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APPLICATIONS OF SEMI-EMBEDDINGS TO THE STUDY OF THE PROJECTIVE TENSOR PRODUCT OF BANACH SPACES

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Abstract. In this paper, we use the theory of semi-embeddings to show that if E is a Banach lattice and X is a Banach space then $E \hat{\otimes} X$, the projective tensor product of E and X, has, respectively, the near Radon–Nikodym property, the analytic Radon–Nikodym property, the analytic complete continuity property, and the property of non-containment of a copy of c_0 whenever both E and X have the same property.

1. Introduction

For Banach spaces X and Y, let $X \hat{\otimes} Y$ denote the projective tensor product of X and Y. Bourgain and Pisier [2] constructed a Banach space X with the Radon–Nikodym property for which $X \hat{\otimes} X$ fails to have the Radon–Nikodym property. This remarkable counter-example shows that the Radon–Nikodym property is, in general, not inherited by the projective tensor products. From the Pisier's famous example that $L^1/H_0^1 \hat{\otimes} L^1/H_0^1$ contains c_0 (and hence fails to have the near Radon–Nikodym property while L^1/H_0^1 has the near Radon–Nikodym property, see [19]), it is shown that the near Radon–Nikodym property and the property of non-containment of a copy of c_0 are, in general, not inherited by the projective tensor products. However, these properties are indeed inherited by the projective tensor products under special circumstances. For instance, Andrews [1] showed that the Radon–Nikodym property is inherited by $X^* \hat{\otimes} Y$ if X^* has the approximation property, and Oja [25] showed that the property of non-containment of a copy of c_0 is inherited by $X \hat{\otimes} Y$

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if X is a weakly compactly generated space with the bounded approximation property and every integral operator from Y^* to X is nuclear.

The theory of semi-embeddings in Banach spaces was introduced by Lotz, Peck, and Porta [23] and then developed by Bourgain and Rosenthal [4]. By using the theory of semi-embeddings, Diestel, Fourie, and Swart [9], [10] showed that the Radon–Nikodym property is inherited by $X \hat{\otimes} Y$ if one of X and Y is a Banach lattice. In this paper, we use the theory of semi-embeddings to show that the near Radon–Nikodym property, the analytic Radon–Nikodym property, the analytic complete continuity property, and the property of non-containment of a copy of c_0 are inherited by $X \hat{\otimes} Y$ if one of X and Y is a Banach lattice.

2. Basic definitions

A continuous linear operator from a Banach space X to a Banach space Y is called a *semi-embedding* if it is one to one and the image of the closed unit ball of X is a closed subset of Y. A Banach space X is said to *semi-embed* into a Banach space Y if there is a semi-embedding from X to Y (see [23]). A Banach space property $\mathcal{P}$ is called (i) *separably determined* if a Banach space X has $\mathcal{P}$ whenever every separable closed subspace of X has $\mathcal{P}$; (ii) *separably semi-embeddably stable* if a separable Banach space X has $\mathcal{P}$ whenever X semi-embeds into a Banach space with $\mathcal{P}$; (iii) *isomorphically stable* if a Banach space X has $\mathcal{P}$ whenever X is isomorphic to a Banach space with $\mathcal{P}$ (see [15], [16]).

A continuous linear operator $T : L_1[0,1] \to X$ is called (i) *representable* if there is a Bochner integrable function $g \in L_\infty([0,1],X)$ such that $T(f) = \int fg\,dm$ for all $f \in L_1[0,1]$; (ii) *Dunford–Pettis* (or *completely continuous*) if T sends weakly null sequences into norm null sequences; (iii) *nearly representable* if for each Dunford–Pettis operator $D : L_1[0,1] \to L_1[0,1]$, the composition $T \circ D$ is representable. A Banach space X is said to have (i) *the Radon–Nikodym property* (RNP for short) if every continuous linear operator from $L_1[0,1]$ to X is representable (see [11, Chapter 3]); (ii) *the near Radon–Nikodym property* (nRNP for short) if every nearly representable operator from $L_1[0,1]$ to X is representable (see [19]); (iii) *the complete continuity property* (CCP for short) if every continuous linear operator from $L_1[0,1]$ to X is completely continuous (see [24]).

Remark 2.1. RNP $\Longrightarrow$ nRNP, and RNP $\Longrightarrow$ CCP. Neither converse is true. For instance, $L_1[0,1]$ has nRNP (see [19]) but fails to have RNP, and Bourgain–Rosenthal space (see [3]) has CCP but fails to have RNP. Moreover, RNP (see [4]), nRNP (see [19]), and CCP (see [31]) are separably semi-embeddably stable, separably determined, and isomorphically stable.

Let X be a complex Banach space, $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle of $\mathbb{C}$, $\mathcal{B}$ be the σ-algebra of Borel subsets of $\mathbb{T}$, and λ be the normalized

Lebesgue measure on $\mathbb{T}$. The *Fouries coefficients* of a countably additive X-valued measure μ of bounded variation are defined to be

$$\hat{\mu}(n) = \int_{\mathbb{T}} e^{-int}\, d\mu(t), \quad n \in \mathbb{Z}.$$

A countably additive X-valued measure μ of bounded variation is called *analytic* if $\hat{\mu}(n) = 0$ for all $n < 0$. A complex Banach space X is said to have (i) the *analytic Radon–Nikodym property* (aRNP for short) if each analytic X-valued measure is differentiable (see [8]); and (ii) the *analytic complete continuity property* (aCCP for short) if each analytic X-valued measure has a relatively compact range (see [30]).

REMARK 2.2. RNP $\Longrightarrow$ aRNP $\Longrightarrow$ aCCP, and CCP $\Longrightarrow$ aCCP. None of the possible converses is true. For instance, $L_1(\mathbb{T})$ has aRNP (see [8]) and aCCP (see [30]) but fails to have RNP and CCP; and the Davis–Figiel–Johnson–Pelczynski interpolation space has aCCP but fails to have aRNP (see [30]). Moreover, aRNP (see [13]) and aCCP (see [31]) are separably semi-embeddably stable, separably determined, and isomorphically stable.

REMARK 2.3. It is known from [14] that the property of non-containment of a copy of c_0 is separably semi-embeddably stable. It is also separably determined and isomorphically stable.

3. Köthe–Bochner function space $E(\mu, X)$

Throughout this paper, for a Banach space X, X^* will denote its topological dual and B_X will denote its closed unit ball. For Banach spaces X and Y, $X \hat{\otimes} Y$ will denote the projective tensor product of X and Y.

Let (Ω, Σ, μ) be a probability measure space and $L_0(\mu, X)$ be the space of all (equivalence classes of) strongly μ-measurable functions from Ω to X. Recall that a Banach space $E(\mu)$ is called a *Köthe function space* over (Ω, Σ, μ) (see [22, p. 28] or [21, p. 149]) if $E(\mu)$ consists of (equivalence classes of) μ-integrable real valued functions on Ω such that

(i) If $|f(\omega)| \le |g(\omega)|$ μ-a.e. on Ω, with f μ-measurable and $g \in E(\mu)$, then $f \in E(\mu)$ and $\|f\|_{E(\mu)} \le \|g\|_{E(\mu)}$.
(ii) For every $A \in \Sigma$, the characteristic function χ_A of A belongs to $E(\mu)$.

Clearly, $E(\mu)$ is a Banach lattice in the obvious order ($f \ge 0$ if $f(\omega) \ge 0$ a.e. on Ω). Let $E'(\mu)$ denote the *Köthe dual* of $E(\mu)$, i.e.,

$$E'(\mu) = \left\{ g \in L_0(\mu, \mathbb{R}) : \int_\Omega \big| f(\omega) g(\omega) \big|\, d\mu(\omega) < \infty \ \forall f \in E(\mu) \right\}.$$

Obviously, $E'(\mu) \subseteq E(\mu)^*$. With the norm induced by $E(\mu)^*$, $E'(\mu)$ is also a Köthe function space on (Ω, Σ, μ) with the norm

$$\|g\|_{E'(\mu)} = \sup \left\{ \left| \int_\Omega f(\omega) g(\omega)\, d\mu(\omega) \right| : f \in B_{E(\mu)} \right\} \quad \forall g \in E'(\mu).$$

Moreover, $E'(\mu) = E(\mu)^*$ if and only if $E(\mu)$ is σ-order continuous (see [22, p. 29]).

Recall that a Köthe function space $E(\mu)$ is said to have the *Fatou property* if

$$f_n \in E(\mu), 0 \le f_n(\omega) \uparrow f(\omega)\text{a.e.}, \sup_n \|f_n\| < \infty \Rightarrow f \in E(\mu), \quad \|f\| = \lim_n \|f_n\|.$$

It is known from [22, p. 30] that for any Köthe function space $E(\mu)$, $E'(\mu)$ has the Fatou property, and that $E(\mu)$ has the Fatou property if and only if $E''(\mu) = E(\mu)$.

For a Banach space X, let $E(\mu, X)$ denote the *Köthe–Bochner function space*, that is,

$$E(\mu, X) = \left\{ f \in L_0(\mu, X) : \left\| f(\cdot) \right\|_X \in E(\mu) \right\}$$

and

$$\|f\|_{E(\mu,X)} = \left\| \left\| f(\cdot) \right\|_X \right\|_{E(\mu)} \quad \forall f \in E(\mu, X).$$

Then $(E(\mu, X), \|\cdot\|_{E(\mu,X)})$ is a Banach space (see [21, Chapter 3]). In particular, if $E(\mu) = L_p(\mu)$ then $E(\mu, X) = L_p(\mu, X)$.

By using the Fatou's lemma we improve Lemma 3.1.22 in [21, p. 158] to the following.

PROPOSITION 3.1. *If $E(\mu)$ has the Fatou property then the inclusion map from $E(\mu)$ to $L_1(\mu)$ is a semi-embedding.*

Proof. Let $f_n \in B_{E(\mu)}$ and $f \in L_1(\mu)$ such that $f_n \to f$ in $L_1(\mu)$. Then $f_n \to f$ in measure and hence, there is a subsequence $\{f_{n_k}\}_1^\infty$ of $\{f_n\}_1^\infty$ such that $f_{n_k}(\omega) \to f(\omega)$ μ-a.e. on Ω. Let $g \in E'(\mu)$. Then $f_{n_k}(\omega)g(\omega) \to f(\omega)g(\omega)$ μ-a.e. on Ω. It follows from the Fatou's lemma that

$$\int_\Omega |fg|\, d\mu \le \underline{\lim}_k \int_\Omega |f_{n_k} g|\, d\mu \le \underline{\lim}_k \|f_{n_k}\|_{E(\mu)} \cdot \|g\|_{E'(\mu)} \le \|g\|_{E'(\mu)}.$$

Thus, $f \in E''(\mu) = E(\mu)$. By [22, p. 29, Proposition 1.b.18], $E'(\mu)$ is a norming subspace of $E(\mu)^*$. Thus,

$$\|f\|_{E(\mu)} = \sup \left\{ \left| \int_\Omega f(\omega) g(\omega)\, d\mu(\omega) \right| : g \in B_{E'(\mu)} \right\} \le 1,$$

that is, $f \in B_{E(\mu)}$. □

Randrianantoanina and Saab [28, Lemma 3] showed that if $E(\mu)$ semi-embeds into $L_1(\mu)$ then $E(\mu, X)$ semi-embeds into $L_1(\mu, X)$. With the help of Proposition 3.1 we reformulate Lemma 3 of [28] as follows.

PROPOSITION 3.2. *If $E(\mu)$ has the Fatou property then the inclusion map from $E(\mu, X)$ to $L_1(\mu, X)$ is a semi-embedding.*

Recall that a Banach lattice is called a *Kantorovich–Banach space* (KB-space) if every monotone norm bounded sequence is norm convergent. It is clear that if $E(\mu)$ is a KB-space then it is order continuous and has the Fatou property.

THEOREM 3.3. *Let $\mathcal{P}$ be a Banach space property which is separably determined, separably semi-embeddably stable, and isomorphically stable. If $E(\mu)$ is a KB-space, then $E(\mu, X)$ has $\mathcal{P}$ whenever $L_1(\mu, X)$ has $\mathcal{P}$.*

Proof. Take any separable closed subspace S of $E(\mu, X)$. Then there are a separable closed subspace $F(\mu)$ of $E(\mu)$ and a separable closed subspace Y of X such that S is a subspace of $F(\mu, Y)$. Since $F(\mu)$ is also a KB-space, $F(\mu)$ has the Fatou property. By Proposition 3.2, $F(\mu, Y)$ semi-embeds into $L_1(\mu, Y)$. Note that $F(\mu, Y)$ is separable and $L_1(\mu, Y)$, as a subspace of $L_1(\mu, X)$, has $\mathcal{P}$. Thus, $F(\mu, Y)$ has $\mathcal{P}$ and hence, S has $\mathcal{P}$. □

Note that a Banach lattice is a KB-space if and only if it contains no copy of c_0 and that aRNP and aCCP are stronger than the property of non-containment of a copy of c_0. Also note that $L_1(\mu, X)$ has, respectively, aRNP (see [12]), aCCP (see [30]), and the property of non-containment of a copy of c_0 (see [18], [20]) whenever X has the same property. Thus, Theorem 3.3 yields the following consequence.

COROLLARY 3.4. *$E(\mu, X)$ has, respectively, aRNP, aCCP, and the property of non-containment of a copy of c_0 whenever both $E(\mu)$ and X have the same property.*

REMARK 3.5. Buhvalov [7] showed that $E(\mu, X)$ has RNP whenever both $E(\mu)$ and X have RNP. Randrianantoanina and Saab [29] showed that $E(\mu, X)$ has nRNP whenever both $E(\mu)$ and X have nRNP. Randrianantoanina [27] showed that if $E(\mu)$ has RNP and X has CCP then $E(\mu, X)$ has CCP. We do not know if $E(\mu, X)$ has CCP whenever both $E(\mu)$ and X have CCP.

4. Properties inherited by the projective tensor products

For a Köthe function space $E(\mu)$ and a Banach space X, let $E_{weak}(\mu, X) := E_w(\mu, X)$ denote (so called) the *weak Köthe–Bochner function space*, that is,

$$E_w(\mu, X) = \{ f \in L_0(\mu, X) : x^* f(\cdot) \in E(\mu) \ \forall x^* \in X^* \}$$

and

$$\|f\|_{E_w(\mu,X)} = \sup\{ \|x^* f(\cdot)\|_{E(\mu)} : x^* \in B_{X^*} \}.$$

Then $(E_w(\mu, X), \|\cdot\|_{E_w(\mu,X)})$ is a normed space. Obviously, $E(\mu, X) \subseteq E_w(\mu, X)$ and $\|\cdot\|_{E_w(\mu,X)} \le \|\cdot\|_{E(\mu,X)}$. The following fact is straightforward from the Hahn–Banach Extension theorem.

PROPOSITION 4.1. *If Y is a closed subspace of X then $E_w(\mu,Y) \subseteq E_w(\mu,X)$ and for each $f \in E_w(\mu,Y)$, $\|f\|_{E_w(\mu,Y)} = \|f\|_{E_w(\mu,X)}$.*

For a Köthe function space $E(\mu)$ and a Banach space X, let $E_{weak^*}(\mu, X^*) := E_{w^*}(\mu, X^*)$ denote (so called) the *weak* Köthe–Bochner function space*, i.e.,

$$E_{w^*}\big(\mu, X^*\big) = \big\{g \in L_0\big(\mu, X^*\big) : g(\cdot)(x) \in E(\mu)\ \forall x \in X\big\}$$

and

$$\|g\|_{E_{w^*}(\mu,X^*)} = \sup\big\{\big\|g(\cdot)(x)\big\|_{E(\mu)} : x \in B_X\big\}.$$

Then $(E_{w^*}(\mu,X^*), \|\cdot\|_{E_{w^*}(\mu,X^*)})$ is a normed space. Obviously, $E_w(\mu,X^*) \subseteq E_{w^*}(\mu,X^*)$ and $\|\cdot\|_{E_{w^*}(\mu,X^*)} \le \|\cdot\|_{E_w(\mu,X^*)}$. Moreover, if $E(\mu)$ has the Fatou property then we use the Principle of Local Reflexivity to show that $E_w(\mu,X^*) = E_{w^*}(\mu,X^*)$ with $\|\cdot\|_{E_{w^*}(\mu,X^*)} = \|\cdot\|_{E_w(\mu,X^*)}$.

PROPOSITION 4.2. *If $E(\mu)$ has the Fatou property, then $E_w(\mu,X^*) = E_{w^*}(\mu,X^*)$ and $\|\cdot\|_{E_{w^*}(\mu,X^*)} = \|\cdot\|_{E_w(\mu,X^*)}$.*

Proof. First, take a countably valued function $h \in E_{w^*}(\mu,X^*)$, say $h = \sum_{i=1}^{\infty} x_i^* \chi_{A_i}$, where $x_i^* \in X^*$, $A_i \in \Sigma$ with $A_i \cap A_j = \phi$ for $i \ne j$. For each $x^{**} \in X^{**}$, each $\varepsilon > 0$, and each $n \in \mathbb{N}$, there exists, by the Principle of Local Reflexivity (see [26]), a one to one linear operator $T : \operatorname{span}\{x^{**}\} \longrightarrow X$ such that $\|T\| < 1+\varepsilon$ and $x_i^*(Tx^{**}) = x^{**}(x_i^*)$ for $i = 1,2,\ldots,n$. Thus,

$$\sum_{i=1}^{n} \big|x^{**}\big(x_i^*\big)\big|\chi_{A_i} = \sum_{i=1}^{n} \big|x_i^*\big(Tx^{**}\big)\big|\chi_{A_i} \le \sum_{i=1}^{\infty} \big|x_i^*\big(Tx^{**}\big)\big|\chi_{A_i} = \big|\big\langle Tx^{**}, h\big\rangle\big|.$$

It follows that

$$\bigg\|\sum_{i=1}^{n} \big|x^{**}\big(x_i^*\big)\big|\chi_{A_i}\bigg\|_{E(\mu)} \le \big\|\big\langle Tx^{**}, h\big\rangle\big\|_{E(\mu)} \le (1+\varepsilon)\big\|x^{**}\big\| \cdot \|h\|_{E_{w^*}(\mu,X^*)}$$

and hence

$$\sup_n \bigg\|\sum_{i=1}^{n} \big|x^{**}\big(x_i^*\big)\big|\chi_{A_i}\bigg\|_{E(\mu)} \le (1+\varepsilon)\big\|x^{**}\big\| \cdot \|h\|_{E_{w^*}(\mu,X^*)}.$$

Note that $\sum_{i=1}^{n} |x^{**}(x_i^*)|\chi_{A_i} \uparrow \sum_{i=1}^{\infty} |x^{**}(x_i^*)|\chi_{A_i} = |x^{**}h|$ and $E(\mu)$ has the Fatou property. Thus $|x^{**}h| \in E(\mu)$ and hence, $h \in E_w(\mu,X^*)$. Moreover,

$$\big\|x^{**}h\big\|_{E(\mu)} = \lim_n \bigg\|\sum_{i=1}^{n} \big|x^{**}\big(x_i^*\big)\big|\chi_{A_i}\bigg\|_{E(\mu)} \le (1+\varepsilon)\big\|x^{**}\big\| \cdot \|h\|_{E_{w^*}(\mu,X^*)}.$$

It follows that $\|h\|_{E_w(\mu,X^*)} \le \|h\|_{E_{w^*}(\mu,X^*)}$.

Now take any function $f \in E_{w^*}(\mu,X^*)$. Since f is strongly μ-measurable, there exists, for each $\varepsilon > 0$, a countably X^*-valued μ-measurable function h such that $\|f(\omega) - h(\omega)\|_{X^*} < \varepsilon$ μ-a.e. on Ω. Thus, $f - h \in E(\mu,X^*) \subseteq$

$E_{w^*}(\mu, X^*)$ and hence, $h \in E_{w^*}(\mu, X^*)$. The first part shows that $h \in E_w(\mu, X^*)$. Again, $f - h \in E(\mu, X^*) \subseteq E_w(\mu, X^*)$ and hence, $f \in E_w(\mu, X^*)$. Moreover,

$$\begin{aligned} \|f\|_{E_w(\mu,X^*)} &\le \|f-h\|_{E_w(\mu,X^*)} + \|h\|_{E_w(\mu,X^*)} \\ &\le \|f-h\|_{E(\mu,X^*)} + \|h\|_{E_{w^*}(\mu,X^*)} \\ &\le \varepsilon \cdot \|\chi_\Omega\|_{E(\mu)} + \|h-f\|_{E_{w^*}(\mu,X^*)} + \|f\|_{E_{w^*}(\mu,X^*)} \\ &\le \varepsilon \cdot \|\chi_\Omega\|_{E(\mu)} + \|h-f\|_{E(\mu,X^*)} + \|f\|_{E_{w^*}(\mu,X^*)} \\ &\le \varepsilon \cdot \|\chi_\Omega\|_{E(\mu)} + \varepsilon \cdot \|\chi_\Omega\|_{E(\mu)} + \|f\|_{E_{w^*}(\mu,X^*)}. \end{aligned}$$

Therefore,

$$\|f\|_{E_w(\mu,X^*)} \le \|f\|_{E_{w^*}(\mu,X^*)}. \qquad \square$$

For a Köthe function space $E(\mu)$ with the Fatou property (that is, $E''(\mu) = E(\mu)$) and a Banach space X, let $E_{strong}(\mu, X) := E_s(\mu, X)$ denote (so called) the *strong Köthe–Bochner function space*, that is,

$$E_s(\mu, X) = \left\{ f \in L_0(\mu, X) : \int_\Omega \bigl|\langle f(\omega), g(\omega)\rangle\bigr| \, d\mu(\omega) < \infty \ \forall g \in E'_{w^*}\bigl(\mu, X^*\bigr) \right\}$$

and

$$\|f\|_{E_s(\mu,X)} = \sup\left\{ \int_\Omega \bigl|\langle f(\omega), g(\omega)\rangle\bigr| \, d\mu(\omega) : g \in B_{E'_{w^*}(\mu,X^*)} \right\}.$$

Then $(E_s(\mu, X), \|\cdot\|_{E_s(\mu,X)})$ is a Banach space and $E_s(\mu, X) \subseteq E(\mu, X)$ with $\|\cdot\|_{E(\mu,X)} \le \|\cdot\|_{E_s(\mu,X)}$ (see [6]). By Proposition 2 of [6] and its proof, we have the following.

Proposition 4.3. *If $E(\mu)$ has the Fatou property, then the inclusion map from $E_s(\mu, X)$ to $E(\mu, X)$ and the inclusion map from $E_s(\mu, X)$ to $L_1(\mu, X)$ are semi-embeddings.*

Lemma 4.4. *If $E(\mu)$ has the Fatou property, then for every $f \in E_s(\mu, X)$ there exists a separable closed subspace Y of X such that $f \in E_s(\mu, Y)$ and $\|f\|_{E_s(\mu,Y)} \le \|f\|_{E_s(\mu,X)}$.*

Proof. Since f is strongly μ-measurable, $f[\Omega]$ is essentially separable. Let Z be the closure of the subspace generated by $f[\Omega]$. Then Z is a separable closed subspace of X. By [17, Proposition 3.4], there exists a separable closed subspace Y of X such that $Z \subseteq Y$ and there exists an isometrical embedding $J : Y^* \longrightarrow X^*$ such that $(Jy^*)(y) = y^*(y)$ for all $y \in Y$ and all $y^* \in Y^*$, and such that $J(Y^*)$ is a norm one complemented subspace of X^*.

Now take any $g \in E'_{w^*}(\mu, Y^*)$. Note that $E'(\mu)$ has the Fatou property. It follows from Proposition 4.2 and then from Proposition 4.1 that $g \in E'_w(\mu, Y^*)$ and

$$Jg \in E'_w\bigl(\mu, J\bigl(Y^*\bigr)\bigr) \subseteq E'_w\bigl(\mu, X^*\bigr) = E'_{w^*}\bigl(\mu, X^*\bigr)$$

with

$$\|Jg\|_{E'_{w^*}(\mu,X^*)} = \|Jg\|_{E'_w(\mu,X^*)} = \|Jg\|_{E'_w(\mu,J(Y^*))}$$
$$= \|g\|_{E'_w(\mu,Y^*)} = \|g\|_{E'_{w^*}(\mu,Y^*)}.$$

Thus,

$$\int_\Omega \big|\langle f(\omega), g(\omega)\rangle\big|\, d\mu(\omega) = \int_\Omega \big|\langle f(\omega), Jg(\omega)\rangle\big|\, d\mu(\omega) < \infty,$$

which implies that $f \in E_s(\mu, Y)$. Moreover,

$$\begin{aligned}
\|f\|_{E_s(\mu,Y)} &= \sup\Big\{\int_\Omega \big|\langle f(\omega), g(\omega)\rangle\big|\, d\mu(\omega) : g \in B_{E'_{w^*}(\mu,Y^*)}\Big\} \\
&= \sup\Big\{\int_\Omega \big|\langle f(\omega), Jg(\omega)\rangle\big|\, d\mu(\omega) : Jg \in B_{E'_{w^*}(\mu,X^*)}\Big\} \\
&\le \sup\Big\{\int_\Omega \big|\langle f(\omega), h(\omega)\rangle\big|\, d\mu(\omega) : h \in B_{E'_{w^*}(\mu,X^*)}\Big\} \\
&= \|f\|_{E_s(\mu,X)}.
\end{aligned}$$

□

Diestel, Fourie, and Swart [9], [10] showed that if $E(\mu)$ is a KB-space and X is a separable Banach space then the projective tensor product $E(\mu)\hat{\otimes} X$ is (isometrically) isomorphic to $E_s(\mu, X)$. With the help of Lemma 4.4, we improve this result to the following theorem by removing the separability from X.

THEOREM 4.5. *Let $E(\mu)$ be a KB-space. Then $f \in E_s(\mu, X)$ if and only if for every $\varepsilon > 0$ there exist a sequence (a_k) in $E(\mu)$ and a sequence (x_k) in X with $\sum_{k=1}^\infty \|a_k\|_{E(\mu)} \cdot \|x_k\| < \infty$ such that*

$$f(\omega) = \sum_{k=1}^\infty a_k(\omega) x_k \quad \text{for almost all } \omega \text{ in } \Omega, \tag{1}$$

where the series $\sum_{k=1}^\infty a_k(\omega)x_k$ converges absolutely in X for almost all ω in Ω. Moreover,

$$\|f\|_{E_s(\mu,X)} \le \sum_{k=1}^\infty \|a_k\|_{E(\mu)} \cdot \|x_k\| \le \|f\|_{E_s(\mu,X)} + \varepsilon. \tag{2}$$

Furthermore, $E(\mu)\hat{\otimes} X$ is isometrically isomorphic to $E_s(\mu, X)$.

Proof. It is straightforward that if f has a representation (1) then $f \in E_s(\mu, X)$ and

$$\|f\|_{E_s(\mu,X)} \le \sum_{k=1}^\infty \|a_k\|_{E(\mu)} \cdot \|x_k\|.$$

On the other hand, if $f \in E_s(\mu, X)$ then by Lemma 4.4, there exists a separable closed subspace Y of X such that $f \in E_s(\mu, Y)$ and $\|f\|_{E_s(\mu,Y)} \le \|f\|_{E_s(\mu,X)}$.

It follows from the proof on pages 95–98 in [9] and [10] that for every $\varepsilon > 0$ there exist a sequence (a_k) in $E(\mu)$ and a sequence (x_k) in Y such that

$$f(\omega) = \sum_{k=1}^{\infty} a_k(\omega) x_k \quad \text{for almost all } \omega \text{ in } \Omega$$

and

$$\sum_{k=1}^{\infty} \|a_k\|_{E(\mu)} \cdot \|x_k\|_Y \le \|P\| \cdot \big(\|f\|_{E_s(\mu,Y)} + \varepsilon\big), \tag{3}$$

where $P : E(\mu)^{**} \to E(\mu)$ is a band projection (since $E(\mu)$ is a KB-space, it follows from [22, p. 34, Theorem 1.c.4] that $E(\mu)$ is a projection band of $E(\mu)^{**}$). Note that

$$\sup_n \left\| \sum_{k=1}^{n} |a_k(\cdot)| \cdot \|x_k\| \right\|_{E(\mu)} \le \sum_{k=1}^{\infty} \|a_k\|_{E(\mu)} \cdot \|x_k\| < \infty.$$

Thus, $\lim_n \sum_{k=1}^{n} |a_k(\cdot)| \cdot \|x_k\|$ exists in $E(\mu)$ and hence, $\sum_{k=1}^{\infty} |a_k(\omega)| \cdot \|x_k\|$ converges for almost all ω in Ω. Since $\|P\| \le 1$, it follows from (3) and Lemma 4.4 that

$$\sum_{k=1}^{\infty} \|a_k\|_{E(\mu)} \cdot \|x_k\| \le \|f\|_{E_s(\mu,Y)} + \varepsilon \le \|f\|_{E_s(\mu,X)} + \varepsilon.$$

By [11, p. 227, Proposition 9], every $u \in E(\mu) \hat{\otimes} X$ has a representation $u = \sum_{k=1}^{\infty} a_k \otimes x_k$ such that

$$\|u\|_{E(\mu) \hat{\otimes} X} \le \sum_{k=1}^{\infty} \|a_k\|_{E(\mu)} \cdot \|x_k\| \le \|u\|_{E(\mu) \hat{\otimes} X} + \varepsilon. \tag{4}$$

Thus, we have established a bijection $\varphi : E(\mu) \hat{\otimes} X \to E_s(\mu, X)$ by $\varphi(u) = f$, where f is defined in (1) (note that f is independent of representations of u). Moreover, it follows from (2) and (4) that $\|\varphi(u)\|_{E_s(\mu,X)} = \|u\|_{E(\mu) \hat{\otimes} X}$ and hence, φ is an isometry. □

THEOREM 4.6. *Let $\mathcal{P}$ be a Banach space property which is separably determined, separably semi-embeddably stable, and isomorphically stable. If $E(\mu)$ is a KB-space, then $E(\mu) \hat{\otimes} X$ has $\mathcal{P}$ whenever $E(\mu, X)$ or $L_1(\mu, X)$ has $\mathcal{P}$.*

Proof. Take any separable closed subspace S of $E(\mu) \hat{\otimes} X$. It follows from [6, Lemma 7] that there are a separable closed subspace $F(\mu)$ of $E(\mu)$ and a separable closed subspace Y of X such that S is a subspace of $F(\mu) \hat{\otimes} Y$. Since $F(\mu)$ is a KB-space, by Proposition 4.3 and Theorem 4.5, $F(\mu) \hat{\otimes} Y$ is isometrically isomorphic to $F_s(\mu, Y)$ which semi-embeds into $F(\mu, Y)$ and $L_1(\mu, Y)$. Note that $F_s(\mu, Y)$ is separable since $F(\mu) \hat{\otimes} Y$ is separable. Also note that $F(\mu, Y)$, as a subspace of $E(\mu, X)$, has $\mathcal{P}$ or $L_1(\mu, Y)$, as a subspace of $L_1(\mu, X)$, has $\mathcal{P}$. It follows that $F_s(\mu, Y)$ has $\mathcal{P}$ and hence, $F(\mu)\hat{\otimes} Y$ has $\mathcal{P}$. Therefore S, as a subspace of $F(\mu)\hat{\otimes} Y$, has $\mathcal{P}$. □

Theorem 4.6 combining with Corollary 3.4 and Remark 3.5 yields the following.

Corollary 4.7. (i) *$E(\mu)\hat{\otimes} X$ has, respectively, RNP (due to [9]), nRNP, aRNP, aCCP, and the property of non-containment of a copy of c_0 whenever both $E(\mu)$ and X have the same property.*

(ii) *If $E(\mu)$ has RNP and X has CCP, then $E(\mu)\hat{\otimes} X$ has CCP.*

Note that a separable order continuous Banach lattice is order isometric to a Köthe function space $E(\mu)$ (see [22, p. 25, Theorem 1.b.14] or [21, p. 150, Theorem 3.1.8]). This yields the following.

Theorem 4.8. *Let E be a Banach lattice and X be a Banach space.*

(i) *$E\hat{\otimes} X$ has, respectively, RNP (due to [9]), nRNP, aRNP, aCCP, and the property of non-containment of a copy of c_0 whenever both E and X have the same property.*

(ii) *If E has RNP and X has CCP, then $E\hat{\otimes} X$ has CCP.*

Remark 4.9. Let X and Y be Banach spaces. Under the condition that one of them has an unconditional basis, $X\hat{\otimes} Y$ has, respectively, RNP, nRNP, aRNP, the property of non-containment of a copy of c_0 (see [5]), CCP, and aCCP (see [16]) whenever both X and Y have the same property. Under the condition that one of them is a Banach lattice, we have results of Theorem 4.8. However, we do not know if $X\hat{\otimes} Y$ has CCP whenever both X and Y have CCP.

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