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GENERALIZATIONS OF PRIMARY ABELIAN C_α GROUPS

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To the memory of Charles K. Megibben (October 22, 1936–March 2, 2010) who defined and investigated the concepts upon which it is based

ABSTRACT. A valuated p^n-socle is C_α n-summable if for every ordinal $\beta < \alpha$, it has a β-high subgroup that is n-summable (i.e., a valuated direct sum of countable valuated groups). This generalizes both the classical concepts of a C_α group due to Megibben and of an n-summable valuated p^n-socle developed by the authors. The notion is first analyzed in the category of valuated p^n-socles and then applied to the category of Abelian p-groups. In particular, results of Nunke on the torsion product and results of Keef on the balanced projective dimension of C_{ω_1} groups are recast into statements involving valuated p^n-socles and their related groups.

0. Terminology and introduction

The term "group" will mean an Abelian p-group, where p is a prime fixed for the duration of the paper. Our terminology and notation will be based upon [4], [5] and [7]. We also make use of concepts related to *valuated groups* and *valuated vector spaces* that can be found, for example, in [24] and [6], and that we briefly review: Let $\mathcal{O}$ be the class of ordinals and $\mathcal{O}_\infty = \mathcal{O} \cup \{\infty\}$, where we agree that $\alpha < \infty$ for all $\alpha \in \mathcal{O}_\infty$. A *valuation* on a group V is a function $| \ |_V : V \to \mathcal{O}_\infty$ such that for every $x, y \in V$, $|x \pm y|_V \geq \min\{|x|_V, |y|_V\}$ and $|px|_V > |x|_V$. As a result, for all $\alpha \in \mathcal{O}_\infty$, $V(\alpha) = \{x \in V : |x|_V \geq \alpha\}$ is a subgroup of V with $pV(\alpha) \subseteq V(\alpha+1)$. We say V is *α-bounded* if $V(\alpha) = \{0\}$; the *length* of V is the least ordinal α such that $V(\alpha) = V(\infty)$.

A homomorphism between two valuated groups is *valuated* if it does not decrease values and an *isometry* if it is bijective and preserves values. If

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$\{V_i\}_{i\in I}$, is a collection of valuated groups, then the usual direct sum, $V=\bigoplus_{i\in I} V_i$, has a natural valuation, where $V(\alpha)=\bigoplus_{i\in I} V_i(\alpha)$ for every $\alpha\in\mathcal{O}_\infty$. If W is any subgroup of V, then restricting $|\ |_V$ to W turns it into a valuated group with $W(\alpha)=W\cap V(\alpha)$ for all $\alpha\in\mathcal{O}_\infty$. A valuated group W with $pW=\{0\}$ is called a *valuated vector space*; so each $W(\alpha)$ will be a subspace of W. We say a valuated vector space is *free* if it is isometric to a valuated direct sum of cyclic groups (of order p). If V is a valuated group, then its socle $V[p]=\{x\in V : px=0\}$ is a valuated vector space, and V is *summable* if $V[p]$ is free. A group G is a valuated group using the height function (also denoted by $|\ |_G$) as its valuation; in this case $G(\alpha)=p^\alpha G$, and G is said to be *separable* if it is ω-bounded, or equivalently, p^ω-bounded. So if n is a fixed positive integer, then the p^n*-socle* of G, written $G[p^n]=\{x\in G : p^n x=0\}$, can be viewed as a valuated group.

In [2], an ∞-bounded valuated group V was defined to be a *valuated* p^n*-socle* if $p^nV=\{0\}$ and for every $x\in V[p^{n-1}]$ and every ordinal $\beta<|x|_V$, there is a $y\in V$ with $x=py$ and $\beta\le|y|_V$. It easily follows that an ∞-bounded valuated vector space is a valuated p-socle. The p^n-socle of a reduced group G is always a valuated p^n-socle. (The parallel requirements that V be ∞-bounded and that G be reduced are convenient, but not strictly speaking necessary.)

A valuated p^n-socle V is said to be n*-summable* if it is isometric to the valuated direct sum of a collection of countable valuated groups (each of which will also be a valuated p^n-socle). It was shown in [2] that the theory of n-summable valuated p^n-socles parallels the theory of direct sums of countable groups (or *dsc groups* for short—see Chapter XII of [5] for standard results on these groups). For example, in [2], Theorem 2.7, which parallels [5], Theorem 78.4, it was shown that two n-summable valuated p^n-socles are isometric iff their Ulm functions agree, where the Ulm function of V is defined by $f_V(\alpha)=r(V(\alpha)[p]/V(\alpha+1)[p])$.

The parallel between n-summable valuated p^n-socles and dsc groups can be extended. A subgroup X of a valuated group V is *nice* if every coset $a+X$ has an element of maximal value (such an element is called *proper*). In [2], Theorem 2.1, which parallels [5], Theorem 81.9, it was proved that the ω_1-bounded n-summable valuated p^n-socles can be characterized using nice systems and nice composition series. Naturally, a group G is n*-summable* if $G[p^n]$ is n-summable as a valuated p^n-socle. In [2], Theorem 3.8, it was shown that G is a dsc group iff it is n-summable for every positive integer n. Various properties of these groups are established in [3], [12], [17], [18] and [19].

In [20], Megibben introduced a generalization of the classical notion of a separable group. If $\lambda\le\omega_1$ is a limit ordinal, then G is a C_λ *group* if $G/p^\beta G$ is p^β-projective for all $\beta<\lambda$. In fact, if G is a C_λ group, then for each $\beta<\lambda$, $G/p^\beta G$ will of necessity be a dsc group. Clearly, every group is a C_ω group. In [16], using an idea due to Nunke [23], this definition was extended in the following manner: If $\alpha \leq \omega_1$, then G is a C_α *group* if for every $\beta < \alpha$, G has a p^β-high subgroup which is a dsc group (where H is p^β-high in G if it is maximal with respect to intersecting $p^\beta G$ trivially). In papers such as [13], [14], [15], etc., it was shown that there is a close relationship between dsc groups, C_α groups and the torsion product.

The purpose of this paper is to extend the above parallel between n-summable valuated p^n-socles and dsc groups. This is done in two stages. In Section 1, we concentrate on valuated p^n-socles. We begin by defining the torsion product of two valuated p^n-socles (Lemma 1.10). We will use the notation $V \bigtriangledown W$ for the torsion product. This notation is considerably more convenient, more compact, and more accurately reflects that this is a *product* which is related to the tensor product $\otimes$. We then define a valuated p^n-socle V to be C_α *n-summable* iff for each $\beta < \alpha$, V has a β-high subgroup which is n-summable (where, again, a subgroup is β-high in V if it is maximal with respect to intersecting $V(\beta)$ trivially). We generalize an important result of [23] by showing that if V and W are valuated p^n-socles, V has length α and $W(\alpha) \neq \{0\}$, then $V \bigtriangledown W$ is n-summable iff V is n-summable and W is C_α n-summable (Theorem 1.19). The critical step in this discussion (Theorem 1.15) constructs a valuated splitting of a particular short exact sequence. This construction is related to the fact that a $p^{\alpha+1}$-pure subgroup of a p^α-pure projective group is, in fact, a summand (cf. the proof of [5], Theorem 82.3). In general, we are forced to use combinatorial arguments to replace the homological machinery of [21] and [22].

In Section 2, the above results are applied to groups, with one important distinction. In Section 1 we treat valuated p^n-socles of length strictly greater than ω_1. On the other hand, it is a classical result that if G is a reduced summable group (in particular, if it is n-summable), then $p^{\omega_1}G = \{0\}$ (see [5], Theorem 84.3). This means that, as in [2] and [12], we can restrict our attention to the ω_1-bounded case.

There are two ways to apply our results on valuated p^n-socles to the category of groups. The obvious one is to start with a group G and simply consider the valuated p^n-socle $G[p^n]$; in particular, we say G is C_α n-summable iff the same can be said of $G[p^n]$. Elementary consequences of this type include Corollaries 2.2, 2.3 and 2.4. In the opposite direction, if we start with a valuated p^n-socle V (or indeed, any valuated group), then using a standard construction from [24], V can be embedded as a nice subgroup in a group $H(V)$ such that the valuation on V agrees with the height function on $H(V)$ and $H(V)/V$ is totally projective. We call such an embedding an *n-cover* of V. (Actually, this construction can be viewed as a type of "left adjoint" to the forgetful functor $G \mapsto G[p^n]$ from the category of groups to the category of valuated p^n-socles.)

In [2], the concept of an *n-balanced exact* sequence of valuated p^n-socles was defined and it was observed that an ω_1-bounded valuated p^n-socle V is

n-summable iff V is n-balanced projective iff $H(V)$ is a dsc group iff $H(V)$ is balanced projective. We generalize this result in two ways. First, we verify that if $\alpha \leq \omega_1$, then V is C_α n-summable iff $H(V)$ is a C_α group (Theorem 2.11). We also show that the n-balanced projective dimension of V in the category of ω_1-bounded valuated p^n-socles will always agree with the balanced projective dimension of $H(V)$ in the category of groups (Theorem 2.14).

Kurepa's Hypothesis (or KH) is the assertion that there is a family $\mathcal{F}$ of subsets of ω_1 such that $|\mathcal{F}| > \aleph_1$ whereas for every $\beta < \omega_1$, the collection $\{X \cap \beta : X \in \mathcal{F}\}$ is countable. It is known that KH holds in the constructible universe, but is independent of ZFC (see [11]). In [15], it was shown that KH (or more specifically, $\neg$ KH) is equivalent to a number of interesting conditions pertaining to the torsion product and to the balanced projective dimension of C_{ω_1} groups. We conclude this paper by extending this equivalence to both the category of C_{ω_1} n-summable valuated p^n-socles and to the category of C_{ω_1} n-summable groups (Theorem 2.19). In fact, [1] also relates KH to valuated vector spaces. On the other hand, not only do our results hold for $n > 1$, but the approach in [1] is at its core a way to rephrase and simplify the arguments in [15], while this work is concerned with significantly different questions.

1. Valuated p^n-socles

If V is a valuated p^n-socle, a subgroup W of V is said to be *n-isotype* if, under the valuation on W induced from V, W is also a valuated p^n-socle. In addition, W is said to be *α-high* if it is maximal with respect to the property $W \cap V(\alpha) = \{0\}$. We review a few facts from [2].

1.1. If W is α-high in V, then it is n-isotype ([2], Corollary 1.4).

An ordinal α is said to be an *n-limit* if it is of the form $\lambda + k$, where λ is an infinite limit ordinal and $0 \leq k < n - 1$; otherwise α is *n-isolated.*

1.2. If V is a valuated p^n-socle and α is n-isolated, then V has a subgroup X such that for all α-high subgroups Y there is a valuated decomposition $V = Y \oplus X$, called a *standard α-decomposition* of V; in addition, if $\alpha = \beta + n - 1$, then $X \subseteq V(\beta)$ ([2], Lemmas 1.8 and 1.9).

Again, a valuated p^n-socle is said to be *n-summable* iff it is the valuated direct sum of countable valuated groups.

1.3. If V is n-summable and W is a valuated summand of V, then W is also n-summable ([2], Proposition 1.1).

A subgroup W of a valuated group V is *nice* if every coset of V/W contains an element of maximal value, and *n-balanced* iff it is both n-isotype and nice. The next statement is ([2], (1.A)).

1.4. If W is n-balanced in V and $|x+W|_{V/W} \stackrel{\text{def}}{=} \max\{|x+w|_V : w \in W\}$, then V/W is a valuated p^n-socle.

The next result is critical; it states that the n-summable valuated p^n-socles are projective with respect to the class of n-balanced exact sequences.

1.5. If W is n-balanced in V and V/W is n-summable, then W is a valuated summand of V ([2], Lemma 1.11).

1.6. Suppose V is a valuated p^n-socle, W is n-isotype in V and V/W is countable. If W is n-summable, then V is n-summable ([2], Theorem 2.4).

We now review some facts from [12].

1.7. If V is a valuated p^n-socle, $\beta = \lambda + k$ is an n-limit with λ a limit ordinal, $0 \le k < n-1$, $f_V(\beta) \ne 0$ and $\delta < \lambda$. Then there is an n-isolated ordinal α with $\delta < \alpha < \lambda$ and $f_V(\alpha) \ne 0$ ([12], Lemma 1.1).

A countable valuated p^n-socle V is called an *n,ω-limit* if there is an n-limit ordinal $\beta = \lambda + k$, where λ is a limit ordinal and $0 \le k < n-1$, and a strictly increasing sequence of n-isolated ordinals $\{\gamma_i\}_{i<\omega}$, with limit λ, such that f_V is the characteristic function of $\{\gamma_i\}_{i<\omega} \cup \{\beta\}$.

1.8. If V is an n-summable valuated p^n-socle, then V is isometric to a valuated direct sum $\bigoplus_{i\in I} V_i$, where each V_i is either cyclic or an n,ω-limit ([12], Corollary 1.6). In particular, if the length of V is a limit ordinal λ, then it is a valuated direct sum of groups whose lengths are strictly less than λ.

1.9. Suppose $\alpha = \lambda + k$ is an ordinal, where λ is a limit and $0 \le k < \omega$. Then α is the length of some n-summable valuated p^n-socle V iff $0 < k < n$ implies that λ has countable finality ([12], Corollary 1.7).

We begin with a simple fact about valuated homomorphisms.

LEMMA 1.10. *Suppose V and W are valuated p^n-socles and $f : V \to W$ is a valuated homomorphism. Then f is an isometry iff it restricts to an isometry $V[p] \to W[p]$.*

Proof. Clearly, if f is an isometry on V, then it is an isometry on $V[p]$. Conversely, suppose f is an isometry on $V[p]$. It easily follows that f must be injective. Next, by the definition of a valuated p^n-socle, for $j < n$, $(p^j V)[p] = V(j)[p]$, and clearly $p^n V = 0$. Since similar statements hold for W, we can conclude that $f(V)$ must be pure in W, so that it is, algebraically, a summand. Since $W[p] \subseteq f(V)$, it follows that f is, in fact, bijective.

We now show by induction on the orders of elements that for every $x \in V$, $|f(x)|_W = |x|_V$. Our hypothesis guarantees that this holds for elements of order p. So suppose it holds for all elements of order less than p^k, x has order p^k, $\beta = |x|_V$ and $y = f(x)$. If $|px|_V = \beta + 1$, then by induction, $\beta + 1 = |px|_V = |py|_W \ge |y|_W + 1 \ge |x|_V + 1 = \beta + 1$. Therefore, $|y|_W = \beta$, as required.

Suppose, then, that $|px|_V > \beta+1$. Find $x' \in V(\beta+1)$ such that $px = px'$. If $y' = f(x')$, then it follows that $|y'|_W > \beta$. Since $x - x' \in V[p]$, we know that $|y-y'|_W = |x-x'|_V = \beta$. And it follows that $|y|_W = |(y-y')+y'|_W = \beta$, completing the proof. □

The next result, which parallels [5], Lemma 64.2, contains within it a definition that will be important.

LEMMA 1.11. *If V and W are valuated p^n-socles, then $V \bigtriangledown W$ is also a valuated p^n-socle, where for every ordinal α, we set $(V \bigtriangledown W)(\alpha) = V(\alpha) \bigtriangledown W(\alpha) \subseteq V \bigtriangledown W$.*

Proof. An element of $(V(\alpha) \bigtriangledown W(\alpha))[p^{n-1}]$ is represented by the sum of a collection of generators of the form (v, p^j, w), where $j \leq n-1$, $v \in V(\alpha)$, $w \in W(\alpha)$ and $p^j v = 0 = p^j w$. So if $\beta < \alpha$, then there are elements $v' \in V(\beta)$ and $w' \in W(\beta)$ such that $pv' = v$ and $pw' = w$. Consequently, (v', p^{j+1}, w') is a generator of $V(\beta) \bigtriangledown W(\beta)$ and $p(v', p^{j+1}, w') = (v, p^j, w)$, giving the result. □

If m is a positive integer, we will say a group is $\mathbb{Z}_{p^m}$*-projective* if it is a projective $\mathbb{Z}_{p^m}$-module, that is, iff it is a direct sum of copies of $\mathbb{Z}_{p^m}$. It is a well-known fact that any $\mathbb{Z}_{p^m}$-projective will also be an injective $\mathbb{Z}_{p^m}$-module, that is, it is algebraically a summand of any $\mathbb{Z}_{p^m}$-module which contains it. In particular, if V is any valuated p^n-socle, W is $n-1$-high in V and $V = W \oplus V'$ is a standard $n-1$-decomposition, then V' will be $\mathbb{Z}_{p^n}$-projective. This means that we will on occasion be able to simplify our proofs by assuming that some valuated p^n-socle is $\mathbb{Z}_{p^n}$-projective as a group. The next observation will provide us with a useful mechanism for constructing n-balanced exact sequences.

LEMMA 1.12. *If α is an ordinal, V and W are valuated p^n-socles, Y is an α-high subgroup of W, κ is the rank of W/Y and V is $\alpha+1$-bounded, then there is an n-balanced exact sequence*

$$0 \to V \bigtriangledown Y \to V \bigtriangledown W \to \bigoplus_{\kappa} V \to 0.$$

Proof. If $\alpha < n-1$, then it is easy to check that this is actually a split exact sequence of $\alpha+1$-bounded groups with the height function as the valuation, so the result is trivial. Assume, therefore, that $\alpha \geq n-1$. If $X = W/Y$, then X is $\mathbb{Z}_{p^n}$-projective, and it follows that $V \bigtriangledown X$ is algebraically isomorphic to $\bigoplus_{\kappa} V$.

If β is an ordinal, we need to show that

$$0 \to (V \bigtriangledown Y)(\beta) \to (V \bigtriangledown W)(\beta) \to \bigoplus_{\kappa} V(\beta) \to 0,$$

is exact. If $\beta \geq \alpha+1$, then all these groups are $\{0\}$, so we may assume $\beta \leq \alpha$.

Suppose next that $\beta+n-1\le\alpha$. If $V=Y'\oplus X'$ is a standard $\beta+n-1$-decomposition of W with $Y'\subseteq Y$, then $X'\subseteq W(\beta)$. It follows that $V=Y'+X'\subseteq Y+W(\beta)\subseteq V$. Therefore, $0\to Y(\beta)\to W(\beta)\to X\to 0$ is exact; and since X is a projective $\mathbb{Z}_{p^n}$-module, algebraically, it splits. This gives another exact sequence

$$0\to V(\beta)\bigtriangledown Y(\beta)\to V(\beta)\bigtriangledown W(\beta)\to V(\beta)\bigtriangledown X\to 0,$$

where $V(\beta)\bigtriangledown X\cong\bigoplus_\kappa V(\beta)$.

Suppose next that $\beta+k=\alpha$, where $k<n-1$. Note that $Y(\beta)$ is a p^k-high subgroup of $W(\beta)$, so there is a decomposition $W(\beta)=Y(\beta)\oplus Z$, where Z maps to an essential subgroup of X. This determines a split exact sequence

$$0\to V(\beta)\bigtriangledown Y(\beta)\to V(\beta)\bigtriangledown W(\beta)\to V(\beta)\bigtriangledown Z\to 0.$$

Note that Z will algebraically be a direct sum of κ terms of the form $\mathbb{Z}_{p^j}$, where $k+1\le j\le n$. On the other hand, since $p^{k+1}V(\beta)\subseteq V(\alpha+1)=\{0\}$, it follows that $V(\beta)$ will be isomorphic to a direct sum of terms of the form $\mathbb{Z}_{p^\ell}$, where $0\le\ell\le k+1$. It follows that $V(\beta)\bigtriangledown Z$ is isomorphic to $\bigoplus_\kappa V(\beta)$, completing the proof. □

Corollary 1.13. *Suppose α is an ordinal, V and W are valuated p^n-socles, $f_V(\beta)=0$ for all $\beta>\alpha$ and $f_W(\beta)=0$ for all $\beta<\alpha$. If W has rank κ, then $V\bigtriangledown W$ is isometric to the valuated direct sum $\bigoplus_\kappa V$.*

Proof. In this case, in Lemma 1.12 we have $Y=\{0\}$, so that the sequence reduces to the indicated isometry. □

We pause for a technical observation regarding nice subgroups of n,ω-limit groups.

Lemma 1.14. *If C is a valuated p^n-socle that is an n,ω-limit of length $\lambda+k$, where λ is a limit ordinal and $0<k<n$, then $N=\{x\in C: px\in C(\lambda)\}\subseteq C[p^{k+1}]$ is a nice subgroup of C containing $C[p]$.*

Proof. It can be verified that if $\alpha<\lambda$, then $C/C(\alpha)$ is finite, and that this implies that λ is the only limit point of $\{|x|_C : x\in C-\{0\}\}$. So if $y\in C$ and $\{y+x_m\}_{m<\omega}$ is a collection of nonzero elements of the coset $y+N$ with $|y+x_m|_C<|y+x_{m+1}|_C$ for all $m<\omega$, then we can conclude that these values converge to λ. However, since $\phi: C\to C/C(\lambda)$ given by $\phi(x)=px+C(\lambda)$ is a valuated homomorphism with kernel N, we must have $\phi(y)\in(C/C(\lambda))(\lambda)=\{0\}$, so that $y\in N$. In this case, $0\in y+N$ is obviously proper. □

This brings us to one of the main steps in our inquiry.

Theorem 1.15. *Suppose V and W are valuated p^n-socles, α is the length of V and $W(\alpha)\ne\{0\}$. If $V\bigtriangledown W$ is n-summable, then V is n-summable.*

Proof. We may clearly assume α is infinite. Suppose first that α is n-isolated. There is an n-isolated ordinal $\beta \geq \alpha$ such that $f_W(\beta) \neq 0$. [In fact, if we choose β to be the smallest ordinal such that $\beta \geq \alpha$ and $f_W(\beta) \neq 0$, then 1.7 implies that β is n-isolated.] If Y is β-high in W and $W = Y \oplus U$ is a standard β-decomposition, then algebraically, $U \cong \bigoplus_\kappa \mathbb{Z}_{p^n}$, where $\kappa \neq 0$. By Corollary 1.13, $V \bigtriangledown U$ is isometric to $\bigoplus_\kappa V$. It follows that V is isometric to a summand of $V \bigtriangledown W$, so that it is n-summable by 1.3.

We may therefore assume that α is an n-limit; let $\alpha = \lambda + k$, where λ is a limit ordinal and $k < n - 1$. Let Y be α-high in W, $\kappa > 0$ be the rank of $X \overset{\text{def}}{=} W/Y$ and $\pi : W \to X$ be the canonical epimorphism. By Lemma 1.12, there is an n-balanced exact sequence

$$0 \to V \bigtriangledown Y \overset{\mu}{\to} V \bigtriangledown W \to \bigoplus_\kappa V \to 0,$$

where we interpret μ as an inclusion. We claim that the above sequence must split (in the category of valuated p^n-socles). Once we have established this, it follows that V will be a valuated summand of $V \bigtriangledown W$, so that it is n-summable. So we need to construct a valuated homomorphism $\eta : V \bigtriangledown W \to V \bigtriangledown Y$ such that $\eta \circ \mu = 1_{V \bigtriangledown Y}$.

As valuated p^{k+1}-socles, $Y[p^{k+1}]$ is $\alpha = \lambda + (k+1) - 1$-high in $W[p^{k+1}]$. It follows that there is a standard α-decomposition $W[p^{k+1}] = Y[p^{k+1}] \oplus Z$, where $Z \subseteq W[p^{k+1}](\lambda)$. Let $f : W[p^{k+1}] \to Y[p^{k+1}]$ be the corresponding valuated projection and $g \overset{\text{def}}{=} 1_{V[p^{k+1}]} \bigtriangledown f : (V \bigtriangledown W)[p^{k+1}] \to (V \bigtriangledown Y)[p^{k+1}]$; so g is valuated, as well. In particular, g restricts to the identity on $(V \bigtriangledown Y)[p^{k+1}]$.

If $\beta < \lambda$, then the decomposition $W[p^{k+1}] = Y[p^{k+1}] \oplus Z$ extends to an algebraic decomposition $W = Y \oplus Z_\beta$, where $Z \subseteq Z_\beta \subseteq W(\beta)$. If $f_\beta : W \to Y$ is the corresponding algebraic projection, then f_β restricts to f on $W[p^{k+1}]$. In addition, if $z \in W$, then $z = f_\beta(z) + u$, where $u \in W(\beta)$. This implies that for all $\gamma \leq \beta$, we have $f_\beta(W(\gamma)) \subseteq Y \cap W(\gamma) = Y(\gamma)$. If $\beta < \lambda$, let $g_\beta = 1_V \bigtriangledown f_\beta$; so for all $\gamma \leq \beta$,

$$g_\beta\big((V \bigtriangledown W)(\gamma)\big) \subseteq (V \bigtriangledown Y)(\gamma). \tag{$*$}$$

Clearly, g_β restricts to g on $(V \bigtriangledown W)[p^{k+1}]$.

By 1.8, $V \bigtriangledown W$ is the valuated direct sum $\bigoplus_{i \in I} C_i$, where each C_i is either cyclic or an n, ω-limit. Since V has length α, so does $V \bigtriangledown W$, and hence each C_i has length at most α. For each $i \in I$, we define a valuated homomorphism $\tau_i : C_i \to V \bigtriangledown Y$ as follows:

Case 1. C_i has length $\beta < \lambda$: Let τ_i agree with $g_\beta = 1_V \bigtriangledown f_\beta$ on C_i. Since $C_i(\beta) = \{0\}$, by $(*)$ we can infer that τ_i is valuated on C_i (even though g_β is not necessarily valuated on all of $V \bigtriangledown W$).

Case 2. C_i has length β with $\lambda \leq \beta \leq \lambda + k$: Note that C_i will have to be an n, ω-limit, so that, in fact, $\lambda < \beta$. Let N be defined as in Lemma 1.14; so $C_i[p] \subseteq N_i \subseteq C[p^{k+1}]$. Therefore, g is defined and valuated on N_i. And since

N_i is nice in C_i and C_i/N_i is countable, it follows that g restricted to N_i can be extended to a valuated homomorphism $\tau_i : C_i \to V \bigtriangledown Y$ (the justification of this assertion mirrors the corresponding one for groups, for example, [5], Corollary 81.4).

Let $\tau : V \bigtriangledown W \to V \bigtriangledown Y$ be the valuated homomorphism which restricts to τ_i on each summand C_i. We next verify that τ is the identity when restricted to $(V \bigtriangledown Y)[p] \subseteq V \bigtriangledown W$: All of the homomorphisms g_β, for $\beta < \lambda$, agree with g on $(V \bigtriangledown W)[p]$, which is the identity on $(V \bigtriangledown Y)[p]$. Therefore, on each $C_i[p]$, τ restricts to g; and it follows that on all of $(V \bigtriangledown W)[p]$, τ agrees with g, giving the statement.

So $\nu \stackrel{\text{def}}{=} \tau \circ \mu : V \bigtriangledown Y \to V \bigtriangledown Y$ is a valuated homomorphism that is the identity on $(V \bigtriangledown Y)[p]$. It follows from Lemma 1.10 that ν must be an isometry. If $\eta = \nu^{-1} \circ \tau : V \bigtriangledown W \to V \bigtriangledown Y$, then $\eta \circ \mu = \nu^{-1} \circ \tau \circ \mu = \nu^{-1} \circ \nu = 1_{V \bigtriangledown Y}$. Therefore, $V \bigtriangledown Y$ is a valuated summand of $V \bigtriangledown W$, establishing the result. □

If λ is a limit ordinal and V is a valuated p^n-socle, then the *λ-topology* on V uses $\{V(\beta)\}_{\beta<\lambda}$ as a neighborhood base of 0. If W is n-isotype in V, then the λ-topology on V induces the λ-topology on W; furthermore, W will be n-balanced in V iff for every limit ordinal λ, $W/W(\lambda)$ embeds as a closed subgroup of $V/V(\lambda)$ in the λ-topology. It is a slight variation on a standard result that if λ has uncountable cofinality and V is a valuated direct sum $\bigoplus_{i\in I} V_i$, where each V_i has length strictly less than λ, then V is complete in the λ-topology. (See, for example, the proof of [5], Theorem 84.3.)

The next result generalizes ([2], Corollary 1.10) to the case of n-limit ordinals.

THEOREM 1.16. *Suppose V is a valuated p^n-socle and α is an ordinal. If one α-high subgroup of V is n-summable, then all α-high subgroups of V are n-summable.*

Proof. We may assume α is an n-limit, so $\alpha = \lambda + k$, where λ is a limit and $k < n-1$. Let Y be α-high in V.

Suppose first that λ has uncountable cofinality and Y is n-summable. By 1.9, we can conclude that $Y(\lambda) = \{0\}$. Let Z be a $\lambda + n - 1$-high subgroup of V containing Y, so that Y is dense in Z in the λ-topology. Since Y is complete in the λ-topology, we can conclude that $Z = Y + Z(\lambda)$. However, since Z/Y will be $\mathbb{Z}_{p^n}$-projective and $p^{n-1}Z(\lambda) = \{0\}$, it follows that $Z = Y$ is n-summable and $f_V(\lambda + j) = 0$ for $0 \le j < n-1$. Therefore, any subgroup that is α-high will also be $\lambda + n - 1$-high, and hence n-summable.

Suppose next that λ has countable cofinality. By 1.9 there is a countable valuated p^n-socle W of length $\alpha + 1$. If we apply Lemma 1.12 and 1.5, we can infer that $V \bigtriangledown W$ is isometric to $(Y \bigtriangledown W) \oplus (\bigoplus W)$.

If Y is n-summable, then so is $Y \bigtriangledown W$, and hence, so is $V \bigtriangledown W$. On the other hand, if $V \bigtriangledown W$ is n-summable, then so is $Y \bigtriangledown W$. Utilizing Theorem 1.15, this implies that Y is n-summable. Since the summability of $V \bigtriangledown W$ is independent of which Y is chosen, the result follows. □

COROLLARY 1.17. *If V is an n-summable valued p^n-socle and α is an ordinal, then any α-high subgroup of V is n-summable.*

Proof. Applying Theorem 1.16, we need only find one α-high subgroup which is n-summable. Suppose V is isometric to $\bigoplus_{i\in I} C_i$, where each C_i is countable. For each $i \in I$, let Y_i be α-high in C_i. Then clearly $Y = \bigoplus_{i\in I} Y_i$ will be n-summable and α-high in V. □

Recall that a valued p^n-socle V is C_α *n-summable* if for every $\beta < \alpha$, one, and hence every, β-high subgroup of V is n-summable. Clearly, if V is C_α n-summable then it is C_β n-summable for all $\beta < \alpha$, and if α is a limit ordinal, then this necessary condition is also sufficient. If α is isolated, then by Corollary 1.17, V is C_α n-summable iff it has an $\alpha - 1$-high subgroup that is n-summable. We note in passing the following fact.

PROPOSITION 1.18. *Suppose V is a valued p^n-socle, α is an ordinal and $V(\alpha)$ is countable. Then V is n-summable iff it is $C_{\alpha+1}$ n-summable.*

Proof. Certainly, if V is n-summable, then it is $C_{\alpha+1}$ n-summable. Conversely, if V is $C_{\alpha+1}$ n-summable and W is α-high in V, then W is n-summable. Since $V(\alpha)$ maps to an essential subgroup of V/W, this quotient is countable. By 1.6, we can conclude that V is n-summable, as required. □

This brings us to the main result of this section, which builds upon Theorem 1.15. It parallels [16], Theorem 1, which is a reformulation and extension of a result from [23].

THEOREM 1.19. *Suppose V and W are valued p^n-socles, V has length α and $W(\alpha) \neq \{0\}$. Then $V \bigtriangledown W$ is n-summable iff V is n-summable and W is C_α n-summable.*

Proof. Note that in either direction, by Theorem 1.15, we can infer that V is n-summable. With that assumption, we induct on α to show $V \bigtriangledown W$ is n-summable iff W is C_α n-summable.

First, if α is a limit, then employing 1.8, V will be isometric to a direct sum $\bigoplus_{\beta<\alpha} V_\beta$, where $V_\beta(\beta) = \{0\}$ for each β. So by induction, $V \bigtriangledown W$ is n-summable iff each $V_\beta \bigtriangledown W$ is n-summable iff W is C_β n-summable for each β iff W is C_α n-summable.

Assume, then, that $\alpha = \gamma + 1$ is isolated. By Lemma 1.12, if Y is γ-high in W and κ is the rank of W/Y, then there is an n-balanced exact sequence

$$0 \to V \bigtriangledown Y \to V \bigtriangledown W \to \bigoplus_\kappa V \to 0.$$

Since V is n-summable, this sequence splits. Therefore, $V \bigtriangledown W$ is n-summable iff $V \bigtriangledown Y$ is n-summable. And by Theorem 1.15, this is true iff Y is also n-summable, that is, W is C_α n-summable. □

The next result parallels [16], Theorem 2.

COROLLARY 1.20. *Suppose W is a valuated p^n-socle and α is an ordinal that is not of the form $\lambda+k$, where λ is a limit ordinal of uncountable cofinality and $0<k<n$. Then the following are equivalent*:

(a) *W is C_α n-summable*;
(b) *For every α-bounded n-summable valuated p^n-socle V, $V \bigtriangledown W$ is n-summable*;
(c) *For some n-summable valuated p^n-socle V of length α, $V \bigtriangledown W$ is n-summable.*

Proof. Note that 1.9 says that there is, in fact, an n-summable valuated p^n-socle of length α. Clearly, (b) implies (c). Next, suppose C is some countable valuated p^n-socle with $C(\alpha) \neq \{0\}$. Then W will be C_α n-summable iff $W \oplus C$ has this property, and if V is n-summable, then $V \bigtriangledown W$ is n-summable iff $V \bigtriangledown (W \oplus C)$ is n-summable. Replacing W by $W \oplus C$, then, we may assume that $W(\alpha) \neq \{0\}$. However, in this case, (a) implies (b) and (c) implies (a) follow directly from Theorem 1.19. □

We now aim to provide a way to produce examples of C_α n-summable valuated p^n-socles of length α, at least for ordinals of countable cofinality, that parallels the usual way of constructing separable groups by locating them between a basic subgroup and its torsion completion. We first note that this is trivial for some ordinals.

PROPOSITION 1.21. *Suppose $\alpha = \lambda + k$, where λ is a limit ordinal, $k \geq n$ and V is an α-bounded valuated p^n-socle. Then V is C_α n-summable iff it is n-summable.*

Proof. Certainly, if V is n-summable, then it is C_α n-summable. Conversely, if it is C_α n-summable, then let $V = B \oplus X$ be a standard $\alpha-1$-decomposition. Since V is C_α n-summable, B will be n-summable, and since $V(\alpha) = \{0\}$, X will also be n-summable, giving the result. □

If α is an ordinal and V is a valuated p^n-socle, then a subgroup W of V will be said to be α, n-*dense* if

(1.A) W is n-isotype in V, that is, it is a valuated p^n-socle;
(1.B) For all $\beta < \alpha$, $V[p] = W[p] + V(\beta)[p]$.

It is easy to verify that the property of being α, n-dense is transitive, and that an α-high subgroup will always be α, n-dense.

By an α, n-*basic subgroup* of V, we will mean an n-summable, α, n-dense subgroup B of V. If B' is α-high in B, then by Corollary 1.17, B' will also be n-summable. Therefore, if we wish, me may assume that an α, n-basic subgroup is α-bounded.

Proposition 1.22. *Suppose $\alpha = \lambda + k$ is an ordinal, λ is a limit ordinal of countable cofinality, $k < \omega$ and V is a valuated p^n-socle. Then V has an α, n-basic subgroup iff it is C_α n-summable.*

Proof. Suppose first that V has an α, n-basic subgroup B. If $\beta < \alpha$ and Y is β-high in B, then by Corollary 1.17, Y is n-summable, and by (1.B), Y is also β-high in V. Therefore, V is C_α n-summable.

Conversely, suppose V is C_α n-summable. First, if α is isolated, then let B be any $\alpha - 1$-high subgroup of V. Since V is C_α n-summable, B will be n-summable. And since $V[p]$ will be the valuated direct sum of $B[p]$ and $V(\alpha - 1)[p]$, (1.B) will follow, as well.

Consider next the case where $\alpha = \lambda$ is a limit. Let $\{\alpha_j\}_{j<\omega}$ be a strictly increasing sequence of n-isolated ordinals whose limit is α. Construct an ascending sequence of α_j-high subgroups W_j of V. It follows that each W_j is a valuated summand of V, as well as being n-summable. Let $B_0 = W_0$, and for $0 < j < \omega$, let W_j be the valuated direct sum $B_j \oplus W_{j-1}$. Clearly, $B \stackrel{\text{def}}{=} \bigoplus_{j<\omega} B_j$ will be n-summable and n-isotype in V. In addition, if $\beta < \alpha$, then for some $j < \omega$, $\beta < \alpha_j$. Consequently, $V[p] = W_{\alpha_j}[p] + V(\alpha_j)[p] \subseteq B[p] + V(\beta)[p]$, showing that (1.B) holds, and completing the proof. $\square$

Suppose $\alpha = \lambda + k$, where λ is a limit ordinal of countable cofinality, $n = k + m$, $0 < m$ and V is an α-bounded valuated p^n-socle. Let $L_\lambda V$ be the completion of V in the λ-topology, so that $L_\lambda V$ is the inverse limit of $V/V(\beta)$ over all $\beta < \lambda$. There is clearly a homomorphism $\nu : V \to L_\lambda V$ whose kernel is $V(\lambda)$. Let $N_\alpha V = \nu(V) + (L_\lambda V)[p^m] \subseteq L_\lambda V$ and $M_\alpha V = N_\alpha V/\nu(V)$. We pause for the following observation.

Lemma 1.23. *With the above notation, $M_\alpha V$ is $\mathbb{Z}_{p^m}$-projective, and there is a natural commutative diagram with algebraically splitting rows*:

$$\begin{array}{ccccccccc}
0 & \to & \nu(V)[p^m] & \to & (L_\lambda V)[p^m] & \to & M_\alpha V & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & \nu(V) & \to & N_\alpha V & \to & M_\alpha V & \to & 0
\end{array}$$

Proof. Splitting off a bounded summand, we may clearly assume that V is $\mathbb{Z}_{p^n}$-projective. Let $\{\alpha_j\}_{j<\omega}$, be a strictly ascending sequence of n-isolated ordinals with limit λ and $\{W_j\}_{j<\omega}$, be an ascending chain of α_j-high subgroups of V. If $B_0 = W_0$ and $W_{j+1} = W_j \oplus B_j$, then it is easily checked that $L_\lambda V$ can be identified with $\prod_{j<\omega} B_j$, so that it, too, is $\mathbb{Z}_{p^n}$-projective.

Note that $V(\lambda) \cong \bigoplus_{0<j\leq k} X_j$, where X_j is $\mathbb{Z}_{p^j}$-projective. It follows that $\nu(V) \cong V/V(\lambda) \cong \bigoplus_{m\leq \ell\leq n} Y_\ell$, where again, Y_ℓ is $\mathbb{Z}_{p^\ell}$-projective; so $\nu(V)[p^m]$ is $\mathbb{Z}_{p^m}$-projective.

The existence of the commutative diagram follows from $(L_\lambda V)[p^m] \cap \nu(V) = \nu(V)[p^m]$. Since the upper row consists of $\mathbb{Z}_{p^m}$-modules and $\nu(V)[p^m]$ is $\mathbb{Z}_{p^m}$-projectively, it must split. Therefore, $M_\alpha V$ is also $\mathbb{Z}_{p^m}$-projective and the lower row splits. □

With the above notation, we will say that the α-bounded C_α n-summable valuated p^n-socle V is *α, n-torsion complete* if $\nu(V) = N_\alpha V$. Alternatively, we could require that $M_\alpha V = \{0\}$, or that $\nu(V)[p^m]$ is complete in the (induced) λ-topology. The following shows that most of the techniques utilized in the theory of separable groups can be translated in a natural way to the theory of α-bounded C_α n-summable valuated p^n-socles.

THEOREM 1.24. *Suppose λ is a limit ordinal of countable cofinality, $k < n < \omega$, $m = n - k$ and $\alpha = \lambda + k$.*

(a) *If W is an arbitrary α-bounded C_α n-summable valuated p^n-socle, and B is α, n-basic in W, then $L_\lambda B$ can be identified with $L_\lambda W$ so that $\nu(W)$ is identified with a summand of $N_\alpha B$ containing $\nu(B)$.*
(b) *Suppose W' is another α-bounded C_α n-summable valuated p^n-socle with B as an α, n-basic subgroup and corresponding homomorphism $\nu' : W' \to L_\lambda B$. If $\nu'(W') = \nu(W)$, then W and W' are isometric over B.*
(c) *If B is an α-bounded n-summable valuated p^n-socle, and X is a summand of $N_\alpha B$ containing $\nu(B)$, then there is an α-bounded C_α n-summable valuated p^n-socle W containing B as an α, n-basic subgroup for which $\nu(W) = X$.*
(d) *If W is an arbitrary α-bounded C_α n-summable valuated p^n-socle, then W is an α, n-dense subgroup of an α, n-torsion complete valuated p^n-socle V.*
(e) *If V_0 and V_1 are α, n-torsion-complete valuated p^n-socles, then V_0 and V_1 are isometric iff they have the same Ulm function.*

Proof. After discarding an n-bounded summand, there is clearly no loss of generality in assuming that B, W and W' are $\mathbb{Z}_{p^n}$-projective groups.

Starting with (a), since for every $\beta < \lambda$ there is a natural isomorphism $B/B(\beta) \cong W/W(\beta)$, it follows that $L_\lambda B$ and $L_\lambda W$ are naturally isomorphic. There is an algebraic decomposition $W = B \oplus U$ where $W(\lambda) = B(\lambda) \oplus p^m U$. It follows that $\nu(W) = \nu(B) + \nu(U) \subseteq \nu(B) + (L_\lambda B)[p^m] = N_\alpha B$ and $\nu(W)/\nu(B) \cong U/p^m U$ is $\mathbb{Z}_{p^m}$-projective. By Lemma 1.23, $M_\alpha B$ is $\mathbb{Z}_{p^m}$-projective, so that $\nu(W)/\nu(B)$ is a summand of $M_\alpha B$. Since $\nu(B)$ is a summand of $N_\alpha B$, $\nu(W)$ will be a summand of $N_\alpha B$, which establishes (a).

Turning to (b), there are algebraic decompositions $W = B \oplus U$ and $W' = B \oplus U'$, where U and U' are $\mathbb{Z}_{p^n}$-projective, $W(\lambda) = B(\lambda) \oplus p^m U$ and $W'(\lambda) = B(\lambda) \oplus p^m U'$. If $\nu' : W' \to L_\lambda B$ is the natural homomorphism, then our hypotheses guarantee that $\nu(W) = \nu'(W')$. Since U is a projective $\mathbb{Z}_{p^n}$-module, there is an algebraic isomorphism $\phi : W = B \oplus U \cong B \oplus U' = W'$ which is the identity on B such that $\nu = \nu' \circ \phi$. This latter condition implies that ϕ preserves all values strictly less than λ (as ν and ν' have this property). Since ϕ also induces a group isomorphism $W(\lambda) = \ker \nu \cong \ker \nu' = W'(\lambda)$, and the valuations here are simply λ plus the height functions on these subgroups, it follows that ϕ is actually an isometry, establishing (b).

As to (c), there is an algebraic decomposition $X = \nu(B) \oplus X'$. Let U be a $\mathbb{Z}_{p^n}$-projective of the same rank as X', so there is a homomorphism $\gamma : U \to X' \subseteq N_\alpha W$ with kernel $p^m U$. We then algebraically set $W = B \oplus U$; we still need to define a valuation on W. Mimicking the above, if $b \in B$, $u \in U$, let

$$\big|(b,u)\big|_W = \begin{cases} |\nu(b) + \gamma(u)|_{L_\lambda B}, & \text{if } b + u \notin B(\lambda) \oplus p^m U, \\ \lambda + |(b,u)|_{B(\lambda)\oplus p^m U}, & \text{otherwise.} \end{cases}$$

A straightforward (and somewhat tedious) verification shows that this makes W into a valuated p^n-socle with the required properties.

Next, for (d), suppose W corresponds to the summand $X \subseteq N_\alpha B$. It follows that there is an algebraic decomposition $N_\alpha B = X \oplus X'$, and we again let U be $\mathbb{Z}_{p^n}$-projective of the same rank as X'. It follows that there is a homomorphism $\gamma : U \to X'$ with kernel $p^m U$. If we set $V = W \oplus U$ and define a valuation on V as in (c), then it follows that V is α, n-torsion-complete and that it contains W as an α, n-dense subgroup.

Finally, as to (e), the equality of their Ulm functions guarantees that V_0 and V_1 have isometric α, n-basic subgroups. If we identify these, then by (b) they are isometric over this subgroup. □

As mentioned above, if $\alpha = \lambda + k$ where $k < \omega$ and λ is a limit ordinal of countable cofinality, this gives a technique for describing all α-bounded C_α n-summable valuated p^n-socles that generalizes the usual way of constructing separable groups. If $k \geq n$, then by Proposition 1.21 these will all be n-summable. On the other hand, if $k < n$, then we may start with any function f from α to the cardinals that is n-summable, in the sense of [12]. This determines a unique n-summable valuated p^n-socle B. The collection of α-bounded C_α n-summable valuated p^n-socles that contain B as an α, n-basic subgroup are then in one-to-one correspondence with the algebraic summands of $N_\lambda B$ containing $\nu(B)$. The interested reader can verify that the rank of $M_\lambda B$ is given by $\kappa = \inf_{\beta<\lambda} r(B(\beta))^{\aleph_0}$ and the number of such summands is given by 2^κ.

We now consider the case of limit ordinals of uncountable cofinality.

PROPOSITION 1.25. *Let V be a valuated p^n-socle and λ be a limit ordinal of uncountable cofinality. Then the following are equivalent*:

(a) *V is $C_{\lambda+1}$ n-summable*;
(b) *V has a λ, n-basic subgroup*;

and in this case, V is the valuated direct sum, $B \oplus V(\lambda)$, where B is n-summable and λ-high in V (so that $f_V(\lambda+j)=0$ for $0 \le j < n-1$).

Proof. If V is $C_{\lambda+1}$ n-summable, and we let B be λ-high in V, then B is clearly λ, n-basic in V; therefore, (a) implies (b).

Suppose now that B is a λ, n-basic subgroup of V. We may assume $B(\lambda) = \{0\}$, so that B is complete in the λ-topology. As was observed in the proof of Theorem 1.16, this implies that B is $\lambda+n-1$-high in V; in particular, (a) must hold as well. If $V = B \oplus X$ is a standard $\lambda+n-1$-decomposition of V, then $X \subseteq V(\lambda) \subseteq X$, as required. □

Corollary 1.26. *Suppose V is a valuated p^n-socle and λ is a limit ordinal of uncountable cofinality. Then V is $C_{\lambda+1}$ n-summable iff it is $C_{\lambda+\omega}$ n-summable.*

Proof. Suppose V is $C_{\lambda+1}$ n-summable and consider the valuated decomposition $V = B \oplus V(\lambda)$, as above. If B' is a $\lambda+\omega, n$-basic subgroup of $V(\lambda)$, then $B \oplus B'$ will be $\lambda+\omega, n$-basic in V. Therefore, by Proposition 1.22, V will be $C_{\lambda+\omega}$ n-summable. The converse is trivial. □

Corollary 1.27. *Suppose $\alpha = \lambda + k$, where λ is a limit ordinal of uncountable cofinality and $0 < k < \omega$. If V is a C_α n-summable valuated p^n-socle of length α, then V is n-summable.*

Proof. By Corollary 1.26, we can conclude that V is $C_{\alpha+1}$ n-summable, and hence n-summable. □

Suppose λ is a limit ordinal of uncountable cofinality. For every ordinal $\alpha < \lambda$, let $C_\alpha = \langle x_\alpha \rangle$ be a cyclic valuated p^n-socle of order p^n with $|x_\alpha|_{C_\alpha} = \alpha$. Let $W = \bigoplus_{\alpha<\lambda} C_\alpha$. If $Y = \langle y \rangle$ also has order p^n and $|y|_Y = \lambda$, then the mapping $x_\alpha \mapsto y$ determines a valuated homomorphism $f : W \to Y$. Let V be the kernel of f; it is easy to verify that V is n-isotype in W. In addition, if $\alpha < \lambda$, it is fairly easy to check that $V_\alpha = \bigoplus_{\beta<\alpha} \langle x_\beta - x_\alpha \rangle$ will be an n-summable and $\alpha+n-1$-high subgroup of V. Therefore, V is C_λ n-summable. However, if B was a λ, n-basic subgroup of V, then it would be n-summable, and hence complete in the λ-topology. Since B would be dense in V in the λ-topology, and V is clearly dense in W in the λ-topology, we could conclude that $B = V = W$. Since this is not true, we can conclude that V is C_λ n-summable, but that it does not have a λ, n-basic subgroup.

2. Abelian p-groups

We now translate the results from the last section to the category of Abelian p-groups. We say a group G is *n-summable* or C_α *n-summable* if $G[p^n]$ has the corresponding property. Since an n-summable valuated p^n-socle is summable, the following is a variation on a classical result (cf. [5], Theorem 84.3):

2.1. If G is a reduced n-summable group, then $p^{\omega_1}G=\{0\}$. If $\alpha>\omega_1$, then a reduced group G is C_α n-summable iff it is n-summable (and so $p^{\omega_1}G=\{0\}$).

[For the second statement, by Corollary 1.26, we may assume G is $C_{\omega_1+\omega}$ n-summable. If $k<\omega$ with $f_G(\omega_1+k)\neq 0$, then let H be ω_1+k+1-high in G. It follows that H is summable and $p^{\omega_1}H\neq\{0\}$, which contradicts the first sentence.] This implies that when applying these results to groups, there is little loss of generality in restricting our attention to the ω_1-bounded case.

Again, a subgroup K of G is *p^β-high* if it is maximal with respect to $K\cap p^\beta G=\{0\}$. It is easy to check that if K is p^β-high in G, then $K[p^n]$ is β-high in $G[p^n]$, and conversely, if W is β-high in $G[p^n]$, then $W=K[p^n]$ for some p^β-high subgroup K of G. It follows that G is C_α n-summable iff for every $\beta<\alpha$, G has an n-summable p^β-high subgroup.

The following is a direct consequence of Theorem 1.16, Proposition 1.18 and Theorem 1.19.

Corollary 2.2. *Suppose α is an ordinal and G, H are groups.*

(a) *If one p^α-high subgroup of G is n-summable, then all p^α-high subgroups of G are n-summable.*
(b) *If $p^\alpha G$ is countable, then G is n-summable iff it is $C_{\alpha+1}$ n-summable.*
(c) *Suppose α is the length of G and $p^\alpha H\neq\{0\}$. Then $G\bigtriangledown H$ is n-summable iff G is n-summable and H is C_α n-summable.*

If G is a group and α is an ordinal, then a subgroup H of G will be said to be *α,n-basic* in G if

(2.A) H is isotype in G;
(2.B) H is n-summable;
(2.C) For every $\beta<\alpha$ we have $G[p]=H[p]+(p^\beta G)[p]$.

If $\alpha\le\omega$, then any group has an α,n-basic subgroup, for example, a basic subgroup in the usual meaning of the term. By [7], Theorem 93, an α,n-basic subgroup H will be p^α-pure in G, and if α is infinite, G/H will be divisible.

Corollary 2.3. *Let G be a reduced group.*

(a) *If $\alpha<\omega_1$ is an ordinal, then G has an α,n-basic subgroup iff it is C_α n-summable.*
(b) *G has an ω_1,n-basic subgroup iff it is n-summable.*

Proof. Starting with (a), suppose H is an α,n-basic subgroup of G. It is easily checked that $H[p^n]$ will be α,n-basic in $G[p^n]$. Therefore, by Proposition 1.22, $G[p^n]$, and hence G, is C_α n-summable.

Conversely, suppose G is C_α n-summable, and let B be an α,n-basic subgroup of $G[p^n]$. We may clearly assume that $B(\alpha)=\{0\}$. If we choose H to be a subgroup of G containing B that is maximal with respect to $H[p]=B[p]$, then by [7], Theorem 93, H is p^α-pure in G. In particular, H will be isotype in G, so that $H[p^n]=B$, and hence H, will be n-summable. Clearly, (2.C) follows immediately from (1.B).

As to (b), suppose $G[p^n]$ has an ω_1, n-basic subgroup B. By Proposition 1.25, we can conclude that $G[p^n]$ is C_{ω_1+1} n-summable. So by 2.1, G is n-summable. The converse is trivial. □

Let H_α denote the "generalized Prüfer group of length α" (see, for instance, [7], page 59). (In fact, all we need is that H_α is some totally projective group of length α.) The following is an immediate consequence of Corollary 1.20(c).

COROLLARY 2.4. *If $\alpha \le \omega_1$ is an ordinal, then a group G is C_α n-summable iff $G \bigtriangledown H_\alpha$ is n-summable.*

The last result has the following interesting consequence, which generalizes [16], Proposition 2.

PROPOSITION 2.5. *If $\alpha \le \omega_1$ is an ordinal, then a p^α-projective C_α n-summable group is n-summable.*

Proof. If α is finite, then since a p^α-projective group must be p^α-bounded, the result easily follows. So we may assume α is infinite and G is p^α-projective. By [7], Theorem 84, there is a p^α-pure exact sequence $0\to M_\alpha \to H_\alpha \to \mathbb{Z}_{p^\infty} \to 0$, which leads to another p^α-pure exact sequence

$$0 \to G \bigtriangledown M_\alpha \to G \bigtriangledown H_\alpha \to G \to 0.$$

Since G is p^α-projective, there is a splitting $G \bigtriangledown H_\alpha \cong G \oplus (G \bigtriangledown M_\alpha)$. Since G is C_α n-summable, by Corollary 2.4, $G \bigtriangledown H_\alpha$ is n-summable. Therefore, G is also n-summable, proving the result. □

If V is a valuated p^n-socle, then an *n-simply presented cover* (or an *n-cover*, for short) of V is a group H containing V such that $|x|_V = |x|_H$ for all $x \in V$, V is nice in H and H/V is totally projective. The next result shows that, for our purposes, all n-covers are pretty much equivalent.

LEMMA 2.6. *Suppose H_i, for $i=1,2$, are n-covers of the valuated p^n-socle V and T_i is the totally projective group H_i/V. Then $H_1 \oplus T_2 \cong H_2 \oplus T_1$.*

Proof. Consider the commutative "push-out" diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & V & \to & H_1 & \to & T_1 & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & H_2 & \to & Z & \to & T_1 & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & T_2 & = & T_2 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Since by [5], Corollary 81.4, the identity map $V \to V$ extends both to a homomorphism $H_1 \to H_2$ and a homomorphism $H_2 \to H_1$, we have $H_1 \oplus T_2 \cong Z \cong H_2 \oplus T_1$. □

The standard construction from [24] of an n-cover of a valuated p^n-socle V is to define, for every $x \in V^* = V - \{0\}$, a totally projective group, T_x, such that $p^{\alpha_x} T_x = \langle g_x \rangle$, where $\circ(g_x) = \circ(x)$ and $\alpha_x = |x|_V$. We then let

$$H(V) = \left(V \oplus \left(\bigoplus_{x\in V^*} T_x\right)\right) \Big/ \big\langle (x, -g_x) : x \in V^* \big\rangle.$$

Essentially, V is constructed by adjoining a "tree" of the appropriate length to V for each non-zero $x \in V$. Let $T'_x = T_x/\langle g_x\rangle$; it follows that if V has length α, then $H(V)/V \cong \bigoplus_{x\in V^*} T'_x$ will be a totally projective group of length at most α. The following gives a slight variation on this construction.

LEMMA 2.7. *If V is a valuated p^n-socle of length $\lambda + k$, where $k \le n - 1$, then*

$$H = \left(V \oplus \left(\bigoplus_{x\in V-V(\lambda+1)} T_x\right)\right) \Big/ \big\langle (x, -g_x) : x \in V - V(\lambda + 1) \big\rangle$$

is also an n-cover for G for which H/V has length at most λ.

Proof. There is an obvious embedding $H \subseteq H(V)$ which we assume is an inclusion, and we show that H is actually a summand of $H(V)$. We first verify for every $x \in V(\lambda)$, that $|x|_V = |x|_H$. Clearly, $|x|_V = |x|_{H(V)} \ge |x|_H$. To show the reverse inequality, suppose $|x|_V = \lambda + j$, where $0 \le j < k$. Then $x = p^j x'$, where $|x'|_V = \lambda$, so that $|x|_H \ge |x'|_H + j \ge \lambda + j = |x|_V$.

Note that if $x \in V(\lambda+1) - \{0\}$, then by the last paragraph, $g_x \mapsto x$ extends to a homomorphism $T_x \to H$. If we combine these over all $x \in V(\lambda+1) - \{0\}$, we get a projection $\pi : H(V) \to H$, which shows that H is a summand of $H(V)$. This clearly implies that V is also nice in H and that H/V, which will be a summand of $H(V)/V$, is totally projective, completing the proof. □

The next two important observations are consequences of the proof of [2], Theorem 2.1.

2.8. If H is an n-cover of a valuated p^n-socle V and $p^{\omega_1} H = \{0\}$, then H is a dsc group iff V is n-summable.

In the proof of this in [2], the implication ⇒ is established by verifying the next statement, which is then applied to the left exact sequence $0 \to V \to H(V)[p^n] \to (H(V)/V)[p^n]$:

2.9. If W and Z are ω_1-bounded n-summable valuated p^n-socles, V is an n-isotype subgroup of W and the kernel of a valuated homomorphism $W \to Z$, then V is n-summable.

The next statement is [2], Corollary 2.3.

2.10. Suppose V is an n-summable valuated p^n-socle and W is n-isotype in V. If W has countable length, then W is also n-summable.

We have now come to one of our main results, which is another bridge between the last section and the realm of groups. Recall that G is said to be a C_α *group* if for every $\beta < \alpha$, one (and hence every) p^β-high subgroup K of G is a dsc group.

THEOREM 2.11. *If V is a valuated p^n-socle and $\alpha \le \omega_1$ is an ordinal, then the following are equivalent*:

(a) *V is C_α n-summable*;
(b) *Every n-cover H of V is a C_α group*;
(c) *Some n-cover H of V is a C_α group.*

Proof. Note that (b) and (c) are equivalent by Lemma 2.6. We next show that (c) implies (a), so suppose that H is an n-cover of V that is a C_α group. If $\beta < \alpha$ and W is β-high in V, then in H, $W \cap p^\beta H = \{0\}$. Therefore, there is a p^β-high subgroup K of H containing W. Since H is a C_α group, K is a dsc group, so that $K[p^n]$ is n-summable. Consequently, by 2.10, W is n-summable, so that (a) follows.

We now prove the converse by induction on α, so suppose (a) implies (b) and (c) whenever $\alpha' < \alpha$. If α is a limit ordinal, V is C_α n-summable and H is some n-cover of V, then V is $C_{\alpha'}$ n-summable for all $\alpha' < \alpha$. This implies that H is a $C_{\alpha'}$ group for all $\alpha' < \alpha$, and this gives that H is a C_α group.

Thus we may assume α is isolated and V is C_α n-summable; in particular, we must have $\alpha < \omega_1$. Let $\alpha = \lambda + k$, where λ is a limit and $0 < k < \omega$, so there is a $\beta \overset{\text{def}}{=} \lambda + k - 1 = \alpha - 1$-high subgroup W of V that is n-summable. If $k \ge n$, then let $V = W \oplus X$ be a standard β-decomposition of V. It follows that there is a valuated decomposition $X = X_1 \oplus X_2$ such that X_1 is $\alpha + n - 1$-high in X. Note that X_1, and hence $W \oplus X_1$, will also be n-summable (see, for example, [2], Corollary 1.7) and $X_2 \subseteq V(\alpha)$. Let H_1 and H_2 be n-covers of $W \oplus X_1$ and X_2, respectively. Since $W \oplus X_1$ is n-summable, 2.8 implies that H_1 is a dsc group. Since $X_2 \subseteq p^\alpha H_2$ and H_2/X_2 is a dsc group, it follows that H_2 is a C_α group. So $H_1 \oplus H_2$ is an n-cover of $(W \oplus X_1) \oplus X_2 = V$ and a C_α group, establishing (c).

Suppose next that $0 < k < n$ and again, let W be an n-summable $\beta = \alpha - 1$-high subgroup of V. Find a standard $\lambda + n - 1$-decomposition $V = V_1 \oplus X$, where V_1 is $\lambda + n - 1$-high in V containing W. Next, decompose $X = V_2 \oplus V_3$, where V_2 is n-summable and $V_3 = V_3(\alpha)$. Let H_1, H_2 and H_3 be n-covers of V_1, V_2 and V_3, respectively. Since V_2 is n-summable, 2.8 implies that H_2 is a dsc group. Since $V_3 \subseteq p^\alpha H_3$ and H_3/V_3 is a dsc group, it follows that H_3 is a C_α group. Therefore, there is no loss of generality in assuming that $V = V_1$, i.e., $V(\lambda + n - 1) = \{0\}$.

By Lemma 2.7, we can construct an n-cover H of V such that H/V has length λ. Let K be a p^β-high subgroup of H containing W; in particular, K is isotype in H.

Claim. K is an n-cover of W.

We show that if $x \in K$ and $|x+V|_{H/V} = \gamma < \lambda$, then there is an element $x' \in p^\gamma K$ such that $x + W = x' + W$. This will not only verify that W is nice in K, but it will show that K/W is isotype in the dsc group H/V, so that it is also a dsc group (by a classical result of Hill from [8]).

Since V is nice in H, there is a $y \in V$ such that $|x+y|_H = \gamma$. Since W is dense in V in the λ-topology, $y = z + y'$, where $z \in W$ and $|y'|_H > \gamma$. It follows that $x' \stackrel{\text{def}}{=} x + z = x + y - y' \in K \cap p^\gamma H = p^\gamma K$ and $x + K = x' + K$, establishing the claim.

Finally, since W is n-summable, it follows from 2.8 that K must be a dsc group. Thus H is a C_α group, as required. □

We pause for another relatively unsurprising construction.

Lemma 2.12. *If V is an ω_1-bounded valuated p^n-socle, then there is an ω_1-bounded n-balanced projective resolution $0 \to Q \to P \to V \to 0$ (so P and Q are ω_1-bounded valuated p^n-socles, P is n-summable and Q is n-balanced in P).*

Proof. If $x \in V$, it is easy to confirm that there is a countable n-isotype subgroup $C_x \subseteq V$ containing x. If $P = \bigoplus_{x\in V} C_x$, then clearly P is n-summable. If $\pi : P \to V$ is the sum map, then we need to show that Q, the kernel of π, is n-balanced in P. It is easy to see that Q is nice in P. To verify that it is n-isotype, suppose α is an ordinal and $\mathbf{y} \in Q(\alpha+1)[p^{n-1}]$; so $\mathbf{y}$ will be a vector (y_i), where $y_i \in C_{x_i}$ for $i = 1, \dots, k$, and $y_1 + \cdots + y_k = 0$. Each y_i will be in $C_{x_i}(\alpha+1)[p^{n-1}]$ so that $y_i = pz_i$, where $z_i \in C_{x_i}(\alpha)$. Let $z_{k+1} = -(z_1 + \cdots + z_k) \in V(\alpha)[p]$. If $\mathbf{z}' \in P$ has z_i in the C_{x_i} coordinate for $i = 1, \dots, k$ and zeros elsewhere, and $\mathbf{z}''$ has z_{k+1} in the $C_{z_{k+1}}$ coordinate and zeros elsewhere, then $\mathbf{z} \stackrel{\text{def}}{=} \mathbf{z}' + \mathbf{z}'' \in Q(\alpha)$ and $p\mathbf{z} = \mathbf{y}$. □

The last result allows us to refer to the *n-balanced projective dimension* of an ω_1-bounded valuated p^n-socle. We abbreviate the phrase "balanced projective dimension" by bpd.

Corollary 2.13. *If V is a valuated p^n-socle of countable length α, then V has n-bpd at most* 1.

Proof. Let $0 \to Q \to P \to V \to 0$ be an n-balanced exact sequence where P is an n-summable valuated p^n-socle of length α. It follows from 2.10 that Q is also n-summable, which gives the result. □

Observe that if V is a valuated p^n-socle, then Lemma 2.6 implies that all n-covers of V have the same bpd. The next result generalizes 2.8.

THEOREM 2.14. *If V is an ω_1-bounded valuated p^n-socle and H is an n-cover of V, then the n-bpd of V agrees with the bpd of H in the category of groups.*

Proof. Suppose $0 \to Q \to P \to V \to 0$ is an n-balanced projective resolution of V. Let H_0 be a dsc group such that there is a surjection $H_0 \to H(V)$ whose kernel is balanced in H_0. In addition, $P \to V$ extends to a group homomorphism $H(P) \to H(V)$; in this extension, we may assume that if $x \in P$ is proper with respect to Q and $x \mapsto y$, then $T_x \subseteq H(P)$ maps isomorphically onto $T_y \subseteq H(V)$. These two maps determine a surjective a group homomorphism $H(P) \oplus H_0 \to H(V)$, whose kernel we denote by K. Consider the diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & Q & \to & P & \to & V & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & K & \to & H(P) \oplus H_0 & \to & H(V) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & K/Q & \to & H(P)/P \oplus H_0 & \to & H(V)/V & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

We assert that K is an n-cover of Q. Observe first that the middle row is balanced; this follows easily from the fact that for all ordinals α, $(p^\alpha H_0)[p]$ maps onto $(p^\alpha H(V))[p]$ (see, for example, [5], Proposition 80.2). We conclude that the height valuation on K agrees with the valuation on K induced by the height function on $H(P) \oplus H_0$. In addition, since Q is nice in P, P is nice in $H(P) \oplus H_0$ and niceness is transitive in the category of valuated groups, it follows that Q is nice in K.

We next show that the bottom row splits: If $y \in V^*$, then there is an $x \in P$ which maps to y and is proper with respect to Q; so $|y|_V = |x|_P$. The tree $T'_x \subseteq H(P)/P$ maps isomorphically onto the tree $T'_y \subseteq H(V)/V$. The reverse of these mappings over all $y \in V^*$ gives the required splitting. Consequently, we can infer that K/Q is a dsc group; so K is an n-cover of Q.

By 2.8, the n-bpd of V equals 0 iff the bpd of $H(V)$ equals 0. By induction, it follows from our diagram that the n-bpd of V equals the n-bpd of Q plus one, which equals the bpd of K plus one, which equals the bpd of $H(V)$. □

COROLLARY 2.15. *If V is an ω_1-bounded valuated p^n-socle, then the n-bpd of V is at most* 2.

Proof. Let H be an ω_1-bounded n-cover of V. If $0 \to K \to J \to H \to 0$ is a balanced exact sequence with J a dsc group, then K is an ω_1-bounded IT group. By [16], Theorem 21, the bpd of K is at most 1, so that the bpd of H is at most 2. The result, therefore, follows from Theorem 2.14. □

There is another natural way to construct an n-balanced projective resolution of an ω_1-bounded C_{ω_1} n-summable valuated p^n-socle V. Starting with the aforementioned p^{ω_1}-pure exact sequence $0 \to M_{\omega_1} \to H_{\omega_1} \to \mathbb{Z}_{p^\infty} \to 0$, it is easy to check that this determines an n-balanced exact sequence

$$0 \to M_{\omega_1}\left[p^n\right] \bigtriangledown V \to H_{\omega_1}\left[p^n\right] \bigtriangledown V \to V \to 0.$$

By Corollary 1.20(b), $H_{\omega_1}[p^n] \bigtriangledown V$ is n-summable, giving our resolution. In addition, we have the following consequence.

Corollary 2.16. *If V is an ω_1-bounded C_{ω_1} n-summable valuated p^n-socle, then V has n-bpd at most 1 iff $M_{\omega_1}[p^n] \bigtriangledown V$ is n-summable.*

We next turn to a useful result related to 2.8.

Lemma 2.17. *Suppose V and W are C_{ω_1} n-summable valuated p^n-socles with n-covers G and H, respectively, and $p^{\omega_1}G = p^{\omega_1}H = \{0\}$. If $G \bigtriangledown H$ is n-summable, then $V \bigtriangledown W$ is n-summable.*

Proof. Let $P = G/V$ and $Q = H/W$, so P and Q are dsc groups. There is a left exact sequence

$$0 \to V \bigtriangledown W \to G \bigtriangledown H \to (P \bigtriangledown H) \oplus (G \bigtriangledown Q).$$

Since the right two groups have the height valuation, the right map is trivially valuated. It is easy to check that $V \bigtriangledown W$ is n-isotype in $(G \bigtriangledown H)[p^n]$ which is n-summable. By Theorem 2.11(b), G and H will be C_{ω_1} groups. So by [16], Theorem 2, $P \bigtriangledown H$, and similarly $G \bigtriangledown Q$, is a dsc group. Hence, $((P \bigtriangledown H) \oplus (G \bigtriangledown Q))[p^n]$ is n-summable. And by 2.9, $V \bigtriangledown W$ is n-summable. □

The next observation parallels [10], Theorem 6, and [16], Theorem 23.

Corollary 2.18. *Suppose V and W are ω_1-bounded C_{ω_1} n-summable valuated p^n-socles.*

(a) *If V and W have cardinality at most $\aleph_1$, then $V \bigtriangledown W$ is n-summable.*
(b) *If V and W have n-bpd at most 1, then $V \bigtriangledown W$ is n-summable.*

Proof. There are ω_1-bounded n-covers G and H of V and W, respectively. In (a), we may assume G and H have cardinality at most $\aleph_1$, and [10], Theorem 6, implies $G \bigtriangledown H$ is a dsc group. In (b), Theorem 2.14 implies G and H have bpd at most 1 and [16], Theorem 23, again implies $G \bigtriangledown H$ is a dsc group. In either case, by Lemma 2.17, $V \bigtriangledown W$ is n-summable. □

Lemma 2.17 is exactly what is needed to prove our final result, which can be viewed as an extension of [15], Theorem 13, and is one of the main points of this section.

THEOREM 2.19. *The following are equivalent*:

(a) *Kurepa's Hypothesis fails*;
(b) *If* V *and* W *are any* ω_1*-bounded* C_{ω_1} n*-summable valued* p^n*-socles*, *then* $V \bigtriangledown W$ *is* n*-summable*;
(c) *If* G *and* H *are any* p^{ω_1}*-bounded* C_{ω_1} n*-summable groups*, *then* $G \bigtriangledown H$ *is* n*-summable*;
(d) *If* W *is any* ω_1*-bounded* C_{ω_1} n*-summable valued* p^n*-socle*, *then the* n*-bpd of* W *is at most* 1.
(e) *If* G *is any* p^{ω_1}*-bounded* C_{ω_1} n*-summable group*, *then the bpd of* G *is at most* 1.
(f) *If* G *is any* p^{ω_1}*-bounded* C_{ω_1} *group*, *then the bpd of* G *is at most* 1.

Proof. Appealing to [15], Theorem 13, (a) and (f) are equivalent, so we show that they are also equivalent to the other statements.

Suppose first that Kurepa's Hypothesis fails and that V and W are ω_1-bounded C_{ω_1} n-summable valued p^n-socles. Let G and H be p^{ω_1}-bounded n-covers of V and W, respectively. By Theorem 2.11, G and H are C_{ω_1} groups. Therefore, in view of [15], Theorem 13, $G \bigtriangledown H$ is a dsc group. So, by Lemma 2.17, $V \bigtriangledown W$ is n-summable, showing that (a) implies (b).

Next, assuming that (b) holds, then (c) follows immediately by considering the valuated p^n-socles $G[p^n]$ and $H[p^n]$.

Suppose (c) holds and W is as given in (d). If $V = M_{\omega_1}[p^n]$, then let G and H be n-covers for V and W, respectively. Again, by Theorem 2.11, G and H are C_{ω_1} groups. Consequently, by hypothesis, $G \bigtriangledown H$ is n-summable. So by Lemma 2.17, $V \bigtriangledown W$ is n-summable. And by 2.16, W has n-bpd at most 1.

Assuming that (d) holds, let G be as given in (e). If H is a dsc group and $0 \to Q \to H \to G \to 0$ is a balanced projective resolution of G, then $0 \to Q[p^n] \to H[p^n] \to G[p^n] \to 0$ is an n-balanced projective resolution of $G[p^n]$. So, by hypothesis, $Q[p^n]$ is n-summable, and hence summable (= 1-summable).

By the main result from [9], we can conclude that Q, as a summable and isotype subgroup of the dsc group H, is also a dsc group. This, however, implies that G has bpd at most 1, so that (d) implies (e).

Finally, since any C_{ω_1} group is C_{ω_1} n-summable, we can conclude that (e) implies (f), concluding the proof. □

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References

[1] D. Cutler and R. Dimitric, *Valuated vector spaces, Kurepa's hypothesis and abelian p-groups*, Rocky Mountain J. Math. **23** (1993), no. 4, 1253–1266. MR 1256447

[2] P. Danchev and P. Keef, *n-summable valuated p^n-socles and primary abelian groups*, Comm. Algebra **38** (2010), 3137–3153. MR 2724211

[3] P. Danchev and P. Keef, *Nice elongations of primary abelian groups*, Publ. Mat. **54** (2010), no. 2, 317–339.

[4] L. Fuchs, *Infinite abelian groups, vol. I*, Academic Press, New York, 1970. MR 0255673

[5] L. Fuchs, *Infinite abelian groups, vol. II*, Academic Press, New York, 1973. MR 0349869

[6] L. Fuchs, *Vector spaces with valuations*, J. Algebra **35** (1975), 23–38. MR 0371995

[7] P. Griffith, *Infinite abelian group theory*, Univ. Chicago Press, Chicago and London, 1970. MR 0289638

[8] P. Hill, *Isotype subgroups of direct sums of countable abelian groups*, Illinois J. Math. **13** (1969), 281–290. MR 0240198

[9] P. Hill, *The recovery of some abelian groups from their socles*, Proc. Amer. Math. Soc. **86** (1982), no. 4, 553–560. MR 0674080

[10] P. Hill, *The balance of Tor*, Math. Z. **182** (1983), no. 2, 179–188. MR 0689295

[11] T. Jech, *Set theory*, 3rd Millennium ed., Springer, Berlin, 2002. MR 1492987

[12] P. Keef, *Realization theorems for n-summable valuated p^n-socles*, Rend. Semin. Mat. Univ. Padova **126**, 151–173. MR 2918204

[13] P. Keef, *On the Tor functor and some classes of abelian groups*, Pacific J. Math. **132** (1988), no. 1, 63–84. MR 0929583

[14] P. Keef, *A theorem on the closure of Ω-pure subgroups of C_Ω groups in the Ω-topology*, J. Algebra **125** (1989), no. 1, 150–163. MR 1012668

[15] P. Keef, *On set theory and the balanced-projective dimension of C_Ω-groups*, Contemporary Mathematics, vol. 87, Amer. Math. Soc., Providence, RI, 1989, pp. 31–42. MR 0995262

[16] P. Keef, *On iterated torsion products of abelian p-groups*, Rocky Mountain J. Math. **21** (1991), no. 3, 1035–1055. MR 1138150

[17] P. Keef, *On ω_1-$p^{\omega+n}$-projective primary abelian groups*, J. Algebra and Number Theory Academia **1** (2010), no. 1, 53–87.

[18] P. Keef and P. Danchev, *On m,n-balanced projective and m,n-totally projective primary abelian groups*, J. Korean Math. Soc. **50** (2013), no. 2, 307–330. MR 3031881

[19] P. Keef and P. Danchev, *On n-simply presented primary abelian groups*, Houston J. Math. **38** (2012), no. 4, 1027–1050. MR 3019021

[20] C. Megibben, *A generalization of the classical theory of primary groups*, Tohoku Math. J. **22** (1970), 347–356. MR 0294491

[21] R. Nunke, *Purity and subfunctors of the identity*, Topics in abelian groups, Scott, Foresman and Co., Chicago, IL, 1962, pp. 121–171. MR 0169913

[22] R. Nunke, *Homology and direct sums of countable abelian groups*, Math. Z. **101** (1967), no. 3, 182–212. MR 0218452

[23] R. Nunke, *On the structure of Tor, II*, Pacific J. Math. **22** (1967), 453–464. MR 0214659

[24] F. Richman and E. Walker, *Valuated groups*, J. Algebra **56** (1979), no. 1, 145–167. MR 0527162

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