Illinois Journal of Mathematics
Volume 56, Number 3, Fall 2012, Pages 731–757
S 0019-2082

LATTICE POINTS IN LARGE CONVEX PLANAR DOMAINS OF FINITE TYPE

JINGWEI GUO

ABSTRACT. Let $\mathcal{B}$ be a compact convex planar domain with smooth boundary of finite type and $\mathcal{B}_\theta$ its rotation by an angle θ. We prove that for almost every $\theta \in [0, 2\pi]$ the remainder $P_{\mathcal{B}_\theta}(t)$ is of order $O_\theta(t^{2/3-\zeta})$ with a positive number ζ independent of the domain.

1. Introduction

If $\mathcal{B}$ is a compact domain in $\mathbb{R}^2$, the number of lattice points $\mathbb{Z}^2$ in the dilated domain $t\mathcal{B}$ is approximately area($t\mathcal{B}$) and the lattice point problem is to estimate the remainder, $P_{\mathcal{B}}(t)$, in the equation

$$P_{\mathcal{B}}(t) = \#\big(t\mathcal{B} \cap \mathbb{Z}^2\big) - \operatorname{area}(\mathcal{B})t^2 \quad \text{for } t \geq 1.$$

It is geometrically evident that $P_{\mathcal{B}}(t) = O(t)$. See [11] for the history and fundamental results and methods of this problem.

If $\mathcal{B}$ has sufficiently smooth boundary with nonzero curvature the standard estimate is $P_{\mathcal{B}}(t) = O(t^{2/3})$. With various sophisticated methods this bound has been improved and the best known bound is due to Huxley [6]. See [5] for an introduction to his method. Notice that the conjecture $P_{\mathcal{B}}(t) = O(t^{1/2+\varepsilon})$ is still open.

If we weaken the curvature condition on the boundary, the remainder may become much larger. For instance, if the boundary is of finite type ω, $\omega \geq 3$, (i.e. the maximal order of vanishing of the curvature is $\omega - 2$), Colin de Verdière [2] showed that

$$P_{\mathcal{B}}(t) = O\big(t^{1-1/\omega}\big).$$

At an earlier time, Randol [20] proved the same bound for a particular domain $\{(x_1, x_2) : x_1^\omega + x_2^\omega \leq 1\}$ with $\omega \geq 4$ an even integer. Furthermore, he showed that the exponent is the best possible. The sharpness of this bound

Received June 6, 2011; received in final form November 3, 2011.
2010 *Mathematics Subject Classification.* Primary 11P21, 11L07. Secondary 42B10.

is due to the fact that for this particular domain the normals at boundary points with curvature zero are parallel to the coordinate axes. If we consider $\mathcal{B}_\theta$, the rotation of $\mathcal{B}$ by an angle $\theta \in [0, 2\pi]$ about the origin, however, we expect a substantially better estimate for most choices of θ. Colin de Verdière [2] showed that if θ satisfies a certain Diophantine condition then $P_{\mathcal{B}_\theta}(t) = O_\theta(t^{2/3})$. Moreover, he showed as a consequence of the Diophantine condition that

$$P_{\mathcal{B}_\theta}(t) = O_\theta\big(t^{2/3}\big) \quad \text{for a.e. } \theta. \tag{1.1}$$

Tarnopolska-Weiss [25] obtained the same bound for almost every rotation of a planar domain of finite type which is star-like with respect to the origin.

Iosevich [7] further developed this type of results by weakening the curvature condition, and proved

$$P_{\mathcal{B}_\theta}(t) = O_\theta\big(t^{2/3} \log^{\delta(p_0)}(t)\big) \quad \text{for a.e. } \theta,$$

with $\delta(p_0) > 1/p_0$ for a given $p_0 > 1$, for a certain class of convex planar domains whose curvature is allowed to vanish (at isolated points) to infinite order. His result was extended, in the paper [1] by Brandolini, Colzani, Iosevich, Podkorytov, and Travaglini, to arbitrary convex planar domains with no curvature or regularity assumption on the boundary.

One can also develop Colin de Verdière's result in another direction—to improve the exponent $2/3$ under certain curvature conditions. The first result of this kind can be found in Müller and Nowak [16], where they considered a compact planar domain bounded by a closed smooth Jordan curve determined by an analytic function. They evaluated the contributions of boundary points with curvature zero to the remainder $P_{\mathcal{B}}(t)$ and distinguished the cases where the tangent at such a point has rational or irrational slope (see also Nowak [18], [19]). Under certain assumptions about the Diophantine approximation of irrational slope, the asymptotic formula they gave for $P_{\mathcal{B}}(t)$ contains an error term of order $O(t^\gamma)$, $\gamma < 2/3$ unspecified. As a consequence they obtained

$$P_{\mathcal{B}_\theta}(t) = O_\theta\big(t^\gamma\big) \quad \text{for a.e. } \theta,$$

where the $\gamma < 2/3$ depends on the order of vanishing of the curvature, but not on θ.

Later on, Müller and Nowak [17] improved their previous results by using more sophisticated methods from analytic number theory. In particular, they obtained

$$\big|P_{\mathcal{B}_\theta}(t)\big| \le C_\theta \max\big(t^{\gamma(\omega)}, t^{7/11}(\log t)^{45/22}\big) \quad \text{for a.e. } \theta, \tag{1.2}$$

where $\gamma(\omega) = 2/3 - 1/(9\omega - 12)$ and $\omega - 2$ is the maximal order of vanishing of the curvature.

Note that $\gamma(\omega)$ tends to $2/3$ as ω goes to infinity. The goal of this paper is to prove a bound $P_{\mathcal{B}_\theta}(t) = O_\theta(t^{2/3-\zeta})$ for almost every rotation with a $\zeta > 0$ that does not depend on ω. In exchange, we have to give up some information in terms of the Diophantine condition as given in those papers by Müller and Nowak. Specifically we obtain the following theorem.

THEOREM 1.1. *Let* $\zeta = 1/3831$. *If* $\mathcal{B}$ *is a compact convex planar domain with smooth boundary of finite type which contains the origin as an interior point*, *then*

$$P_{\mathcal{B}_\theta}(t) = O_\theta\big(t^{2/3-\zeta}\big) \quad \text{for a.e. } \theta.$$

This bound is better than (1.1), and it is better than (1.2) if $\omega > 427$. Theorem 1.1 follows easily from the following theorem, which contains an improved but more technical statement.

THEOREM 1.2. *Let* $\zeta = 1/3831$. *If* $\mathcal{B}$ *is a compact convex planar domain with smooth boundary of finite type* ω *which contains the origin as an interior point*, *then*

$$\sup_{t\geq 2} \log^{-b}(t) t^{-2/3+\zeta+\sigma(\omega)} \big|P_{\mathcal{B}_\theta}(t)\big| \in L^1\big(S^1\big),$$

where $b > 1$ *and*

$$\sigma(\omega) = \frac{832}{1277(1277\omega - 2496)}.$$

In this paper, we focus on the remainder for rotated planar domains. For other interesting related results (like the mean square lattice point discrepancy for rotated planar domains, or the remainder for rotated high dimensional domains), see [9], [10], [22], [8], etc.

Notations: We use the usual Euclidean norm $|x|$ for a point $x \in \mathbb{R}^2$. $B(x,r) \subset \mathbb{R}^2$ represents the Euclidean ball centered at x with radius r. The norm of a matrix $A \in \mathbb{R}^{2\times 2}$ is given by $\|A\| = \sup_{|x|=1} |Ax|$. We set $e(f(x)) = \exp(-2\pi i f(x))$, $\mathbb{Z}^2_* = \mathbb{Z}^2 \setminus \{0\}$, and $\mathbb{R}^2_* = \mathbb{R}^2 \setminus \{0\}$. The Fourier transform of any function $f \in L^1(\mathbb{R}^2)$ is $\widehat{f}(\xi) = \int f(x) e(\langle x, \xi\rangle)\, dx$.

We fix χ_0 to be a smooth cut-off function whose value is 1 on $B(0,1/2)$ and 0 on the complement of $B(0,1)$. For a set $E \subset \mathbb{R}^2$ and a positive number a, we define $E_{(a)}$ to be the larger set

$$E_{(a)} = \big\{x \in \mathbb{R}^2 : \operatorname{dist}(E,x) < a\big\}.$$

We use the differential operators

$$D^\nu_x = \frac{\partial^{|\nu|}}{\partial x_1^{\nu_1}\, \partial x_2^{\nu_2}} \quad \big(\nu = (\nu_1,\nu_2) \in \mathbb{N}_0^2, |\nu| = \nu_1 + \nu_2\big)$$

and the gradient operator ∇_x. We often omit the subscript if no ambiguity occurs.

For functions f and g with g taking nonnegative real values, $f \lesssim g$ means $|f| \leq Cg$ for some constant C. If f is nonnegative, $f \gtrsim g$ means $g \lesssim f$. The Landau notation $f = O(g)$ is equivalent to $f \lesssim g$. The notation $f \asymp g$ means that $f \lesssim g$ and $g \lesssim f$.

Structure of the paper: After giving some geometric facts related to convex planar domains of finite type, we will study the support function of such domains in Section 3. We will establish the nonvanishing of certain 2×2 determinants. This result is a two dimensional refinement to Müller's [15, Lemma 3] with more precise bounds given. We will then prove in Section 4 our main analytical tool, the asymptotic formula of the Fourier transform of certain indicator functions. In Section 5, we give an estimation of exponential sums under certain hypotheses that can be verified (in the lattice point problem) based on the results from Section 3. One important difference between the results (from Sections 3–5) and their analogues in the nonvanishing curvature case is that we keep track of the curvature terms in all bounds given here. In Section 6, we put all these ingredients together to prove our main theorem. In the Appendix, we collect several standard results mainly from the oscillatory integral theory.

2. Some geometric facts

In the rest of this paper, unless otherwise stated $\mathcal{B}$ will always denote a compact convex planar domain with smooth boundary of finite type ω. In particular we assume, only in Section 6, that it contains the origin as an interior point.

Since $\partial\mathcal{B}$ is compact and of finite type, it is easy to see that $\partial\mathcal{B}$ can contain only finitely many points with curvature zero. Assume $\{P_i\}_{i=1}^{\Xi}$ are all such points and the curvature of $\partial\mathcal{B}$ at P_i vanishes of order $\omega_i - 2$. Each ω_i must be an even integer greater than three due to the convexity of $\mathcal{B}$.

The Gauss map of $\partial\mathcal{B}$, denoted by $\vec{n}$, maps each boundary point $x \in \partial\mathcal{B}$ to a unit exterior normal $\vec{n}(x) \in S^1$. It is bijective since $\mathcal{B}$ is convex and of finite type. At each boundary point with nonzero curvature, there exists a neighborhood on which the Gauss map is a diffeomorphism. Denote by $\vec{t}(x)$ the unit tangent vector at $x \in \partial\mathcal{B}$ such that $\{\vec{t}(x), -\vec{n}(x)\}$ has the same orientation as $\{e_1, e_2\}$.

When we express $\partial\mathcal{B}$ by a parametric equation, we always assume the orientation is counterclockwise. Hence, the signed curvature is always nonnegative.

For each nonzero $\xi \in \mathbb{R}^2$, there exists a unique point $x(\xi) \in \partial\mathcal{B}$ where the exterior normal is ξ. Denote by K_ξ the curvature of $\partial\mathcal{B}$ at $x(\xi)$. Denote the 2×2 rotation matrix by

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

and its transpose by R_θ^t. Then $\mathcal{B}_\theta = R_\theta\mathcal{B}$. Define $x^\theta(\xi) = R_\theta x(R_\theta^t\xi)$ and $K_\xi^\theta = K_{R_\theta^t\xi}$. Then $x^\theta(\xi)$ is the unique point on $\partial\mathcal{B}_\theta$ where the exterior normal is ξ and K_ξ^θ is the curvature of $\partial\mathcal{B}_\theta$ at $x^\theta(\xi)$.

If v_1 and v_2 are vectors $\in \mathbb{R}^2$, denote by $\mathfrak{A}_{v_1,v_2}$ the angle between them that is in $[0,\pi]$. By Taylor's formula, it is easy to prove

LEMMA 2.1. *For each* $1 \le i \le \Xi$ *there exists a ball*[1] B_i *in* $\partial\mathcal{B}$ *about* P_i *such that for any* $P \in B_i$

$$\mathfrak{A}_{\vec{n}(P_i),\vec{n}(P)} \asymp \left(K_{\vec{n}(P)}\right)^{\frac{\omega_i-1}{\omega_i-2}}, \tag{2.1}$$

where the implicit constants depend only on $\mathcal{B}$.

A consequence of this lemma is the following result which will be needed in Section 3. This result is proved for $\xi \in S^1$, however, it can be easily extended to $\mathbb{R}^2_*$ since K_ξ is positively homogeneous of degree zero.

LEMMA 2.2. *There exists a constant* $c_1 > 0$ *such that for any* $\xi \in S^1$ *with* $K_\xi > 0$

$$K_\eta \asymp K_\xi \quad \text{if } \eta \in B\big(\xi, c_1(K_\xi)^{3/2}\big).$$

The constant c_1 *and implicit constants depend only on* $\mathcal{B}$.

Proof. It suffices to prove this result for $\xi \in S^1$ such that $x(\xi)$ is in a small neighborhood in $\partial\mathcal{B}$ about a boundary point with curvature zero, say, P_i. Otherwise it follows easily from the mean value theorem. If $\eta \in B(\xi, c_1(K_\xi)^{3/2})$ then

$$\mathfrak{A}_{\eta,\xi} \le \frac{\pi}{2}\sin(\mathfrak{A}_{\eta,\xi}) \le \frac{\pi}{2} c_1 (K_\xi)^{\frac{\omega_i-1}{\omega_i-2}}.$$

To get the last inequality, we use $K_\xi < 1$ and $\omega_i \ge 4$. But (2.1) implies

$$\mathfrak{A}_{\xi,\vec{n}(P_i)} \asymp (K_\xi)^{\frac{\omega_i-1}{\omega_i-2}}.$$

Hence, if c_1 is sufficiently small then $\mathfrak{A}_{\eta,\xi} \le \mathfrak{A}_{\xi,\vec{n}(P_i)}/2$, which implies $1/2 \le \mathfrak{A}_{\eta,\vec{n}(P_i)}/\mathfrak{A}_{\xi,\vec{n}(P_i)} \le 3/2$. By (2.1) again, we get $K_\eta/K_\xi \asymp 1$. □

3. Nonvanishing 2×2 determinants

In this section, we will give lower bounds of determinants of certain 2×2 matrices (see Lemma 3.4 below). This result is a refinement to Müller's [15, Lemma 3] in two dimensional case with more precise bounds given. It is obtained based on Müller's original proof.

The support function of $\mathcal{B}$ is given by $H(\xi) = \sup_{y \in \mathcal{B}} \langle \xi, y \rangle$ for any nonzero $\xi \in \mathbb{R}^2$. Then $H(\xi) = \langle \xi, x(\xi) \rangle$. It is positively homogeneous of degree one, i.e. $H(\lambda\xi) = \lambda H(\xi)$ if $\lambda > 0$. Denote

$$\mathcal{H} = \mathbb{R}^2 \setminus \left\{ r\vec{n}(P_i) : \forall r \ge 0, i = 1, \dots, \Xi \right\}.$$

[1] The balls in $\partial\mathcal{B}$ are the intersection with $\partial\mathcal{B}$ of the usual balls in the space $\mathbb{R}^2$.

LEMMA 3.1. *H is smooth in $\mathcal{H}$ and for every $\xi \in \mathcal{H}$*

$$H(\xi) \lesssim |\xi|,$$
$$D^\nu H(\xi) \lesssim 1 \quad \textit{for } |\nu| = 1,$$

and

$$D^\nu H(\xi) \lesssim |\xi|^{1-|\nu|}(K_\xi)^{3-2|\nu|} \quad \textit{for } |\nu| \geq 2.$$

All implicit constants may depend only on $|\nu|$ and $\mathcal{B}$.

Proof. Assume $\vec{r}(s) = x_1(s)e_1 + x_2(s)e_2$ is a parametrization of $\partial\mathcal{B}$ by arc length s. For every $\xi \neq 0$, there exists a unique $s(\xi)$ such that $x(\xi) = x_1(s(\xi))e_1 + x_2(s(\xi))e_2$, which leads to

$$H(\xi) = \xi_1 x_1\big(s(\xi)\big) + \xi_2 x_2\big(s(\xi)\big).$$

Since

$$\xi/|\xi| = x_2'\big(s(\xi)\big)e_1 - x_1'\big(s(\xi)\big)\big)e_2,$$

we get $\xi_1 x_1'(s(\xi)) + \xi_2 x_2'(s(\xi)) = 0$. Note that if $\xi \in \mathcal{H}$ then

$$K_\xi = x_1'\big(s(\xi)\big)x_2''\big(s(\xi)\big) - x_1''\big(s(\xi)\big)x_2'\big(s(\xi)\big) \neq 0.$$

It follows from the implicit function theorem that $s = s(\xi)$ is smooth at ξ. Hence H is smooth in $\mathcal{H}$.

We can now estimate derivatives of H by the implicit differentiation. Assume $\xi \in S^1 \cap \mathcal{H}$. Differentiating $\xi_1 x_1'(s(\xi)) + \xi_2 x_2'(s(\xi)) = 0$ yields

$$\frac{\partial s}{\partial \xi_i}(\xi) = \frac{x_i'(s(\xi))}{x_1'(s(\xi))x_2''(s(\xi)) - x_1''(s(\xi))x_2'(s(\xi))} \quad (i = 1, 2).$$

Continuing differentiating these two formulas, we get, by induction,

$$D_\xi^\nu s(\xi) \lesssim (K_\xi)^{1-2|\nu|} \quad \text{for } |\nu| \geq 1, \tag{3.1}$$

where the implicit constant depends only on $|\nu|$ and $\mathcal{B}$. Differentiating H gives

$$\frac{\partial H}{\partial \xi_i} = x_i\big(s(\xi)\big) \lesssim 1 \quad (i = 1, 2).$$

Hence, the bounds for H follow from (3.1) and the homogeneity of H. □

REMARK 3.2. The support function of $\mathcal{B}_\theta$ is given by $H_\theta(\xi) = \sup_{y \in \mathcal{B}_\theta} \langle \xi, y \rangle$. Denote $\mathcal{H}_\theta = R_\theta \mathcal{H}$. Since $H_\theta(\xi) = H(R_\theta^t \xi)$, we can easily get bounds for H_θ in the same form as in Lemma 3.1 (with $\mathcal{H}$ and K_ξ replaced by $\mathcal{H}_\theta$ and K_ξ^θ).

The following result is concerning the Hessian matrix of H and will be needed in the proof of next lemma. The proof is easy and we omit it.

LEMMA 3.3. *For any $\xi \in \mathcal{H}$, the matrix $\nabla_{\xi\xi}^2 H(\xi)$ has two eigenvalues 0 and $(|\xi| K_\xi)^{-1}$.*

Given vectors v_1, $v_2 \in \mathbb{R}^2$, by writing $V=(v_1,v_2)$ we mean V is the matrix with column vectors v_1, v_2. If $y \neq 0$ define $F_\theta(u_1,u_2)=H_\theta(y+u_1v_1+u_2v_2)$, $u_1,u_2 \in \mathbb{R}$. For $q \in \mathbb{N}$ let

$$h_q^\theta(y,v_1,v_2)=\det(g_{i,j})_{1\le i,j\le 2},$$

where

$$g_{i,j}=\frac{\partial^{q+2}F_\theta}{\partial u_1\,\partial u_i\,\partial u_j\,\partial u_2^{q-1}}(0).$$

The main estimate in this section is the following lemma—the key preliminary for our (later) application of the method of stationary phase with nondegenerate critical points. This result is proved for $\xi \in S^1 \cap \mathcal{H}_\theta$, but can be easily extended to $\mathcal{H}_\theta$ due to the homogeneity of H_θ.

LEMMA 3.4. *For every $\xi \in S^1 \cap \mathcal{H}_\theta$, there exist two orthogonal vectors $v_1^*(\xi)$, $v_2^*(\xi) \in \mathbb{Z}^2$ such that*

$$|v_1^*|=|v_2^*| \asymp \left(K_\xi^\theta\right)^{-4q} \quad \text{and} \quad \left\|\left(v_1^*,v_2^*\right)^{-1}\right\| \lesssim \left(K_\xi^\theta\right)^{4q}, \tag{3.2}$$

and a constant $c_2>0$ (depending only on q and $\mathcal{B}$) such that for any $\eta \in B(\xi, c_2(K_\xi^\theta)^{4q+2})$

$$\left|h_q^\theta\left(\eta,v_1^*,v_2^*\right)\right| \gtrsim \left(K_\xi^\theta\right)^{-8q^2-16q-2}, \tag{3.3}$$

$$D^\nu H_\theta(\eta) \lesssim 1 \quad \text{for } 0\le|\nu|\le 1, \tag{3.4}$$

and

$$D^\nu H_\theta(\eta) \lesssim \left(K_\xi^\theta\right)^{3-2|\nu|} \quad \text{for } |\nu|\ge 2. \tag{3.5}$$

The constants implicit in (3.2) and (3.3) depend only on q and $\mathcal{B}$. Those implicit in (3.4) and (3.5) depend only on $|\nu|$ and $\mathcal{B}$.

Proof. We will follow the proof of Müller's [15, Lemma 3] (with some minor modification) and establish these inequalities through four steps for an arbitrarily fixed $\xi=(\xi_1,\xi_2)^t \in S^1 \cap \mathcal{H}_\theta$.

Step 1. Denote $v_1=(-\xi_2,\xi_1)^t$ and $v_2=(\xi_1,\xi_2)^t$. We will first prove

$$h_q^\theta(\xi,v_1,v_2)=-q!^2\left(K_\xi^\theta\right)^{-2}. \tag{3.6}$$

For $y=(y_1,y_2)^t$, set $\widetilde{H}_\theta(y)=H_\theta(My)$ where $M=(v_2,v_1)$ is an orthogonal matrix. Since H_θ is smooth at ξ, so is $\widetilde{H}_\theta$ at e_1. The Hessian matrix of $\widetilde{H}_\theta$ is

$$\nabla^2\widetilde{H}_\theta(y)=M^t\nabla^2H_\theta(My)M.$$

Since $\nabla^2H_\theta(\xi)$ has two eigenvalues 0 and $(K_\xi^\theta)^{-1}$ by Lemma 3.3, so does $\nabla^2\widetilde{H}_\theta(e_1)$. Note that

$$H_\theta(\xi+u_1v_1+u_2v_2)=\widetilde{H}_\theta(e_1+u_1e_2+u_2e_1).$$

We use the latter expression to compute $h^\theta_q(\xi, v_1, v_2)$ since the following two equalities (derived from the homogeneity of $\widetilde{H}_\theta$; see the proof of Müller's [15, Lemma 3]) can simplify the computation:

$$(\widetilde{H}_\theta)_{1j}(e_1) = (\widetilde{H}_\theta)_{j1}(e_1) = 0 \quad (1 \le j \le 2); \tag{3.7}$$

$$\frac{\partial^{q+2}\widetilde{H}_\theta}{\partial y_1^q\,\partial y_i\,\partial y_j}(e_1) = (-1)^q q!(\widetilde{H}_\theta)_{ij}(e_1) \quad (1 \le i, j \le 2). \tag{3.8}$$

The equality (3.7) implies that $(\widetilde{H}_\theta)_{22}(e_1) = (K^\theta_\xi)^{-1}$. This, combined with (3.8), implies

$$\frac{\partial^{q+2}}{\partial u_1\,\partial u_i\,\partial u_2\,\partial u_2^{q-1}}\big(H_\theta(\xi + u_1 v_1 + u_2 v_2)\big)(0) = \delta_{1i}(-1)^q q!\big(K^\theta_\xi\big)^{-1},$$

where δ_{ij} is the Kronecker notation. This equality easily leads to (3.6).

Step 2. For any $N \in \mathbb{N}$ there exist two integers N_l $(l = 1, 2)$ such that $|\xi_l - N_l/N| \le 1/N$. Denote $\widetilde{v}_1 = (-N_2/N, N_1/N)^t$ and $\widetilde{v}_2 = (N_1/N, N_2/N)^t$. Then $|v_l - \widetilde{v}_l| \le \sqrt{2}/N$. If $N \ge 2\sqrt{2}$ then $1/2 \le |\widetilde{v}_1| = |\widetilde{v}_2| \le 3/2$. By the mean value theorem and Lemma 3.1 we get

$$\big|h^\theta_q(\xi, \widetilde{v}_1, \widetilde{v}_2) - h^\theta_q(\xi, v_1, v_2)\big| \le C_1 N^{-1}\big(K^\theta_\xi\big)^{-4q-2},$$

where C_1 depends only on q and $\mathcal{B}$. Let N be the smallest integer not less than $2C_1 q!^{-2}(K^\theta_\xi)^{-4q}$. Then

$$\big|h^\theta_q(\xi, \widetilde{v}_1, \widetilde{v}_2)\big| \ge q!^2\big(K^\theta_\xi\big)^{-2}/2.$$

Step 3. Set $v_1^* = N\widetilde{v}_1$ and $v_2^* = N\widetilde{v}_2$. Then v_1^* and v_2^* are two orthogonal integral vectors such that $|v_1^*| = |v_2^*| \asymp_{q,\mathcal{B}} (K^\theta_\xi)^{-4q}$ and

$$\big|h^\theta_q\big(\xi, v_1^*, v_2^*\big)\big| = N^{2q+4}\big|h^\theta_q(\xi, \widetilde{v}_1, \widetilde{v}_2)\big| \gtrsim_{q,\mathcal{B}} \big(K^\theta_\xi\big)^{-8q^2-16q-2}.$$

Since $(v_1^*, v_2^*) = N(\tilde{v}_1, \tilde{v}_2)$ its inverse matrix is

$$\big(v_1^*, v_2^*\big)^{-1} = N^{-1}\big(\text{adjugate matrix of } (\tilde{v}_1, \tilde{v}_2)\big)/\det(\tilde{v}_1, \tilde{v}_2),$$

followed by $\|(v_1^*, v_2^*)^{-1}\| \lesssim_{q,\mathcal{B}} (K^\theta_\xi)^{4q}$.

Step 4. Assume $\eta \in B(\xi, c_2(K^\theta_\xi)^{4q+2})$ with c_2 chosen below. If c_2 is sufficiently small, Lemma 2.2 implies $K^\theta_\eta \asymp K^\theta_\xi$. Recalling also Remark 3.2 we immediately get (3.4) and (3.5).

By the mean value theorem and the assumption $|\eta - \xi| \le c_2(K^\theta_\xi)^{4q+2}$ we get

$$\big|h^\theta_q\big(\eta, v_1^*, v_2^*\big) - h^\theta_q\big(\xi, v_1^*, v_2^*\big)\big| \le C_2 c_2\big(K^\theta_\xi\big)^{-8q^2-16q-2},$$

where C_2 depends only on q and $\mathcal{B}$. The inequality (3.3) follows if c_2 is sufficiently small. This finishes the proof. □

4. The Fourier transform of certain indicator functions

If $\mathcal{B}$ is a compact convex planar domain with smooth boundary and *positive* curvature, Hörmander's [4, Corollary 7.7.15] gives an asymptotic formula for the Fourier transform of the indicator function $\chi_{\mathcal{B}}$ for *every* $\xi \in S^1$ (see Lemma A.4). If $\partial\mathcal{B}$ contains points with curvature zero, however, the error term of that formula is not good (although the leading terms are).

Randol [21] studied the Fourier transform of the indicator function of a compact (not necessarily convex) planar domain $\mathcal{B}$ of finite type. In particular, he gave an upper bound for

$$\Phi(\xi) = \sup_{r>0} r^{3/2} \left|\widehat{\chi}_{\mathcal{B}}(r\xi)\right|, \quad \xi \in S^1.$$

His Theorem 1 says that $\Phi(\xi)$ is always bounded, except in neighborhoods of those points of S^1 corresponding to exterior or interior normals to $\partial\mathcal{B}$ at points with curvature zero. In a neighborhood of such a point $\xi_0 \in S^1$,

$$\Phi(\xi) \lesssim (\mathfrak{A}_{\xi,\xi_0})^{-\frac{\omega_0-2}{2(\omega_0-1)}}, \tag{4.1}$$

where ω_0 is the largest type at those points of $\partial\mathcal{B}$ at which the exterior normal is either ξ_0 or $-\xi_0$. For convex domains of finite type, (4.1) also follows easily from Lemma 2.1 and the argument on [24, p. 19].

In the rest of this section, we will consider slightly more general planar domains and prove an asymptotic formula for $\widehat{\chi}_{\mathcal{B}_\theta}$. We first prove a preliminary.

LEMMA 4.1. *Let $\mathcal{B}$ be a compact strictly convex planar domain with smooth boundary. Then there exist two positive constants c and c_3* (*both depending only on $\mathcal{B}$*) *such that, for any $\xi \in S^1$ and $r \le c_3$,*

$$\left|\langle \vec{n}(x), \xi\rangle\right| \le 1 - cr^2\left(\min(K_\xi, K_{-\xi})\right)^4 \tag{4.2}$$

if x is in $\partial\mathcal{B} \setminus (B(x(\xi), rK_\xi) \cup B(x(-\xi), rK_{-\xi}))$.

Proof. Note that there exists a $C_0 > 0$ such that, for any $\xi \in S^1$, the boundary $\partial\mathcal{B}$ in a neighborhood of $x(\xi)$ can be parametrized by

$$\vec{r}(u) = x(\xi) + u\vec{t}\big(x(\xi)\big) + h(u,\xi)(-\xi), \quad u \in I = [-C_0, C_0], \tag{4.3}$$

where $h(\cdot,\xi) \in C^\infty(I)$ for all $\xi \in S^1$ such that $h(0,\xi) = 0$. Note that (4.3) implies $h'_u(0,\xi) = 0$ and $h''_u(0,\xi) = K_\xi$. Since the map

$$\xi \in S^1 \mapsto h(\cdot,\xi)$$

is continuous and its domain is compact we have

$$\left|\partial_u^j h(u,\xi)\right| \le C_1 \quad \text{for any } \xi \in S^1, j \in \mathbb{N}_0, \text{ and } u \in I. \tag{4.4}$$

Denote by $K_{\max}$ the largest curvature of $\partial\mathcal{B}$. Let

$$c_3 = \min\left(\frac{C_0}{K_{\max}}, \frac{1}{C_1}, \frac{2\sqrt{6}}{9K_{\max}^2}\right)$$

and $r \le c_3$. Due to the symmetry and monotonicity it suffices to prove (4.2) for a fixed $x \in \partial\mathcal{B} \cap \partial B(x(\xi), rK_\xi)$. Since $r \le C_0/K_{\max}$ there exists a $u_* \in I$ such that $x = \vec{r}(u_*)$ with $rK_\xi(1+C_1^2)^{-1/2} \le |u_*| \le rK_\xi$.

Since $r \le 1/C_1$, Taylor's formula and the size of u_* yields

$$rK_\xi^2\big(1+C_1^2\big)^{-1/2}/2 \le u_*K_\xi/2 \le \big|h'_u(u_*,\xi)\big| \le 3u_*K_\xi/2 \le 3rK_\xi^2/2. \tag{4.5}$$

By Taylor's formula again,

$$\big\langle \vec{n}(x), \xi\big\rangle = \big(1+h'_u(u_*,\xi)^2\big)^{-1/2} = 1 - h'_u(u_*,\xi)^2/2 + \mathrm{R}$$

with a remainder $|\mathrm{R}| \le 3h'_u(u_*,\xi)^4/8 \le h'_u(u_*,\xi)^2/4$. The last inequality follows from (4.5) and $r \le 2\sqrt{6}/9K_{\max}^2$. By (4.5) again we get

$$\big\langle \vec{n}(x), \xi\big\rangle \le 1 - cr^2K_\xi^4,$$

where $c = 1/(16+16C_1^2)$. □

THEOREM 4.2. *Let $\mathcal{B}$ be a compact strictly convex planar domain with smooth boundary, s the arc length on $\partial\mathcal{B}$, and n_l $(l=1,2)$ the lth component of the Gauss map of $\partial\mathcal{B}$. For $\xi \in S^1$ with $\delta_\xi = \min(K_\xi, K_{-\xi}) > 0$, we have*

$$\begin{aligned}\widehat{n_l ds}(\lambda\xi) = {} & \big(e^{\pi i/4}\xi_l(K_\xi)^{-1/2}e^{-2\pi i\lambda H(\xi)} \\ & + e^{-\pi i/4}(-\xi_l)(K_{-\xi})^{-1/2}e^{2\pi i\lambda H(-\xi)}\big)\lambda^{-1/2} \\ & + O\big(\lambda^{-3/2}(\delta_\xi)^{-7/2} + \lambda^{-N}(\delta_\xi)^{-4N}\big) \quad \textit{for } \lambda > 0,\end{aligned}$$

where $N \in \mathbb{N}$ and the implicit constant depends only on N and $\mathcal{B}$.

Proof. As in the proof of Lemma 4.1, the boundary $\partial\mathcal{B}$ in a neighborhood of $x(\xi)$ can be parametrized by (4.3) with a uniform upper bound as in (4.4) and we assume C_0, C_1, c_3, and $K_{\max}$ are constants appearing there. Let

$$c_4 = \min\left(\frac{C_0}{K_{\max}}, \frac{3}{2C_1(1+C_0)}, 2c_3\right).$$

Decompose n_l as a sum $n_l = \psi_1 + \psi_2 + \psi_3$ where

$$\psi_1(x,\xi) = n_l(x)\chi_0\left(\frac{x - x(\xi)}{c_4K_\xi}\right) \quad \text{and} \quad \psi_2(x,\xi) = n_l(x)\chi_0\left(\frac{x - x(-\xi)}{c_4K_{-\xi}}\right).$$

We first estimate $\widehat{\psi_1 ds}$ and by (4.3)

$$\widehat{\psi_1 ds}(\lambda\xi) = e^{-2\pi i\lambda\langle\xi, x(\xi)\rangle}\int \tau(u,\xi)e^{2\pi i\lambda h(u,\xi)}\,du, \tag{4.6}$$

where $\tau(u,\xi) = \psi_1(\vec{r}(u),\xi)(1+h'_u(u,\xi)^2)^{1/2}$ such that

$$\tau(\cdot,\xi) \in C_c^\infty(-c_4K_\xi, c_4K_\xi)$$

and

$$\big|\partial_u^j\tau(u,\xi)\big| \le C(\chi_0, C_1)(c_4K_\xi)^{-j}.$$

Denote the integral in (4.6) by $\Delta(\xi)$. By a change of variable,

$$\Delta(\xi) = K_\xi \int \tau(K_\xi u, \xi) e^{2\pi i \lambda h(K_\xi u, \xi)}\, du.$$

By Taylor's formula,

$$h(K_\xi u, \xi) = (K_\xi)^3 u^2 \big(1 + \varepsilon(u,\xi)\big)/2, \quad u \in [-c_4, c_4],$$

where $\varepsilon(u,\xi) = u \int_0^1 \partial_u^3 h(K_\xi u m, \xi)(1-m)^2\, dm$. Since $1/2 \le 1 + \varepsilon(u,\xi) \le 3/2$ (due to $c_4 \le 3/(2C_1)$), we can define $v = u(1+\varepsilon(u,\xi))^{1/2}$. Since $\partial_u v(u,\xi) > \sqrt{2}/4$ (due to $c_4 \le \min(C_0/K_{\max}, 3/2C_1(1+C_0)))$, then $u \mapsto v$ is a smooth invertible mapping from $(-c_4, c_4)$ to a neighborhood of 0 in v-space such that $|\partial_u^j v| \le C(C_0, C_1)$, $|\partial_v^j u| \le C(C_0, C_1)$, and

$$h(K_\xi u, \xi) = (K_\xi)^3 v^2/2.$$

Then

$$\Delta(\xi) = K_\xi \int \tilde{\tau}(v,\xi) e^{i\tilde{\lambda} v^2/2}\, dv,$$

where $\tilde{\lambda} = 2\pi (K_\xi)^3 \lambda$ and $\tilde{\tau}(v,\xi) = \tau(K_\xi u(v), \xi)\partial_v u$. Applying Lemma A.3 (with $k = 1$ there) to the integral above yields an asymptotic expansion, which in turn gives

$$\widehat{\psi_1 ds}(\lambda\xi) = e^{\pi i/4} \xi_l (K_\xi)^{-1/2} e^{-2\pi i \lambda \langle \xi, x(\xi)\rangle} \lambda^{-1/2} + O\big(\lambda^{-3/2}(K_\xi)^{-7/2}\big),$$

where the implicit constant depends only on $\mathcal{B}$.

Since $\widehat{\psi_2 ds}$ is similar, it remains to estimate $\widehat{\psi_3 ds}$. Assume $\vec{r}: s \in [0, L] \mapsto \vec{r}(s) \in \partial\mathcal{B}$ is a parametrization of $\partial\mathcal{B}$ by arc length and $\vec{r}(0) = x(\xi)$. Then

$$\widehat{\psi_3 ds}(\lambda\xi) = \int \tau_1(s,\xi) e^{-2\pi i \lambda f(s,\xi)}\, ds,$$

where $\tau_1(s,\xi) = \psi_3(\vec{r}(s), \xi)$ and $f(s,\xi) = \langle \vec{r}(s), \xi\rangle$. Note that

$$f_s'(s,\xi) = \big\langle \vec{t}\big(\vec{r}(s)\big), \xi\big\rangle.$$

But $|\langle \vec{t}(\vec{r}(s)), \xi\rangle|^2 + |\langle \vec{n}(\vec{r}(s)), \xi\rangle|^2 = 1$ and Lemma 4.1 (with $c_4 \le 2c_3$) yields, for any s such that $\tau_1(s,\xi) \ne 0$, that

$$\big|\big\langle \vec{n}\big(\vec{r}(s)\big), \xi\big\rangle\big| \le 1 - c c_4^2 (\delta_\xi)^4/4.$$

Hence, $|\langle \vec{n}(\vec{r}(s)), \xi\rangle|^2 \le 1 - c c_4^2 (\delta_\xi)^4/4$. It follows that

$$\big|f_s'(s,\xi)\big| \ge \sqrt{c}\, c_4 (\delta_\xi)^2/2.$$

Note that $\partial_s^j f \lesssim 1$ and $\partial_s^j \tau_1 \lesssim (\delta_\xi)^{-j}$, thus by Lemma A.2 we get

$$\widehat{\psi_3 ds}(\lambda\xi) \lesssim \lambda^{-N} (\delta_\xi)^{-4N},$$

where the implicit constant depends only on N and $\mathcal{B}$. □

As a consequence of the Gauss–Green formula, we get the following corollary.

Corollary 4.3. *Let $\mathcal{B}$ be a compact strictly convex planar domain with smooth boundary. For $\xi \in S^1$ with $\delta_\xi^\theta = \min(K_\xi^\theta, K_{-\xi}^\theta) > 0$, we have*

$$\begin{aligned}\widehat{\chi}_{\mathcal{B}_\theta}(\lambda\xi) = \big(&(2\pi)^{-1} e^{3\pi i/4} \big(K_\xi^\theta\big)^{-1/2} e^{-2\pi i \lambda H_\theta(\xi)} \\ &+ (2\pi)^{-1} e^{-3\pi i/4} \big(K_{-\xi}^\theta\big)^{-1/2} e^{2\pi i \lambda H_\theta(-\xi)}\big) \lambda^{-3/2} \\ &+ O\big(\lambda^{-5/2} \big(\delta_\xi^\theta\big)^{-7/2} + \lambda^{-N-1} \big(\delta_\xi^\theta\big)^{-4N}\big) \quad \text{for } \lambda > 0,\end{aligned}$$

where $N \in \mathbb{N}$ and the implicit constant depends only on N and $\mathcal{B}$.

Remark 4.4. In Section 6, we will apply this result ($N = 1$) to convex planar domains of finite type. The error term becomes $O(\lambda^{-2}(\delta_\xi^\theta)^{-4})$.

Proof of Corollary 4.3. This result follows easily from

$$\widehat{\chi}_{\mathcal{B}_\theta}(\lambda\xi) = \widehat{\chi}_{\mathcal{B}}\big(\lambda R_\theta^t \xi\big),$$
$$2\pi i \lambda \xi_l \widehat{\chi}_{\mathcal{B}}(\lambda\xi) = -\widehat{n_l ds}(\lambda\xi),$$

and Theorem 4.2. □

5. Estimate of exponential sums

Let $M_* > 1$ and $T > 0$ be parameters. In this section, we consider two-dimensional exponential sums of the form

$$S(T, M_*; G, F) = \sum_{m \in \mathbb{Z}^2} G(m/M_*) e\big(TF(m/M_*)\big),$$

where $G : \mathbb{R}^2 \to \mathbb{R}$ is C^∞ smooth, compactly supported, and bounded above by a constant, and $F : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ is C^∞ smooth on an open convex domain Ω such that

$$\operatorname{supp}(G) \subset \Omega \subset c_0 B(0,1),$$

where $c_0 > 0$ is a fixed constant. Here we quote the author's [3, Lemma 2.2] ($d = 2$), that is, Müller's [15, Lemma 1] (in a slightly different form).

Lemma 5.1. *Let $q \in \mathbb{N}$, $Q = 2^q$, and $r_1, \dots, r_q \in \mathbb{Z}^2$ be nonzero integral vectors with $|r_i| \lesssim 1$. Furthermore, let H be a real parameter which satisfies $1 < H \lesssim M_*$. Set $H_l = H_{q,l} = H^{2^{l-q}}$ for $l = 1, \dots, q$. Then*

$$\big|S(T, M_*; G, F)\big|^Q \lesssim \frac{M_*^{2Q}}{H} + \frac{M_*^{2(Q-1)}}{H_1 \cdot \dots \cdot H_q} \sum_{\substack{1 \le h_i < H_i \\ 1 \le i \le q}} \big|S\big(\mathscr{H} T M_*^{-q}, M_*; G_q, F_q\big)\big|,$$

where $\mathscr{H} = \prod_{l=1}^q h_l$ and functions G_q, F_q are defined as follows:

$$G_q(x) = G_q(x, h_1, \dots, h_q) = \prod_{\substack{u_i \in \{0,1\} \\ 1 \le i \le q}} G\left(x + \sum_{l=1}^q \frac{h_l}{M_*} u_l r_l\right)$$

and

$$F_q(x) = F_q(x, h_1, \dots, h_q) \\ = \int_{(0,1)^q} \langle r_1, \nabla\rangle \cdots \langle r_q, \nabla\rangle F\left(x + \sum_{l=1}^{q} \frac{h_l}{M_*} u_l r_l\right) du_1 \cdots du_q.$$

The integral representation of F_q *is well defined on the open convex set* $\Omega_q = \Omega_q(h_1, \dots, h_q) = \{x \in \Omega : x + \sum_{l=1}^{q}(h_l/M_*)u_l r_l \in \Omega$ *for all* $u_l \in \{0,1\}, l = 1, \dots, q\}$. *Moreover,* $\operatorname{supp}(G_q) \subset \Omega_q \subset \Omega$.

PROPOSITION 5.2. *Let* $q \in \mathbb{N}$, $Q = 2^q$, *and* $K < 1$ *be a positive parameter. Assume that*

$$\operatorname{dist}\big(\operatorname{supp}(G), \Omega^c\big) \gtrsim K^{4q+2}, \tag{5.1}$$

and that for all $\nu \in \mathbb{N}_0^2$ *and* $y \in \Omega$

$$D^\nu G(y) \lesssim K^{-(4q+2)|\nu|}, \tag{5.2}$$

$$D^\nu F(y) \lesssim \begin{cases} 1 & \text{if } 0 \le |\nu| \le 1, \\ K^{3-2|\nu|} & \text{if } |\nu| \ge 2 \end{cases} \tag{5.3}$$

and for $\mu = (1, q-1)$

$$\big|\det\big(\nabla^2 D^\mu F(y)\big)\big| \gtrsim K^{-2}. \tag{5.4}$$

If $M_* \ge K^{-4q-2}$ *and* T *is restricted to*

$$T \ge K^{(8q+4)/Q-5} M_*^{q-1+2/Q}, \tag{5.5}$$

then

$$S(T, M_*; G, F) \lesssim \big(K^{-12q-1} T M_*^{6Q-q-6}\big)^{1/(3Q-2)} + R, \tag{5.6}$$

where

$$R = K^{-(20q+4)/Q} M_*^{2-2/Q}\big[K^{-(12q+4)/Q} \\ + \big(K^{12q+1} T^{-1} M_*^{q+2}\big)^{1/(3Q-2)} \\ \times \big(\big(K^{-20q-7} T^{-1} M_*^{q}\big)^{1/Q} + (\log M_*)^{1/Q}\big)\big]. \tag{5.7}$$

The constant implicit in (5.6) *depends only on* q, c_0, *and constants implicit in* (5.1), (5.2), (5.3), *and* (5.4).

REMARK 5.3. This result is similar with [15, Theorem 2] and [3, Proposition 2.4 and 2.5], but here there is an extra parameter K in various bounds.

Proof of Proposition 5.2. Let $1 < H \le c_5 K^{4q+2} M_*$ with $c_5 < 1$ chosen (later) to be sufficiently small. Then $H \le M_*$. We use Lemma 5.1 with $r_1 = e_1$ and $r_j = e_2$ $(j = 2, \dots, q)$. Applying to $S_4 := S(\mathscr{H} T M_*^{-q}, M_*; G_q, F_q)$

the Poisson summation formula followed by a change of variables $x = K^2 M_* z$ yields

$$S_4 = \sum_{p\in\mathbb{Z}^2} \int_{\mathbb{R}^2} G_q(x/M_*) e\big(\mathscr{H} T M_*^{-q} F_q(x/M_*) - \langle p, x\rangle\big)\, dx$$

$$= \sum_{p\in\mathbb{Z}^2} K^4 M_*^2 \int_{\mathbb{R}^2} \Psi_q(z) e\big(\mathscr{H} T M_*^{-q} F_q\big(K^2 z\big) - K^2 M_* \langle p, z\rangle\big)\, dz,$$

where $\Psi_q(z) = G_q(K^2 z)$. It is obvious that

$$\operatorname{supp}(\Psi_q) \subset K^{-2}\Omega_q \subset c_0 K^{-2} B(0,1). \tag{5.8}$$

By (5.1), we also have

$$\operatorname{dist}\big(\operatorname{supp}(\Psi_q), \big(K^{-2}\Omega_q\big)^c\big) \gtrsim K^{4q}. \tag{5.9}$$

By the assumption (5.3), there exists a constant A_1 such that

$$\big|\nabla_z\big(F_q\big(K^2 z\big)\big)\big| \le (A_1/2) K^{3-2q}.$$

We split S_4 into two parts

$$S_4 = \sum_{|p|<A_1 K^{1-2q}\mathscr{H} T M_*^{-q-1}} + \sum_{|p|\ge A_1 K^{1-2q}\mathscr{H} T M_*^{-q-1}} =: S_5 + R_5.$$

We will prove the following lemma (later) by integration by parts.

LEMMA 5.4.

$$R_5 \lesssim K^{-12q-6} M_*^{-1}.$$

Next, we will estimate S_5. Define $\lambda_1 = K^{3-2q}\mathscr{H} T M_*^{-q}$ and

$$\Phi_q(z,p) = \big(\mathscr{H} T M_*^{-q} F_q\big(K^2 z\big) - K^2 M_* \langle p, z\rangle\big)/\lambda_1,$$

then

$$S_5 = K^4 M_*^2 \sum_{|p|<A_1 K^{1-2q}\mathscr{H} T M_*^{-q-1}} \int_{\mathbb{R}^2} \Psi_q(z) e\big(\lambda_1 \Phi_q(z,p)\big)\, dz. \tag{5.10}$$

For all $z \in K^{-2}\Omega_q$, by (5.2), (5.3), and (5.4),

$$D_z^\nu \Psi_q(z) \lesssim K^{-4q|\nu|}, \tag{5.11}$$

$$D_z^\nu \Phi_q(z,p) \lesssim \begin{cases} K^{-2} & \text{for } \nu = 0, \\ 1 & \text{for } |\nu| \ge 1 \end{cases} \tag{5.12}$$

and

$$\big|\det\big(\nabla_{zz}^2 \Phi_q(z,p)\big)\big| \gtrsim K^{4q}. \tag{5.13}$$

To prove this lower bound (5.13) we first note, by using the definition of F_q and the mean value theorem, that for $\mu = (1, q-1)$

$$\frac{\partial^2}{\partial z_{l_1}\, \partial z_{l_2}}\big(\Phi_q(z,p)\big) = K^{2q+1} \frac{\partial^2 D^\mu F}{\partial x_{l_1}\, \partial x_{l_2}}\big(K^2 z\big) + O\bigg(K^{-2}\frac{H}{M_*}\bigg).$$

The two terms on the right are $\lesssim 1$ and c_5K^{4q}, respectively. Thus,

$$\det\big(\nabla^2_{zz}\big(\Phi_q(z,p)\big)\big)=K^{4q+2}\det\big(\nabla^2 D^\mu F\big(K^2z\big)\big)+O\big(c_5K^{4q}\big).$$

By (5.4), we get (5.13) if we pick a sufficiently small c_5.

With (5.8), (5.9), (5.11), (5.12), and (5.13), we can estimate the integrals in sum S_5. Let us fix an arbitrary $|p|<A_1K^{1-2q}\mathscr{H}TM_*^{-q-1}$. We first estimate the number of critical points of the phase function Φ_q. Denote $\widetilde{p}=K^2M_*p/\lambda_1$ and $F(z)=K^{2q-3}\nabla_z(F_q(K^2z))$, then $\nabla_z\Phi_q(z,p)=F(z)-\widetilde{p}$ and the critical points are determined by the equation

$$F(z)=\widetilde{p}\quad\text{for } z\in K^{-2}\Omega_q.$$

The bounds (5.12) and (5.13) imply that the mapping $F=(F_1,F_2)$ satisfies

$$D^\nu F_j(z)\lesssim 1\quad\text{for } |\nu|\le 2, j=1,2,$$

and

$$\big|\det\big(\nabla_z F(z)\big)\big|\gtrsim K^{4q}.$$

By (5.9), we know that $\operatorname{supp}(\Psi_q)$ is strictly smaller than $K^{-2}\Omega_q$ and the distance between their boundary is larger than a_1K^{4q} for some positive constant a_1. Let $r_0=a_1K^{4q}/2$. By Taylor's formula, there exists a positive constant a_2 $(<a_1/2)$ such that if $\tilde{z}$ is a critical point in $(\operatorname{supp}(\Psi_q))_{(r_0)}$ then

$$\big|\nabla_z\Phi_q(z,p)\big|\gtrsim K^{4q}|z-\tilde{z}|,\quad\text{for any } z\in B\big(\tilde{z},a_2K^{4q}\big). \tag{5.14}$$

Applying Lemma A.1 to F with r_0 as above yields two positive constants a_3 $(<a_2/2)$ and a_4 such that if $r_1=a_3K^{4q}$, $r_2=a_4K^{8q}$, then F is bijective from $B(z,2r_1)$ to an open set containing $B(F(z),2r_2)$ for any $z\in(\operatorname{supp}(\Psi_q))_{(r_0)}$. It follows, simply by a size estimate, that the number of critical points in $(\operatorname{supp}(\Psi_q))_{(r_1)}$ is $\lesssim (K^{-2}/r_1)^2\lesssim K^{-8q-4}$.

Let Z_j $(j=1,\dots,J(p))$ be all critical points in $(\operatorname{supp}(\Psi_q))_{(r_1)}$ corresponding to the p we fixed. Let $\chi_j(z)=\chi_0((z-Z_j)/(c_6r_1))$ with c_6 chosen (below). Then

$$\int\Psi_q(z)e\big(\lambda_1\Phi_q(z,p)\big)\,dz=\sum_{j=1}^{J(p)}\int\chi_j(z)\Psi_q(z)e\big(\lambda_1\Phi_q(z,p)\big)\,dz+R_6, \tag{5.15}$$

where

$$R_6=\int\left[1-\sum_{j=1}^{J(p)}\chi_j(z)\right]\Psi_q(z)e\big(\lambda_1\Phi_q(z,p)\big)\,dz.$$

For each $j=1,\dots,J(p)$, we consider a new phase function $\phi_j(z,p)=\Phi_q(z,p)-\Phi_q(Z_j,p)$ satisfying $D^\nu_z\phi_j(z,p)\lesssim 1$. By Lemma A.5 (with $\delta=K^{4q}$), if c_6 is sufficiently small then

$$\begin{aligned}&\left|\int\chi_j(z)\Psi_q(z)e\big(\lambda_1\Phi_q(z,p)\big)\,dz\right|\\&\quad=\left|\int\chi_j(z)\Psi_q(z)e\big(\lambda_1\phi_j(z,p)\big)\,dz\right|\lesssim\lambda_1^{-1}K^{-2q}.\end{aligned} \tag{5.16}$$

We will prove (later), by integration by parts, that

LEMMA 5.5.

$$R_6 \lesssim K^{-32q-8}\lambda_1^{-2}. \tag{5.17}$$

Using (5.10), (5.15), (5.16), and (5.17), we get

$$\begin{aligned} S_5 &\lesssim K^4 M_*^2\big(1+\big(A_1K^{1-2q}\mathscr{H}TM_*^{-q-1}\big)^2\big)\left(\frac{\lambda_1^{-1}K^{-2q}}{K^{8q+4}}+\frac{\lambda_1^{-2}}{K^{32q+8}}\right)\\ &\lesssim K^{-12q-1}\mathscr{H}TM_*^{-q}+R_7, \end{aligned}$$

where

$$R_7 = K^{-8q-3}(\mathscr{H}T)^{-1}M_*^{q+2}+K^{-28q-10}(\mathscr{H}T)^{-2}M_*^{2q+2}+K^{-32q-8}.$$

Recall that $R_5 \lesssim K^{-12q-6}M_*^{-1}$, hence $R_5 \lesssim K^{-32q-8}$ and

$$S_4 = S_5 + R_5 \lesssim K^{-12q-1}\mathscr{H}TM_*^{-q}+R_7.$$

Plugging this bound into the inequality in Lemma 5.1 gives

$$\big|S(T,M_*;G,F)\big|^Q \lesssim M_*^{2Q}H^{-1}+K^{-12q-1}TM_*^{2Q-q-2}H^{2-2/Q}+\mathrm{E}, \tag{5.18}$$

where

$$\begin{aligned} \mathrm{E} = M_*^{2(Q-1)}\big(&K^{-8q-3}T^{-1}M_*^{q+2}H^{-2+2/Q}\log H\\ &+K^{-32q-8}+K^{-28q-10}T^{-2}M_*^{2q+2}H^{-2+2/Q}\big). \end{aligned}$$

In order to balance the first two terms on the right-hand side of (5.18), we let

$$H = c_5\big(K^{12q+1}T^{-1}M_*^{q+2}\big)^{Q/(3Q-2)}.$$

The assumption (5.5) implies $H \le c_5K^{4q+2}M_*$. We also have $1 < H$ since we can assume

$$T < c_7K^{12q+1}M_*^{q+2}$$

with a sufficiently small c_7 (otherwise the trivial bound of $S(T,M_*;G,F)$, i.e. M_*^2, is better than (5.6)). With the choice of H as above, (5.18) leads to (5.6). □

Proof of Lemma 5.4. Let $\lambda_2 = \lambda_2(p) = M_*|p|$ and

$$\Gamma_q(z,p) = \big(\mathscr{H}TM_*^{-q}F_q\big(K^2z\big)-K^2M_*\langle p,z\rangle\big)/\lambda_2,$$

then

$$R_5 = K^4M_*^2\sum_{|p|\ge A_1K^{1-2q}\mathscr{H}TM_*^{-q-1}}\int\Psi_q(z)e\big(\lambda_2\Gamma_q(z,p)\big)\,dz.$$

For $z\in K^{-2}\Omega_q$, we have $D^\nu\Psi_q(z)\lesssim K^{-4q|\nu|}$ and $D_z^\nu\Gamma_q(z,p)\lesssim 1$. We also have

$$\big|\nabla_z\Gamma_q(z,p)+K^2p/|p|\big|\le K^2/2,$$

which implies $|\nabla_z \Gamma_q(z,p)| \geq K^2/2$. By Lemma A.2, we have for any $N \in \mathbb{N}$

$$\int \Psi_q(z) e\big(\lambda_2 \Gamma_q(z,p)\big)\, dz \lesssim K^{-(4q+2)N-4} M_*^{-N} |p|^{-N}.$$

The case $N=3$ gives the desired bound for R_5. □

Proof of Lemma 5.5. Denote $\lambda_3 = K^{-2}\lambda_1$, $g(z,p) = K^2 \Phi_q(z,p)$, and

$$u(z) = \left[1 - \sum_{j=1}^{J(p)} \chi_j(z)\right] \Psi_q(z).$$

By (5.12), we have

$$D_z^\nu g(z,p) \lesssim 1.$$

Since $\mathrm{supp}(u)$ is away from critical points, we get $|\nabla_z \Phi_q(z,p)| \gtrsim K^{8q}$ if $z \in \mathrm{supp}(u)$ by (5.14) (see the proof of the author's [3, Proposition 2.4] for more details), which gives

$$\big|\nabla_z g(z,p)\big| \gtrsim K^{8q+2} \quad \text{if } z \in \mathrm{supp}(u).$$

Since χ_j's have disjoint support and $D^\nu \chi_j \lesssim K^{-4q|\nu|}$, we get

$$D^\nu u(z) \lesssim K^{-4q|\nu|}.$$

By Lemma A.2 for any $N \in \mathbb{N}$

$$R_6 = \int u(z) e\big(\lambda_3 g(z,p)\big)\, dz \lesssim K^{-(16q+2)N-4} \lambda_1^{-N}.$$

In particular, we get (5.17) if we let $N=2$. □

6. Proof of Theorem 1.2

Let $\rho \in C_0^\infty(\mathbb{R}^2)$ such that $\int_{\mathbb{R}^2} \rho(y)\, dy = 1$. The central question in the lattice point problem is how to estimate the sum

$$\sum_{k \in \mathbb{Z}_*^2} \widehat{\chi}_{\mathcal{B}}(tk) \widehat{\rho}(\varepsilon k). \tag{6.1}$$

If $\partial\mathcal{B}$ has positive curvature, by Hörmander's formula in Lemma A.4, the estimate of (6.1) is reduced to an exponential sum. Then one can use the classical Van der Corput methods for exponential sums. For such treatment the reader could consult, for example, Krätzel and Nowak [12], [13], Müller [15], the author [3], etc.

If the curvature is allowed to vanish, this cannot be done directly. One may then replace $\mathcal{B}$ by $\mathcal{B}_\theta$ and consider the integral of (6.1) over all rotations, namely

$$\int_0^{2\pi} \left| \sum_{k \in \mathbb{Z}_*^2} \widehat{\chi}_{\mathcal{B}_\theta}(tk) \widehat{\rho}(\varepsilon k) \right| d\theta.$$

For example Iosevich [7] estimated such an integral by putting absolute value on each term in the sum. Rather than doing that, we properly split the sum into two parts: one with more terms, one with less. We put absolute value on each term in the latter part. To the former part, we apply the asymptotic formula of $\widehat{\chi}_{\mathcal{B}_\theta}(\xi)$ away from those points ξ corresponding to small curvature. This is where we need Corollary 4.3 with an error term containing curvature explicitly. Then the estimate is reduced to an exponential sum, to which we can apply similar methods used in [12], [13], [15], and [3]. The former part is where we gain and the reason why we achieve a sharper bound. We carry out this idea in the proof of the following lemma.

LEMMA 6.1. *Let $\zeta = 1/3831$ and $\rho \in C_0^\infty(\mathbb{R}^2)$ such that $\int_{\mathbb{R}^2} \rho(y)\,dy = 1$. If $\mathcal{B}$ is a compact convex planar domain with smooth boundary of finite type ω which contains the origin as an interior point, then for $j \in \mathbb{N}$ we have*

$$\int_0^{2\pi} \sup_{2^{j-1} \le t < 2^{j+2}} \left| t^{4/3+\zeta+\sigma(\omega)} \sum_{k \in \mathbb{Z}^2_*} \widehat{\chi}_{\mathcal{B}_\theta}(tk)\widehat{\rho}(\varepsilon k) \right| d\theta \lesssim 1,$$

where $\varepsilon = \varepsilon(j,\omega) = 2^{-j\alpha(\omega)}$,

$$\alpha(\omega) = \frac{426\omega - 832}{1277\omega - 2496}, \quad \textit{and} \quad \sigma(\omega) = \frac{832}{1277(1277\omega - 2496)}.$$

Before we prove this result, we first apply it to prove the following lemma, which easily implies Theorem 1.2 (see Iosevich [7, p. 27] for this argument).

LEMMA 6.2. *Under the same hypothesis as in Lemma 6.1, for $j \in \mathbb{N}$ we have*

$$\int_0^{2\pi} \sup_{2^j \le t < 2^{j+1}} t^{-2/3+\zeta+\sigma(\omega)} \left| P_{\mathcal{B}_\theta}(t) \right| d\theta \le C,$$

where C is independent of j.

Proof. Define $\rho_\varepsilon(y) = \varepsilon^{-2}\rho(\varepsilon^{-1}y)$ and

$$N_{\varepsilon,\theta}(t) = \sum_{k \in \mathbb{Z}^2} \chi_{t\mathcal{B}_\theta} * \rho_\varepsilon(k).$$

By the Poisson summation formula

$$N_{\varepsilon,\theta}(t) = t^2 \sum_{k \in \mathbb{Z}^2} \widehat{\chi}_{\mathcal{B}_\theta}(tk)\widehat{\rho}(\varepsilon k) = \operatorname{area}(\mathcal{B})t^2 + R_{\varepsilon,\theta}(t),$$

where

$$R_{\varepsilon,\theta}(t) = t^2 \sum_{k \in \mathbb{Z}^2_*} \widehat{\chi}_{\mathcal{B}_\theta}(tk)\widehat{\rho}(\varepsilon k).$$

Müller proved in [14] that if $C_1 > 0$ satisfies $B(0, 1/C_1) \subset \mathcal{B}$ then

$$N_{\varepsilon,\theta}(t - C_1\varepsilon) \le \#\{\mathbb{Z}^2 \cap t\mathcal{B}_\theta\} = \sum_{k \in \mathbb{Z}^2} \chi_{t\mathcal{B}_\theta}(k) \le N_{\varepsilon,\theta}(t + C_1\varepsilon),$$

which implies

$$P_{\mathcal{B}_\theta}(t) \lesssim \left| R_{\varepsilon,\theta}(t + C_1\varepsilon) \right| + \left| R_{\varepsilon,\theta}(t - C_1\varepsilon) \right| + t\varepsilon.$$

Then

$$\sup_{2^j \le t < 2^{j+1}} t^{-2/3+\zeta+\sigma(\omega)} \left| P_{\mathcal{B}_\theta}(t) \right| \lesssim \sup_{2^j \le t < 2^{j+1}} t^{-2/3+\zeta+\sigma(\omega)} t\varepsilon + \sup_{2^j \le t < 2^{j+1}} t^{-2/3+\zeta+\sigma(\omega)} \left| R_{\varepsilon,\theta}(t \pm C_1\varepsilon) \right|.$$

The first term on the right-hand side is bounded by a constant, and the second one is in $L^1(S^1)$ due to Lemma 6.1. □

Proof of Lemma 6.1. Let $t \in [2^{j-1}, 2^{j+2}]$ and

$$\delta = \delta(j,\omega) = 2^{-j\beta(\omega)} \quad \text{with } \beta(\omega) = \frac{\omega - 2}{1277\omega - 2496}.$$

For any $\theta \in [0, 2\pi]$, we have the following splitting

$$\sum_{k \in \mathbb{Z}^2_*} \widehat{\chi}_{\mathcal{B}_\theta}(tk) \widehat{\rho}(\varepsilon k) = \text{sum I} + \text{sum II},$$

where

$$\text{sum I} = \sum_{k \in D_1(\delta,\theta)} \widehat{\chi}_{\mathcal{B}_\theta}(tk) \widehat{\rho}(\varepsilon k),$$

$$\text{sum II} = \sum_{k \in D_2(\delta,\theta)} \widehat{\chi}_{\mathcal{B}_\theta}(tk) \widehat{\rho}(\varepsilon k),$$

and $D_1(\delta,\theta)$, $D_2(\delta,\theta)$ are two regions defined as follows: if

$$D_2(\delta, 0) = \left\{ \xi \in \mathbb{R}^2_* : K_\xi \le \delta \text{ or } K_{-\xi} \le \delta \right\},$$

and $D_1(\delta, 0) = \mathbb{R}^2_* \setminus D_2(\delta, 0)$, then

$$D_2(\delta,\theta) = R_\theta D_2(\delta, 0) \quad \text{and} \quad D_1(\delta,\theta) = R_\theta D_1(\delta, 0).$$

Note that $D_2(\delta, 0)$ is the union of finitely many planar double cones (symmetric about the origin)[2] minus the origin. If t is large, these cones intersect only at the origin.

For sum II, we have the following claim.

Claim 6.3.

$$\int_0^{2\pi} \sup_{2^{j-1} \le t < 2^{j+2}} t^{4/3+\zeta+\sigma(\omega)} |\text{sum II}| \, d\theta \lesssim 1. \tag{6.2}$$

We defer the proof of this claim until later. Next, we will prove

$$\sup_{2^{j-1} \le t < 2^{j+2}} t^{4/3+\zeta+\sigma(\omega)} |\text{sum I}| \lesssim 1 \tag{6.3}$$

with an implicit constant depending only on $\mathcal{B}$. The conclusion in Lemma 6.1 follows easily from (6.2) and (6.3).

---

[2] A planar double cone symmetric about the origin is, for example, the (smaller) region bounded between $y = x$ and $y = 1.001x$.

Note if $\xi \in D_1(\delta,\theta)$ then $K^\theta_{\pm\xi} \geq \delta$. Applying Corollary 4.3 to sum I yields

$$\text{sum I} = (2\pi)^{-1} e^{3\pi i/4} S_1 + (2\pi)^{-1} e^{-3\pi i/4} \widetilde{S}_1 + R_1, \tag{6.4}$$

where

$$S_1 = S_1(t,\varepsilon,\delta,\theta) = t^{-3/2} \sum_{k\in D_1(\delta,\theta)} |k|^{-3/2} \big(K^\theta_k\big)^{-1/2} \widehat{\rho}(\varepsilon k) e\big(tH_\theta(k)\big),$$

$$\widetilde{S}_1 = \widetilde{S}_1(t,\varepsilon,\delta,\theta) = t^{-3/2} \sum_{k\in D_1(\delta,\theta)} |k|^{-3/2} \big(K^\theta_{-k}\big)^{-1/2} \widehat{\rho}(\varepsilon k) e\big(-tH_\theta(-k)\big),$$

and

$$R_1 \lesssim_{\mathcal{B}} \delta^{-4} t^{-2} \sum_{k\in\mathbb{Z}^2_*} |k|^{-2} |\widehat{\rho}(\varepsilon k)| \lesssim \delta^{-4} t^{-2} \log\big(\varepsilon^{-1}\big) \lesssim t^{-4/3-\zeta-\sigma(\omega)}. \tag{6.5}$$

We will only estimate S_1 since $\widetilde{S}_1$ is similar. Denote $\mathscr{C}_1 = \{\xi \in \mathbb{R}^2 : 1/2 \leq |\xi| \leq 2\}$. Let us introduce a dyadic decomposition and a partition of unity.

Assume $\varphi \in C^\infty_0(\mathbb{R}^2)$ is a real radial function such that $\operatorname{supp}(\varphi) \subset \mathscr{C}_1$, $0 \leq \varphi \leq 1$, and

$$\sum_{l_0=-\infty}^{\infty} \varphi\left(\frac{\xi}{2^{l_0}}\right) = 1 \quad \text{for } \xi \in \mathbb{R}^2 \setminus \{0\}.$$

Denote

$$S_{1,M} = \sum_{k\in D_1(\delta,\theta)} \varphi\big(M^{-1}k\big) |k|^{-3/2} \big(K^\theta_k\big)^{-1/2} \widehat{\rho}(\varepsilon k) e\big(tH_\theta(k)\big),$$

then $S_1 = t^{-3/2} \sum_{l_0=0}^\infty S_{1,2^{l_0}}$. We will estimate $S_{1,M}$ for a fixed $M = 2^{l_0}$, $l_0 \in \mathbb{N}_0$.

Let $q \in \mathbb{N}$. For each $\xi \in S^1 \cap \mathcal{H}$ there exists a cone

$$\mathfrak{C}\big(\xi, 2r(\xi)\big) := \bigcup_{l>0} lB\big(\xi, 2r(\xi)\big),$$

where $r(\xi) = c_2 (K_\xi)^{4q+2}/2$ and c_2 is the constant appearing in the statement of Lemma 3.4. Note that $K_\eta \asymp K_\xi$ if $\eta \in \mathfrak{C}(\xi, 2r(\xi))$. From the family of cones $\{\mathfrak{C}(\xi, r(\xi)/2) : \xi \in S^1 \cap \mathcal{H}\}$, we can choose, by a Vitali procedure, a sequence $\{\mathfrak{C}(\xi_i, r(\xi_i)/2)\}_{i=1}^\infty$ such that these cones still cover $\mathcal{H}$ and that $\{\mathfrak{C}(\xi_i, r(\xi_i))\}_{i=1}^\infty$ satisfy the bounded overlap property. Denote

$$\mathfrak{C}^\theta_i = R_\theta \mathfrak{C}\big(\xi_i, r(\xi_i)\big).$$

Then the collection $\{\mathfrak{C}^\theta_i\}_{i=1}^\infty$ forms an open cover of $\mathcal{H}_\theta$. We can construct a partition of unity $\{\psi_i\}_{i=1}^\infty$ such that

(i) $\sum_i \psi_i \equiv 1$ on $\mathcal{H}_\theta$, and $\psi_i \in C^\infty_0(\mathfrak{C}^\theta_i)$;
(ii) each ψ_i is homogeneous of degree zero;
(iii) $|D^\nu \psi_i| \lesssim_{|\nu|} (K_{\xi_i})^{-(4q+2)|\nu|}$ on $\mathscr{C}_1$.

From the family $\{\mathfrak{C}_i^\theta\}_{i=1}^\infty$ we can find a subfamily $\{\mathfrak{C}_i^\theta\}_{i\in\mathscr{A}}$ covering $D_1(\delta,\theta)$, where $\mathscr{A}=\mathscr{A}(\delta)$ is an index set such that $i\in\mathscr{A}$ if and only if $\mathfrak{C}_i^\theta$ intersects $D_1(\delta,\theta)$. Since $r(\xi_i)\gtrsim\delta^{4q+2}$ for any $i\in\mathscr{A}$, a size estimate gives that $\#\mathscr{A}\lesssim\delta^{-4q-2}$. Define

$$S_{1,M}^*=\sum_{i\in\mathscr{A}}S_{2,i}, \tag{6.6}$$

where

$$S_{2,i}=\sum_{k\in\mathbb{Z}^2}U_i^\theta(k)e\big(tH_\theta(k)\big)$$

and

$$U_i^\theta(k)=\psi_i\big(M^{-1}k\big)\varphi\big(M^{-1}k\big)|k|^{-3/2}\big(K_k^\theta\big)^{-1/2}\widehat{\rho}(\varepsilon k).$$

Instead of $S_{1,M}$ we will estimate $S_{1,M}^*$. It turns out that the error

$$R_{2,M}=S_{1,M}^*-S_{1,M} \tag{6.7}$$

is relatively small and this will be clear later (see Claim 6.4 below). To estimate $S_{1,M}^*$, it suffices to estimate $S_{2,i}$ for any fixed $i\in\mathscr{A}$.

By Lemma 3.4 and the homogeneity of H_θ, there exist two orthogonal vectors $v_1^*=v_1^*(R_\theta\xi_i)$, $v_2^*=v_2^*(R_\theta\xi_i)\in\mathbb{Z}^2$ such that $|v_1^*|=|v_2^*|\asymp(K_{\xi_i})^{-4q}$, $\|(v_1^*,v_2^*)^{-1}\|\lesssim(K_{\xi_i})^{4q}$, and

$$\big|h_q^\theta\big(\eta,v_1^*,v_2^*\big)\big|\gtrsim(K_{\xi_i})^{-8q^2-16q-2}\quad\text{if }\eta\in\bigcup_{1/4\le l\le 4}lB\big(R_\theta\xi_i,2r(\xi_i)\big). \tag{6.8}$$

Let $L=[\mathbb{Z}^2:\mathbb{Z}v_1^*\oplus\mathbb{Z}v_2^*]$ be the index of the lattice spanned by v_1^*, v_2^* in the lattice $\mathbb{Z}^2$. Then

$$L=\big|\det\big(v_1^*,v_2^*\big)\big|\lesssim(K_{\xi_i})^{-8q},$$

and there exist vectors $b_l\in\mathbb{Z}^2$ $(l=1,\dots,L)$ such that $|b_l|\lesssim(K_{\xi_i})^{-4q}$ and

$$\mathbb{Z}^2=\biguplus_{l=1}^{L}\big(\mathbb{Z}v_1^*+\mathbb{Z}v_2^*+b_l\big).$$

Let $N\in\mathbb{N}$ be arbitrarily fixed. Applying this decomposition, for any $k\in\mathbb{Z}^2$ we can write $k=m_1v_1^*+m_2v_2^*+b_l$ where $m_s\in\mathbb{Z}$ $(s=1,2)$. Hence,

$$\begin{aligned}S_{2,i}&=\sum_{l=1}^{L}\sum_{m\in\mathbb{Z}^2}U_i^\theta\big(m_1v_1^*+m_2v_2^*+b_l\big)e\big(tH_\theta\big(m_1v_1^*+m_2v_2^*+b_l\big)\big)\\&=(K_{\xi_i})^{-1/2}M^{-3/2}(1+M\varepsilon)^{-N}\sum_{l=1}^{L}S(T,M_*;G_l,F_l),\end{aligned} \tag{6.9}$$

where $T=tM$, $M_*=K^{4q}M$, $K=K_{\xi_i}$,

$$G_l(y)=K^{1/2}M^{3/2}(1+M\varepsilon)^NU_i^\theta\big(M_*y_1v_1^*+M_*y_2v_2^*+b_l\big),$$

and

$$F_l(y) = H_\theta\big(y_1 K^{4q} v_1^* + y_2 K^{4q} v_2^* + M^{-1} b_l\big).$$

We consider the function F_l restricted to the convex domain

$$\Omega_l = \Big\{(y_1, y_2)^t \in \mathbb{R}^2 : \\ y_1 K^{4q} v_1^* + y_2 K^{4q} v_2^* + M^{-1} b_l \in \bigcup_{1/4 \le l \le 4} lB\big(R_\theta \xi_i, 2r(\xi_i)\big)\Big\}.$$

The support of G_l satisfies

$$\operatorname{supp}(G_l) \subset \big\{(y_1, y_2)^t \in \mathbb{R}^2 : y_1 K^{4q} v_1^* + y_2 K^{4q} v_2^* + M^{-1} b_l \in \overline{\mathscr{C}_1 \cap \mathfrak{C}_i^\theta}\big\} \subset \Omega_l.$$

We want to apply to $S(T, M_*; G_l, F_l)$ Proposition 5.2 with $G = G_l$, $F = F_l$, $\Omega = \Omega_l$, and $q = 3$.

Since $1 \gtrsim K_{\xi_i} \gtrsim \delta$ for $i \in \mathscr{A}$, there exist positive constants C_2 and C_3 such that the assumptions of Proposition 5.2 are satisfied if $C_3 \delta^{-26} \le M \le C_2 t^{4/5}$. This follows from Lemma 3.4, (6.8) and the following facts: if $(K_{\xi_i})^{-4q} \lesssim M$ then $\Omega_l \subset c_0 B(0,1)$ for a constant c_0 (depending only on q, $\mathcal{B}$);

$$\operatorname{dist}\bigg(\Big(\bigcup_{1/4 \le l \le 4} lB\big(R_\theta \xi_i, 2r(\xi_i)\big)\Big)^c, \overline{\mathscr{C}_1 \cap \mathfrak{C}_i^\theta}\bigg) \ge c_2 (K_{\xi_i})^{4q+2}/8;$$

and

$$D^\nu U_i^\theta \lesssim (K_{\xi_i})^{-(4q+2)|\nu| - 1/2} M^{-|\nu| - 3/2} (1 + M\varepsilon)^{-N}.$$

Thus by Proposition 5.2 (with $q = 3$), we get

$$S(T, M_*; G_l, F_l) \lesssim_{\mathcal{B}} (K_{\xi_i})^{24 - 97/22} t^{1/22} M^{20/11} + R, \tag{6.10}$$

where

$$R \lesssim_{\mathcal{B}} K_{\xi_i}^{24}\big[K_{\xi_i}^{-16} M^{7/4} + K_{\xi_i}^{-921/88} t^{-15/88} M^{24/11} \\ + K_{\xi_i}^{-145/22} t^{-1/22} M^{85/44} (\log M)^{1/8}\big].$$

Using (6.6), (6.9), (6.10), $K_{\xi_i} \gtrsim \delta$, and bounds for $\#\mathscr{A}$ and L, we get

$$S_{1,M}^* \lesssim \delta^{-29/2} M^{-3/2} (1 + M\varepsilon)^{-N} \big[\delta^{-97/22} t^{1/22} M^{20/11} \\ + \delta^{-16} M^{7/4} + \delta^{-921/88} t^{-15/88} M^{24/11} \\ + \delta^{-145/22} t^{-1/22} M^{85/44} (\log M)^{1/8}\big]. \tag{6.11}$$

Now we can estimate S_1. By (6.7),

$$S_1 = t^{-3/2}\bigg(\sum_{C_3 \delta^{-26} \le 2^{l_0} \le C_2 t^{4/5}} S_{1,2^{l_0}}^* + R_2 + R_3 + R_4\bigg), \tag{6.12}$$

where

$$R_2 = -\sum_{C_3\delta^{-26}\le 2^{l_0}\le C_2 t^{4/5}} R_{2,2^{l_0}},$$

$$R_3 = \sum_{2^{l_0}<C_3\delta^{-26}} S_{1,2^{l_0}}, \quad \text{and} \quad R_4 = \sum_{2^{l_0}>C_2 t^{4/5}} S_{1,2^{l_0}}.$$

Taking (6.11) and sizes of δ and ε into account, we get

$$\begin{aligned}(6.13)\qquad &\sum_{C_3\delta^{-26}\le 2^{l_0}\le C_2 t^{4/5}} S^*_{1,2^{l_0}} \\ &\lesssim_{\mathcal{B}} \delta^{-208/11}t^{1/22}\varepsilon^{-7/22} + \delta^{-2197/88}t^{-15/88}\varepsilon^{-15/22} \\ &\quad + \delta^{-61/2}\varepsilon^{-1/4} + \delta^{-232/11}t^{-1/22}\varepsilon^{-19/44}\log t \\ &\lesssim_{\mathcal{B}} \delta^{-208/11}t^{1/22}\varepsilon^{-7/22}.\end{aligned}$$

Claim 6.4.

$$\max\bigl(|R_2|,|R_3|,|R_4|\bigr) \lesssim_{\mathcal{B}} \delta^{-208/11}t^{1/22}\varepsilon^{-7/22}.$$

Hence (6.3) follows from (6.4), (6.5), (6.12), (6.13), Claim 6.4, and sizes of δ and ε. □

Remarks 6.5. (1) Our proof works for convex planar domains of finite type. If $\omega = 2$, the curvature does not vanish and $D_2(\delta,\theta)$ is empty. The method we used is essentially the same as those used in [3], [15], and will produce the same bound $O(t^{2/3-1/87})$.

(2) With essentially the same proof, we can actually prove that if $1 \le p < 2 + 2/(\omega-2)$ then

$$\sup_{t\ge 2} \log^{-a}(t)t^{-2/3+\Upsilon(\omega,p)}\bigl|P_{\mathcal{B}_\theta}(t)\bigr| \in L^p\bigl(S^1\bigr)$$

where $a > 1/p$ and

$$\Upsilon(\omega,p) = \frac{(2-p)\omega + 2p - 2}{3(1219p+58)\omega - 3(2438p+58)}.$$

(3) We can possibly improve the exponent in Theorem 1.1 by iterating the Van der Corput method.

Proof of Claim 6.3. To begin with, we estimate

$$\begin{aligned}(*) &:= \int_0^{2\pi} 1_{D_2(\delta,\theta)}(k)\Bigl(\sup_{2^{j-1}\le t<2^{j+2}} |tk|^{3/2}\bigl|\widehat{\chi}_{\mathcal{B}_\theta}(tk)\bigr|\Bigr)\,d\theta \\ &= \int_0^{2\pi} 1_{D_2(\delta,0)}\bigl(|k|(\cos\theta,\sin\theta)\bigr)\Bigl(\sup_{t\sim 2^j}\bigl|tk\bigr|^{3/2}\bigl|\widehat{\chi}_{\mathcal{B}}\bigl(t|k|(\cos\theta,\sin\theta)\bigr)\bigr|\Bigr)\,d\theta.\end{aligned}$$

Recall the definition of $D_2(\delta,0)$, we are only integrating over finitely many (no more than 2Ξ) small arcs on S^1. By Lemma 2.1, the length of each arc is $\lesssim \delta^{(\omega_i-1)/(\omega_i-2)}$ $(i=1,\dots,\Xi)$. The inequality (4.1) gives explicit upper bounds for

$$\sup |tk|^{3/2}\big|\widehat{\chi}_{\mathcal{B}}\big(t|k|(\cos\theta,\sin\theta)\big)\big|$$

over these small arcs. Hence, the estimate is reduced to the following integral

$$\int_0^{c\delta^{(\omega_i-1)/(\omega_i-2)}} \theta^{-\frac{\omega_i-2}{2(\omega_i-1)}}\,d\theta \lesssim \delta^{1/2+1/(\omega_i-2)}.$$

It follows that

$$(*) \lesssim \sum_{i=1}^{\Xi} \delta^{1/2+1/(\omega_i-2)} \lesssim \delta^{1/2+1/(\omega-2)}.$$

Hence, the left-hand side of (6.2) is bounded by

$$\begin{aligned}&\lesssim \big(2^j\big)^{-1/6+\zeta+\sigma(\omega)} \sum_{k\in\mathbb{Z}^2_*} \big|\widehat{\rho}(\varepsilon k)\big||k|^{-3/2}(*)\\ &\lesssim \big(2^j\big)^{-1/6+\zeta+\sigma(\omega)}\varepsilon^{-1/2}\delta^{1/2+1/(\omega-2)} \lesssim 1.\end{aligned}$$

In the last step, we use the definition of ε and δ. □

Proof of Claim 6.4. We first estimate R_2. The trivial estimate gives

$$|R_{2,M}| \le \sum_{k\in\mathbb{Z}^2\setminus D_1(\delta,\theta)} \sum_{i\in\mathscr{A}} \big|\psi_i(k/M)\big|\big|\varphi(k/M)\big||k|^{-3/2}\big(K_k^\theta\big)^{-1/2}\big|\widehat{\rho}(\varepsilon k)\big|.$$

If $k\in(\mathbb{Z}^2\setminus D_1(\delta,\theta))\cap\mathfrak{C}_i^\theta$ for any $i\in\mathscr{A}$, there are only two possibilities as to the size of K_k^θ: (i) If $K_k^\theta\gtrsim 1$, then k is contained in finitely many (no more than Ξ) cones with angles $\lesssim\delta^{(\omega-1)/(\omega-2)}$; (ii) If $\delta\lesssim K_k^\theta\lesssim\delta^{1/2}$ then k is contained in 2Ξ cones with angles $\lesssim\delta^{2q+1}$. Based on these two cases, we split the sum above into two parts as follows:

$$|R_{2,M}| \le \sum_{\substack{k\in\mathbb{Z}^2\setminus D_1(\delta,\theta)\\ K_k^\theta\gtrsim 1}} \sum_{i\in\mathscr{A}} + \sum_{\substack{k\in\mathbb{Z}^2\setminus D_1(\delta,\theta)\\ \delta\lesssim K_k^\theta\lesssim\delta^{1/2}}} \sum_{i\in\mathscr{A}}.$$

Using this splitting, we get

$$\begin{aligned}|R_2| &\le \sum_{C_3\delta^{-26}\le 2^{l_0}\le C_2 t^{4/5}} |R_{2,2^{l_0}}| \lesssim \delta^{(\omega-1)/(\omega-2)}\varepsilon^{-1/2} + \delta^{2q+1/2}\varepsilon^{-1/2}\\ &\lesssim \delta^{-208/11}t^{1/22}\varepsilon^{-7/22}.\end{aligned}$$

As to R_3, by a trivial estimate of $S_{1,2^{l_0}}$, we get

$$|R_3| \le \delta^{-1/2} \sum_{2^{l_0}<C_3\delta^{-26}} \sum_{k\in\mathbb{Z}^2_*} \varphi\big(2^{-l_0}k\big)|k|^{-3/2}\big|\widehat{\rho}(\varepsilon k)\big| \lesssim \delta^{-27/2}.$$

Similarly, $R_4 = O(\delta^{-1/2}t^{2/5-4N/5}\varepsilon^{-N})$ for any $N\in\mathbb{N}$. Both are smaller than $\delta^{-208/11}t^{1/22}\varepsilon^{-7/22}$. □

Appendix: Several lemmas

Here we give a quantitative version of the inverse function theorem (see the Appendix in [3]). It is routine to prove it by following a standard proof of the inverse function theorem.

Lemma A.1. *Suppose f is a $C^{(k)}$ $(k \geq 2)$ mapping from an open set $\Omega \subset \mathbb{R}^d$ into $\mathbb{R}^d$ and $b = f(a)$ for some $a \in \Omega$. Assume $|\det(\nabla f(a))| \geq c$ and for any $x \in \Omega$,*

$$\left|D^\nu f_i(x)\right| \leq C \quad \text{for } |\nu| \leq 2, 1 \leq i \leq d.$$

If $r_0 \leq \sup\{r > 0 : B(a,r) \subset \Omega\}$, then f is bijective from $B(a, r_1)$ to an open set containing $B(b, r_2)$ where

$$r_1 = \min\left\{\frac{c}{2d^2 d! C^d}, r_0\right\}, \qquad r_2 = \frac{c}{4d! C^{d-1}} r_1.$$

The inverse mapping f^{-1} is also in $C^{(k)}$.

Hörmander's [4, Theorem 7.7.1] gives the following estimate obtained by integration by parts.

Lemma A.2. *Let $K \subset \mathbb{R}^d$ be a compact set, X an open neighborhood of K and k a nonnegative integer. If $u \in C_0^k(K)$, real $f \in C^{k+1}(X)$, then*

$$\left|\int u(x) e^{i\lambda f(x)}\, dx\right| \leq C|K|\lambda^{-k} \sum_{|\nu| \leq k} \sup\left|D^\nu u\right| |\nabla f|^{|\nu| - 2k}, \quad \lambda > 0.$$

Here C is bounded when f stays in a bounded set in $C^{k+1}(X)$.

The following three lemmas are various results of the method of stationary phase. The first one is the one dimensional version of Hörmander's [4, Lemma 7.7.3]. The second one is Hörmander's [4, Corollary 7.7.15] $(d = 2)$. And the third one is Sogge and Stein's [23, Lemma 2].

Lemma A.3. *If $u \in \mathscr{S}(\mathbb{R})$, then for every $k \in \mathbb{N}$*

$$\left|\int u(x) e^{-i\lambda x^2/2}\, dx - (2\pi)^{1/2} e^{-\pi i/4} \lambda^{-1/2} \sum_{j=0}^{k-1} (2i\lambda)^{-j} u^{(2j)}(0)/j!\right|$$
$$\leq \left(2^{1-k}\sqrt{\pi}/k!\right)\lambda^{-k-1/2}\left(\left\|u^{(2k)}\right\|_{L^2} + \left\|u^{(2k+1)}\right\|_{L^2}\right).$$

Lemma A.4. *If $\mathcal{B} \subset \mathbb{R}^2$ is a compact convex domain with smooth boundary and positive curvature, then for any $\xi \in S^1$, $\lambda > 1$*

$$\widehat{\chi}_{\mathcal{B}}(\lambda\xi) = (2\pi)^{-1}\big[e^{3\pi i/4} K_\xi^{-1/2} e^{-2\pi i\lambda\langle \xi, x(\xi)\rangle}$$
$$+ e^{-3\pi i/4} K_{-\xi}^{-1/2} e^{2\pi i \lambda \langle -\xi, x(-\xi)\rangle}\big]\lambda^{-3/2} + O\left(\lambda^{-5/2}\right).$$

LEMMA A.5. *Suppose ϕ and ψ are smooth functions in $B(0,\delta) \subset \mathbb{R}^d$ with ϕ real-valued. Assume that $|(\partial/\partial x)^\nu \phi| \le C_1$, $|\nu| \le d+2$ and $|(\partial/\partial x)^\nu \psi| \le C_2 \delta^{-|\nu|}$, $|\nu| \le d$. We also suppose that $(\nabla\phi)(0) = 0$, but $|\det \nabla^2 \phi(0)| \ge \delta$. Then there exists a positive constant c_1 (independent of δ), which is sufficiently small, so that if ψ is supported in $B(0, c_1\delta)$ we can assert that*

$$\left| \int_{\mathbb{R}^d} \psi e^{i\lambda\phi}\, dx \right| \le C\lambda^{-d/2}\delta^{-1/2}.$$

Acknowledgments. The subject of this paper was suggested by Professor Andreas Seeger. I would like to express my gratitude to him for his valuable advice and great help during the work.

REFERENCES

[1] L. Brandolini, L. Colzani, A. Iosevich, A. Podkorytov and G. Travaglini, *Geometry of the Gauss map and lattice points in convex domains*, Mathematika **48** (2001), 107–117. MR 1996364

[2] Y. Colin de Verdière, *Nombre de points entiers dans une famille homothétique de domaines de* $\mathbb{R}$, Ann. Sci. École Norm. Sup. (4) **10** (1977), 559–575. MR 0480399

[3] J. W. Guo, *On lattice points in large convex bodies*, Acta Arith. **151** (2012), 83–108. MR 2853046

[4] L. Hörmander, *The analysis of linear partial differential operators. I. Distribution theory and Fourier analysis*, Springer-Verlag, Berlin, 1983.

[5] M. N. Huxley, *Area, lattice points, and exponential sums*, The Clarendon Press, Oxford Univ. Press, New York, 1996. MR 1420620

[6] M. N. Huxley, *Exponential sums and lattice points III*, Proc. London Math. Soc. (3) **87** (2003), 591–609. MR 2005876

[7] A. Iosevich, *Lattice points and generalized Diophantine conditions*, J. Number Theory **90** (2001), 19–30. MR 1850870

[8] A. Iosevich, E. Sawyer and A. Seeger, *Two problems associated with convex finite type domains*, Publ. Mat. **46** (2002), 153–177. MR 1904861

[9] A. Iosevich, E. Sawyer and A. Seeger, *Mean square discrepancy bounds for the number of lattice points in large convex bodies*, J. Anal. Math. **87** (2002), 209–230. MR 1945283

[10] A. Iosevich, E. Sawyer and A. Seeger, *Mean lattice point discrepancy bounds, II: Convex domains in the plane*, J. Anal. Math. **101** (2007), 25–63. MR 2346539

[11] E. Krätzel, *Lattice points*, Kluwer Academic Publishers Group, Dordrecht, 1988. MR 0998378

[12] E. Krätzel and W. G. Nowak, *Lattice points in large convex bodies*, Monatsh. Math. **112** (1991), 61–72. MR 1122105

[13] E. Krätzel and W. G. Nowak, *Lattice points in large convex bodies II*, Acta Arith. **62** (1992), 285–295. MR 1197422

[14] W. Müller, *On the average order of the lattice rest of a convex body*, Acta Arith. **80** (1997), 89–100. MR 1450420

[15] W. Müller, *Lattice points in large convex bodies*, Monatsh. Math. **128** (1999), 315–330. MR 1726766

[16] W. Müller and W. G. Nowak, *On lattice points in planar domains*, Math. J. Okayama Univ. **27** (1985), 173–184. MR 0833466

[17] W. Müller and W. G. Nowak, *Lattice points in planar domains: Applications of Huxley's "discrete Hardy–Littlewood method"*, Number-theoretic analysis (Vienna,

1988–89), Lecture Notes in Math., vol. 1452, Springer, Berlin, 1990, pp. 139–164. MR 1084644

[18] W. G. Nowak, *Zur Gitterpunktlehre der euklidischen Ebene*, Indag. Math. **46** (1984), 209–223. MR 0749533

[19] W. G. Nowak, *Zur Gitterpunktlehre der euklidischen Ebene. II*, Österreich. Akad. Wiss. Math.-Natur. Kl. Sitzungsber. II **194** (1985), 31–37. MR 0833889

[20] B. Randol, *A lattice point problem*, Trans. Amer. Math. Soc. **121** (1966), 257–268. MR 0201407

[21] B. Randol, *On the Fourier transform of the indicator function of a planar set*, Trans. Amer. Math. Soc. **139** (1969), 271–278. MR 0251449

[22] B. Randol, *On the asymptotic behavior of the Fourier transform of the indicator function of a convex set*, Trans. Amer. Math. Soc. **139** (1969), 279–285. MR 0251450

[23] C. D. Sogge and E. M. Stein, *Averages of functions over hypersurfaces in* $\mathbb{R}^n$, Invent. Math. **82** (1985), 543–556. MR 0811550

[24] I. Svensson, *Estimate for the Fourier transform of the characteristic function of a convex set*, Ark. Mat. **9** (1971), 11–22. MR 0328471

[25] M. Tarnopolska-Weiss, *On the number of lattice points in planar domains*, Proc. Amer. Math. Soc. **69** (1978), 308–311. MR 0486837

Jingwei Guo, Department of Mathematics, University of Illinois at Urbana-Champaign, Urbana, IL 61801, USA

E-mail address: jwguo@illinois.edu