval on which $\mu_s(b)$ is constant, taking a value in the interval I. Now from (5.6) and (5.7), we see that this value in I lies in the point spectrum of b. In both cases the spectral projection e of b for the interval, $[\mu_{t_1+\varepsilon_1}(b), \mu_{t_0-\varepsilon_0}(b)]$ is nonzero. By hypothesis there exists a nonzero self-adjoint element $w \in eMe$ so that $\mathbb{E}_A(w) = 0$, and by scaling we may assume that $\|w\| < \delta/2$. Note that $\mathbb{E}_A(b \pm w) = x$. We now establish that $b \pm w \in \Delta_{x,z}$ which will contradict the assumption that b is an extreme point. By symmetry, we need only consider $b + w$. There are several cases.

Since $\mu_s(b)$ is nonincreasing and right continuous, there exists $r \in [0, t_0 - \varepsilon_0]$ so that

$$\mu_s(b) \le \mu_{t_0-\varepsilon_0}(b) + \delta/2, \quad s \in [r, t_0 - \varepsilon_0], \tag{5.14}$$

while $\mu_s(b) > \mu_{t_0-\varepsilon_0}(b) + \delta/2$ for $s \in [0, r)$. If $t \in [r, t_0 - \varepsilon_0]$, then

$$\begin{aligned} \|b+w\|_{(t)} &= \|b+w\|_{(r)} + \int_r^t \mu_s(b+w)\,ds \le \|b\|_{(r)} + \int_r^t \mu_s(b) + \delta/2\,ds \\ &\le \|z\|_{(r)} + \int_r^t \mu_{t_0-\varepsilon_0}(b) + \delta\,ds \le \|z\|_{(r)} + \int_r^t \mu_{t_0-\varepsilon_0}(z)\,ds \\ &\le \|z\|_{(r)} + \int_r^t \mu_s(z)\,ds = \|z\|_{(t)}, \end{aligned} \tag{5.15}$$

where we have used, respectively, $\|w\| \le \delta/2$, (5.14), (5.13), and the fact that $\mu_s(z)$ is nonincreasing.

If $r = 0$, then (5.15) has already handled the interval $[0, t_0 - \varepsilon_0]$, so we assume that $r > 0$ and we now examine the interval $[0, r)$. Fix a value s in this interval, and let f be the spectral projection of b for the interval $[0, \mu_s(b)]$. By [6, Proposition 2.2], $\tau(f) \ge 1 - s$. Thus $\mu_s(b + \delta e/2) \le \|(b + \delta e/2)f\|$. Since e is supported on $\sigma(b) \cap [0, \mu_s(b) - \delta/2]$, we have $\|(b+\delta e/2)f\| = \|bf\| = \mu_s(b)$, which, together with [6, Lemma 2.5(iii)] implies that $\mu_s(b+w) \le \mu_s(b+\delta e/2) \le \mu_s(b)$. It follows by integrating these inequalities that $\|b + w\|_{(t)} \le \|b\|_{(t)} \le \|z\|_{(t)}$ for $t \in [0, r)$.

On $[t_0 - \varepsilon_0, t_1 + \varepsilon_1]$, using $\|w\| \le \delta/2$ and (5.12), we have

$$\|b+w\|_{(t)} \le \|b\|_{(t)} + \delta/2 \le \|z\|_{(t)}, \tag{5.16}$$

so it remains to consider $[t_1 + \varepsilon_1, 1]$, which is handled in a similar manner to $[0, t_0 - \varepsilon_0]$. Let $r' \in (t_1 + \varepsilon_1, 1]$ be the maximum value so that $\mu_s(b) > \mu_{t_1+\varepsilon_1}(b) - \delta/2$ for $s \in [t_1 + \varepsilon_1, r')$. This value exists since $\mu_s(b)$ is monotone nonincreasing and right continuous. First, consider the case $r' < 1$. Then the inequality $\mu_s(b) \le \mu_{t_1+\varepsilon_1}(b) - \delta/2$ holds for $s \in [r', 1]$. We take $s \in [r', 1]$. Then the spectral projection g of b for the interval $[0, \mu_s(b)]$ is orthogonal to e and has trace at least $1 - s$, by [6, Proposition 2.2]. Thus

$$\mu_s(b + w) = \big\|(b + \delta e/2)g\big\| = \|bg\| = \mu_s(b), \tag{5.17}$$

for $s \in [r', 1]$. Thus if $t \in [r', 1]$, then (see [6, Proposition 2.7])

$$\begin{aligned}\int_t^1 \mu_s(b + w)\, ds = \int_t^1 \mu_s(b)\, ds = \tau(b) - \|b\|_{(t)} \\ \ge \tau(z) - \|z\|_{(t)} = \int_t^1 \mu_s(z)\, ds.\end{aligned} \tag{5.18}$$

Since $\tau(b + w) = \tau(b) = \tau(z)$, we have $\|b + w\|_{(t)} \le \|z\|_{(t)}$ on this interval.

Let $s \in [t_1 + \varepsilon_1, r')$. Then

$$\begin{aligned}\mu_s(b + w) &\ge \mu_s(b - \delta e/2) \ge \mu_s(b) - \delta/2 \\ &\ge \mu_{t_1+\varepsilon_1}(b) - \delta \ge \mu_{t_1+\varepsilon_1}(z) \ge \mu_s(z).\end{aligned} \tag{5.19}$$

If $t \in [t_1 + \varepsilon_1, r')$, then

$$\begin{aligned}\int_t^1 \mu_s(b + w)\, ds &= \int_t^{r'} \mu_s(b + w)\, ds + \int_{r'}^1 \mu_s(b + w)\, ds \\ &\ge \int_t^{r'} \mu_s(z)\, ds + \int_{r'}^1 \mu_s(z)\, ds = \int_t^1 \mu_s(z)\, ds,\end{aligned} \tag{5.20}$$

where we have used (5.18) with $t = r'$. Since $\tau(b + w) = \tau(b) = \tau(z)$, we have the inequality $\|b + w\|_{(t)} \le \|z\|_{(t)}$ on this interval also.

This shows that $b + w \in \Delta_{x,z}$, completing the proof in the case $r' < 1$. If $r' = 1$, then the proof of (5.20) for $t \in [t_1 + \varepsilon_1, 1)$ is exactly as before, and this suffices to prove $b + w \in \Delta_{x,z}$. □

As a consequence of Theorem 5.3, we can immediately deduce two corollaries whose proofs are so similar to those of Theorems 3.1, 3.4 and 4.1 that we omit the details. The only minor change is that when passing from the augmented algebras back to the original ones, instead of observing that a certain element is countably supported we need this for a countable set of elements, which is of course true.

Corollary 5.4. *Let A be either a generator masa or the radial masa in $L(\mathbb{F}_n)$, $2 \le n < \infty$. If $x \in A^+$, $z \in L(\mathbb{F}_n)^+$ and $x \preceq z$, then $x \in \mathbb{E}_A(\mathcal{O}(z))$.*

Corollary 5.5. *If A is the Cartan masa in the hyperfinite factor R and $x \in A^+$, $z \in R^+$ satisfy $x \preceq z$, then there is a trace preserving automorphism θ of A so that $\theta(x) \in \mathbb{E}_A(\mathcal{O}(z))$.*

In [1], $\{x : x \preceq z\}$ was shown to be the σ-SOT closure of $\mathbb{E}_A(\mathcal{O}(z))$ for general masas A. In the case of the Cartan masa, we can improve this to norm closure. We will need a simple preliminary lemma.

LEMMA 5.6. *Let A be the Cartan masa in the hyperfinite factor R. Given two sets of orthogonal projections $\{p_1,\dots,p_n\}$ and $\{q_1,\dots,q_n\}$ in A satisfying $\tau(p_i)=\tau(q_i)$, $1\le i\le n$, there exists a unitary normalizer u of A so that $up_iu^*=q_i$, $1\le i\le n$.*

Proof. We proceed by induction on the number n of projections. The case $n=1$ is proved in [16] (see also [18, Lemma 6.2.6]), so suppose that the result is true for $n-1$ projections. Choose a unitary normalizer u_1 of A so that $u_1p_1u_1^*=q_1$, and consider the sets of projections $\{p_1,\dots,p_n\}$ and $\{u_1^*q_1u_1,\dots,u_1^*q_nu_1\}$. Since $A(1-p_1)$ is a Cartan masa in $(1-p_1)R(1-p_1)$, we may apply the induction hypothesis to the sets of projections $\{p_2,\dots,p_n\}$ and $\{u_1^*q_2u_1,\dots,u_1^*q_nu_1\}$ in $A(1-p_1)$ to obtain a unitary normalizer $w\in(1-p_1)R(1-p_1)$ of $A(1-p_1)$ so that $wp_iw^*=u_1^*q_iu_1$ for $2\le i\le n$. This extends to a unitary normalizer $v=w+p_1$ of A in R. The proof is completed by defining u to be u_1v, so that $up_iu^*=q_i$ for $1\le i\le n$. $\square$

COROLLARY 5.7. *Let A be the Cartan masa in the hyperfinite factor R. If $x\in A^+$, $z\in R^+$ and $x\preceq z$, then $x\in\overline{\mathbb{E}_A(\mathcal{O}(z))}$ (norm closure).*

Proof. By Corollary 5.5 there exists a trace preserving automorphism θ of A so that $\theta(x)\in\mathbb{E}_A(\mathcal{O}(z))$. Given $\varepsilon>0$, there exist projections $p_i\in A$, $1\le i\le n$ and positive constants λ_i, $1\le i\le n$, so that

$$\left\|x-\sum_{i=1}^n\lambda_ip_i\right\|<\varepsilon \tag{5.21}$$

and such that $\sum_{i=1}^n p_i=1$. Since $\tau(p_i)=\tau(\theta(p_i))$, Lemma 5.6 gives a unitary normalizer u of A satisfying

$$u^*p_iu=\theta(p_i),\quad 1\le i\le n. \tag{5.22}$$

Choose unitaries $v_n\in R$ such that

$$\lim_{n\to\infty}\left\|\theta(x)-\mathbb{E}_A\left(v_nzv_n^*\right)\right\|=0. \tag{5.23}$$

From (5.21), (5.22) and (5.23) it follows that

$$\limsup_{n\to\infty}\left\|x-u\mathbb{E}_A\left(v_nzv_n^*\right)u^*\right\|\le\varepsilon. \tag{5.24}$$

But uniqueness of the conditional expectation gives $u\mathbb{E}_A(v_nzv_n^*)u^*=\mathbb{E}_A(uv_nzv_n^*u^*)$, and the result follows from (5.24). $\square$

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