Illinois Journal of Mathematics
Volume 56, Number 3, Fall 2012, Pages 931–955
S 0019-2082

COMPOSITION OF q-QUASICONFORMAL MAPPINGS AND FUNCTIONS IN ORLICZ–SOBOLEV SPACES

STANISLAV HENCL AND LUDĚK KLEPRLÍK

ABSTRACT. Let $\Omega \subset \mathbb{R}^n$, $q \geq n$ and $\alpha \geq 0$ or $1 < q \leq n$ and $\alpha \leq 0$. We prove that the composition of q-quasiconfomal mapping f and function $u \in WL^q \log^\alpha L_{\mathrm{loc}}(f(\Omega))$ satisfies $u \circ f \in WL^q \log^\alpha L_{\mathrm{loc}}(\Omega)$. Moreover, each homeomorphism f which introduces continuous composition operator from $WL^q \log^\alpha L$ to $WL^q \log^\alpha L$ is necessarily a q-quasiconformal mapping. As a new tool, we prove a Lebesgue density type theorem for Orlicz spaces.

1. Introduction

Let $\Omega_1, \Omega_2 \subset \mathbb{R}^n$ be domains and let $f : \Omega_1 \to \Omega_2$ be a homeomorphism. Given a function space X we would like to characterize mappings f for which the composition operator $T_f : T_f(u) = u \circ f$ maps $X(\Omega_2)$ into $X(\Omega_1)$ continuously. This problem has been studied for many function spaces and one the most important is the following well-known result: The composition operator $T_f : T_f(u) = u \circ f$ maps $W^{1,n}_{\mathrm{loc}}(\Omega_2)$ into $W^{1,n}_{\mathrm{loc}}(\Omega_1)$ continuously if $f : \Omega_1 \to \Omega_2$ is a quasiconformal mapping ([21], [24], [17, Lemma 5.13]). Moreover, each homeomorphism f which maps $W^{1,n}_{\mathrm{loc}}(\Omega_2)$ into $W^{1,n}_{\mathrm{loc}}(\Omega_1)$ continuously is necessarily a quasiconfomal mapping. Similarly, it is possible to characterize homeomorphism for which the composition operator is continuous from $W^{1,q}_{\mathrm{loc}}$ to $W^{1,q}_{\mathrm{loc}}$ and we obtain a class of q-quasiconformal mappings [5] (see also [15]).

Here homeomorphism $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ is called a q-quasiconformal mapping if there is a constant $K \geq 1$ such that

$$|Df(x)|^q \leq K|J_f(x)| \quad \text{for a.e. } x \in \Omega. \tag{1.1}$$

Received October 19, 2011; received in final form April 2, 2013.

The first author was supported by the RSJ algorithmic trading grant and the work of the second author was supported by the Grant SVV-2011-263316.

2010 *Mathematics Subject Classification.* 46E35, 46E30, 30C65.

For the properties and further applications of n-quasiconformal mappings, see [1], [13], [17] and [21]. Let us note that we do not assume that $J_f \geq 0$ a.e. as usual, that is, on the right-hand side we have $|J_f|$ and not J_f. This does not seem to be an essential restriction since all homeomorphisms f that are regular enough satisfy either $J_f \geq 0$ a.e. in Ω or $J_f \leq 0$ a.e. in Ω (see [12] for details). That is up to a simple reflection we have the usually considered definition.

In general, one could expect that different function spaces have a different class of morphisms unless the answer is somehow trivial. Surprisingly this is not the case as many examples indicate. For example n-quasiconformal mappings serve as the best class of morphisms not only for $W^{1,n}_{\mathrm{loc}}$ functions but also for other function spaces that are "close" to $W^{1,n}_{\mathrm{loc}}$. Let us mention for example the stability under quasiconformal mappings for the BMO space [20], fractional Sobolev spaces $\dot{M}^s_{n/s,q}$, $s \in (0,1]$, [16, Theorem 1.3] (see also [22] and [10]), absolutely continuous functions of several variables AC^n_λ [7] or exponential Orlicz space $\exp L(\Omega)$ in the plane [3]. We would like to explore this in detail and we would like to know if there is some general principle that "somehow close" spaces have the same class of morphisms.

We show that the same phenomenon occurs for some Orlicz–Sobolev spaces (see Preliminaries for the definition and basic properties) that are close to $W^{1,q}$ and that q-quasiconfomal mappings are the best class of morphisms also for some of those function spaces. In particular (see Sections 3 and 4 for the general statement), we prove the following theorem.

THEOREM 1.1. *Let $q \geq n$ and $\alpha \geq 0$ or $1 < q \leq n$ and $\alpha \leq 0$ and suppose that $f : \Omega_1 \to \Omega_2$ is a q-quasiconformal mapping. Then the operator T_f maps $WL^q \log^\alpha L_{\mathrm{loc}}(\Omega_2) \cap C(\Omega_2)$ into $WL^q \log^\alpha L_{\mathrm{loc}}(\Omega_1)$ for $q > n$ and T_f maps $WL^q \log^\alpha L_{\mathrm{loc}}(\Omega_2)$ into $WL^q \log^\alpha L_{\mathrm{loc}}(\Omega_1)$ for $q \leq n$.*

Moreover, the q-quasiconformal mappings are the best class of homeomorphisms for these Orlicz–Sobolev spaces if $q \geq n$ and $\alpha \geq 0$ or in the second case $q \leq n$ and $\alpha \leq 0$.

THEOREM 1.2. *Let $f : \Omega_1 \to \Omega_2$ be a homeomorphism, $q \geq 1$ and let $\alpha \in \mathbb{R}$. For $q \leq n-1$ we moreover assume that f is differentiable a.e. Suppose that T_f maps $WL^q \log^\alpha L(\Omega_2)$ into $WL^q \log^\alpha L(\Omega_1)$ continuously, that is*

$$\|Du \circ f\|_{L^q \log^\alpha L(\Omega_1)} \leq C \|Du\|_{L^q \log^\alpha L(\Omega_2)}$$

for every $u \in WL^q \log^\alpha L(\Omega_2) \cap C(\Omega_2)$. Then f is a q-quasiconfomal mapping.

It follows that also in the remaining cases $q < n$ and $\alpha > 0$ or $q > n$ and $\alpha < 0$ we get that the morphisms of $WL^q \log^\alpha L$ spaces are subclass of q-quasiconformal mappings.

On the other hand for each $q < n$ and $\alpha > 0$ or for each $q > n$ and $\alpha < 0$, we give an explicit construction of q-quasiconformal mapping f and a function $u \in WL^q \log^\alpha L \cap C$ such that $u \circ f \notin WL^q \log^\alpha L_{\mathrm{loc}}$. Thus, an analogy of Theorem 1.1 does not hold for these values of parameters and the exact description of the class of morphisms must be different.

Let us note that the assumption that f is differentiable a.e. (for $q \ge n-1$) or that T_f is continuous in Theorem 1.2 is not necessary as was shown recently in [14]. The general version of the statement is shown there: Let X be a rearrangement invariant function space somehow close to L^q. Then each homeomorphism f such that T_f maps W^1X into W^1X (not necessarily continuously) is q-quasiconformal. It was also shown in [8] that each homeomorphism f such that T_f maps the Sobolev–Lorentz space $WL^{n,q}$ into $WL^{n,q}$ must be bilipschitz. This shows that the characterization of the composition operator for the spaces $WL^{n,q}$ and $WL^n \log^\alpha L$ is entirely different although both spaces are close to $W^{1,n}$.

For the proof of the Theorem 1.2, we use the usual approach inspired by [5]. We construct a suitable test functions in the small neighborhood of the point x and after passing to the limit we use a Lebesgue density type theorem to conclude that the derivative satisfies (1.1). In the proof of the Theorem 1.2, we use only the simpler conclusion (1.2) but we believe that this Lebesgue density theorem for Orlicz functions is of independent interest and may find application elsewhere.

THEOREM 1.3. *Suppose that Φ is a Young function and let $f \in L^\Phi(\Omega)$ be nonnegative. Then*

$$\liminf_{r\to 0+} \frac{\|f\chi_{B(x,r)}\|_{L^\Phi}}{\|\chi_{B(x,r)}\|_{L^\Phi}} \ge f(x) \quad \textit{for almost every } x \in \Omega. \tag{1.2}$$

If we moreover assume that our Φ satisfies

$$\Phi(ab) \le C\Phi(a)\Phi(b) \quad \textit{for every } a, b \ge 0, \tag{1.3}$$

then

$$\lim_{r\to 0+} \frac{\|f\chi_{B(x,r)}\|_{L^\Phi}}{\|\chi_{B(x,r)}\|_{L^\Phi}} = f(x) \quad \textit{for almost every } x \in \Omega. \tag{1.4}$$

Surprisingly, we cannot have (1.4) for general Young functions, because the term $\|\chi_{B(x,r)}\|$ does not necessarily scale well for small r. For $\alpha < 0$, we construct a function $f \in L^q \log^\alpha L$ such that the limit in (1.4) is infinite everywhere. The additional condition (1.3) is the so called Δ'-condition and it is known to be important for other properties in the theory of Orlicz spaces (see [19]).

This paper is organized as follows. In Section 2, we recall some basic properties of quasiconformal mappings and Orlicz spaces. In Section 3, we prove Theorem 1.3 and we also give a simple counterexample to such a statement for $L^q \log^\alpha L$ spaces for $\alpha < 0$. We prove a general version of Theorem 1.1 in Section 4 and general version of Theorem 1.2 in Section 5. Finally in Section 6, we construct examples showing that q-quasiconformal mappings do not map $WL^q \log^\alpha L$ to $WL^q \log^\alpha L$ for all values of q and α.

2. Preliminaries

We use the usual convention that C denotes a generic positive constant whose exact value may change from line to line. For two functions $g, h : \Omega \mapsto [0,\infty)$, we write $g \lesssim h$ on I, if there is $C > 0$ such that $g(x) \le Ch(x)$ for every $x \in \Omega$. If $g \lesssim h$ and $h \lesssim g$, we write $g \sim h$.

For a function $h : \Omega \to \mathbb{R}$, we denote by $\operatorname{supp} h$ its support. By $A \subset\subset \Omega$ we denote the fact that the closure of A lies inside Ω, that is, $\overline{A} \subset \Omega$. The Lebesgue measure of a set A is denoted by $\mathcal{L}^n(A)$ or for short $|A|$.

2.1. Orlicz spaces. A function $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ is a Young function if $\Phi(0) = 0$, Φ is increasing and convex.

Denote by $L^\Phi(A)$ the corresponding Orlicz space with Young function Φ on a set A with measure $\mathcal{L}^n$. This space is equipped with the Luxemburg norm

$$\|f\|_{L^\Phi(A)} = \inf\left\{\lambda > 0 : \int_A \Phi\big(|f(x)|/\lambda\big)\, dx \le 1\right\}. \tag{2.1}$$

For $q \ge 1$ and $\alpha \in \mathbb{R}$, we denote by $L^q \log^\alpha L(A)$ the Orlicz space with a Young function such that

$$\lim_{t\to\infty} \frac{\Phi(t)}{t^q \log^\alpha t} = 1.$$

For an introduction to Orlicz spaces see, for example, [19].

We define the Orlicz–Sobolev space $WL^\Phi(A)$ as the set

$$WL^\Phi(A) := \big\{u : u, |Du| \in L^\Phi(A)\big\}$$

equipped with the norm

$$\|u\|_{WL^\Phi(A)} := \|u\|_{L^\Phi(A)} + \|Du\|_{L^\Phi(A)},$$

where Du is the weak derivative of u.

Let Φ be a Young function and let us define

$$H(t) = \frac{1}{\Phi(\frac{1}{t})} = \left(\frac{1}{\Phi^{-1}(\frac{1}{t})}\right)^{-1}. \tag{2.2}$$

The standard computation gives us

$$\|\chi_E\|_{L^\Phi} = \frac{1}{\Phi^{-1}(\frac{1}{|E|})} = H^{-1}\big(|E|\big) \tag{2.3}$$

for any measurable set $E \subset \Omega$.

We say that a function Φ satisfies the Δ_2-condition, if there is $C_\Delta > 1$ such that

$$\Phi(2t) \le C_\Delta \Phi(t) \quad \text{whenever } t \ge 0.$$

Analogously, we say that a function Φ satisfies the ∇_2-condition if there is $C_\nabla > 2$ such that

$$\Phi(2t) \ge C_\nabla \Phi(t) \quad \text{whenever } t \ge 0.$$

It is not difficult to show that if Φ satisfies Δ_2 condition, then

$$\|h_k\|_{L^\Phi(\Omega)} \overset{k\to\infty}{\to} 0 \quad \Longleftrightarrow \quad \int_\Omega \Phi\big(|h_k|\big)\,dx \overset{k\to\infty}{\to} 0. \tag{2.4}$$

2.2. On q-quasiconformal mapping. We will need the following version of the derivative of composed function (see [15, Theorem 1.3] for special choice $p = q$).

THEOREM 2.1. *Let $q \ge 1$ and let $f : \Omega_1 \to \Omega_2$ be a homeomorphism of finite q-distortion. Then the operator T_f is continuous from $W^{1,q}_{\text{loc}}(\Omega_2) \cap C(\Omega_2)$ to $W^{1,q}_{\text{loc}}(\Omega_1)$ for $q > n$ and continuous from $W^{1,q}_{\text{loc}}(\Omega_2)$ to $W^{1,q}_{\text{loc}}(\Omega_1)$ for $q \le n$.*

Moreover for every $u \in W^{1,q}_{\text{loc}}(\Omega_2)$, we have

$$D(u \circ f)(x) = Du\big(f(x)\big)Df(x) \quad \textit{for a.e. } x \in \Omega_1 \tag{2.5}$$

if we use the convention that $Du(f(x)) \cdot 0 = 0$ even if Du does not exist or it is infinity at $f(x)$.

It is easy to see from the definition of q-quasiconformal mappings that

$$\big|Df(x)\big|^q \le K\big|J_f(x)\big| \le K\big|Df(x)\big|^n \tag{2.6}$$

and therefore each such a map lies in $W^{1,\infty}$ for $q > n$. By [15], we know that q-quasiconformal mapping f for $q < n$ satisfies Luzin (N^{-1}) condition, that is, f^{-1} maps sets of zero measure onto sets of zero measure. Therefore for $q < n$ it cannot happen that $J_f = 0$ on a set of positive measure and we can use (2.6) to obtain $\frac{1}{|Df|} \in L^\infty$. It is also well known that each quasiconformal mappings has better integrability (see, e.g., [17]).

THEOREM 2.2. *Let Ω be an open set. Suppose that f is n-quasiconformal mapping on Ω. Then there exist $p > n$ and $r > 0$ such that $|Df|^p \in L^1_{\text{loc}}(\Omega)$ and $\frac{1}{|Df|^r} \in L^1_{\text{loc}}(\Omega)$.*

2.3. Volume derivative. Let us denote the volume derivative by

$$f'_v(x) = \lim_{r\to 0} \frac{|f(B(x,r))|}{|B(x,r)|}.$$

We shall need the following connection between f'_v and the Jacobian of f [23, Theorem 24.2 and Theorem 24.4].

THEOREM 2.3. *Let $f:\Omega\to\mathbb{R}^n$ be a homeomorphism. Then f'_v is a measurable function and $f'_v<\infty$ almost everywhere. Moreover, $f'_v(x)=|J_f(x)|$ for every point x where f is differentiable.*

2.4. Area formula. We will use the well-known area formula for homeomorphisms in $W^{1,1}_{\mathrm{loc}}(\Omega)$. It is known that each $f\in W^{1,1}_{\mathrm{loc}}(\Omega)$ is approximatively differentiable almost everywhere [2, Theorem 3.1.4] and that the set of approximative differentiability can be exhausted up to a set of measure zero by sets the restriction to which of f is Lipschitz [2, Theorem 3.1.8]. Hence, we can decompose Ω into pairwise disjoint sets

$$\Omega = Z\cup\bigcup_{k=1}^{\infty}\Omega_k \tag{2.7}$$

such that $|Z|=0$ and $f|_{\Omega_i}$ is Lipschitz. Let $f\in W^{1,1}_{\mathrm{loc}}(\Omega;\mathbb{R}^n)$ be a homeomorphism and let $B\subset\Omega$ be a Borel set. Let η be a nonnegative Borel measurable function on $\mathbb{R}^n$. Without any additional assumption, we have

$$\int_B \eta\big(f(x)\big)\big|J_f(x)\big|\,dx\le\int_{f(B)}\eta(y)\,dy. \tag{2.8}$$

This follows from the area formula for Lipschitz mappings and (2.7).

3. Lebesgue density theorem for Orlicz spaces

Let us note that a Young function such that

$$\Phi(t)\sim t^n\log^{\alpha}(e+t)\quad\text{for some }\alpha\ge 0$$

clearly satisfies (1.3) since

$$\log(e+ab)\le\log\big((e+a)(e+b)\big)\le 2\log(e+a)\log(e+b)\quad\text{for every } a,b\ge 0.$$

LEMMA 3.1. *Suppose that a Young function Φ satisfies (1.3). Then*

$$H\big(\|h\|_{L^{\Phi}(\Omega)}\big)\le C\int_{\operatorname{supp} h}\Phi\big(|h(x)|\big)\,dx$$

for every function $h\in L^{\Phi}(\Omega)$.

Proof. Let us denote $\lambda=\|h\|_{L^{\Phi}(\Omega)}$. By the definition of the Luxemburg norm and (1.3), we obtain

$$1=\int_{\Omega}\Phi\bigg(\frac{|h(x)|}{\lambda}\bigg)\,dx\le C\int_{\operatorname{supp} h}\Phi\big(|h(x)|\big)\Phi\bigg(\frac{1}{\lambda}\bigg)\,dx. \tag{3.1}$$

Using (2.2) and (3.1), we get

$$H(\lambda)=\frac{1}{\Phi(\frac{1}{\lambda})}\le C\int_{\operatorname{supp} h}\Phi\big(|h(x)|\big)\,dx.$$

□

Proof of Theorem 1.3. Let us first prove (1.2) for arbitrary Φ. By the Jensen's inequality and the definition of the Luxemburg norm, we obtain

$$\begin{aligned}\Phi\biggl(\frac{1}{|B(x,r)|}\int_{B(x,r)}\frac{f}{\|f\chi_{B(x,r)}\|_{L^\Phi}}\biggr) &\le \frac{1}{|B(x,r)|}\int_{B(x,r)}\Phi\biggl(\frac{f}{\|f\chi_{B(x,r)}\|_{L^\Phi}}\biggr)\\ &\le \frac{1}{|B(x,r)|}.\end{aligned}$$

By (2.3), we now have

$$\frac{1}{|B(x,r)|}\int_{B(x,r)}|f| \le \|f\chi_{B(x,r)}\|_{L^\Phi}\Phi^{-1}\biggl(\frac{1}{|B(x,r)|}\biggr) = \frac{\|f\chi_{B(x,r)}\|_{L^\Phi}}{\|\chi_{B(x,r)}\|_{L^\Phi}}.$$

Since almost every point is a Lebesgue point of density for L^1 we obtain (1.2).

Now assume that our Φ satisfies (1.3) and let us prove (1.4). Let us fix $\alpha > 0$. We want to show that the measure of the set

$$S_\alpha = \biggl\{x\in\Omega:\ \limsup_{r\to 0+}\biggl|\frac{\|f\chi_{B(x,r)}\|_{L^\Phi}}{\|\chi_{B(x,r)}\|_{L^\Phi}} - f(x)\biggr| > 2\alpha\biggr\}$$

is zero. It is easy to see that (1.4) is valid for every continuous function. The Δ'-condition implies that our Φ satisfies the Δ_2-condition (see [19, Chapter 2.2]) and therefore continuous functions are dense in L^Φ. Hence, we can find g continuous such that

$$f(x) = g(x) + h(x) \quad\text{and}\quad \|h\|_{L^\Phi(\Omega)} < \varepsilon.$$

Clearly,

$$\frac{\|f\chi_{B(x,r)}\|_{L^\Phi}}{\|\chi_{B(x,r)}\|_{L^\Phi}} - f(x) \le \frac{\|g\chi_{B(x,r)}\|_{L^\Phi}}{\|\chi_{B(x,r)}\|_{L^\Phi}} - g(x) + \frac{\|h\chi_{B(x,r)}\|_{L^\Phi}}{\|\chi_{B(x,r)}\|_{L^\Phi}} - h(x)$$

and

$$\frac{\|f\chi_{B(x,r)}\|_{L^\Phi}}{\|\chi_{B(x,r)}\|_{L^\Phi}} - f(x) \ge \frac{\|g\chi_{B(x,r)}\|_{L^\Phi}}{\|\chi_{B(x,r)}\|_{L^\Phi}} - g(x) - \frac{\|h\chi_{B(x,r)}\|_{L^\Phi}}{\|\chi_{B(x,r)}\|_{L^\Phi}} - h(x).$$

Since (1.4) is valid for g it is easy to see that

$$S_\alpha \subset N_\alpha \cup M_\alpha,$$

where

$$\begin{aligned} N_\alpha &= \bigl\{x\in\Omega:\ \bigl|h(x)\bigr| \ge \alpha\bigr\} \quad\text{and}\\ M_\alpha &= \biggl\{x\in\Omega:\ \limsup_{r\to 0+}\frac{\|h\chi_{B(x,r)}\|_{L^\Phi}}{\|\chi_{B(x,r)}\|_{L^\Phi}} > \alpha\biggr\}.\end{aligned}$$

It is easy to estimate the measure of N_α by

$$|N_\alpha| = \frac{1}{\Phi(\alpha)}\int_{N(\alpha)}\Phi(\alpha)\,dx \le \frac{1}{\Phi(\alpha)}\int_\Omega \Phi\bigl(h(x)\bigr)\,dx. \tag{3.2}$$

It remains to estimate $|M_\alpha|$. Using a Besicovitch covering theorem, we obtain balls $B_i(r_i)$ such that

$$M_\alpha \subset \bigcup_i B_i(r_i), \quad \sum_i \chi_{B_i(r_i)} \leq C \text{ and } \frac{\|h\chi_{B_i(r_i)}\|_{L^\Phi}}{\|\chi_{B_i(r_i)}\|_{L^\Phi}} > \alpha.$$

From (2.3) and the last inequality, we obtain

$$\alpha H^{-1}\big(|B_i(r_i)|\big) = \alpha \|\chi_{B_i(r_i)}\|_{L^\Phi} < \|h\chi_{B_i(r_i)}\|_{L^\Phi}.$$

Using (2.2) twice and (1.3), we thus get

$$\begin{aligned} H\big(\|h\chi_{B_i(r_i)}\|_{L^\Phi}\big) > H\big(\alpha H^{-1}\big(|B_i(r_i)|\big)\big) &= \frac{1}{\Phi(\frac{1}{\alpha H^{-1}(|B_i(r_i)|)})} \\ &\geq \frac{C}{\Phi(\frac{1}{\alpha})\Phi(\frac{1}{H^{-1}(|B_i(r_i)|)})} = \frac{C}{\Phi(\frac{1}{\alpha})}\big|B_i(r_i)\big|. \end{aligned}$$

Using Lemma 3.1, we now obtain that

$$\begin{aligned} |M_\alpha| &\leq \sum_i \big|B_i(r_i)\big| \leq C\Phi\left(\frac{1}{\alpha}\right)\sum_i H\big(\|h\chi_{B_i(r_i)}\|_{L^\Phi}\big) \\ &\leq C\Phi\left(\frac{1}{\alpha}\right)\sum_i \int_{B_i(r_i)} \Phi\big(|h(x)|\big)\,dx \leq C\Phi\left(\frac{1}{\alpha}\right)\int_\Omega \Phi\big(|h(x)|\big)\,dx. \end{aligned}$$

Using this estimate and (3.2), we may use $\|h\|_{L^\Phi} < \varepsilon$ and (2.4) to obtain

$$|S_\alpha| \leq |N_\alpha| + |M_\alpha| \overset{\varepsilon\to 0+}{\to} 0. \qquad \square$$

3.1. Counterexample.

EXAMPLE 3.2. Let $q \geq 1$ and $\alpha < 0$. Then there is $f \in L^q \log^\alpha L(0,1)$ such that

$$\lim_{r\to 0+} \frac{\|f\chi_{B(x,r)}\|_{L^q \log^\alpha L}}{\|\chi_{B(x,r)}\|_{L^q \log^\alpha L}} = \infty \quad \text{for every } x \in (0,1).$$

Proof. Let us consider the Young function such that

$$\Phi(t) \sim t^q \log^\alpha(t) \quad \text{for } t \geq 2.$$

Let us fix $0 < \varepsilon < -\alpha$ and set

$$r_k = 2^{-q2^k} 2^{-k(\alpha+1)} \frac{1}{k^{1+\varepsilon}}$$

and define the function $f : [0,1] \to \mathbb{R}$ by

$$f(x) = 2^{2^k} \quad \text{for } x \in \left[\frac{j}{2^k}, \frac{j}{2^k} + r_k\right], k \in \mathbb{N} \text{ and } j \in \big\{0, 1, \ldots, 2^k - 1\big\}.$$

If the two intervals intersect for different k_1 and k_2 then we define f as the bigger number. It is easy to see that $f \in L^{\Phi}([0,1])$ since

$$\begin{aligned}\int_0^1 \Phi(f) &\leq \sum_{k=1}^{\infty} 2^k r_k \Phi\big(2^{2^k}\big)\\ &\leq C\sum_{k=1}^{\infty} 2^k 2^{-q2^k} 2^{-k(\alpha+1)} \frac{1}{k^{1+\varepsilon}} 2^{q2^k} \log^{\alpha}\big(2^{2^k}\big) = \sum_{k=1}^{\infty} \frac{1}{k^{1+\varepsilon}} < \infty.\end{aligned}$$

Now let us fix $x \in (0,1)$ and pick a radius $r = 2^{-k_0}$. It is not difficult to see that $\Phi^{-1}(t) \sim t^{\frac{1}{q}} \log^{-\frac{\alpha}{q}} t$ for large t and hence we can use (2.3) to get

$$\|\chi_{B(x,r)}\|_{L^{\Phi}} = \frac{1}{\Phi^{-1}\big(\frac{1}{|B(x,r)|}\big)} \leq C(2r)^{\frac{1}{q}} \log^{\frac{\alpha}{q}} \frac{1}{2r} \leq C2^{-\frac{k_0}{q}} k_0^{\frac{\alpha}{q}}. \tag{3.3}$$

Let us denote $\lambda = \|f\chi_{B(x,r)}\|_{L^{\Phi}}$ and we may assume that r is so small that $\lambda < 1$. For $k \geq k_0$ we have at most $C2^{k-k_0}$ points of the type $\frac{j}{2^k}$ in the interval $(x-r, x+r)$. Using definition of Luxemburg norm and $\alpha < 0$, we thus get

$$\begin{aligned}1 = \int_{B(x,r)} \Phi\Big(\frac{f}{\lambda}\Big) &\leq C\sum_{k=k_0}^{\infty} 2^{k-k_0} r_k \Phi\Big(\frac{2^{2^k}}{\lambda}\Big)\\ &\leq C\sum_{k=k_0}^{\infty} 2^{-k_0} 2^{-k\alpha} \frac{1}{k^{1+\varepsilon}} \frac{1}{\lambda^q} \log^{\alpha}\Big(\frac{2^{2^k}}{\lambda}\Big)\\ &\leq C\sum_{k=k_0}^{\infty} 2^{-k_0} 2^{-k\alpha} \frac{1}{k^{1+\varepsilon}} \frac{1}{\lambda^q} \log^{\alpha}\big(2^{2^k}\big) = \frac{C2^{-k_0}}{\lambda^q} \sum_{k=k_0}^{\infty} \frac{1}{k^{1+\varepsilon}}.\end{aligned}$$

This inequality implies an estimate of λ which gives us

$$\log\Big(\frac{2^{2^k}}{\lambda}\Big) \leq C\log\big(2^{2^k}\big) \quad \text{for each } k \geq k_0. \tag{3.4}$$

Moreover for each $k \geq k_0$ we have at least $C2^{k-k_0}$ points of the type $\frac{j}{2^k}$ in the interval $(x-r, x+r)$. Further, the value 2^{2^k} is attained on each interval $[\frac{j}{2^k}, \frac{j}{2^k} + r_k]$ on a set of measure at least

$$r_k - \sum_{j=k+1}^{\infty} 2^{j-k} r_j \geq \frac{r_k}{2}.$$

Using all these estimates and the definition of Luxemburg norm, we get

$$\begin{aligned}1 = \int_{B(x,r)} \Phi\Big(\frac{f}{\lambda}\Big) &\geq C\sum_{k=k_0}^{\infty} 2^{k-k_0} \frac{r_k}{2} \Phi\Big(\frac{2^{2^k}}{\lambda}\Big)\\ &\geq C\sum_{k=k_0}^{\infty} 2^{-k_0} 2^{-k\alpha} \frac{1}{k^{1+\varepsilon}} \frac{1}{\lambda^q} \log^{\alpha}\Big(\frac{2^{2^k}}{\lambda}\Big)\end{aligned}$$

$$\geq C \sum_{k=k_0}^{\infty} 2^{-k_0} 2^{-k\alpha} \frac{1}{k^{1+\varepsilon}} \frac{1}{\lambda^q} \log^{\alpha}\left(2^{2^k}\right) = \frac{C 2^{-k_0}}{\lambda^q} \sum_{k=k_0}^{\infty} \frac{1}{k^{1+\varepsilon}}.$$

It follows that

$$\|f\chi_{B(x,r)}\|_{L^\Phi} = \lambda \geq C 2^{-\frac{k_0}{q}} k_0^{-\frac{\varepsilon}{q}}.$$

Now we can use this estimate, (3.3) and $\varepsilon < -\alpha$ to obtain

$$\lim_{r\to 0+} \frac{\|f\chi_{B(x,r)}\|_{L^\Phi}}{\|\chi_{B(x,r)}\|_{L^\Phi}} \geq C \lim_{k_0\to\infty} \frac{2^{-\frac{k_0}{q}} k_0^{-\frac{\varepsilon}{q}}}{2^{-\frac{k_0}{q}} k_0^{\frac{\alpha}{q}}} = C \lim_{k_0\to\infty} k_0^{-\frac{\varepsilon+\alpha}{q}} = \infty. \qquad \square$$

4. Stability of WL^Φ under q-quasiconformal mappings

THEOREM 4.1. *Let $\Omega_1, \Omega_2 \subset \mathbb{R}^n$ be domains, $q \geq n$ and let $f \in W^{1,1}_{\mathrm{loc}}(\Omega_1, \Omega_2)$ be a q-quasiconformal homeomorphism. Suppose that Φ is a Young function such that $\Phi(t) = t^q \alpha(t)$ where*

$$\text{(4.1)} \qquad \begin{aligned} &\text{(i)} && \alpha \textit{ is non-decreasing},\\ &\text{(ii)} && \lim_{t\to\infty} \frac{\alpha(t)}{t^\delta} = 0 \textit{ for every } \delta > 0,\\ &\text{(iii)} && \alpha\left(t^\beta\right) \leq C(\beta)\alpha(t) \textit{ for every } \beta \geq 1 \textit{ and } t \geq 1. \end{aligned}$$

Then the operator T_f is continuous from $WL^\Phi_{\mathrm{loc}}(\Omega_2)$ into $WL^\Phi_{\mathrm{loc}}(\Omega_1)$, i.e. for every open set $A \subset\subset \Omega_1$ we have

$$\text{(4.2)} \qquad \|DT_f u\|_{L^\Phi(A)} \leq C\|Du\|_{L^\Phi(f(A))}$$

for every $u \in WL^\Phi(\Omega_2) \cap C(\Omega_2)$ or for every $u \in WL^\Phi(\Omega_2)$ for $q = n$.

Proof. Let $u \in WL^\Phi_{\mathrm{loc}}(\Omega_2) \cap C(\Omega_2)$ and $A \subset\subset \Omega_1$. By (4.1)(i) we know $u \in W^{1,q}_{\mathrm{loc}}(\Omega_2)$ and therefore we may use Theorem 2.1 to conclude that $u \circ f \in W^{1,q}_{\mathrm{loc}}(\Omega_1)$ and

$$D(u\circ f) = \big((Du)\circ f\big) \cdot Df.$$

To obtain (4.2), it is enough to show that the modular of $D(u \circ f)$ is bounded for each u such that $\|Du\|_{L^\Phi} \leq 1$. We have

$$\text{(4.3)} \qquad \begin{aligned} &\int_A |Du\circ f|^q \alpha\big(|Du\circ f|\big)\\ &\quad \leq \int_A \big|Du\big(f(x)\big)\big|^q \big|Df(x)\big|^q \alpha\Big(\big|Du\big(f(x)\big)\big|\big|Df(x)\big|\Big)\,dx. \end{aligned}$$

From Section 2.2, we know that there is $p > q$ such that $f \in W^{1,p}(A, \mathbb{R}^n)$. Let us fix $q < s < p$ and we will divide integral into two integrals over sets

$$U = \Big\{x \in A : \big|Du\big(f(x)\big)\big| \geq \big|Df(x)\big|^{\frac{s-q}{q}}\Big\} \quad \text{and}$$
$$V = \Big\{x \in A : \big|Du\big(f(x)\big)\big| \leq \big|Df(x)\big|^{\frac{s-q}{q}}\Big\}.$$

We use the definition of q-quasiconformal mappings, (4.1) and the area formula (2.8) to bound the left-hand side of (4.3) by

$$\begin{aligned}
&\int_U \big|Du\big(f(x)\big)\big|^q \big|Df(x)\big|^q \alpha\big(\big|Du\big(f(x)\big)\big|\big|Df(x)\big|\big)\,dx \\
&\qquad + \int_V \big|Du\big(f(x)\big)\big|^q \big|Df(x)\big|^q \alpha\big(\big|Du\big(f(x)\big)\big|\big|Df(x)\big|\big)\,dx \\
&\quad \le \int_U \big|Du\big(f(x)\big)\big|^q K\big|J_f(x)\big|\alpha\big(\big|Du\big(f(x)\big)\big|^{\frac{s}{s-q}}\big)\,dx \\
&\qquad + \int_V \big|Df(x)\big|^s \alpha\big(\big|Df(x)\big|^{\frac{s}{q}}\big)\,dx \\
&\quad \le CK \int_{f(A)} \big(1+\big|Du(y)\big|^q \alpha\big(\big|Du(y)\big|\big)\big)\,dy + C\int_A \big(\big|Df(x)\big|^p+1\big)\,dx.
\end{aligned}$$

and the result follows. □

The most important step in the proof of Theorem 1.1 and its generalization is to show that $u \circ f \in W^{1,1}_{\mathrm{loc}}$. This follows quite easily from known facts in most of the cases with the exception of the important case $q=n$ and $\alpha<0$. In this case analogously to [9] we need to construct some approximation sequence with the help of the following lemma.

LEMMA 4.2. *Let $B \subset \mathbb{R}^n$ be an open ball and suppose that $u \in W^{1,1}(3B)$. Then for all Lebesgue points $x, y \in B$ of function u we have*

$$\big|u(x)-u(y)\big| \le C(n)|x-y|\big(M|Du|(x)+M|Du|(y)\big),$$

where $Mh(x)$ denotes the Hardy–Littlewood maximal operator of $h: 3B \to \mathbb{R}$

$$Mh(x) = \sup_{B(x_0,r)\subset 3B} \frac{1}{\mathcal{L}^n(B(x_0,r))}\int_{B(x_0,r)} \big|h(x)\big|\,dx.$$

LEMMA 4.3. *Let $L^\Phi(A)$ be an Orlicz space where Φ satisfies Δ_2 and ∇_2 condition and let $h \in L^\Phi(\Omega)$. Then we have*

$$\lim_{\lambda\to\infty} \Phi(\lambda)\mathcal{L}^n\big(\{x : Mh(x)>\lambda\}\big)=0.$$

Proof. From [4], we know that the maximal operator M is continuous from L^Φ to L^Φ and hence we get

$$\Phi(\lambda)\mathcal{L}^n\big(\{Mh>\lambda\}\big) = \int_{\{Mh>\lambda\}} \Phi(\lambda) \le \int_{\{Mh>\lambda\}} \Phi(Mh) \to 0.$$

□

LEMMA 4.4. *Let Φ be an Young function satisfying Δ_2 and ∇_2 condition and let $u \in WL^\Phi(B(x_0,3r))$. There is a sequence of functions u_k with Lipschitz constant Ck and sequence of measurable sets F_k such that $F_k \subset \{u=u_k\}$, $F_k \subset F_{k+1}$*

$$\lim_{k\to\infty} \mathcal{L}^n\big(B(x_0,r)\setminus F_k\big)=0 \quad \text{and} \quad u_k \overset{k\to\infty}{\to} u \quad \text{in } WL^\Phi\big(B(x_0,r)\big).$$

Proof. Let $B=B(x_0,r)$ and for $k>0$ we set

$$F_k=\{x\in B: M(|Du|)\le k \text{ and } |u|\le k\}\\ \cap\{x: x \text{ is Lebesgue point of function } u\}.$$

It is easy to see that $\mathcal{L}^n(B\setminus F_k)\overset{k\to\infty}{\to}0$.

From Lemma 4.2 we obtain, that the mapping u is Lipschitz continuous on F_k. By the classical McShane extension theorem, there exists a function $u_k:\mathbb{R}^n\to\mathbb{R}$ with Lipschitz constant Ck such that $u_k=u$ on F_k and $|u|\le k$.

Since u_k is Lipschitz function, there exists a derivative almost everywhere and we can estimate $|\nabla u_k|\le Ck$. Additionally for this extension we have $\nabla u_k=Du$ almost everywhere on F_k. Indeed, if x is a density point of F_k, where the derivative of u_k exists and the approximative differential of u is equal to the weak derivative, then it is not difficult to show that $\nabla u_k(x)=Du(x)$.

First we prove, that the functions ∇u_k converge to Du in $L^\Phi(B)$:

$$\begin{aligned}\int_B\Phi(|\nabla u_k-Du|)&=\int_{B\setminus F_k}\Phi(|\nabla u_k-Du|)\\&\le C\int_{B\setminus F_k}\Phi(|\nabla u_k|)+C\int_{B\setminus F_k}\Phi(|Du|)\\&\le C\mathcal{L}^n(B\setminus F_k)\Phi(k)+C\int_{B\setminus F_k}\Phi(|Du|)\overset{k\to\infty}{\to}0,\end{aligned}$$

where we have used Lemma 4.3 which together with Chebychev's inequality easily implies $\Phi(k)\mathcal{L}^n(B\setminus F_k)\overset{k\to\infty}{\to}0$.

Now we want to show that u_k converges to u in $L^\Phi(B)$.

$$\begin{aligned}\int_B\Phi(|u_k-u|)=\int_{B\setminus F_k}\Phi(|u_k-u|)&\le C\int_{B\setminus F_k}\Phi(|u_k|)+C\int_{B\setminus F_k}\Phi(|u|)\\&\le C\int_{B\setminus F_k}\Phi(k)+C\int_{B\setminus F_k}\Phi(|u|)\overset{k\to\infty}{\to}0,\end{aligned}$$

where we have again used the estimate $\Phi(k)\mathcal{L}^n(B\setminus F_k)\overset{k\to\infty}{\to}0$. □

THEOREM 4.5. *Let $\Omega_1,\Omega_2\subset\mathbb{R}^n$ be domains, $1<q\le n$ and let $f\in W^{1,1}_{\mathrm{loc}}(\Omega_1,\Omega_2)$ be a q-quasiconformal homeomorphism. Suppose that Φ is a Young function such that $\Phi(t)=t^q\alpha(t)$ where*

$$\begin{aligned}&\text{(i)}\quad \alpha \text{ is non-increasing},\\ (4.4)\qquad &\text{(ii)}\quad \lim_{t\to\infty}\alpha(t)t^\delta=\infty \text{ for every } \delta>0,\\ &\text{(iii)}\quad \alpha(t^\beta)\le C(\beta)\alpha(t) \text{ for every } \beta\le 1 \text{ and } t\ge 1.\end{aligned}$$

Then the operator T_f *is continuous from* $WL^{\Phi}_{\mathrm{loc}}(\Omega_2)$ *into* $WL^{\Phi}_{\mathrm{loc}}(\Omega_1)$, *i.e. for every open set* $A \subset\subset \Omega_1$ *we have*

$$\|DT_f u\|_{L^{\Phi}(A)} \leq C\|Du\|_{L^{\Phi}(f(A))} \tag{4.5}$$

for every $u \in WL^{\Phi}(\Omega_2)$.

Proof. Let $A \subset\subset \Omega_2$ be an arbitrary. Suppose that we already know that $u \circ f \in W^{1,1}_{\mathrm{loc}}$ and that

$$D(u \circ f) = \big((Du) \circ f\big) Df \tag{4.6}$$

holds. Then we can estimate the modular by

$$\begin{aligned} &\int_A |Du \circ f|^q \alpha\big(|Du \circ f|\big) \\ &\qquad \leq \int_A \big|Du\big(f(x)\big)\big|^q \big|Df(x)\big|^q \alpha\big(\big|Du\big(f(x)\big)\big|\big|Df(x)\big|\big)\, dx. \end{aligned} \tag{4.7}$$

From Section 2.2, we know that there is $r > 0$ such that $\frac{1}{|Df|^r} \in L^1_{\mathrm{loc}}$. Let us fix $0 < s < r$ and define

$$U = \left\{ x \in A : \big|Du\big(f(x)\big)\big| \geq \frac{1}{|Df(x)|^{\frac{q+s}{q}}} \right\} \quad \text{and}$$
$$V = \left\{ x \in A : \big|Du\big(f(x)\big)\big| \leq \frac{1}{|Df(x)|^{\frac{q+s}{q}}} \right\}.$$

We use the definition of q-quasiconformal mappings, (4.4)(iii), (ii) and the area formula (2.8) to bound the left-hand side of (4.7) by

$$\begin{aligned} &\int_U \big|Du\big(f(x)\big)\big|^q \big|Df(x)\big|^q \alpha\big(\big|Du\big(f(x)\big)\big|\big|Df(x)\big|\big)\, dx \\ &\qquad + \int_V \big|Du\big(f(x)\big)\big|^q \big|Df(x)\big|^q \alpha\big(\big|Du\big(f(x)\big)\big|\big|Df(x)\big|\big)\, dx \\ &\quad \leq \int_U \big|Du\big(f(x)\big)\big|^q K\big|J_f(x)\big|\alpha\big(\big|Du\big(f(x)\big)\big|^{1-\frac{q}{q+s}}\big)\, dx \\ &\qquad + \int_V \frac{1}{|Df|^s}\alpha\big(|Df|^{\frac{-s}{q}}\big) \\ &\quad \leq CK \int_{f(A)} \big(1 + \big|Du(y)\big|^q \alpha\big(\big|Du(y)\big|\big)\big)\, dy + C\int_A \left(\frac{1}{|Df|^r} + 1\right) \end{aligned} \tag{4.8}$$

and the result follows once we verify $u \circ f \in W^{1,1}_{\mathrm{loc}}$ and (4.6).

If $1 < q < n$, we obtain from Section 2.2 that $|Df| > C$ and thus we can use the definition of q-quasiconformal mapping to obtain that f is also 1-quasiconformal. We know that $u \in W^{1,1}_{\mathrm{loc}}(\Omega_2)$ and hence Theorem 2.1 implies $u \circ f \in W^{1,1}_{\mathrm{loc}}(\Omega_1)$ and (4.6).

It remains to treat the case $q = n$. It is not difficult to see that our Φ satisfies Δ_2 and ∇_2 condition. To obtain (4.6), we will approximate u using the previous lemma. Let $x_0 \in \Omega_1$ and fix a ball B and $r > 0$ such that $3B \subset\subset \Omega_2$ and $f(B(x_0,r)) \subset B$. We have to show that $u \circ f \in W^{1,1}(B(x_0,r))$. By applying Lemma 4.4 we find a sequence u_k of functions with Lipschitz constant Ck and a sequence of measurable sets $F_k \subset B$ such that

$$u_k = u \quad \text{on } F_k, F_k \subset F_{k+1} \quad \text{and} \quad \lim_{k\to\infty} \mathcal{L}^n(F_k) = \mathcal{L}^n(B).$$

Set $g_j = u_j \circ f$ for each $j \in \mathbb{N}$. Since u_j are Lipschitz functions, we obtain from Theorem 2.1, that $g_j \in W^{1,1}(B(x_0,r))$.

We will show that ∇g_j is a Cauchy sequence in $L^\Phi(B(x_0,r),\mathbb{R}^n)$. Let v be a Lipschitz function. Then (4.6) holds and thanks to (4.7) and (4.8) we get

$$\big\|D(v \circ f)\big\|_{L^\Phi(\Omega_1)} \le C\|Dv\|_{L^\Phi(\Omega_2)}.$$

If we apply this estimate to the function $v = u_j - u_k$, we easily get, that the sequence $D(u_j \circ f) = Dg_j$ is Cauchy in $L^\Phi(B(x_0,r),\mathbb{R}^n)$. Hence there exists a mapping $h \in L^\Phi(B(x_0,r),\mathbb{R}^n)$ such that

$$Dg_j \overset{j\to\infty}{\to} h \quad \text{in } L^\Phi\big(B(x_0,r),\mathbb{R}^n\big). \tag{4.9}$$

By [15], we know that our mapping f satisfies Luzin (N^{-1}) condition, that is, f^{-1} maps sets of zero measure onto sets of zero measure. Since $\mathcal{L}^n(B \setminus F_j)$ converge to 0, we obtain, that sets $A_j := B(x_0,r) \cap f^{-1}(F_j)$ satisfy

$$\begin{aligned}\lim_{j\to\infty} \mathcal{L}^n(A_j) &= \mathcal{L}^n\left(B(x_0,r) \cap f^{-1}\left(\bigcup_{j=1}^\infty F_j\right)\right)\\ &= \mathcal{L}^n\left(B(x_0,r) \setminus f^{-1}\left(B \setminus \bigcup_{j=1}^\infty F_j\right)\right)\\ &= \mathcal{L}^n\big(B(x_0,r)\big).\end{aligned}$$

Hence, we can find j_0 such that $\mathcal{L}^n(A_{j_0}) \ge \frac{1}{2}\mathcal{L}^n(B(x_0,r))$. From the definition of g_j we have $g_j(x) = u \circ f(x)$ for all $x \in A_{j_0}$, and hence $g_j(x) - g_i(x) = 0$ on A_{j_0} for all $i, j \ge j_0$. Denote $g = g_i - g_j$. It follows from the Poincaré inequality, $g_{A_{j_0}} = 0$ and $\mathcal{L}^n(A_{j_0}) \ge \frac{1}{2}\mathcal{L}^n(B(x_0,r))$ that

$$\begin{aligned}&\int_{B(x_0,r)} |g_i - g_j|\\ &\quad = \int_{B(x_0,r)} |g| = \int_{B(x_0,r)} \big|g(x) - g_{A_{j_0}}\big|\,dx\\ &\quad \le \int_{B(x_0,r)} \big|g(x) - g_{B(x_0,r)}\big|\,dx + \mathcal{L}^n\big(B(x_0,r)\big)|g_{A_{j_0}} - g_{B(x_0,r)}|\end{aligned}$$

$$
\begin{aligned}
&\le C(n)r\int_{B(x_0,r)}|\nabla g| + \frac{\mathcal{L}^n(B(x_0,r))}{\mathcal{L}^n(A_{j_0})}\int_{A_{j_0}}\big|g(x)-g_{B(x_0,r)}\big|\,dx\\
&\le C(n,r)\int_{B(x_0,r)}|\nabla g| = C(n,r)\int_{B(x_0,r)}|\nabla g_j-\nabla g_i|.
\end{aligned}
$$

Since $\{\nabla g_j\}$ is a Cauchy sequence in $L^1(B(x_0,r),\mathbb{R}^n)$ we obtain that $\{g_j\}$ is a Cauchy sequence in $L^1(B(x_0,r))$. And due to convergence of g_j to $u\circ f$ in points of $\bigcup_{j=1}^{\infty}A_j$, that is, almost everywhere, we have $g_j\to u\circ f$ in $L^1(B(x_0,r))$. The definition of the weak derivative gives us

$$
\int_{B(x_0,r)}\nabla g_j(x)\phi(x) = -\int_{B(x_0,r)}g_j(x)\nabla\phi(x)
$$

for each function $\phi\in C^\infty(B(x_0,r))$ with compact support. Since $\nabla g_j\to h$ in $L^1(B(x_0,r),\mathbb{R}^n)$ and $g_j\to u\circ f$ in $L^1(B(x_0,r))$, by passing j to infinity we get

$$
\int_{B(x_0,r)}h(x)\phi(x) = -\int_{B(x_0,r)}u\circ f(x)\nabla\phi(x). \tag{4.10}
$$

This means, that h is the weak gradient of $u\circ f$ on $B(x_0,r)$ and hence $u\circ f\in W^{1,1}_{\mathrm{loc}}$.

It remains to show that the familiar formula (4.6) holds a.e. in Ω_1. Our f satisfies Luzin (N^{-1}) condition and hence $\nabla u_k(f(x))$ is well defined a.e. in $B(x_0,r)$. Passing to a subsequence (still denoted as u_k), we may assume that $\nabla u_k(x)\to Du(x)$ on $B\setminus N$, where N is a Borel measurable set of zero measure. It easily follows that

$$
\nabla u_k\big(f(x)\big)Df(x)\overset{k\to\infty}{\to} Du\big(f(x)\big)Df(x) \tag{4.11}
$$

on $B(x_0,r)\setminus f^{-1}(N)$, that is, almost everywhere. From (4.9) and (4.10), we know that $D(u_k\circ f)=\nabla u_k\circ f\cdot Df$ converges to $D(u\circ f)$. The condition (4.6) a.e. now follows easily. ☐

Remark 4.6. For $q>n$ it is not necessary to assume (iii) in (4.1). In this case, we know that $|Df|<C$ and hence we may estimate (4.3) by

$$
\int\big|Du\big(f(x)\big)\big|^q\big|Df(x)\big|^q\alpha\big(C\big|Du\big(f(x)\big)\big|\big)\,dx \tag{4.12}
$$

and we can finish similarly to the estimate on U.

Analogously for $1<q<n$ it is not necessary to assume (iii) in (4.4). We know that $|Df|>C$ and α is nonincreasing so we can again estimate by (4.12).

Proof of Theorem 1.1. The result follows from Theorem 4.1 and Theorem 4.5. ☐

5. Necessity of q-quasiconformal mappings

In this section, we will use ideas of Gold'stein, Gurov and Romanov [5]. They proved that a homeomorphism $F:\Omega\to\mathbb{R}^n$ which induces a bounded operator from $W^{1,q}(\Omega_2)$ to $W^{1,q}(\Omega_1)$ is a q-quasiconformal mapping (see [5] for details and [6] for history of similar problems).

Lemma 5.1. *Suppose that a homeomorphism $f:\Omega_1\to\Omega_2$ induces the operator $T_f: WL^{\Phi}(\Omega_2)\to WL^{\Phi}(\Omega_1)$, then f is in $WL^{\Phi}_{\mathrm{loc}}(\Omega_1)$. If we moreover assume that $L^{\Phi}_{\mathrm{loc}}(\Omega_1)$ is embedded into $L^p_{\mathrm{loc}}(\Omega_1)$ for some $p>n-1$, then f is differentiable a.e.*

Proof. Fix $R>0$. Mapping f is a homeomorphism and therefore the set

$$A_R:=\big\{x\in\Omega_1: f(x)\in B(0,R)\big\}=f^{-1}\big(B(0,R)\big)$$

is open. Fix $1\le i\le n$. Plainly there is a Lipschitz function $u:\Omega_2\to\mathbb{R}$ such that

$$u(x)=\begin{cases}x_i & \text{for } x\in\Omega_2, |x|<R\\ 0 & \text{for } x\in\Omega_2, |x|>R+1.\end{cases}$$

Hence, $u\in W^{1,\infty}_{\mathrm{loc}}(\Omega_2)\subset WL^{\Phi}_{\mathrm{loc}}(\Omega_2)$ implies $T_f(u)=u\circ f\in WL^{\Phi}_{\mathrm{loc}}(\Omega_1)$. If $|f(x)|<R$, then $u\circ f=f_i(x)$ and thus $f_i(x)\in WL^{\Phi}(A_R)$.

It is well known that each homeomorphism in the Sobolev space $W^{1,p}_{\mathrm{loc}}$ is differentiable a.e. if $p>n-1$ (see, e.g., [18]). From the embedding of L^{Φ}_{loc} into L^p_{loc}, we thus obtain that f is differentiable a.e. ☐

In the proof of Theorem 5.3, we will need the following elementary lemma [5, Lemma 3.5].

Lemma 5.2. *Let $f:\Omega\to\mathbb{R}^n$ be a continuous mapping and $G\subset\mathbb{R}^k$. Suppose that $\{K_y\}_{y\in G}$ is a family of pairwise disjoint compact sets such that $K_y\subset f(\Omega)$. Then $\mathcal{L}^n(f^{-1}(K_y))=0$ for all $y\in G$ except possibly a countable subset of G.*

Theorem 5.3. *Let $q\ge 1$ and suppose that Φ is a Young function such that*

$$\liminf_{s\to\infty}\frac{\Phi^{-1}(s)}{\Phi^{-1}(Ks)}\le CK^{-\frac{1}{q}}\quad\text{for every } K>0. \tag{5.1}$$

Suppose that a homeomorphism $f:\Omega_1\to\Omega_2$ induces the bounded operator $T_f: WL^{\Phi}(\Omega_2)\to WL^{\Phi}(\Omega_1)$, then

$$\big|Df_j(x_0)\big|^q\le Cf_v'(x_0)\quad\text{for almost all } x_0\in\Omega_1. \tag{5.2}$$

If we moreover assume that f is differentiable a.e., then f is q-quasiconformal.

Proof. By Theorem 2.3, we know that $f'_v(x) < \infty$ a.e. Fix $\varepsilon > 0$ and a point $x_0 \in \Omega_1$ such that $f'_v(x_0) < \infty$. There is r_0 such that for all $r \in (0, r_0)$ we have

$$\big|f\big(B(x_0, 2r)\big)\big| \leq \big(f'_v(x_0) + \varepsilon\big)\big|B(x_0, 2r)\big| = \big(f'_v(x_0) + \varepsilon\big)2^n|B(x_0, r)|. \tag{5.3}$$

Set $M = (f'_v(x_0) + \varepsilon)2^n$. Let us call a cube Qh-regular if all its edges are parallel to the coordinate axes, the length of the edge is h and every vertex has the form $[k_1h, k_2h, \dots, k_nh]$ where $k_1, k_2, \dots, k_n$ are integers. Fix $r < r_0$ and choose $h > 0$ such that

$$h < \frac{1}{2\sqrt{n}} \operatorname{dist}\big(f\big(S(x_0, 2r)\big), f\big(S(x_0, r)\big)\big).$$

Let A be the union of all h-regular cubes Q such that $Q \cap f(B(x_0, r)) \neq \emptyset$. It is evident that

$$f\big(B(x_0, r)\big) \subset A \subset f\big(B(x_0, 2r)\big).$$

Fix $j \in \{1, \dots, n\}$ and let us focus on the jth coordinate. Denote the hyperplanes $x_j = th$ by L_t. The hyperplanes L_m (m is an integer) divide $\mathbb{R}^n$ into the layers

$$Z_m = \big\{x \in \mathbb{R}^n : mh < x_j < (m+1)h\big\}.$$

Put $A_m = Z_m \cap A$.

For every A_m , we construct three functions:

$$\psi_{m,1} = x_j - mh, \qquad \psi_{m,2} = (m+1)h - x_j,$$
$$\psi_{m,3} = \frac{h}{2} - \operatorname{dist}\big(P_j(x), P_j(A_m)\big).$$

Here $P_j : \mathbb{R}^n \to \mathbb{R}^{n-1}_j$ is the orthogonal projection of $\mathbb{R}^n$ onto $\mathbb{R}^{n-1}_j$. Consider the functions

$$\psi_m = \max\big\{0, \min\{\psi_{m,1}, \psi_{m,2}, \psi_{m,3}\}\big\} \quad \text{and} \quad \psi = \sum_m \psi_m.$$

Put $E = \{x \in G : \psi(x)$ is not differentiable at the point $x\}$. It follows from the definition of ψ that

(1) $\operatorname{supp}(\psi) \subset f(B(x_0, 2r))$;
(2) ψ is Lipschitz with constant 1;
(3) $\psi \in WL^\Phi(f(\Omega))$;
(4) ψ is differentiable almost everywhere;
(5) $\psi(x) = \pm x_j + \text{const}$ in all components of the set $f(B(x_0, r)) \setminus E$.

The set $E \cap f(B(x_0, r))$ belongs to a union of hyperplanes $L_{t_1}, L_{t_2}, \dots, L_{t_s}$ where $2t_i$ is an integer. By Lemma 5.2 for almost all small translations τ_y parallel to the axis x_j, we have

$$\left|f^{-1}\left(\tau_y\left(\bigcup_{i=-\infty}^{\infty} L_{\frac{i}{2}}\right) \cap f\big(\overline{B(x_0, r)}\big)\right)\right| = 0.$$

Thus we can assume without loss of generality that

$$\big|f^{-1}\big(E\cap f\big(B(x_0,r)\big)\big)\big|=0. \tag{5.4}$$

Otherwise it is possible to change the jth coordinate of the point $[0,0,\dots,0]$ at the beginning of the construction of ψ.

By the assumption of the theorem $T_f(\psi)=\psi\circ f\in WL^{\Phi}(\Omega_1)$. It follows from (5) and (5.4) that

$$(\psi\circ f)(x)=\pm f_j(x)+\mathrm{const}$$

for almost all $x\in B(x_0,r)$. This fact and the continuity of T_f give us

$$\begin{aligned}\|Df_j\|_{L^{\Phi}(B(x_0,r))}=\big\|D(\psi\circ f)\big\|_{L^{\Phi}(B(x_0,r))}&\le C\|D\psi\|_{L^{\Phi}(f(B(x_0,2r)))}\\&\le C\|1\|_{L^{\Phi}(f(B(x_0,2r)))},\end{aligned}$$

because ψ is Lipschitz with constant 1 and supported in $f(B(x_0,2r))$. Hence, we can use (2.3), (5.3) and (5.1) to obtain

$$\begin{aligned}\liminf_{r\to 0_+}\frac{\|Df_j\|_{L^{\Phi}(B(x_0,r))}}{\|1\|_{L^{\Phi}(B(x_0,r))}}&\le C\liminf_{r\to 0_+}\frac{\|1\|_{L^{\Phi}(f(B(x_0,2r)))}}{\|1\|_{L^{\Phi}(B(x_0,r))}}\\&\le C\liminf_{r\to 0_+}\frac{\Phi^{-1}\big(\frac{1}{|B(x_0,r)|}\big)}{\Phi^{-1}\big(\frac{1}{|f(B(x_0,2r))|}\big)}\\&\le C\liminf_{r\to 0_+}\frac{\Phi^{-1}\big(\frac{1}{|B(x_0,r)|}\big)}{\Phi^{-1}\big(\frac{1}{M|B(x_0,r)|}\big)}\le CM^{\frac{1}{q}}.\end{aligned}$$

Theorem 1.3 now gives us

$$\big|Df_j(x_0)\big|\le CM^{\frac{1}{q}}=C\big(f'_v(x_0)+\varepsilon\big)^{\frac{1}{q}}$$

for almost all $x_0\in\Omega$ and by letting $\varepsilon\to 0$ we obtain (5.2). If we know that f is differentiable a.e., we may use Theorem 2.3 to conclude that f is q-quasiconformal. □

Proof of Theorem 1.2. We know that $\Phi(t)\sim t^q\log^{\alpha}(e+t)$ for large t and thus

$$\Phi^{-1}(s)\sim s^{\frac{1}{q}}\log^{-\frac{\alpha}{q}}(e+s)$$

for large values of s. Therefore, we obtain

$$\lim_{s\to\infty}\frac{\Phi^{-1}(s)}{\Phi^{-1}(Ks)}\le CK^{-\frac{1}{q}}\lim_{s\to\infty}\bigg(\frac{\log(e+s)}{\log(e+Ks)}\bigg)^{-\frac{\alpha}{q}}=CK^{-\frac{1}{q}}$$

and the statement now follows easily from Theorem 5.3. The requirement that f is differentiable a.e. for $q>n-1$ is verified by Theorem 5.1 and for $q\le n-1$ we have assumed it. □

6. Construction of examples

In the theory of n-quasiconformal mappings or their generalization one often uses a radial stretching $f(x) = \frac{x}{|x|}\rho(|x|)$ as a counterexample. This f maps spheres of radius r to spheres of radius $\rho(r)$ (or cubes to cubes if $|\cdot|$ denotes maximum norm). However, these maps are too symmetric and thus not critical for q-quasiconformal maps. Instead we need to use mappings that map rectangles to rectangles and are inspired by some construction from [11, Section 5].

6.1. Canonical transformation. If $c \in \mathbb{R}^n$, $a, b > 0$, we use the notation

$$Q(c,a,b) := [c_1 - a, c_1 + a] \times \cdots \times [c_{n-1} - a, c_{n-1} + a] \times [c_n - b, c_n + b]$$

for the interval with center at c and halfedges a in the first $n-1$ coordinates and b in the last coordinate. If $Q = Q(c,a,b)$, the affine mapping

$$\varphi_Q(y) = (c_1 + ay_1, \dots, c_{n-1} + ay_{n-1}, c_n + by_n)$$

is called the canonical parametrization of the interval Q. Let P, P' be concentric intervals, $P = Q(c,a,b)$, $P' = Q(c,a',b')$, where $0 < a < a'$ and $0 < b < b'$. We set

$$\varphi_{P,P'}(t,y) = (1-t)\varphi_P(y) + t\varphi_{P'}(y), \quad t \in [0,1], y \in \partial[-1,1]^n.$$

This mapping is called the canonical parametrization of the rectangular annulus $P' \setminus P^\circ$, where P° is the interior of P.

Now, we consider two rectangular annuli, $P' \setminus P^\circ$, and $\tilde{P}' \setminus \tilde{P}^\circ$, where $P = Q(c,a,b)$, $P' = Q(c,a',b')$, $\tilde{P} = Q(\tilde{c},\tilde{a},\tilde{b})$ and $\tilde{P}' = Q(\tilde{c},\tilde{a}',\tilde{b}')$, The mapping

$$h = \varphi_{\tilde{P},\tilde{P}'} \circ (\varphi_{P,P'})^{-1}$$

is called the *canonical transformation* of $P' \setminus P^\circ$ onto $\tilde{P}' \setminus \tilde{P}^\circ$.

We will need the estimate of the derivate of h on $P' \setminus P^\circ$. Let us assume that

$$a' \le C_0 a, \qquad \tilde{a}' \le C_0 \tilde{a}, \qquad b' \le C_0 b \quad \text{and} \quad \tilde{b}' \le C_0 \tilde{b}. \tag{6.1}$$

It can be computed (see [11, Section 5] for details) that

$$\bigl|Dh(x)\bigr| \sim \max\left\{\frac{\tilde{a}'}{a'}, \frac{\tilde{b}' - \tilde{b}}{b' - b}\right\} \quad \text{and} \quad J_h(x) \sim \left(\frac{\tilde{a}'}{a'}\right)^{n-1} \frac{\tilde{b}' - \tilde{b}}{b' - b} \tag{6.2}$$

for a.e. x in parts A of Figure 1 and similarly we get that

$$\begin{aligned} \bigl|Dh(x)\bigr| &\sim \max\left\{(n-2)\frac{\tilde{a}'}{a'}, \frac{\tilde{a}' - \tilde{a}}{a' - a}, \frac{\tilde{b}'}{b'}\right\} \quad \text{and} \\ J_h(x) &\sim \left(\frac{\tilde{a}'}{a'}\right)^{n-2} \frac{\tilde{a}' - \tilde{a}}{a' - a} \frac{\tilde{b}'}{b'} \end{aligned} \tag{6.3}$$

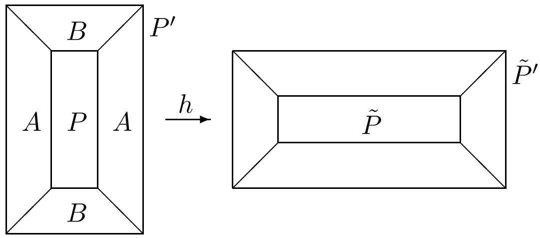

FIGURE 1. The canonical transformation of $P' \setminus P^\circ$ onto $\tilde{P}' \setminus \tilde{P}^\circ$ for $n = 2$.

for a.e. x in parts B of Figure 1. All the constants involved depend only on the dimension n and the constant C_0 from (6.1). Moreover, we can easily estimate the volume of these sets as

$$\mathcal{L}^n(A) \sim \big(a'\big)^{n-1}\big(b'-b\big) \quad \text{and} \quad \mathcal{L}^n(B) \sim \big(a'\big)^{n-2}\big(a'-a\big)b'. \tag{6.4}$$

EXAMPLE 6.1. Let $q < n$ and $\alpha > 0$. Then there is a q-quasiconformal homeomorphism $f \in W^{1,q}((-1,1)^n, (-1,1)^n)$ such that $f \notin WL^q \log^\alpha L_{\mathrm{loc}}$. It follows that the composition with the identity mapping $u(x) = x$ satisfies $u \circ f \notin WL^q \log^\alpha L_{\mathrm{loc}}$.

Proof. If $q = 1$, then we set

$$f(x_1, x_2, \dots, x_n) = \bigg(\frac{g(|x_1|)\operatorname{sgn} x_1}{g(1)}, x_2, \dots, x_n\bigg),$$

where

$$g(s) = \int_0^s \frac{1}{t \log^{1+\frac{\alpha}{2}} \frac{t}{2}}\, dt.$$

It is easy to see that f is 1-quasiconformal and $|Df| \notin L^1 \log^\alpha L_{\mathrm{loc}}$.

Suppose now that $1 < q < n$. Set

$$\begin{aligned} &\beta = 1, \qquad \gamma = \frac{q-1}{n-1}, \qquad \delta = \frac{\alpha}{2}, \\ &\zeta = \frac{1+\alpha-q\delta}{n-1}, \qquad \eta = \frac{\delta(q-1)-1}{n-1} + \zeta. \end{aligned} \tag{6.5}$$

With the help of (6.5), it is not difficult to verify that

$$\begin{aligned} \beta(q-1) - \gamma(n-1) &= 0, \\ \delta(q-1) + (\zeta-\eta)(n-1) - 1 &= 0, \\ \delta q + \zeta(n-1) - \alpha &= 1. \end{aligned} \tag{6.6}$$

Let us set

$$a_k = \frac{1}{(k+1)^\gamma \log^\zeta(e+k)}, \qquad b_k = \frac{1}{(k+1)^\beta},$$
$$\tilde{a}_k = \frac{1}{\log^\eta(e+k)} \quad \text{and} \quad \tilde{b}_k = \frac{1}{\log^\delta(e+k)}.$$

Our mapping f will be defined as the corresponding canonical transformation from

$$P_k := Q(0, a_k, b_k) \setminus Q(0, a_{k+1}, b_{k+1}) \quad \text{onto} \quad Q(0, \tilde{a}_k, \tilde{b}_k) \setminus Q(0, \tilde{a}_{k+1}, \tilde{b}_{k+1})$$

for every $k \in \mathbb{N}_0$. It is easy to check that f is a homeomorphism, absolutely continuous on almost all lines parallel to coordinate axes and differentiable a.e. To get our conclusion, it is now enough to show that the corresponding integrals of the derivative are finite or infinite.

Clearly,

$$\frac{1}{(k+1)^\omega \log^\eta(e+k)} - \frac{1}{(k+2)^\omega \log^\eta(e+k+1)} \sim \frac{1}{(k+1)^{\omega+1} \log^\eta(e+k)}$$

and

$$\frac{1}{\log^\omega(e+k)} - \frac{1}{\log^\omega(e+k+1)} \sim \frac{1}{(k+1)\log^{\omega+1}(e+k)}$$

for every $\omega > 0$. From (6.5), we obtain that $\gamma < \beta$ and hence we can use (6.2) and (6.3) to estimate

$$|Df(x)| \lesssim \frac{(k+1)^\beta}{\log^\delta(e+k)} \quad \text{and} \quad Jf \sim \left(\frac{(k+1)^\gamma}{\log^{\eta-\zeta}(e+k)}\right)^{n-1} \frac{(k+1)^\beta}{\log^{\delta+1}(e+k)} \tag{6.7}$$

for a.e. $x \in P_k$. Note that the value of constant C_0 in (6.1) does not depend on k and thus the constants in $\lesssim$ and $\sim$ above are independent of k. Now we may use (6.6) to obtain

$$K_q(x) = \frac{|Df(x)|^q}{J_f(x)} \lesssim \frac{(k+1)^{\beta(q-1)-\gamma(n-1)}}{\log^{\delta(q-1)+(\zeta-\eta)(n-1)-1}(e+k)} = 1. \tag{6.8}$$

From (6.4), we know that

$$\mathcal{L}^n(P_k) \sim \frac{1}{(k+1)^{(n-1)\gamma+\beta+1} \log^{\zeta(n-1)}(e+k)} \tag{6.9}$$

and thus we use (6.7) and (6.6) to obtain

$$\begin{aligned}\int_{Q(0,a_1,b_1)} |Df|^q \, dx &\lesssim \sum_{k=0}^\infty \int_{P_k} |Df|^p \, dx \\ &\lesssim \sum_{k=0}^\infty \frac{1}{(k+1)^{(n-1)\gamma+\beta+1} \log^{\zeta(n-1)}(e+k)} \frac{(k+1)^{\beta q}}{\log^{\delta q}(e+k)} \\ &\lesssim \sum_{k=0}^\infty \frac{1}{(k+1)\log^{1+\alpha}(e+k)} < \infty.\end{aligned}$$

It follows that $f \in W^{1,q}$ and by (6.8) we know that f is q-quasiconformal mapping. Note that in (6.7) we have not only $\lesssim$ (on both parts A and B in Figure 1) but also $\sim$ on a set of measure comparable to (6.9) (on part A). Hence, analogously as above we may use (6.6) to get

$$\begin{aligned}\int_{Q(0,a_1,b_1)} |Df|^q \log^\alpha\bigl(e+|Df|\bigr) &\sim \sum_{k\in\mathbb{N}_0} \frac{\log^\alpha(e+k)}{(k+1)\log^{\delta q+\zeta(n-1)}(e+k)}\\ &= \sum_{k\in\mathbb{N}_0} \frac{1}{(k+1)\log(e+k)} = \infty.\end{aligned}$$ □

EXAMPLE 6.2. Let $q > n$ and $\alpha < 0$. There are q-quasiconformal homeomorphism $f \in W^{1,1}((-1,1)^n,(-1,1)^n)$ and $u \in WL^q\log^\alpha L((-1,1)^n,(-1,1)^n)$ such that $u\circ f \notin WL^q\log^\alpha L_{\mathrm{loc}}$.

Proof. Set

$$\begin{aligned}&A = \frac{1}{n-1}, \qquad B = \frac{1-\alpha}{q-1}, \qquad H = \frac{q-1-\alpha}{q-1},\\ &E = \frac{-\alpha}{2q}, \qquad \gamma = q-1 \quad\text{and}\quad \delta = n-1.\end{aligned} \tag{6.10}$$

With the help of (6.10) and it is not difficult to verify that

$$\begin{aligned}(n-1)\gamma-(q-1)\delta &= 0,\\ q-1-H(q-1)-A(n-1)+B(q-1) &= 0,\\ A(n-1)-B(q-1)+qE &< 0,\\ qE-H(q-1)+q-\alpha &> 1.\end{aligned} \tag{6.11}$$

Let us set

$$\begin{aligned}&a_k = \frac{1}{\log^A(e+k)}, \qquad b_k = \frac{1}{\log^B(e+k)},\\ &\tilde a_k = \frac{1}{(k+1)^\gamma} \quad\text{and}\quad \tilde b_k = \frac{1}{(k+1)^\delta\log^H(e+k)}.\end{aligned} \tag{6.12}$$

Our mapping f will be defined as the corresponding canonical transformation from

$$P_k := Q(0,a_k,b_k)\setminus Q(0,a_{k+1},b_{k+1})$$

onto

$$\tilde P_k := Q(0,\tilde a_k,\tilde b_k)\setminus Q(0,\tilde a_{k+1},\tilde b_{k+1})$$

for every $k \in \mathbb{N}_0$. It is easy to check that f is a homeomorphism, absolutely continuous on almost all lines parallel to coordinate axes and differentiable a.e.

From (6.10), we obtain $\gamma > \delta$ and hence analogously to the previous example we can use (6.2) and (6.3) to estimate

$$\begin{aligned} \big|Df(x)\big| &\lesssim \frac{\log^{B+1-H}(e+k)}{(k+1)^{\delta}} \quad \text{and} \\ Jf &\sim \left(\frac{\log^{A}(e+k)}{(k+1)^{\gamma}}\right)^{n-1} \frac{\log^{B+1-H}(e+k)}{(k+1)^{\delta}} \end{aligned} \tag{6.13}$$

for a.e. $x \in P_k$. In fact it is easy to see that for every $x \in (-a_{k+1}, a_{k+1})^{n-1} \times (b_{k+1}, b_k)$ we have

$$\left|\frac{\partial f(x)}{\partial x_n}\right| \sim \frac{\log^{B+1-H}(e+k)}{(k+1)^{\delta}}. \tag{6.14}$$

Now we may use (6.11) to obtain

$$K_q(x) = \frac{|Df(x)|^q}{J_f(x)} \lesssim \frac{\log^{q-1-H(q-1)-A(n-1)+B(q-1)}(e+k)}{(k+1)^{\delta(q-1)-\gamma(n-1)}} = 1. \tag{6.15}$$

By (6.4) we get

$$\mathcal{L}^n(P_k) \sim \frac{1}{(k+1)\log^{1+(n-1)A+B}(e+k)} \tag{6.16}$$

and thus we use (6.13) and (6.11) to obtain

$$\begin{aligned} &\int_{Q(0,a_1,b_1)} |Df|^q \, dx \\ &\quad \lesssim \sum_{k=0}^{\infty} \frac{1}{(k+1)\log^{1+(n-1)A+B}(e+k)} \frac{\log^{(B+1-H)q}(e+k)}{(k+1)^{\delta q}} < \infty. \end{aligned}$$

It follows that $f \in W^{1,q}$ and by (6.15) we know that f is q-quasiconformal mapping.

It is easy to see that for every $k \in \mathbb{N}$ we can construct a function u_k such that

$$\begin{aligned} &\text{(i)} && u_k \in W_0^{1,\infty}\big([-2\tilde{a}_k, 2\tilde{a}_k]^{n-1} \times [\tilde{b}_{k+1}, \tilde{b}_k]\big), \\ &\text{(ii)} && |Du_k| \le \frac{(k+1)^{\delta}}{\log^{E-H+1}(e+k)}, \\ &\text{(iii)} && \frac{\partial u}{\partial x_n} = \pm \frac{(k+1)^{\delta}}{\log^{E-H+1}(e+k)} \quad \text{a.e. on } [-\tilde{a}_k, \tilde{a}_k]^{n-1} \times [\tilde{b}_{k+1}, \tilde{b}_k], \\ &\text{(iv)} && \frac{\partial u}{\partial x_j} = 0, \quad j = 1, \dots, n-1 \text{ on } [-\tilde{a}_k, \tilde{a}_k]^{n-1} \times [\tilde{b}_{k+1}, \tilde{b}_k]. \end{aligned} \tag{6.17}$$

We set

$$u = \sum_{k=1}^{\infty} u_k(x).$$

With the help of (6.11) and (6.12) this implies that $u \in WL^q \log^\alpha L$ since

$$\begin{aligned}
&\int_{\mathbb{R}^n} |Du|^q \log^\alpha |Du| \\
&\quad = \sum_{k=1}^\infty \int_{[-2\tilde{a}_k, 2\tilde{a}_k]^{n-1} \times [\tilde{b}_{k+1}, \tilde{b}_k]} |Du|^q \log^\alpha \big(|Du|\big) \\
&\quad \lesssim \sum_{k=1}^\infty \frac{1}{k^{(n-1)\gamma+\delta+1} \log^H (e+k)} \frac{(k+1)^{q\delta}}{\log^{q(E-H+1)}(e+k)} \log^\alpha (e+k) \\
&\quad = \sum_{k=1}^\infty \frac{1}{k \log^{H+(E-H+1)q-\alpha}(k)} < \infty.
\end{aligned}$$

It is easy to see that

$$[-\tilde{a}_k, \tilde{a}_k]^{n-1} \times [\tilde{b}_{k+1}, \tilde{b}_k] \supset f\big([-a_{k+1}, a_{k+1}]^{n-1} \times [b_{k+1}, b_k]\big)$$

and on $[-a_{k+1}, a_{k+1}]^{n-1} \times [b_{k+1}, b_k]$ we have with the help of (6.14) and (6.17)(iii) that

$$\left| \frac{\partial}{\partial x_n} (u \circ f)(x) \right| = \left| \frac{\partial u(f(x))}{\partial x_n} \cdot \frac{\partial f(x)}{\partial x_n} \right| \sim \log^{B-E}(e+k).$$

Together with (6.12), this gives us

$$\begin{aligned}
&\int \big|D(u \circ f)\big|^q \log^\alpha \big|D(u \circ f)\big| \\
&\quad \geq \sum_{k=1}^\infty \int_{[-a_{k+1}, a_{k+1}]^{n-1} \times [b_{k+1} - b_k]} \left| \frac{\partial u \circ f}{\partial x_n} \right|^q \log^\alpha \left(\left| \frac{\partial u \circ f}{\partial x_n} \right| \right) \\
&\quad \gtrsim \sum_{k=1}^\infty \frac{1}{(k+1) \log^{1+(n-1)A+B}(e+k)} \log^{(B-E)q}(e+k) \log^\alpha \big(\log(e+k)\big).
\end{aligned}$$

From (6.11), we easily see that this sum diverges. □

References

[1] K. Astala, T. Iwaniec and G. Martin, *Elliptic partial differential equations and quasiconformal mappings in the plane*, Princeton Mathematical Series, vol. 48, Princeton University Press, Princeton, NJ, 2009. MR 2472875

[2] H. Federer, *Geometric measure theory*, Die Grundlehren der mathematischen Wissenschaften, vol. 153, Springer-Verlag, New York, 1969 (Second edition 1996). MR 0257325

[3] F. Farroni and R. Giova, *Quasiconformal mappings and exponentially integrable functions*, Studia Math. **203** (2011), 195–203. MR 2784024

[4] D. Gallardo, *Weighted weak type integral inequalities for the Hardy-Littlewood maximal operator*, Israel J. Math. **67** (1989), no. 1, 95–108. MR 1021364

[5] V. Gold'stein, L. Gurov and A. Romanov, *Homeomorphisms that induce monomorphisms of Sobolev spaces*, Israel J. Math. **91** (1995), 31–60. MR 1348304

[6] V. Gold'stein and Y. G. Reshetnyak, *Quasiconformal mappings and Sobolev spaces*, Kluwer Academic Publishers, Dordrecht, 1990. MR 1136035

[7] S. Hencl, *Absolutely continuous functions of several variables and quasiconformal mappings*, Z. Anal. Anwendungen **22** (2003), no. 4, 767–778. MR 2036929

[8] S. Hencl, L. Kleprlík and J. Malý, *Composition operator and Sobolev–Lorentz spaces* $WL^{n,q}$, preprint MATH-KMA-2012/404, available at `http://www.karlin.mff.cuni.cz/kma-preprints/`.

[9] S. Hencl and P. Koskela, *Mappings of finite distortion: Composition operator*, Ann. Acad. Sci. Fenn. Math **33** (2008), 65–80. MR 2386837

[10] S. Hencl and P. Koskela, *Composition of quasiconformal mappings and functions in fractional Triebel–Lizorkin spaces*, Math. Nachr. **286** (2013), 669–678. MR 3060838

[11] S. Hencl, P. Koskela and J. Malý, *Regularity of the inverse of a Sobolev homeomorphism in space*, Proc. Roy. Soc. Edinburgh Sect. A **136** (2006), no. 6, 1267–1285. MR 2290133

[12] S. Hencl and J. Malý, *Jacobians of Sobolev homeomorphisms*, Calc. Var. Partial Differential Equations **38** (2010), 233–242. MR 2610531

[13] T. Iwaniec and G. Martin, *Geometric function theory and nonlinear analysis*, Oxford Mathematical Monographs, Clarendon Press, Oxford, 2001. MR 1859913

[14] L. Kleprlík, *Composition operators on W^1X are necessarily induced by quasiconformal mappings*, to appear in Cent. Eur. J. Math.

[15] L. Kleprlík, *The zero set of the Jacobian and composition of mappings*, J. Math. Anal. Appl. **386** (2012), 870–881. MR 2834794

[16] P. Koskela, D. Yang and Y. Zhou, *Pointwise characterization of Besov and Triebel–Lizorkin spaces and quasiconformal mappings*, Adv. Math. **226** (2011), no. 4, 3579–3621. MR 2764899

[17] P. Koskela, *Lectures on quasiconformal and quasisymmetric mappings*, available at `http://users.jyu.fi/~pkoskela/`.

[18] J. Onninen, *Differentiability of monotone Sobolev functions*, Real Anal. Exchange **26** (2000), no. 2, 761–772. MR 1844392

[19] M. M. Rao and Z. D. Ren, *Theory of Orlicz spaces*, Monographs and Textbooks in Pure and Applied Mathematics, vol. 146, Marcel Dekker, New York, 1991. MR 1113700

[20] H. M. Reimann, *Functions of bounded mean oscillation and quasiconformal mappings*, Comment. Math. Helv. **49** (1974), 260–276. MR 0361067

[21] S. Rickman, *Quasiregular mappings*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 26, Springer-Verlag, Berlin, 1993. MR 1238941

[22] P. Tukia and J. Väisäla, *Quasiconformal extension from dimension n to $n+1$*, Ann. of Math. (2) **115** (1982), no. 2, 331–348. MR 0647809

[23] J. Väisäla, *Quasi-symmetric embeddings in Euclidian spaces*, Trans. Amer. Math. Soc. **264** (1981), 191–204. MR 0597876

[24] W. P. Ziemer, *Weakly differentiable functions*, Graduate Texts in Mathematics, vol. 120, Springer-Verlag, New York, 1989. MR 1014685

Stanislav Hencl, Department of Mathematical Analysis, Charles University, Sokolovská 83, 186 00 Prague 8, Czech Republic

E-mail address: `hencl@karlin.mff.cuni.cz`

Luděk Kleprlík, Department of Mathematical Analysis, Charles University, Sokolovská 83, 186 00 Prague 8, Czech Republic

E-mail address: `kleprlik@karlin.mff.cuni.cz`