Illinois Journal of Mathematics
Volume 56, Number 3, Fall 2012, Pages 861–883
S 0019-2082

STRICTLY SINGULAR OPERATORS IN ASYMPTOTIC ℓ_p BANACH SPACES

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ABSTRACT. We present a condition on higher order asymptotic behavior of basic sequences in a Banach space ensuring the existence of bounded noncompact strictly singular operators on a subspace. Applications concern asymptotic ℓ_p spaces, $1 \leq p < \infty$, in particular convexified mixed Tsirelson spaces and related asymptotic ℓ_p HI spaces.

Introduction

The research on conditions ensuring the existence of nontrivial strictly singular operators on/in Banach spaces increased in the last years, in connection with the famous "scalar-plus-compact" problem and following constructions of spaces with "few operators." The "scalar-plus-compact" problem asks if there is an infinite dimensional Banach space on which any bounded operator is a compact perturbation of a multiple of the identity. An important step towards solving this problem was made by Gowers and Maurey [17], who constructed the first HI (hereditarily indecomposable) space, X_{GM}, that is, a space without closed infinite dimensional subspaces which can be written as a direct sum of a pair of further closed infinite dimensional subspaces. Moreover, any operator on a subspace of X_{GM} is a strictly singular perturbation of an inclusion operator. An operator between Banach spaces is *strictly singular*, if none of its restrictions to an infinite dimensional subspace is an isomorphism. The construction of X_{GM} was followed by a class of asymptotic ℓ_1 HI spaces, started with X_{AD} by Argyros and Deliyanni [6], and by a class of asymptotic ℓ_p HI spaces [2], [13]. However, X_{GM} was shown to admit bounded strictly singular noncompact operators first on a subspace [18], and later—on

Received September 28, 2011; received in final form July 31, 2012.
The research supported by the Polish Ministry of Science and Higher Education grant N N201 421739.
2010 *Mathematics Subject Classification.* 46B20, 46B06.

the whole space [5]. Also [16], [11] gave some conditions on parameters of the constructed asymptotic ℓ_p HI spaces, ensuring the existence of nontrivial strictly singular operators on the space. Finally the "scalar-plus-compact" problem was solved positively by Argyros and Haydon [9] in the celebrated construction of an HI $\mathscr{L}_\infty$-space with "very few operators."

A hereditary version of the "scalar-plus-compact" problem, concerning operators on infinite dimensional subspaces of a given space, remains open. Construction of nontrivial strictly singular operators in a Banach space X is based usually on different types of asymptotic behavior of basic sequences in X with respect to an auxiliary basic sequence (e_n): local representation of (e_n) in X, provided for example by Krivine theorem in Lemberg's version [20], on one side, and "strong" domination of a spreading model of some basic sequence in X by (e_n) on the other [3], [24], [4], which ensures strict singularity of the constructed operator. In case of (e_n) equal to the usual basis of ℓ_1 the asymptotic "strong" domination appears whenever X contains a weakly null basic sequence not generating ℓ_1-spreading model [3]. Construction of nontrivial strictly singular operators based on the higher order representability of ℓ_1 in a space was studied in [24]. The operators on the whole space demands specific asymptotic properties of basic sequences in the dual space [5], [16], [11]. In the last two cases, strict singularity is related closely to the hereditary indecomposability of the considered space.

We present in this paper a general criterion (Theorem 4.2) ensuring the existence of nontrivial operators in a Banach space in terms of higher order asymptotic behavior of basic sequences with respect to an auxiliary basic sequence with some regularity properties, under partial unconditionality assumptions. To this end, we introduce and study α-strong domination, extending to higher order Schreier families the notion used in [24], [4]. Next, we apply the general construction in case of any asymptotic ℓ_p space X (Corollary 4.4), providing, as a counterpart of Krivine theorem, "local" lower estimates of basic sequences in X by the usual basis of the p-convexified Tsirelson-type space $T^{(p)}[\mathcal{S}_1,\theta]$ with θ related to asymptotic constants of X (Theorem 2.2). The further application brings nontrivial strictly singular operators on subspaces of convexified mixed Tsirelson spaces and asymptotic ℓ_p HI spaces of types constructed in [2], [13] under mild conditions on parameters defining the spaces (Corollaries 4.4, 4.7).

The paper is organized as follows: in Section 1 we recall basic notions, in Section 2 we focus on properties of asymptotic ℓ_p spaces, proving the "local" lower Tsirelson-type estimates. Section 3 is devoted to the study of α-strong domination, for limit $\alpha < \omega_1$, and in Section 4 we apply developed tools to construct nontrivial operators in general setting and in asymptotic ℓ_p spaces, with application to convexified mixed Tsirelson spaces and HI spaces.

1. Preliminaries

We recall the basic definitions and standard notation. By a *tree* we shall mean a nonempty partially ordered set $(\mathcal{T},\preceq)$ such that any set of the form $\{y\in\mathcal{T}: y\preceq x\}$, $x\in\mathcal{T}$, is linearly ordered and finite. If $\mathcal{T}'\subseteq\mathcal{T}$, then we say that $(\mathcal{T}',\preceq)$ is a *subtree* of $(\mathcal{T},\preceq)$. The smallest element of a tree (if it exists) is called its *root*, the maximal elements are called *terminal* nodes of a tree. A *branch* in a tree $\mathcal{T}$ is a maximal linearly ordered set in $\mathcal{T}$. The *height* of a finite tree is the maximal length of its branches. The *immediate successors* of $t\in\mathcal{T}$, denoted by $\mathrm{succ}(t)$, are all the nodes $s\in\mathcal{T}$ such that $t\prec s$ but there is no $r\in\mathcal{T}$ with $t\prec r\prec s$. An *order* of a node t of the tree with a root is defined as $\mathrm{ord}(t)=\#\{s\in\mathcal{T}: s\preceq t\}$.

We write $E<F$, for $E,F\subset\mathbb{N}$, if $\max E<\min F$. For any $J\subset\mathbb{N}$ by $[J]^{<\infty}$ we denote the family of finite subsets of J. A family $\mathcal{F}\subset[\mathbb{N}]^{<\infty}$ is *regular*, if it is *hereditary*, that is, for any $G\subset F$, $F\in\mathcal{F}$ also $G\in\mathcal{F}$, *spreading*, that is, for any integers $n_1<\cdots<n_k$ and $m_1<\cdots<m_k$ with $n_i\le m_i$, $i=1,\dots,k$, if $\{n_1,\dots,n_k\}\in\mathcal{F}$ then also $\{m_1,\dots,m_k\}\in\mathcal{F}$, and *compact* in the product topology of $2^{\mathbb{N}}$.

Let $\mathcal{F}$ be a compact family of finite subset of $\mathbb{N}$ endowed with the product topology of $2^{\mathbb{N}}$. We let $\mathcal{F}^0=\mathcal{F}$, for any ordinal α we set $\mathcal{F}^{\alpha+1}=\{F\in\mathcal{F}: F\text{—a limit point of }\mathcal{F}^\alpha\}$ and for any limit ordinal α we set $\mathcal{F}^\alpha=\bigcap_{\beta<\alpha}\mathcal{F}^\beta$. The Cantor–Bendixson index of $\mathcal{F}$, denoted by $\mathrm{CB}(\mathcal{F})$, is defined as the least α for which $\mathcal{F}^\alpha=\emptyset$.

Schreier families $(\mathcal{S}_\alpha)_{\alpha<\omega_1}$, introduced in [1], are defined by induction:

$$\mathcal{S}_0=\big\{\{k\}: k\in\mathbb{N}\big\}\cup\{\emptyset\},$$
$$\mathcal{S}_{\alpha+1}=\{F_1\cup\cdots\cup F_k: k\le F_1<\cdots<F_k, F_1,\dots,F_k\in\mathcal{S}_\alpha\},\quad \alpha<\omega_1.$$

If α is a limit ordinal, choose $\alpha_n\nearrow\alpha$ and set

$$\mathcal{S}_\alpha=\{F: F\in\mathcal{S}_{\alpha_n}\text{ and } n\le F\text{ for some } n\in\mathbb{N}\}.$$

It is well known that the Schreier families $\mathcal{S}_\alpha$, $\alpha<\omega_1$, are regular and $\mathrm{CB}(\mathcal{S}_\alpha)=\omega^\alpha+1$, $\alpha<\omega_1$ (cf. [1]). For any regular family $\mathcal{F}$ let

$$\mathcal{S}_1(\mathcal{F})=\{F_1\cup\cdots\cup F_k: k\le F_1,\dots,F_k\in\mathcal{F}, F_1,\dots,F_k\text{ pairwise disjoint}\}.$$

By an easy adaptation of the argument in Lemma 2.1 [21], one can show that $\mathcal{S}_1(\mathcal{S}_\alpha)=\mathcal{S}_{\alpha+1}$, $\alpha<\omega_1$ (cf. also [8]). We say that a sequence $E_1,\dots,E_k$ of subsets of $\mathbb{N}$ is $\mathcal{S}_\alpha$-*admissible*, $\alpha<\omega_1$, if $E_1<\cdots<E_k$ and $(\min E_i)_{i=1}^k\in\mathcal{S}_\alpha$.

Definition 1.1 ($\mathcal{S}_1$-admissible tree). We call $\mathcal{S}_1$-*admissible tree* of finite subsets of $\mathbb{N}$ any collection $(E_t)_{t\in\mathcal{T}}$, indexed by a finite tree $\mathcal{T}$ with a root 0, such that for any nonterminal node $t\in\mathcal{T}$ the sequence $(E_s)_{s\in\mathrm{succ}(t)}$ is $\mathcal{S}_1$-admissible and $E_t=\bigcup_{s\in\mathrm{succ}(t)}E_s$.

Remark 1.2. Any $\mathcal{S}_1$-admissible tree is a tree ordered by inclusion. By the definition of families $(\mathcal{S}_n)$ for any $\mathcal{S}_M$-admissible sequence $(E_k)_k$ of finite subsets of $\mathbb{N}$, $M\in\mathbb{N}$, there is an $\mathcal{S}_1$-admissible tree $(E_t)_{t\in\mathcal{T}}$ of height at most M with $E_0=\bigcup_k E_k$ and $(E_t)_{t\in\mathcal{T},t\,\mathrm{terminal}}=(E_k)_k$.

Given a Banach space X by B_X denote the closed unit ball of X. Let now X be a Banach space with a basis (e_i). The *support* of a vector $x=\sum_i x_ie_i$ is the set $\operatorname{supp}x=\{i\in\mathbb{N}: x_i\neq 0\}$. We write $x<y$ for vectors $x,y\in X$, if $\operatorname{supp}x<\operatorname{supp}y$. Any sequence $(x_n)\subset X$ with $x_1<x_2<\cdots$ is called a *block sequence*, a closed subspace spanned by an infinite block sequence (x_n) is called a *block subspace* and denoted by $[x_n]$. We say that a sequence (x_n) is seminormalized, if $0<\inf_n\|x_n\|\leq\sup_n\|x_n\|<\infty$. A basic sequence (x_n) *C-dominates* a basic sequence (y_n), $C\geq 1$, if for any $(a_n)\in c_{00}$ we have

$$\Big\|\sum_n a_ny_n\Big\|\leq C\Big\|\sum_n a_nx_n\Big\|.$$

Two basic sequences (x_n) and (y_n) are *C-equivalent*, $C\geq 1$, if (x_n) C-dominates (y_n) and (y_n) C-dominates (x_n). We shall use also the following notion of partial unconditionality [14] and equivalence of basic sequences.

Definition 1.3. Let $\mathcal{F}$ be a family of finite subsets of $\mathbb{N}$.

[14] A basic sequence (x_i) is *$\mathcal{F}$-unconditional*, if $\|\sum_{i\in F}a_ie_i\|\leq C\|\sum_i a_ie_i\|$ for any $(a_i)\in c_{00}$, any $F\in\mathcal{F}$ for some universal $C\geq 1$.

We say that basic sequences (x_i) and (y_i) are *$\mathcal{F}$-equivalent*, if $(x_i)_{i\in F}$ and $(y_i)_{i\in F}$ are C-equivalent for any $F\in\mathcal{F}$ for some universal $C\geq 1$.

In the language above a basic sequence (x_i) generates a spreading model (e_i) [12], iff for any $\varepsilon>0$ for some $n\in\mathbb{N}$ sequences $(e_i)_{i>n}$ and $(x_i)_{i>n}$ are $\mathcal{S}_1$-equivalent with constant $1+\varepsilon$. A basic sequence (x_i) generates an ℓ_1^α-spreading model, $\alpha<\omega_1$ [7], iff it is $\mathcal{S}_\alpha$-equivalent to the unit vector basis (abbreviated in the sequel as the u.v.b.) of ℓ_1.

We recall that a Banach space X with a basis is *ℓ_p-asymptotic*, $1\leq p\leq\infty$, if any normalized block sequence $n\leq x_1<\cdots<x_n$ is C-equivalent to the u.v.b. of ℓ_p^n, for any $n\in\mathbb{N}$ for some universal $C\geq 1$.

Finally, we say that a sequence $x_1<\cdots<x_n$ is *$\mathcal{S}_\alpha$-admissible*, $\alpha<\omega_1$, if $(\operatorname{supp}x_i)_{i=1}^n$ is $\mathcal{S}_\alpha$-admissible.

Definition 1.4 (p-convexified mixed Tsirelson space). [13] Fix $1\leq p<\infty$, a set $N\subset\mathbb{N}$ and scalars $(\theta_n)_{n\in N}\subset(0,1)$. Define a norm $\|\cdot\|$ on c_{00} as the unique norm on c_{00} satisfying the equation

$$\|x\|=\max\bigg\{\|x\|_\infty,\sup\bigg\{\theta_n^{1/p}\bigg(\sum_i\|E_ix\|^p\bigg)^{1/p}:(E_i)\ \mathcal{S}_n\text{-admissible},n\in N\bigg\}\bigg\}.$$

The *p-convexified mixed Tsirelson space* $T^{(p)}[(\mathcal{S}_n,\theta_n)_{n\in N}]$ is the completion of $(c_{00},\|\cdot\|)$.

Take $1<q\leq\infty$ with $\frac{1}{p}+\frac{1}{q}=1$. It is standard to verify that $\|x\|=\sup\{f(x): f\in K\}$, $x\in c_{00}$, where $K\subset c_{00}$ is the smallest set such that

(K1) $(\pm e_i^*)_i \subset K$,

(K2) for any $\mathcal{S}_n$-admissible $(f_i) \subset K$, $n \in N$, and any $(\gamma_i) \in B_{\ell_q}$ we have $\theta_n^{1/p} \sum_i \gamma_i f_i \in K$.

In case $p=1$, we obtain the classical mixed Tsirelson space $T[(\mathcal{S}_n, \theta_n)_{n\in N}]$, introduced in [6]. Notice that for any $p>1$ the space $T^{(p)}[(\mathcal{S}_n, \theta_n)_{n\in N}]$ is the p-convexification of $T[(\mathcal{S}_n, \theta_n)_{n\in N}]$ [13] and is ℓ_p-asymptotic. It follows immediately by the definition of the space that the u.v.b. (e_n) is 1-unconditional in $T^{(p)}[(\mathcal{S}_n, \theta_n)_{n\in N}]$.

If $N = \{n\}$, we obtain the classical p-convexified Tsirelson-type space $T^{(p)}[\mathcal{S}_n, \theta]$. The space $T[\mathcal{S}_1, 1/2]$ is the famous Tsirelson space. For $\theta = 1$, we have $T^{(p)}[\mathcal{S}_n, 1] = \ell_p$. We will shorten the notation by denoting any space $T^{(p)}[\mathcal{S}_1, \theta]$ by $T_\theta^{(p)}$. We recall Lemma 4.13 [23]: for any sequence $(\theta_n) \subset (0,1]$, with $\theta_{n+m} \geq \theta_n \theta_m$, $n, m \in \mathbb{N}$, $\lim_{n\to\infty} \theta_n^{1/n}$ exists and is equal to $\sup_n \theta_n^{1/n}$.

Notation 1.5. A space $T^{(p)}[(\mathcal{S}_n, \theta_n)_{n\in\mathbb{N}}]$ with $\theta_n \searrow 0$ and $\theta_{n+m} \geq \theta_n \theta_m$ is called a *regular* space. In this case we define $\theta = \lim_n \theta_n^{1/n} \in (0,1]$.

Remark 1.6. It follows straightforward that any convexified mixed Tsirelson space $T^{(p)}[(\mathcal{S}_n, \theta_n)_{n\in N}]$, with infinite $N \subset \mathbb{N}$ and $\theta_n \to 0$, is isometric to a regular space $T^{(p)}[(\mathcal{S}_n, \bar\theta_n)_{n\in\mathbb{N}}]$, with $\bar\theta_n = \sup\{\prod_{i=1}^l \theta_{n_i} : \sum_{i=1}^l n_i \geq n, n_1, \dots, n_l \in N\}$, $n \in \mathbb{N}$.

The following notion provides a useful tool for estimating norms in convexified mixed Tsirelson spaces.

Definition 1.7 (The tree-analysis of a norming functional). Let $f \in K$, where K is the norming set of a convexified mixed Tsirelson space $T^{(p)}[(\mathcal{S}_n, \theta_n)_{n\in N}]$. By a *tree-analysis* of f we mean a finite family $(f_t)_{t\in\mathcal{T}}$ indexed by a tree $\mathcal{T}$ with a unique root $0 \in \mathcal{T}$ satisfying the following:

(1) $f_0 = f$ and $f_t \in K$ for all $t \in \mathcal{T}$,

(2) $t \in \mathcal{T}$ is terminal if and only if $f_t \in (\pm e_n^*)$,

(3) for any nonterminal $t \in \mathcal{T}$ there is some $n \in N$ such that $(f_s)_{s\in \mathrm{succ}(t)}$ is an $\mathcal{S}_n$-admissible sequence and $f_t = \theta_n^{1/p}(\sum_{s\in\mathrm{succ}(t)} \gamma_s f_s)$ for some $(\gamma_s)_{s\in\mathrm{succ}(t)} \in B_{\ell_q} \setminus \{0\}$. In such a case the character of f_t is defined as $\mathrm{char}(f_t) = n$.

The character order of a node f_t, $t \in \mathcal{T}$, is defined as the sum of characters of all nodes preceding f_t. The set the character order of f_0 to be equal 0.

Notice that any $f \in K$ admits a tree-analysis, not necessarily unique.

2. Lower Tsirelson-type estimate in asymptotic ℓ_p spaces

Throughout this section, we assume that X is an asymptotic ℓ_p space, $1 \leq p \leq \infty$, with a basis.

For any $n \in \mathbb{N}$ define the lower asymptotic constant $\theta_n = \theta_n(X) \in (0,1]$ (in case $p=1$ cf. [23]) as the biggest constant such that for any $\mathcal{S}_n$-admissible block sequence $n \leq x_1 < \cdots < x_k \in X$ we have $\|x_1 + \cdots + x_k\|^p \geq \theta_n(\|x_1\|^p + \cdots + \|x_k\|^p)$. It follows easily that $\theta_{n+m} \geq \theta_n\theta_m$, $n,m \in \mathbb{N}$. Let $\theta = \lim_n \theta_n^{1/n} \in (0,1]$. We will not make at this point the standard stabilization of the constants over block subspaces, or tail subspaces, as it will be done later to satisfy more restrictive conditions.

The model space for the above situation is a regular convexified mixed Tsirelson space $T^{(p)}[(\mathcal{S}_n,\theta_n)_{n\in\mathbb{N}}]$. Indeed, by the Fact 2.1 below and the definition of the space (θ_n) is the sequence of its lower asymptotic constants.

FACT 2.1. *Let $Z = T^{(p)}[(\mathcal{S}_n,\theta_n)_{n\in\mathbb{N}}]$ be a regular p-convexified mixed Tsirelson space. Then for any $n \in \mathbb{N}$ and $\varepsilon > 0$ there is a vector $x = \sum_{i\in F} a_i e_i$ with $F \in \mathcal{S}_n$ such that $\|x\| \leq (\theta_n^{1/p} + \varepsilon)(\sum_{i\in F} |a_i|^p)^{1/p}$.*

Proof. By Lemma 1.6 [6] for any $n \in \mathbb{N}$ and $\delta > 0$ there is $(b_i)_{i\in F} \subset (0,1)$, $F \in \mathcal{S}_n$, such that $\sum_{i\in F} b_i = 1$ and $\sum_{i\in G} b_i < \delta$ for any $G \in \mathcal{S}_{n-1}$. Let $x = \sum_{i\in F} b_i^{1/p} e_i$. Take a norming functional $f \in K$ with a tree-analysis $(f_t)_{t\in\mathcal{T}}$ and let G be the set of all terminal nodes of $\mathcal{T}$ with character order smaller than n. Then $G \in \mathcal{S}_{n-1}$ and by Hölder inequality and regularity of (θ_n)

$$\begin{aligned} f(x) &= f\Big(\sum_{i\in G\cap F} b_i^{1/p} e_i\Big) + f\Big(\sum_{i\in F\setminus G} b_i^{1/p} e_i\Big) \\ &\leq \Big(\sum_{i\in G\cap F} b_i\Big)^{1/p} + \theta_n^{1/p}\Big(\sum_{F\setminus G} b_i\Big)^{1/p} < \delta^{1/p} + \theta_n^{1/p}. \end{aligned}$$

Taking $\delta = \varepsilon^p$ we finish the proof. □

In the sequel, we will generalize some of the estimates known for Z [19] to the case of arbitrary asymptotic ℓ_p space X. The following theorem generalizes Lemma 2.14 [19] (in case of mixed Tsirelson spaces) and Proposition 3.3 [7] (in case of $\theta = 1$), providing also block sequences with supports of uniformly bounded admissibility. One can view this result in context of Krivine theorem in Lemberg's version [20], stating that for any basic sequence (x_i) there is some $1 \leq p \leq \infty$, such that for any $M \in \mathbb{N}$ and $\delta > 0$ there is a block sequence $(x_i^{(n)})$ such that any its subsequence of length M is $(1+\delta)$-equivalent to the u.v.b. of ℓ_p. In case of asymptotic ℓ_p spaces we increase the order of sequences uniformly "representing" (more precisely dominating) the u.v.b. of some $T^{(p)}[\mathcal{S}_1,\theta]$ from sequences of fixed length to $\mathcal{S}_M$-admissible.

THEOREM 2.2. *Let X be an asymptotic ℓ_p space, $1 \leq p < \infty$, with lower asymptotic constants (θ_n). Let $\theta = \lim_n \theta_n^{1/n}$. Then for every $M \in \mathbb{N}$ and $\delta > 0$, there is a normalized block sequence $(x_i) \subset X$ satisfying for any $G \in \mathcal{S}_M$*

and scalars $(a_i)_{i\in G}$

$$\Big\|\sum_{i\in G} a_i x_i\Big\| \geq \frac{1}{2}(1-\delta)\Big\|\sum_{i\in G} a_i e_{\mathrm{minsupp}\, x_i}\Big\|_{T_\theta^{(p)}}.$$

Moreover (x_i) *can be chosen to satisfy* $\mathrm{supp}\, x_i \in \mathcal{S}_r$ *for all* i, *for some* $r\in\mathbb{N}$.

In order to achieve the "Moreover..." statement in the above proposition, we introduce more precise lower asymptotic constants measuring the asymptoticity on block sequences with supports of the same admissibility.

For any normalized block sequence $\mathbf{x}=(x_i)\subset X$ and any $n\in\mathbb{N}$ let $\widetilde{\eta}_n(\mathbf{x})\in(0,1]$ be the biggest constant such that for any $\mathcal{S}_n$-admissible block subsequence $x_{i_1}<\cdots<x_{i_k}$ and any scalars $(a_i)_{i=1}^k$ we have $\|a_1x_{i_1}+\cdots+a_kx_{i_k}\|^p\geq \widetilde{\eta}_n(\mathbf{x})(|a_1|^p+\cdots+|a_k|^p)$. Then let

$$\eta_n(\mathbf{x})=\sup_{k\in\mathbb{N}}\widetilde{\eta}_n\big((x_i)_{i\geq k}\big)$$

and finally for any $n\in\mathbb{N}$ let

$$\eta_n=\inf\big\{\eta_n(\mathbf{x}) : \mathbf{x}=(x_i)\text{—a normalized block sequence}\\ \text{with } \mathrm{supp}\, x_i\in\mathcal{S}_{r_{\mathbf{x}}} \text{ for all } i, \text{ for some } r_{\mathbf{x}}\in\mathbb{N}\big\}.$$

It is clear that $\eta_{n+m}\geq\eta_n\eta_m$, $n,m\in\mathbb{N}$. Let $\eta=\lim_n\eta_n^{1/n}\in(0,1]$. As $\eta_n\geq\theta_n$ for any $n\in\mathbb{N}$ we have also $\eta\geq\theta$, therefore it will be sufficient to prove the estimate in Theorem 2.2 for $T_\eta^{(p)}$ instead of $T_\theta^{(p)}$.

The proof of Theorem 2.2 is based on the following facts.

LEMMA 2.3. *For any* $M\in\mathbb{N}$ *there is a block sequence* $(x_i)\subset X$ *such that for any* $1\leq j<M$ *there is some* $\mathcal{S}_j$*-admissible* (E_k) *with* $\|x_i\|^p\leq 2\eta^j\sum_k\|E_kx_i\|^p$, $i\in\mathbb{N}$, *and* $\mathrm{supp}\, x_i\in\mathcal{S}_r$ *for all* i, *for some* $r\in\mathbb{N}$.

Proof. Notice first that for any $M\in\mathbb{N}$ we have

$$\big(\sqrt[m]{\eta_m}\big)^M\leq\sqrt[m]{\eta_{Mm}}\leq\sqrt[m]{\eta^{mM}},$$

thus $\lim_{m\to\infty}\sqrt[m]{\eta_{Mm}}=\eta^M$. Fix $M\in\mathbb{N}$ and by the above pick $m\in\mathbb{N}$ such that $2^{1/m}\eta_{mM}^{1/m}<2\theta_M^{1/m}\eta^M$. By definition of η_{mM} pick a block sequence $(y_i)\subset X$ with $\|y_i\|^p\leq 2\eta_{mM}\sum_k\|F_ky_i\|^p$ for some $\mathcal{S}_{mM}$-admissible (F_k) and $\mathrm{supp}\, y_i\in\mathcal{S}_r$ for some $r\in\mathbb{N}$.

Fix $i\in\mathbb{N}$, let $y=y_i$ and assume that for any $z\in X$ with $\mathrm{supp}\, z\subset\mathrm{supp}\, y$ there is some $1\leq j<M$ such that $\|z\|^p>2\eta^j\sum_k\|E_kz\|^p$ for any $\mathcal{S}_j$-admissible (E_k). Notice that if we arrive to contradiction, as $i\in\mathbb{N}$ is arbitrary, we will finish the proof of the lemma.

Take an $\mathcal{S}_1$-admissible tree $(F_t)_{t\in\mathcal{T}}$ associated to $(F_k)_k$ as in Remark 1.2. We will choose inductively some subtree $\mathcal{R}\subset\mathcal{T}$ with the same root such that

(1) $\mathrm{ord}_{\mathcal{T}}(t)>(m-1)M$ for any terminal $t\in\mathcal{R}$,

(2) if $t \in \mathcal{R}$ is nonterminal, then for some $1 \le j_t \le M$ the sequence $(F_s)_{s\in \mathrm{succ}_{\mathcal{R}}(t)}$ is $\mathcal{S}_{j_t}$-admissible and $\|F_t y\|^p \ge 2\eta^{j_t}\sum_{s\in \mathrm{succ}_{\mathcal{R}}(t)}\|F_s y\|^p$.

Notice first that the length of the branch linking any terminal node t of $\mathcal{R}$ and the root is at least m and $\|F_t y\|^p \ge \theta_M \sum_{F_k\subset F_t}\|F_k y\|^p$ as $(F_k)_{F_k\subset F_t}$ is $\mathcal{S}_{mM-\mathrm{ord}_{\mathcal{T}}(t)}$-admissible and thus also $\mathcal{S}_M$-admissible by (1). Therefore,

$$\begin{aligned} 2\eta_{mM}\sum_k \|F_k y\|^p \ge \|y\|^p &\ge 2^m \sum_{t\in\mathcal{R}, t\,\mathrm{terminal}} \eta^{\mathrm{ord}_{\mathcal{T}}(t)}\|F_t y\|^p \\ &\ge 2^m \sum_{t\in\mathcal{R}, t\,\mathrm{terminal}} \eta^{\mathrm{ord}_{\mathcal{T}}(t)}\theta_M \sum_{F_k\subset F_t}\|F_k y\|^p \\ &\ge 2^m \eta^{mM}\theta_M \sum_k \|F_k y\|^p, \end{aligned}$$

hence $2\eta_{mM} \ge 2^m\theta_M\eta^{mM}$ which contradicts the choice of m.

We proceed to define the tree $\mathcal{R}$. By our assumption on y, considering $z=y$ we have $\|y\|^p \ge 2\eta^{j_0}\sum_{s\in\mathcal{T},\mathrm{ord}(s)=j_0}\|F_s y\|^p$ for some $1\le j_0\le M$. Let $\mathrm{succ}_{\mathcal{R}}(0)=\{s\in\mathcal{T},\mathrm{ord}_{\mathcal{T}}(s)=j_0\}$. Assume we have defined $t\in\mathcal{R}$ with order $\le (m-1)M$. By our assumption on y, considering $z=F_t y$, we can pick some $1\le j_t\le M$ with $\|F_t y\|^p \ge 2\eta^{j_t}\sum_{s\in\mathcal{T},\mathrm{ord}_{\mathcal{T}}(s)=\mathrm{ord}_{\mathcal{T}}(t)+j_t, F_s\subset F_t}\|F_s y\|^p$. Let $\mathrm{succ}_{\mathcal{R}}(t)=\{s\in\mathcal{T},\mathrm{ord}_{\mathcal{T}}(s)=\mathrm{ord}_{\mathcal{T}}(t)+j_t, F_s\subset F_t\}$ and thus we finish the construction of $\mathcal{R}$. □

FACT 2.4. *For any $G\in\mathcal{S}_M$ and any $z=\sum_{i\in G}c_ie_i\in T_\eta^{(p)}$ there is an $\mathcal{S}_1$-admissible tree $\mathcal{R}$ of height at most M, with terminal nodes $\{i\}$, $i\in F$ for some $F\subset G$, of orders $(l_i)_{i\in F}\subset\{1,\dots,M\}$ satisfying $\|z\|^p_{T_\eta^{(p)}}\le 2^p\sum_{i\in F}\eta^{l_i}|c_i|^p$.*

Proof. Take a norming functional $g=\sum_{i\in G}\eta^{k_i/p}\gamma_ie_i^*$ with $(\gamma_i)_{i\in G}\in B_{\ell_q}$ and tree-analysis $(g_t)_{t\in\mathcal{T}}$ satisfying $g(z)=\|z\|_{T_\eta^{(p)}}$. Let $I=\{i\in G: k_i\le M\}$. Let g_1 be the restriction of g to I and $g_2=g-g_1$. If $g_1(z)\ge g_2(z)$, then

$$g(z)\le 2g_1(z)\le 2\sum_{i\in I}\eta^{k_i/p}|\gamma_ic_i|\le 2\Big(\sum_{i\in I}\eta^{k_i}|c_i|^p\Big)^{1/p},$$

and we take the tree $\mathcal{R}=(\mathrm{supp}\, g_t\cap I)_{t\in\mathcal{R}}$. If $g_1(z)\le g_2(z)$, compute

$$g(z)\le 2g_2(z)\le 2\eta^{M/p}\sum_{i\in G\setminus I}|\gamma_ic_i|\le 2\eta^{M/p}\Big(\sum_{i\in G}|c_i|^p\Big)^{1/p},$$

and we take a tree $\mathcal{R}$ associated to $\mathcal{S}_M$-admissible $(\{i\})_{i\in G}$ by Remark 1.2. □

Proof of Theorem 2.2. The proof follows the idea of the proof of Lemma 2.14 [19]. Assume the contrary. As in the proof of Lemma 2.3 for any $M\in\mathbb{N}$, we have $\lim_{m\to\infty}\sqrt[m]{\eta_{Mm}}=\eta^M$. Pick $m\in\mathbb{N}$ such that $\eta_{Mm}^{1/m}>2^{1/m}(1-\delta)^p\eta^M$. Take a block sequence $(x_i^0)_i$ according to Lemma 2.3 for $mM\in\mathbb{N}$, with $(\mathrm{supp}\, x_i^0)\subset\mathcal{S}_r$, for some $r\in\mathbb{N}$.

Since the assertion fails there is an infinite sequence $(G^1_k)_k$ of successive elements of $\mathcal{S}_M$ and coefficients $(a^1_i)_{i\in G^1_k,k}$ such that

$$\Big\|\sum_{i\in G^1_k} a^1_i x^0_i\Big\| < \frac{1}{2}(1-\delta)\Big\|\sum_{i\in G^1_k} a^1_i \|x^0_i\| e_{m^0_i}\Big\|_{T^{(p)}_\eta}, \quad \text{for each } k\in\mathbb{N},$$

where $m^0_i = \operatorname{minsupp} x^0_i$ for each i. For any $k\in\mathbb{N}$ set $x^1_k = \sum_{i\in G^1_k} a^1_i x^0_i$ and by Fact 2.4 take an $\mathcal{S}_1$-admissible tree $\mathcal{R}^1_k$ with the root $F^1_k \subset G^1_k$ and terminal nodes $(\{i\})_{i\in F^1_k}$, $F^1_k\subset G^1_k$, of orders $(l^1_i)_{i\in F^1_k}\subset\{1,\dots,M\}$ satisfying

$$\Big\|\sum_{i\in G^1_k} a^1_i \|x^0_i\| e_{m^0_i}\Big\|^p_{T^{(p)}_\eta} \le 2^p \sum_{i\in F^1_k} \eta^{l^1_i} |a^1_i|^p \|x^0_i\|^p.$$

Assume that we have defined $(x^{j-1}_k)_k$ and $(\mathcal{R}^{j-1}_k)_k$ with terminal nodes of orders $(l^{j-1}_i)_{i\in F^{j-1}_k,k}$ for some $j\le m$. Then the failure of the assertion implies the existence of a sequence $(G^j_k)_k$ of successive elements of $\mathcal{S}_M$ and a sequence $(a^j_i)_{i\in G^j_k,k}$ such that for any $k\in\mathbb{N}$

$$\Big\|\sum_{i\in G^j_k} a^j_i x^{j-1}_i\Big\| < \frac{1}{2}(1-\delta)\Big\|\sum_{i\in G^j_k} a^j_i \|x^{j-1}_i\| e_{m^{j-1}_i}\Big\|_{T^{(p)}_\eta},$$

where $m^{j-1}_i = \operatorname{minsupp} x^{j-1}_i$ for each i. For any $k\in\mathbb{N}$ set $x^j_k = \sum_{i\in G^j_k} a^j_i x^{j-1}_i$ and by Fact 2.4 take an $\mathcal{S}_1$-admissible tree $\mathcal{R}^j_k$ with terminal nodes $(\{i\})_{i\in F^j_k}$, $F^j_k\subset G^j_k$, of orders $(l^j_i)_{i\in F^j_k}\subset\{1,\dots,M\}$ satisfying

$$\Big\|\sum_{i\in G^j_k} a^j_i \|x^{j-1}_i\| e_{m^{j-1}_i}\Big\|^p_{T^{(p)}_\eta} \le 2^p \sum_{i\in F^j_k} \eta^{l^j_i} |a^j_i|^p \|x^{j-1}_i\|^p \quad \text{for each } k\in\mathbb{N}.$$

The inductive construction ends once we get sequences $(x^m_k)_k$ and $(\mathcal{R}^m_k)_k$.

By the construction for any $1\le j\le m, k\in\mathbb{N}$, we have

$$\|x^j_k\|^p < (1-\delta)^p \sum_{i\in G^j_k} \eta^{l^j_i} |a^j_i|^p \|x^{j-1}_i\|^p. \tag{2.1}$$

Put $G_k = \bigcup_{k_{m-1}\in G^m_k} \bigcup_{k_{m-2}\in G^{m-1}_{k_{m-1}}} \cdots \bigcup_{k_1\in G^2_{k_2}} G^1_{k_1}$, and analogously define F_k, for each $k\in\mathbb{N}$. Fix $k\in\mathbb{N}$ and inductively, beginning from $\mathcal{R}^m_k$ produce an $\mathcal{S}_1$-admissible tree $\mathcal{R}_k$ by substituting each terminal node $\{i\}$ of $\mathcal{R}^j_{k_j}$, $j=1,\dots,m$, by the tree $\mathcal{R}^{j-1}_i$. Let $(\{i\})_{i\in F_k}$ be the collection of terminal nodes of $\mathcal{R}_k$ with orders $(l_i)_{i\in F_k}$. Notice that $l_i\le mM$ for any $i\in F_k$, as each $l^j_i\le M$. We compute the norm of x^m_k, which is of the form

$$x^m_k = \sum_{k_{m-1}\in G^m_k} \sum_{k_{m-2}\in G^{m-1}_{k_{m-1}}} \cdots \sum_{k_1\in G^2_{k_2}} \sum_{i\in G^1_{k_1}} a^m_{k_{m-1}} \cdots a^1_i x^0_i = \sum_{i\in G_k} b_i x^0_i.$$

By the choice of (x_i^0), for any $i\in\mathbb{N}$ there is an $\mathcal{S}_{mM-l_i}$-admissible sequence $(E_l)_{l\in L_i}$ with $\|x_i^0\|^p\le 2\eta^{mM-l_i}\sum_{l\in L_i}\|E_l x_i^0\|^p$.

For each $k\in\mathbb{N}$, we have on one hand by repeated use of (2.1)

$$\begin{aligned}\|x_k^m\|^p &\le (1-\delta)^{pm}\sum_{i\in F_k}\eta^{l_i}b_i^p\|x_i^0\|^p\\ &\le (1-\delta)^{pm}2\sum_{i\in F_k}\eta^{l_i}b_i^p\eta^{mM-l_i}\sum_{l\in L_i}\|E_l x_i^0\|^p\\ &= (1-\delta)^{pm}2\eta^{mM}\sum_{i\in F_k}b_i^p\sum_{l\in L_i}\|E_l x_i^0\|^p.\end{aligned}$$

On the other hand for each $k\in\mathbb{N}$ the sequence $(E_l)_{l\in L_i,i\in F_k}$ is $\mathcal{S}_{mM}$-admissible by the definition of $\mathcal{R}_k$. Consider the block sequence $(E_l x_i^0)_{l\in L_i,i\in F_k,k\in\mathbb{N}}$ and notice that $E_l\cap\operatorname{supp}x_i^0\in\mathcal{S}_r$, for each $l\in L_i, i\in F_k, k\in\mathbb{N}$, by the choice of (x_i^0). Thus by definition of η_{mM} for some $k_0\in\mathbb{N}$ we have

$$\|x_{k_0}^m\|^p\ge\eta_{mM}\sum_{i\in F_{k_0}}b_i^p\sum_{l\in L_i}\|E_l x_i^0\|^p,$$

which brings $\eta_{mM}\le(1-\delta)^{pm}2\eta^{mM}$, a contradiction with the choice of m. $\square$

Remark 2.5. In case of ℓ_p^α-asymptotic spaces, $1\le p<\infty$, $\alpha<\omega_1$, where all normalized $\mathcal{S}_\alpha$-admissible sequences are uniformly equivalent to the u.v.b. of ℓ_p of suitable size, one can define lower asymptotic constants tested on $\mathcal{S}_{\alpha n}$-admissible sequences (in case $p=1$ studied in [23]). In this setting, one obtains analogous results with Tsirelson-type spaces $T^{(p)}[\mathcal{S}_\alpha,\theta]$. Since the reasoning in this general case follows exactly the argument in case $\alpha=1$ above, just by replacing families $(\mathcal{S}_n)$ by $(\mathcal{S}_{\alpha n})$, for simplicity we present only this last case.

3. α-strong domination

We examine in this section properties of α-strong domination, a higher order counterpart of "strong domination" in [24] or "domination on small coefficients" in [4]. Throughout this section, we fix a limit ordinal $\alpha<\omega_1$.

For a pair of seminormalized basic sequences (x_i), (y_i) consider conditions:

(★) there are regular families $(\mathcal{F}_n)$ on $\mathbb{N}$ with $\mathcal{F}_n\subset\mathcal{F}_{n+1}$, $n\in\mathbb{N}$, and $\mathrm{CB}(\mathcal{F}_n)\nearrow\omega^\alpha$, such that $\Delta_0<\infty$ and $\Delta_n\to 0$, where for any $n\in\mathbb{N}$

$$\Delta_n=\sup\Big\{\Big\|\sum_i a_ix_i\Big\| : \max_{F\in\mathcal{F}_n}\Big\|\sum_{i\in F}a_iy_i\Big\|\le\frac{1}{2^n},\Big\|\sum_i a_iy_i\Big\|\le 1,(a_i)\in c_{00}\Big\}.$$

(▲) there are regular families $(\mathcal{F}_n)$ on $\mathbb{N}$ with $\mathcal{F}_n\subset\mathcal{F}_{n+1}$, $n\in\mathbb{N}$, and $\mathrm{CB}(\mathcal{F}_n)\nearrow\omega^\alpha$, such that for some (δ_n) with $\delta_n\searrow 0$

$$\Big\|\sum_i a_ix_i\Big\|\le\max_{n\in\mathbb{N}}\delta_n\max_{n\le F\in\mathcal{F}_n}\Big\|\sum_{i\in F}a_iy_i\Big\|\quad\text{for any } (a_i)\in c_{00}.$$

REMARK 3.1. Take (α_n) used to define $\mathcal{S}_\alpha$. By Proposition 3.10 [23] for any $\mathcal{F}$ with $\mathrm{CB}(\mathcal{F}) < \omega^\alpha$ there are infinite $J \subset \mathbb{N}$ and $n \in \mathbb{N}$ with $\mathcal{F} \cap [J]^{<\infty} \subset \mathcal{S}_{\alpha_n}$. Therefore, ($\bigstar$) and ($\blacktriangle$) imply that for some infinite $J = (j_n) \subset \mathbb{N}$ and $(k_n) \subset \mathbb{N}$, subsequences $(x_i)_{i\in J}$ and $(y_i)_{i\in J}$ satisfy analogous properties with families $(\mathcal{S}_{\alpha_{k_n}} \cap [(j_l)_{l>n}]^{<\infty})$.

DEFINITION 3.2. Fix two seminormalized basic sequences (x_i), (y_i). We say that (y_i) *α-strongly dominates* (x_i) if (y_i) is $\mathcal{S}_\alpha$-unconditional, $[y_i]$ does not contain c_0 and the pair (x_i), (y_i) satisfies ($\bigstar$).

As $\mathcal{F}_0$ is hereditary and spreading, it contains $\mathcal{S}_0 \cap \{k, k+1, \dots\}$ for some k and thus α-strong domination, by $\Delta_0 < \infty$, implies domination. The next observation provides a suitable setting for the above definition by Remark 3.1.

FACT 3.3. *Let (y_i) be a seminormalized $\mathcal{S}_\alpha$-unconditional basic sequence with $[y_i]$ not containing c_0. Then for any $\beta \le \alpha$ and $\varepsilon > 0$, every block subspace $W \subset [y_i]$ contains a vector $w = \sum_i a_i y_i$ with $\max_{F\in\mathcal{S}_\beta} \|\sum_{i\in F} \pm a_i y_i\| < \varepsilon \|w\|$.*

Proof. We show the fact by induction on $\beta \le \alpha$, following the idea of Lemma 3.6 [23]. Assume that (y_i) is $\mathcal{S}_\alpha$-unconditional with constant 1. For $n = 0$ the statement is obvious. Assume the statement holds for $\gamma < \beta$ for fixed $\beta \le \alpha$.

If β is limit, take (β_n) used to define $\mathcal{S}_\beta$ and pick a normalized block sequence $(z_k) \subset W$, $z_k = \sum_{i\in I_k} a_i y_i$, $k \in \mathbb{N}$, such that

$$\max_{G\in\mathcal{S}_{\beta_n}, G\subset I_k} \left\| \sum_{i\in G} \pm a_i y_i \right\| \le \frac{1}{2^k}, \quad n \le \max I_{k-1}, k \in \mathbb{N}.$$

Pick any $F \in \mathcal{S}_\beta$, then $n \le F \in \mathcal{S}_{\beta_n}$ for some n. Let $k_0 = \min\{k \in \mathbb{N} : I_k \cap F \neq \emptyset\}$ and compute, using $n \le \operatorname{maxsupp} z_{k_0}$ and the $\mathcal{S}_\alpha$-unconditionality (provided $\min I_1$ is big enough to ensure $F \cap I_{k_0} \in \mathcal{S}_\alpha$),

$$\left\| \sum_{i\in F} \pm a_i y_i \right\| \le \left\| \sum_{i\in F\cap I_{k_0}} \pm a_i y_i \right\| + \sum_{k>k_0} \left\| \sum_{i\in F\cap I_k} \pm a_i y_i \right\| \le 1 + \sum_{k>k_0} \frac{1}{2^k} \le 2.$$

Consider the family $A = \{\sum_{k\in L} \pm z_k : L \in [\mathbb{N}]^{<\infty}\}$. As $[y_i]$ does not contain c_0, $\sup_{w\in A} \|w\| = \infty$ and thus some $w \in A$ satisfies the desired estimate.

If $\beta = \gamma + 1$, pick a normalized block sequence $(z_k) \subset W$, $z_k = \sum_{i\in I_k} a_i y_i$, $k \in \mathbb{N}$, such that

$$\max_{G\in\mathcal{S}_\gamma, G\subset I_k} \left\| \sum_{i\in G} \pm a_i y_i \right\| \le 1/\big(2^k \max I_{k-1}\big), \quad k \in \mathbb{N}.$$

Pick any $F \in \mathcal{S}_\beta$, write F as $F = F_1 \cup \dots \cup F_m$, for some $m \le F_1 < \dots < F_m \in \mathcal{S}_\gamma$, let $k_0 = \min\{k \in \mathbb{N} : I_k \cap F \neq \emptyset\}$ and compute, using the $\mathcal{S}_\alpha$-unconditionality (provided $\min I_1$ is big enough to ensure $F \cap I_{k_0} \in \mathcal{S}_\alpha$)

$$\Big\|\sum_{i\in F} \pm a_i y_i\Big\| \le \Big\|\sum_{i\in F\cap I_{k_0}} \pm a_i y_i\Big\| + \sum_{k>k_0}\sum_{j=1}^{m}\Big\|\sum_{i\in F_j\cap I_k} \pm a_i y_i\Big\| \le 1 + \sum_{k>k_0}\frac{1}{2^k} \le 2.$$

As in the previous case, we obtain a suitable $w \in W$ and finish the proof. ☐

However the α-strong domination appears to be a stronger notion than domination without equivalence, in case of ℓ_1 the situation is simpler.

LEMMA 3.4. *Let (x_i) be a normalized $\mathcal{S}_\alpha$-unconditional basic sequence. Then either some subsequence of (x_i) is $\mathcal{S}_\alpha$-equivalent to the u.v.b. of ℓ_1 or some subsequence of (x_i) is α-strongly dominated by the u.v.b. of ℓ_1.*

Proof. Let (x_i) be $\mathcal{S}_\alpha$-unconditional with constant 1. Pick (α_n) used to define $\mathcal{S}_\alpha$. Assume none of subsequences of (x_i) is α-strongly dominated by the u.v.b. of ℓ_1. Then there are $\delta > 0$ and infinite $L \subset \mathbb{N}$ such that for any infinite $J \subset L$ and any $n \in \mathbb{N}$ there is $k_n > n$ and $(a_i) \in c_{00}(J)$ such that $\max_{F\in\mathcal{S}_{\alpha_{k_n}}} \|\sum_{i\in F\cap J} a_i\| \le 1/2^{k_n}$, $\sum_i |a_i| \le 1$ and $\|\sum_i a_i x_i\| > 2\delta$. Notice that for any $n \ge 4$ with $n/2^n < \delta$ we have $\|\sum_{i\le k_n} a_i x_i\| \le k_n/2^{k_n} < \delta$. Thus, for any such n and $J \subset L$ there is $k_n > n$ and $(a_i) \in c_{00}(J \cap \{k_n+1, k_n+2, \dots\})$ with $\max_{F\in\mathcal{S}_{\alpha_{k_n}}} \|\sum_{i\in F\cap J} a_i\| \le 1/2^{k_n}$, $\sum_i |a_i| \le 1$ and $\|\sum_i a_i x_i\| > \delta$. By $\mathcal{S}_\alpha$-unconditionality of (x_i) we may assume that $(a_i) \subset [0,1]$.

Let (x_i^*) be the biorthogonal functionals to (x_i). Pick (a_i) as above. Take $(b_i) \subset [0,1]$ with $\sum_i b_i a_i \ge \delta$ and $\|\sum_i b_i x_i^*\| = 1$. Let $G_0 = \{i \in J : i > k_n, b_i > \frac{\delta}{4}\}$. Notice that $G_0 \notin \mathcal{S}_{\alpha_{k_n}}$, otherwise we arrive to contradiction by the following

$$\delta \le \sum_i b_i a_i \le \sum_{i\notin G_0} b_i a_i + \sum_{i\in G_0} b_i a_i \le \frac{\delta}{4} + \frac{1}{2^{k_n}} \le \frac{\delta}{2}.$$

Pick any $G_1 \subset G_0$ with $G_1 \in \mathcal{S}_{\alpha_{k_n+1}} \setminus \mathcal{S}_{\alpha_{k_n}}$. For any $(c_i)_{i\in G_1} \subset [0,1]$, we have $\|\sum_{i\in G_1} c_i x_i\| \ge \sum_{i\in G_1} b_i c_i \ge \frac{\delta}{4}\sum_{i\in G_1} c_i$, thus by $\mathcal{S}_\alpha$-unconditionality $(x_i)_{i\in G_1}$ is $4/\delta$-equivalent to the u.v.b. of $\ell_1^{\#G_1}$.

Let $\mathcal{G}$ be the collection of all finite $G \subset L$ such that $(x_i)_{i\in G}$ is $4/\delta$-equivalent to the u.v.b. of $\ell_1^{\#G}$. Obviously $\mathcal{G}$ is hereditary. By the above $\mathcal{G} \cap [J]^{<\infty} \not\subset \mathcal{S}_{\alpha_n}$ for any infinite $J \subset L$ and any $n \in \mathbb{N}$. Therefore by dichotomy [15], there are $L \supset J_0 \supset J_1 \supset \cdots$ with $\mathcal{S}_{\alpha_n} \cap [J_n]^{<\infty} \subset \mathcal{G}$, $n \in \mathbb{N}$. It follows that the subsequence $(x_i)_{i\in N}$, where $N = (\min J_n)$, is $\mathcal{S}_\alpha$-equivalent to the u.v.b. of ℓ_1. ☐

A typical example of ω-strong domination is formed by convexified mixed Tsirelson spaces and Tsirelson-type spaces, as the following observation shows.

LEMMA 3.5. *Assume $Z=T^{(p)}[(\mathcal{S}_n,\theta_n)_{n\in\mathbb{N}}]$ is a regular p-convexified mixed Tsirelson space with $\theta_n/\theta^n\to 0$, where $\theta=\lim_n \theta_n^{1/n}$. Then the u.v.b. of $T_\theta^{(p)}$ ω-strongly dominates the u.v.b. of Z.*

Proof. As $T_\theta^{(p)}$ is reflexive, if $\theta<1$ or $p>1$, and $T_1^{(1)}=\ell_1$, in all cases $T_\theta^{(p)}$ does not contain c_0. To prove condition ($\bigstar$), notice first that by definition $\|x\|_Z\le\|x\|_{T_\theta^{(p)}}$ for any $x\in c_{00}$. Pick $(a_i)\in c_{00}$ with $\|\sum_i a_ie_i\|_{T_\theta^{(p)}}=1$ and $\|\sum_{i\in F}a_ie_i\|_{T_\theta^{(p)}}\le\frac{1}{2^n}$ for any $F\in\mathcal{S}_n$. Let $\|\sum_i a_ie_i\|_Z=\sum_{i\in I}(\prod_j\theta_{l_{i,j}}^{1/p})\gamma_i|a_i|$ for some $(l_{i,j})\subset\mathbb{N}$ and $(\gamma_i)\in B_{\ell_q}$. Let $l_i=\sum_j l_{i,j}$, for any $i\in I$, and $K=\{i\in I: l_i\le n\}$, notice that $K\in\mathcal{S}_n$ and compute, by regularity of Z,

$$\begin{aligned}\Big\|\sum_{i\in I}a_ie_i\Big\|_Z&\le\Big\|\sum_{i\in K}a_ie_i\Big\|_Z+\sum_{i\in I\setminus K}\theta_{l_i}^{1/p}\gamma_i|a_i|\\&\le\frac{1}{2^n}+\Big(\max_{l\ge n}\frac{\theta_l}{\theta^l}\Big)^{1/p}\sum_{i\in I\setminus K}\theta^{l_i/p}\gamma_i|a_i|\\&\le\frac{1}{2^n}+\Big(\max_{l\ge n}\frac{\theta_l}{\theta^l}\Big)^{1/p}\Big\|\sum_i a_ie_i\Big\|_{T_\theta^{(p)}},\end{aligned}$$

which by assumption on (θ_n) shows condition ($\bigstar$) for (e_i) in Z and (e_i) in $T_\theta^{(p)}$ with families $(\mathcal{S}_n)$. □

Next, two lemmas provide characterization of α-strong domination and its invariance (up to taking subsequences) under $\mathcal{S}_\alpha$-equivalence. Their proofs follow the reasoning of Proposition 2.3 and Lemma 2.4 [24], however additional technique is needed in order to deal with higher order families.

LEMMA 3.6. *Fix two seminormalized basic sequences (x_i), (y_i), with (y_i) unconditional. Then*

(1) *if the pair (x_i), (y_i) satisfies ($\blacktriangle$), then it also satisfies ($\bigstar$),*
(2) *if $[y_i]$ does not contain uniformly c_0^n's, and the pair (x_i), (y_i) satisfies ($\bigstar$), then for some infinite $J\subset\mathbb{N}$ the pair $(x_i)_{i\in J}$, $(y_i)_{i\in J}$ satisfies ($\blacktriangle$) with $\delta_n=1/4^n$, $n\in\mathbb{N}$.*

Proof. (1) We can assume that (y_i) is 1-unconditional. Fix $n_0\in\mathbb{N}$, take $(a_i)\in c_{00}$ with $\|\sum_i a_iy_i\|=1$ and $\|\sum_{i\in F}a_iy_i\|\le\frac{1}{2^{n_0}}$ for any $F\in\mathcal{F}_{n_0}$ and compute by the condition ($\blacktriangle$)

$$\begin{aligned}\Big\|\sum_i a_ix_i\Big\|&\le\max_n\delta_n\max_{F\in\mathcal{F}_n}\Big\|\sum_{i\in F}a_iy_i\Big\|\\&\le\max\Big\{\max_{F\in\mathcal{F}_{n_0}}\Big\|\sum_{i\in F}a_iy_i\Big\|,\delta_{n_0}\max_{\substack{n>n_0\\F\in\mathcal{F}_n}}\Big\|\sum_{i\in F}a_iy_i\Big\|\Big\}\le\max\Big\{\frac{1}{2^{n_0}},\delta_{n_0}\Big\}.\end{aligned}$$

(2) We can assume that (y_i) is 1-unconditional and 1-dominates (x_i). Pick $(k_n) \subset \mathbb{N}$, $k_n > 3(n+2)$, such that $\Delta_{k_n} < 1/8^{n+1}$, $n \in \mathbb{N}$, where $(\Delta_n)_n$ satisfies the condition ($\bigstar$) for (x_i) and (y_i).

Define a seminormalized basic sequence (w_i) by the formula

$$\Big\|\sum_i a_i w_i\Big\| = \Big\|\sum_i a_i x_i\Big\| + \max_n \frac{1}{2^n} \max_{F \in \mathcal{S}_1(\mathcal{F}_n)} \Big\|\sum_{i \in F} a_i y_i\Big\|.$$

It is clear that (w_i) dominates (x_i), (y_i) 2-dominates (w_i) (as (y_i) 1-dominates (x_i) and is 1-unconditional by the assumption at the beginning of the proof) and the pair (w_i), (y_i) satisfies ($\bigstar$) with $(\mathcal{S}_1(\mathcal{F}_n))$ and $(\bar{\Delta}_n) = (\Delta_n + \frac{1}{2^n})$. Hence, it is enough to show the implication in (2) for sequences (w_i) and (y_i).

As (y_i) is unconditional and its span does not contain uniformly c_0^n's, we have $l_n < \infty$ for any $n \in \mathbb{N}$, where l_n is the supremum of all $l \in \mathbb{N}$ such that for some $(z_1, \ldots, z_l) \in [y_i]$ with pairwise disjoint supports we have $\|z_j\| > 1/2 \cdot 8^{k_n}$, $j = 1, \ldots, l$, and $\|z_1 + \cdots + z_l\| \leq 2^n$. It follows by definition of (w_i) that for any n the constant $4l_n$ dominates the supremum of all $l \in \mathbb{N}$ such that for some vector $w \in [w_i]$ with $\|w\| = 1$ and some pairwise disjoint $(E_1, \ldots, E_l) \subset \mathcal{F}_n$ we have $\|E_j w\| > 1/8^{k_n}$, $j = 1, \ldots, l$.

Let $j_n = \max\{k_n + 1, 4l_n\}$, $n \in \mathbb{N}$, and $J = \{j_n : n \in \mathbb{N}\}$. Take $(a_i) \in c_{00}(J)$, with $\|\sum_i a_i w_i\| = 1$. We define inductively a partition of J into pairwise disjoint (F_n) such that for any $n \in \mathbb{N}$

(F1) $F_n \cap \{j_n, j_{n+1}, \ldots\} \in \mathcal{S}_1(\mathcal{F}_{k_n})$,
(F2) $\|\sum_{i \in G} a_i w_i\| \leq 1/8^{k_{n-1}}$ for any $G \subset F_n$ with $G \in \mathcal{F}_{k_{n-1}}$,
(F3) if $F_n \neq \emptyset$, then F_n contains some $F \in \mathcal{F}_{k_n}$ with $\|\sum_{i \in F} a_i w_i\| > 1/8^{k_n}$,
(F4) $\|\sum_{i \in F} a_i w_i\| \leq 1/8^{k_n}$ for any $F \cap (F_1 \cup \cdots \cup F_n) = \emptyset$ with $F \in \mathcal{F}_{k_n}$.

The first inductive step is similar to the general step, thus we present only the general case. Assume we have $F_1, \ldots, F_{n-1}$ satisfying the above. From $J \setminus (F_1 \cup \cdots \cup F_{n-1})$, we pick a maximal family of pairwise disjoint sets $(F_n^j)_j \subset \mathcal{F}_{k_n}$ with $\|\sum_{i \in F_n^j} a_i w_i\| > 1/8^{k_n}$ for each j. Let $F_n = \bigcup_j F_n^j$. It follows that conditions (F3) and (F4) are satisfied. As there can be at most $4l_n \leq j_n$ many (F_n^j)'s we obtain (F1). Finally, the condition (F4) for $n-1$ implies (F2) for n, which ends the inductive construction. Compute, using (F2)

$$\begin{aligned} 1 = \Big\|\sum_i a_i w_i\Big\| &\leq \sum_n \sum_{i \in F_n, i < j_n} |a_i| + \sum_n \Big\|\sum_{i \in F_n, i \geq j_n} a_i w_i\Big\| \\ &\leq \sum_n \frac{n}{8^{k_{n-1}}} + \sum_n \Big\|\sum_{i \in F_n, i \geq j_n} a_i w_i\Big\|. \end{aligned}$$

It follows that $1/2 \leq \sum_n \|\sum_{i \in F_n, i \geq j_n} a_i w_i\|$ and thus for some n_0 we have

$$\Big\|\sum_{i \in F_{n_0}, i \geq j_{n_0}} a_i w_i\Big\| \geq \frac{1}{2^{n_0+1}}.$$

As (y_i) 2-dominates (w_i) we have $1/2^{k_{n_0-1}} \leq \|\sum_{i\in F_{n_0}, i>j_{n_0}} a_i y_i\|$. On the other hand by (F2) and definition of (w_i) we have $\|\sum_{i\in G} a_i y_i\| < 1/4^{k_{n_0-1}}$ for any $G\subset F_{n_0}$ with $G\in\mathcal{F}_{k_{n_0-1}}$. Thus by (★) for (w_i) and (y_i), we obtain

$$\Big\|\sum_{i\in F_{n_0}, i\geq j_{n_0}} a_i w_i\Big\| \leq \bar{\Delta}_{k_{n_0-1}} \Big\|\sum_{i\in F_{n_0}, i\geq j_{n_0}} a_i y_i\Big\|.$$

Putting the estimates together, by the choice of (k_n) and (F1) we obtain

$$\begin{aligned}\Big\|\sum_i a_i w_i\Big\| = 1 \leq 2^{n_0+1}\Big\|\sum_{i\in F_{n_0}, i>j_{n_0}} a_i w_i\Big\| &\leq \frac{1}{4^{n_0}}\Big\|\sum_{i\in F_{n_0}, i>j_{n_0}} a_i y_i\Big\| \\ &\leq \frac{1}{4^{n_0}} \max_{n_0\leq F\in\mathcal{S}_1(\mathcal{F}_{k_{n_0}})}\Big\|\sum_{i\in F\cap J} a_i y_i\Big\|\end{aligned}$$

which yields (▲) for $(x_i)_{i\in J}$ and $(y_i)_{i\in J}$ with families $(\mathcal{S}_1(\mathcal{F}_{k_n})\cap [J]^{<\infty})$. □

REMARK 3.7. Notice that by a simple modification of the above proof in (2) we can obtain the condition (▲) with arbitrary (δ_n), $\delta_n \searrow 0$.

LEMMA 3.8. *Consider seminormalized basic sequences* (x_i), (z_i), (y_i) *with* (y_i) *unconditional and* $[y_i]$ *not containing uniformly* c_0^n*'s.*

Assume (x_i) *and* (z_i) *are* $\mathcal{S}_\alpha$*-equivalent. Then if the pair* $(z_i), (y_i)$ *satisfies* (★), *then for some infinite* $J\subset\mathbb{N}$ *also* $(x_i)_{i\in J}, (y_i)_{i\in J}$ *satisfies* (★).

Proof. We can assume that the basic sequence (x_i) is bimonotone and (y_i) is 1-unconditional. Let $C\geq 1$ be the $\mathcal{S}_\alpha$-equivalence of $(x_i), (z_i)$ constant. Take (α_n) used to define $\mathcal{S}_\alpha$. Take (Δ_n) satisfying the condition (★) for $(z_i), (y_i)$ and pick (k_n), $k_n\geq n$, such that $\sum_n \Delta_{k_{n-1}} < \infty$. By Remark 3.1, there is $(t_n)\subset\mathbb{N}$ such that $\mathcal{F}_{k_n}\cap[(t_i)_{i>n}]^{<\infty}\subset\mathcal{S}_{\alpha_{t_n}}$ for each $n\in\mathbb{N}$.

Since $[y_i]$ does not contain uniformly c_0^n's, for any n we have $l_n<\infty$, where l_n is the supremum of all $l\in\mathbb{N}$ such that for some disjointly supported $z_1,\dots,z_l\in[y_i]$ with $\|z_j\|>1/2^{k_n}$, $j=1,\dots,l$, we have $\|z_1+\dots+z_l\|\leq 1$.

Pick $J=\{j_n : n\in\mathbb{N}\}\subset\{t_n\}$ with $j_n\geq\max\{k_n+1, l_n, t_n+1\}$, $n\in\mathbb{N}$. Take $(a_i)\in c_{00}(J)$, with $\|\sum_i a_i y_i\|=1$. As in the proof of Lemma 3.6, we define inductively a partition of J into pairwise disjoint (F_n) such that for any $n\in\mathbb{N}$

(F1) $F_n\cap\{j_n, j_{n+1},\dots\}\in\mathcal{S}_1(\mathcal{F}_{k_n})\subset\mathcal{S}_{\alpha_{t_n}+1}$,
(F2) $\|\sum_{i\in G} a_i y_i\|\leq 1/2^{k_{n-1}}$ for any $G\subset F_n$ with $G\in\mathcal{F}_{k_{n-1}}$,
(F3) if $F_n\neq\emptyset$, then F_n contains some $F\in\mathcal{F}_{k_n}$ with $\|\sum_{i\in F} a_i y_i\|>1/2^{k_n}$,
(F4) $\|\sum_{i\in F} a_i y_i\|\leq 1/2^{k_n}$ for any $F\cap(F_1\cup\dots\cup F_n)=\emptyset$ with $F\in\mathcal{F}_{k_n}$.

Now compute

$$\begin{aligned}\Big\|\sum_i a_i x_i\Big\| &\le \sum_{n:F_n\neq\emptyset}\sum_{i\in F_n, i<j_n}|a_i| + \sum_{n:F_n\neq\emptyset}\Big\|\sum_{i\in F_n, i\ge j_n} a_i x_i\Big\|\\ &\le \sum_{n:F_n\neq\emptyset}\frac{n}{2^{k_{n-1}}} + C\sum_{n:F_n\neq\emptyset}\Big\|\sum_{i\in F_n, i\ge j_n} a_i z_i\Big\| \quad \text{by (F2) and (F1)}\\ &\le \sum_{n:F_n\neq\emptyset}\frac{n}{2^{k_{n-1}}} + C\sum_{n:F_n\neq\emptyset}\Delta_{k_{n-1}} \quad \text{by (F2) and } (\bigstar).\end{aligned}$$

Fix $n_0\in\mathbb{N}$ and assume additionally that $\|\sum_{i\in F} a_i y_i\| \le 1/2^{k_{n_0}}$ for any $F\in\mathcal{F}_{k_{n_0}}$. Then by (F3), (F4) and the above computation

$$\Big\|\sum_i a_i x_i\Big\| \le \sum_{n\ge n_0}\frac{n}{2^{k_{n-1}}} + C\sum_{n\ge n_0}\Delta_{k_{n-1}},$$

thus $(\bigstar)$ for $(x_i)_{i\in J}$, $(y_i)_{i\in J}$ is satisfied with families $(\mathcal{F}_{k_n}\cap[J]^{<\infty})$. □

4. Strictly singular noncompact operators

In this section, we apply tools developed in the previous part to give sufficient conditions for the existence of nontrivial strictly singular operators. We note first a version of Theorem 1.1 [24] in $\mathcal{S}_\alpha$-unconditional setting.

PROPOSITION 4.1. *Let (x_i) and (y_i) be two seminormalized basic sequences such that (y_i) α-strongly dominates (x_i), for some limit $\alpha<\omega_1$.*

Then the map $y_i\mapsto x_i$ extends to a bounded noncompact strictly singular operator between $[y_i]$ and $[x_i]$.

Proof. As (y_i) dominates (x_i), the map $y_i\mapsto x_i$ extends to a bounded noncompact operator T between $[x_i]$ and $[y_i]$. To prove the strict singularity, use $(\bigstar)$ and Fact 3.3 with Remark 3.1. □

The next theorem will serve as a base for further applications. We build an operator using block sequences with different asymptotic behavior with respect to an auxiliary basic sequence (e_i). However the situation is analogous to the results in [3], [4], [19], we work on $(\mathcal{S}_{\alpha_n})$-admissible sequences instead of $(\mathcal{A}_n)$-admissible sequences, that is, sequences of length n, $n\in\mathbb{N}$. The sequence (e_i) plays the role of a spreading model in [3], [4], [19], in our setting we require domination of (e_i) by all its subsequences instead of subsymmetry.

THEOREM 4.2. *Let X be a Banach space with an $\mathcal{S}_\alpha$-unconditional basis, for limit $\alpha<\omega_1$. Let E be a Banach space with an unconditional basis (e_i) dominated by all its subsequences, not containing uniformly c_0^n's. Assume that*

(1) *X has a normalized basic sequence (x_i) α-strongly dominated by (e_i),*

(2) *for any* $\beta<\alpha$ *there exists a normalized block sequence* $(x_i^\beta)_i$ *with* $(\operatorname{supp} x_i^\beta)_i\subset\mathcal{S}_{r_\beta}$, *for some* $r_\beta\in\mathbb{N}$, *such that* $(x_i^\beta)_{i\in F}$ C*-dominates* $(e_i)_{i\in F}$ *for any* $F\in\mathcal{S}_\beta$ *and universal* $C\geq 1$.

Then X *admits a bounded strictly singular noncompact operator on a subspace.*

Remark 4.3. In case $E=\ell_1$ theorem above follows by Theorem 1.4, [24], as (1) and (2) imply (a) and (b) in Theorem 1.4. In case of $E=\ell_1$ partial unconditionality of suitable sequences follows by [10]. Comparing to Theorem 1.4 [24], theorem above can be regarded as an extension of Theorem 1.4 in replacing the u.v.b. of ℓ_1 by other basic sequence, however with the price paid on additional assumptions related to partial unconditionality. Recall that by [22] any normalized weakly null sequence admits an $\mathcal{S}_1$-unconditional subsequence, and the result was extended in [4] to special arrays of vectors, but analogous statement does not hold for $\mathcal{S}_\alpha$ with $\alpha>1$.

In the proof the lack of full unconditionality is substituted by $\mathcal{S}_\alpha$-unconditionality and uniform bound on admissibility of supports of each of block sequences $(x_i^{(n)})_i$ in (2). It follows that projections on $[(x_i^\beta)_{i\in F}]$ are bounded uniformly on $F\in\mathcal{S}_\beta$ provided $\min F$ is big enough and $\beta<\alpha$. We produce a block sequence (y_i) from sequences (x_i^β) in the standard way and show that some subsequences $(x_i)_{i\in J}$ and $(y_i)_{i\in J}$ satisfy ($\bigstar$) passing through Lemma 3.6. Since we cannot assure even $\mathcal{S}_\alpha$-unconditionality of (y_i), we need to prove strict singularity of the operator carrying $(y_i)_{i\in J}$ to $(x_i)_{i\in J}$ by hand.

Proof of Theorem 4.2. Take (α_n) used to define $\mathcal{S}_\alpha$. We can assume that X does not contain c_0 and its basis is $\mathcal{S}_\alpha$-unconditional with constant 1. As (e_n) is dominated by all its subsequences, it is also uniformly dominated by its subsequences, and we assume that the uniform domination constant is 1. By Lemma 3.6 and Remark 3.1, for some infinite $J\subset\mathbb{N}$, $(k_n)\subset\mathbb{N}$, we have, letting $\mathcal{F}_n=\mathcal{S}_{\alpha_{k_n}}$,

$$\Big\|\sum_i a_i x_i\Big\|\leq\max_{n\in\mathbb{N}}\frac{1}{4^n}\max_{n\leq F\in\mathcal{F}_n}\Big\|\sum_{i\in F\cap J}a_i e_i\Big\|,\quad (a_i)\in c_{00}(J).$$

Given $(x_i^{\alpha_n})_i\subset X$, $n\in\mathbb{N}$, as in (2) let $y_i^{(n)}=x_i^{\alpha_{k_n}}$ for any $i,n\in\mathbb{N}$. By the assumption on (e_i), passing to subsequences we can assume that $y_1^{(1)}<y_2^{(1)}<y_2^{(2)}<y_3^{(1)}<y_3^{(2)}<y_3^{(3)}<\cdots$ and $r_{\alpha_{k_n}}+k_n<y_i^{(n)}$ for any $i\geq n$. Then

$$\operatorname{supp}\sum_{i\in F}y_i^{(n)}\in\mathcal{S}_\alpha\quad\text{for any } n\leq F\in\mathcal{F}_n, n\in\mathbb{N}. \tag{4.1}$$

By choice of $(y_i^{(n)})_i$, we have for any $(a_i)\in c_{00}(J)$

$$\Big\|\sum_i a_i x_i\Big\|\leq\max_{n\in\mathbb{N}}\frac{C}{4^n}\max_{n\leq F\in\mathcal{F}_n}\Big\|\sum_{i\in F\cap J}a_i y_i^{(n)}\Big\|.$$

Let $y_i = \sum_{n=1}^{i} \frac{1}{2^n} y_i^{(n)}$ for any $i \in I$. Obviously (y_i) is a seminormalized block sequence. Fix now $n_0 \in \mathbb{N}$ and continue the above estimation

$$\begin{aligned}\Big\|\sum_i a_i x_i\Big\| &\leq C \max\Big\{ \max_{\substack{n\leq n_0\\ n\leq F\in\mathcal{F}_n}} \Big\|\sum_{i\in F\cap J} a_i y_i^{(n)}\Big\|, \frac{1}{4^{n_0}} \max_{\substack{n> n_0\\ n\leq F\in\mathcal{F}_n}} \Big\|\sum_{i\in F\cap J} a_i y_i^{(n)}\Big\|\Big\} \\ &\leq C \max\Big\{ \max_{n\leq n_0} \max_{n\leq F\in\mathcal{F}_{n_0}} \Big\|\sum_{i\in F\cap J} a_i y_i^{(n)}\Big\|, \frac{1}{2^{n_0}} \Big\|\sum_i a_i y_i\Big\|\Big\},\end{aligned}$$

where the last inequality follows by (4.1) and $\mathcal{S}_\alpha$-unconditionality of the basis of X. Thus, the following claim holds true.

Claim (A). *For any $n_0 \in \mathbb{N}$ and $(a_i) \in c_{00}(J)$ with $\|\sum_i a_i y_i\| = 1$ we have*

$$\Big\|\sum_i a_i x_i\Big\| \leq C \max\Big\{ \max_{n\leq n_0} \max_{n\leq F\in\mathcal{F}_n} \Big\|\sum_{i\in F\cap J} a_i y_i^{(n)}\Big\|, \frac{1}{2^{n_0}}\Big\}.$$

Taking $n_0 = 0$, we obtain that $(y_i)_{i\in J}$ dominates $(x_i)_{i\in J}$, thus the mapping $y_i \mapsto x_i$ extends to a bounded noncompact operator $T : [(y_i)_{i\in J}] \to [(x_i)_{i\in J}]$. However, we obtained also (★) for the pair $(x_i)_{i\in J}$, $(y_i)_{i\in J}$, without $\mathcal{S}_\alpha$-unconditionality of (y_i) we need to prove the strict singularity of T by hand. First, we adapt Fact 3.3 to our setting.

Claim (B). *Given any $n \in \mathbb{N}$ and $\varepsilon > 0$, any block subspace $W \subset [y_i]$ contains a further block subspace V such that any $w = \sum_i a_i y_i \in V$ satisfies*

$$\max_{F\in\mathcal{F}_n} \Big\|\sum_{i\in F} a_i y_i^{(n)}\Big\| < \varepsilon\|w\|.$$

To prove Claim (B), we first show that for any $\varepsilon > 0$, $n \in \mathbb{N}$, $\beta \leq \alpha_{k_n}$, any block subspace $W \subset [y_i]$ contains a vector $w_\varepsilon = \sum_i a_i y_i$ satisfying $\max_{F\in\mathcal{S}_\beta} \|\sum_{i\in F} a_i y_i^{(n)}\| < \varepsilon\|w_\varepsilon\|$. The proof of this statement follows step by step the proof of Fact 3.3, as we assumed at the beginning that X does not contain c_0. We assume that $W \geq n$, estimate $\|\sum_{i\in F} \pm a_i y_i^{(n)}\|$ instead of $\|\sum_{i\in F} \pm a_i y_i\|$ and use (4.1) to obtain $\|\sum_{i\in G} a_i y_i^{(n)}\| \leq \|\sum_{i\in G} a_i y_i\|$ for any $n \leq G \in \mathcal{F}_n$. Once we have this statement, to complete the proof of Claim (B) let $V = [w_{\varepsilon/2^i}]$.

With the above two claims, we are ready to prove the strict singularity of T. Fix $n_0 \in \mathbb{N}$, take any block subspace $W \subset [y_i]$ and using Claim (B) pick inductively block subspaces $W \supset V_{n_0} \supset V_{n_0-1} \supset \cdots \supset V_0$ such that for any $w = \sum_i a_i y_i \in V_0$ we have $\max_{F\in\mathcal{F}_n} \sum_{i\in F} a_i y_i^{(n)} \leq \frac{1}{2^{n_0}}\|w\|$ for any $n \leq n_0$. Claim (A) ends the proof. □

The model space E in Theorem 4.2 is the p-convexified Tsirelson-type space $T_\theta^{(p)}$, for $1 \leq p < \infty$ and $\theta \in (0, 1]$. As Theorem 2.2 yields condition (2) of Theorem 4.2 in case $\alpha = \omega$ for any asymptotic ℓ_p space X with lower asymptotic constants (θ_n) and $E = T_\theta^{(p)}$, where $\theta = \lim_n \theta_n^{1/n}$, we obtain the following.

COROLLARY 4.4. *Let X be an asymptotic ℓ_p Banach space, $1 \leq p < \infty$, with lower asymptotic constants (θ_n) and an $\mathcal{S}_\omega$-unconditional basis.*

Assume X contains a normalized basic sequence (x_i) ω-strongly dominated by the u.v.b. of $T_\theta^{(p)}$, where $\theta = \lim_n \theta_n^{1/n}$.

Then X admits a bounded strictly singular noncompact operator on a subspace.

By Lemma 3.5, the typical space X for the above situation is a regular p-convexified mixed Tsirelson space $T^{(p)}[(\mathcal{S}_n, \theta_n)_n]$ with $\theta_n/\theta^n \to 0$, where $\theta = \sup_n \theta_n^{1/n}$. However, as conditions (1) and (2) of Theorem 4.2 are invariant under $\mathcal{S}_\omega$-equivalence up to taking subsequences (for (1) use Lemma 3.8), a stronger result, requiring only $\mathcal{S}_\omega$-representation of the regular mixed Tsirelson space, holds true.

COROLLARY 4.5. *Let X be a Banach space with an $\mathcal{S}_\omega$-unconditional basis (x_i).*

Assume the basis (x_i) is $\mathcal{S}_\omega$-equivalent to the u.v.b. of a regular p-convexified mixed Tsirelson space $T^{(p)}[(\mathcal{S}_n, \theta_n)_n]$. Assume also that $\theta_n/\theta^n \to 0$, where $\theta = \lim_n \theta_n^{1/n}$.

Then X admits a bounded strictly singular noncompact operator on a subspace.

REMARK 4.6. By Remark 2.5 above, the corollaries hold for any $\alpha < \omega_1$, in terms of ℓ_p^α-asymptotic spaces, convexified mixed Tsirelson spaces $T^{(p)}[(\mathcal{S}_{\alpha n}, \theta_n)]$ and convexified Tsirelson-type spaces $T^{(p)}[\mathcal{S}_\alpha, \theta]$.

We will recall now construction of spaces based on mixed Tsirelson spaces, initiated in [6], used for building classes of HI asymptotic ℓ_p spaces with different types of properties, see also [8], [2], [13].

Fix $1 \leq p < \infty$, let $1 < q \leq \infty$ satisfy $1/p + 1/q = 1$. Fix infinite sets $N, L \subset \mathbb{N}$ (not necessarily disjoint) and scalars $(\theta_n)_{n \in N}, (\rho_l)_{l \in L} \subset (0,1)$ with $\theta_n \to 0$, $\rho_l \to 0$. Assume that $\theta_n^{1/p} \in \mathbb{Q}$ for any $n \in N$ and $\rho^{1/p} \in \mathbb{Q}$ for any $l \in L$. Assume also the regularity of (θ_n), that is, that $\theta_n \geq \prod_{i=1}^l \theta_{n_i}$ for any $n, n_1, \ldots, n_l \in N$ with $\sum_{i=1}^l n_i \geq n$.

Set $c_{00}(\mathbb{Q}) = c_{00} \cap \mathbb{Q}^{\mathbb{N}}$. Let $\mathcal{W} = \{(f_1, \ldots, f_k) : f_1 < \cdots < f_k \in c_{00}(\mathbb{Q}) \cap B_{\ell_q}, k \in \mathbb{N}\}$ and fix an injective function $\sigma : \mathcal{W} \to N$. For any $D \subset c_{00}(\mathbb{Q})$, define

$$D_n = \left\{ \theta_n^{1/p} \sum_{i=1}^k \gamma_i f_i : f_1, \ldots, f_k \in D, (f_1, \ldots, f_k) \text{ is } \mathcal{S}_n\text{-admissible}, \right.$$
$$\left. (\gamma_i) \in B_{\ell_q} \cap c_{00}(\mathbb{Q}), k \in \mathbb{N} \right\}, \quad n \in N,$$

$$D_l^\sigma = \Bigg\{ \rho_l^{1/p} \sum_{i=1}^k \gamma_i E f_i : f_1,\dots,f_k \in D, (f_1,\dots,f_k) \text{ is } (\sigma,\mathcal{S}_l)\text{-admissible}, \\ (\gamma_i) \in B_{\ell_q} \cap c_{00}(\mathbb{Q}), E \subset \mathbb{N} \text{ interval}, k \in \mathbb{N} \Bigg\}, \quad l \in L,$$

where a block sequence $(f_1,\dots,f_k)$ is $(\sigma,\mathcal{S}_l)$-admissible, if $(f_1,\dots,f_k)$ is $\mathcal{S}_l$-admissible, $f_1 \in \bigcup_{n\in N} D_n$ and $f_{i+1} \in D_{\sigma(f_1,\dots,f_i)}$ for any $i<k$.

Consider a symmetric set $D \subset c_{00}(\mathbb{Q})$ such that

(D1) $(\pm e_n^*)_n \subset D$,
(D2) $D \subset \bigcup_{n\in N} D_n \cup \bigcup_{l\in L} D_l^\sigma$,
(D3) $D_n \subset D$ for any $n \in N$.

Define a norm on c_{00} by $\|x\|_D = \sup\{f(x) : f \in D\}$, $x \in c_{00}$, denote by X_D the completion of $(c_{00}, \|\cdot\|_D)$. Obviously the u.v.b. (e_n) is a basis for X_D.

It follows that $D \subset K_{N\cup L}$, where $K_{N\cup L}$ is the norming set of the p-convexified mixed Tsirelson space defined by all pairs $(\mathcal{S}_n,\theta_n)_{n\in N} \cup (\mathcal{S}_l,\rho_l)_{l\in L}$, thus each functional in D admits a tree-analysis (Definition 1.7). By (D3) also $D \supset K$, where K is the norming set of $T^{(p)}[(\mathcal{S}_n,\theta_n)_{n\in N}]$.

Corollary 4.7. *Let X_D be defined as above. Assume*

$$\lim_{n\in N, n\to\infty} \theta_n/\theta^n = 0, \quad \textit{where } \theta = \sup_{n\in N} \theta_n^{1/n}.$$

Then X_D admits a bounded noncompact strictly singular operator on a subspace.

Proof. It is enough to show that for some $(i_n)_n \subset \mathbb{N}$ the following hold

(1) sequence $(e_{i_n}) \subset X_D$ is $\mathcal{S}_\omega$-unconditional,
(2) sequences $(e_{i_n}) \subset X_D$, $(e_{i_n}) \subset T^{(p)}[(\mathcal{S}_n,\theta_n)_{n\in N}]$ are $\mathcal{S}_\omega$-equivalent.

Indeed, recall that $T^{(p)}[(\mathcal{S}_n,\theta_n)_{n\in N}]$ is isomorphic to a regular space given by $T^{(p)}[(\mathcal{S}_n,\bar\theta_n)_{n\in \mathbb{N}}]$, with $(\bar\theta_n)$ defined as in Remark 1.6. By the regularity of $(\theta_n)_{n\in N}$, we have $\theta_n = \bar\theta_n$ for any $n \in N$. Therefore, the subspace $[e_{i_n}]$ by Lemma 3.5 satisfies the assumption of Corollary 4.5, which ends the proof.

Now we pick $(i_n)_n \subset \mathbb{N}$ with desired properties. Let $Z = T^{(p)}[(\mathcal{S}_n,\theta_n)_{n\in N}]$. We denote by (e_i) the u.v.b. both in X_D and Z. We will show the following.

Claim. *For any $n \in \mathbb{N}$ there is $i_n \in \mathbb{N}$ such that for any $(a_i)_{i\in F}$ with $F \in \mathcal{S}_n$ and $F \geq i_n$ we have $\|\sum_{i\in F} a_i e_i\|_D \leq 4\|\sum_{i\in F} a_i e_i\|_Z$.*

First, notice that claim implies (1) and (2) for (e_{i_n}). Indeed, (2) follows straightforward, as $\|\sum_i a_i e_i\|_D \geq \|\sum_i a_i e_i\|_Z$ for any $(a_i) \in c_{00}$ by the property $K \subset D$. Also by claim for any $(a_i)_{i\in F}$ with $i_n \leq F \in \mathcal{S}_n$, $n \in \mathbb{N}$, there is a norming functional $f \in Z^*$, therefore also $f \in X_D^*$, with $\operatorname{supp} f \subset F$, such that $\|\sum_{i\in F} a_i e_i\|_D \leq 4 f(\sum_{i\in F} a_i e_i)$ in X_D. Thus, we obtain (1) for (e_{i_n}).

We proceed to proof of claim. Fix $n\in\mathbb{N}$. Pick j_n such that $\theta_j\le\frac{1}{2^p}\theta_n$ for any $j_n\le j\in N$ and $\rho_j\le\frac{1}{2^p}\theta_n$ for any $j_n\le j\in L$. By injectivity of σ there is i_n such that $\sigma(f)>j_n$ for any $f\in\mathcal{W}$ with $\max\operatorname{supp} f\ge i_n$.

Take now any $(a_i)_{i\in F}$, $i_n\le F\in\mathcal{S}_n$, with $\|\sum_{i\in F}a_ie_i\|_D=1$. It follows by (D1) and (D3) that $\theta_n\sum_{i\in F}|a_i|^p\le 1$. Take a norming functional $f\in D$ with a tree-analysis $(f_t)_{t\in\mathcal{T}}$ satisfying $f(\sum_{i\in F}a_ie_i)=1$. Let

$$I=\big\{i\in F:\ \operatorname{char}(f_t)<j_n \text{ for any } t\in\mathcal{T} \text{ with } f_t(e_i)\neq 0\big\}.$$

Then by Hölder inequality and choice of j_n

$$\left|f\Big(\sum_{i\in F\setminus I}a_ie_i\Big)\right|\le\frac{1}{2}\theta_n^{1/p}\Big(\sum_{i\in F}|a_i|^p\Big)^{1/p}\le\frac{1}{2}.$$

Thus, $f(\sum_{i\in I}a_ie_i)\ge\frac{1}{2}$. Let $I_1=\{i\in I: a_i>0, f(e_i)>0\}$ and $I_2=\{i\in I: a_i<0, f(e_i)<0\}$. Then either $f(\sum_{i\in I_1}a_ie_i)\ge\frac14$ or $f(\sum_{i\in I_2}a_ie_i)\ge\frac14$. Assume the first case holds and let $x=\sum_{i\in I_1}a_ie_i$. Take any $t\in\mathcal{T}$ with $f_t(x)\neq 0$ and $f_t\in D_l^\sigma$ for some $l\in L$. Then by choice of i_n and I there is at most one $s_t\in\operatorname{succ}(t)$ with $\operatorname{supp} f_{s_t}\cap I\neq\emptyset$.

Given any nonterminal $t\in\mathcal{T}$, with $f_t=\theta_{n_t}^{1/p}\sum_{s\in\operatorname{succ}(t)}\gamma_sf_s$ let $|f_t|=\theta_{n_t}^{1/p}\sum_{s\in\operatorname{succ}(t)}|\gamma_s|f_s$. Construct a functional g replacing in the tree-analysis $(f_t)_{t\in\mathcal{T}}$ each $f_t\in D_l^\sigma$ by $|f_{s_t}|$. Then $g\in K$ as every node of the tree-analysis of g belongs to $\bigcup_n D_n$. For h defined as the restriction of g to I we have $h\in K$ and $h(\sum_{i\in F}a_ie_i)=h(x)\ge f(x)\ge\frac14$, which ends the proof of claim. □

Remark 4.8. Notice that in case $\theta=1$ the sequence $(\bar\theta_n)$ defined in Remark 1.6 also satisfies $\bar\theta=1$, thus the assumption of Corollary 4.5 are satisfied. Therefore, in this case we do not need the regularity of $(\theta_n)_{n\in N}$ in Corollary 4.7.

Corollary 4.9. *Each one of the HI asymptotic ℓ_2 Banach space X_{AB} constructed in [2] and the HI asymptotic ℓ_p Banach spaces $X_{(p)}$, $1<p<\infty$, constructed in [13] admit bounded strictly singular noncompact operators on a subspace.*

Proof. To show the corollary notice that spaces X_{AB} and $X_{(p)}$ are of the form X_D with $N=(n_{2i})$, $L=(n_{2i+1})$, $\theta_{n_{2i}}=\frac{1}{m_{2i}^p}$, $i\in\mathbb{N}$, for suitably chosen (n_i), (m_i), satisfying $\theta_{n_{2i}}^{1/n_{2i}}\to 1$. In case of X_{AB} we have $\rho_{n_{2i+1}}=\frac{1}{m_{2i+1}^2}$, in case of $X_{(p)}$ we have $\rho_{n_{2i+1}}=\frac{2}{2^pm_{2i+1}^p}$. The remark above ends the proof. □

Remark 4.10. Comparing to [11], we obtain here a nontrivial strictly singular operator only on a subspace of considered HI asymptotic ℓ_p spaces, nevertheless—thanks to the applied method—with much less restrictions on sets N, L and parameters (θ_n), (ρ_l) used in the construction of the spaces.

Notice that the HI space in [8] also admits a bounded strictly singular noncompact operators on a subspace by Theorem 1.4 [24], Proposition 3.3 [7] and the fact that its basis does not generate an ℓ_ω-spreading model.

Acknowledgment. We are grateful to the referee for valuable comments.

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