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A NOTE ON UNITAL FULL AMALGAMATED FREE PRODUCTS OF RFD C*-ALGEBRAS

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ABSTRACT. In the paper, we consider the question whether a unital full amalgamated free product of RFD (residually finite dimensional) C*-algebras is RFD again. One example shows that the answer to the general case is no. We give a necessary and sufficient condition such that a unital full amalgamated free product of RFD C*-algebras with amalgamation over a finite dimensional C*-algebra is RFD. Applying this result, we conclude that a unital full free product of two same RFD C*-algebras with amalgamation over a finite-dimensional C*-algebra is always RFD.

1. Introduction

A C*-algebra is said to be residually finite-dimensional (RFD) if it has a separating family of finite-dimensional representations. Also this property is inherited by subalgebras. Choi [6] showed that the full C*-algebra of the free group on two generators is RFD. Later Exel and Loring showed that the unital full free product of two unital RFD C*-algebras is RFD [8]. In the same paper, they gave several equivalent conditions for the RFD property. Armstrong, Dykema, Exel and Li [1] characterized the RFD property of unital full amalgamated free products of finite dimensional C*-algebras, which extends an earlier result by Brown and Dykema [4].

In this paper, we are interested in the question whether a unital full free product of two RFD C*-algebras with amalgamation over a common C*-algebra is, again, an RFD C*-algebra. One example (see Example 2.1) is given to show that the answer to this general question is no. But an affirmative

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answer was given by Exel and Loring [8] when the common C*-subalgebra in a unital full amalgamated free product of RFD algebras is *-isomorphic to a full matrix algebra. In fact, a similar result holds when we consider MF algebras and quasidiagonal C*-algebras (for more information about MF algebras and quasidiagonal C*-algebras, we refer the reader to [2], [5]).

When the common C*-subalgebra is a finite-dimensional C*-algebra, we are able to provide a necessary and sufficient condition such that a unital full amalgamated free product of RFD C*-algebras is RFD again. More specifically, we conclude that a unital full free product of two same RFD C*-algebras with amalgamation over a finite-dimensional C*-algebra is always RFD.

A brief overview of this paper is as follows. In Section 2, we recall the definition of unital full amalgamated free product of unital C*-algebas. We show that a unital full amalgamated free product of unital RFD (or MF, quasidiagonal) C*-algebras is RFD (or MF, quasidiagonal) when the overlap C*-algebra is *-isomorphic to a full matrix algebra. One example is given at the end of the section to show that a unital full amalgamated free product of RFD (or MF, quasidiagonal) C*-algebras may not be RFD (or MF, quasidiagonal) again. Section 3 is devoted to results on unital full free products of RFD C*-algebras with amalgamation over finite-dimensional C*-algebras.

2. Definitions and preliminaries

Recall the definition of full amalgamated free product of unital C*-algebras as follows.

Definition 1. Given C*-algebras $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{D}$ with unital embeddings (injective $*$-homomorphisms) $\psi_{\mathcal{A}}:\mathcal{D}\to\mathcal{A}$ and $\psi_{\mathcal{B}}:\mathcal{D}\to\mathcal{B}$, the corresponding full amalgamated free product C*-algebra is the C*-algebra $\mathcal{C}$, equipped with unital embeddings $\sigma_{\mathcal{A}}:\mathcal{A}\to\mathcal{C}$ and $\sigma_{\mathcal{B}}:\mathcal{B}\to\mathcal{C}$ such that $\sigma_{\mathcal{A}}\circ\psi_{\mathcal{A}}=\sigma_{\mathcal{B}}\circ\psi_{\mathcal{B}}$, such that $\mathcal{C}$ is generated by $\sigma_{\mathcal{A}}(\mathcal{A})\cup\sigma_{\mathcal{B}}(\mathcal{B})$ and satisfying the universal property that whenever $\mathcal{E}$ is a C*-algebra and $\pi_{\mathcal{A}}:\mathcal{A}\to\mathcal{E}$ and $\pi_{\mathcal{B}}:\mathcal{B}\to\mathcal{E}$ are $*$-homomorphisms satisfying $\pi_{\mathcal{A}}\circ\psi_{\mathcal{A}}=\pi_{\mathcal{B}}\circ\psi_{\mathcal{B}}$, there is a $*$-homomorphism $\pi:\mathcal{C}\to\mathcal{E}$ such that $\pi\circ\sigma_{\mathcal{A}}=\pi_{\mathcal{A}}$ and $\pi\circ\sigma_{\mathcal{B}}=\pi_{\mathcal{B}}$. The full amalgamated free product C*-algebra $\mathcal{C}$ is commonly denoted by $\mathcal{A}\underset{\mathcal{D}}{*}\mathcal{B}$.

When $D=\mathbb{C}I$, the above definition is the unital full free product $\mathcal{A}\underset{\mathbb{C}}{*}\mathcal{B}$ of $\mathcal{A}$ and $\mathcal{B}$. The following result can be found in [11]. But we offer a new proof, which is perhaps more elementary.

Theorem 1. *Suppose that $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{D}$ are unital C*-algebras. Then*

$$(\mathcal{A}\otimes_{\max}\mathcal{D})\underset{\mathcal{D}}{*}(\mathcal{B}\otimes_{\max}\mathcal{D})\cong\big(\mathcal{A}\underset{\mathbb{C}}{*}\mathcal{B}\big)\otimes_{\max}\mathcal{D}.$$

Proof. Let $I_{\mathcal{A}}$ and $I_{\mathcal{B}}$ be the identity in $\mathcal{A}$ and $\mathcal{B}$, respectively. From the definition of unital full free product, we can get two natural unital embeddings

$$\pi_1:\mathcal{A}\otimes_{\max}\mathcal{D}\to\big(\mathcal{A}\underset{\mathbb{C}}{*}\mathcal{B}\big)\otimes_{\max}\mathcal{D}$$

and

$$\pi_2 : \mathcal{B} \otimes_{\max} \mathcal{D} \to \big(\mathcal{A} \underset{\mathbb{C}}{*} \mathcal{B}\big) \otimes_{\max} \mathcal{D}$$

from $\mathcal{A} \otimes_{\max} \mathcal{D}$ and $\mathcal{B} \otimes_{\max} \mathcal{D}$ into $(\mathcal{A} \underset{\mathbb{C}}{*} \mathcal{B}) \otimes_{\max} \mathcal{D}$, respectively. It is clear that the restrictions of π_1 on $I_{\mathcal{A}} \otimes \mathcal{D}$ and π_2 on $I_{\mathcal{B}} \otimes \mathcal{D}$ agree, i.e., $\pi_1|_{I_{\mathcal{A}} \otimes \mathcal{D}} = \pi_2|_{I_{\mathcal{B}} \otimes \mathcal{D}}$. Suppose $\mathcal{K}$ is a C*-algebra acting on a Hilbert space $\mathcal{H}$ such that there are two *-homomorphisms $q_1 : \mathcal{A} \otimes_{\max} \mathcal{D} \to \mathcal{K}$ and $q_2 : \mathcal{B} \otimes_{\max} \mathcal{D} \to \mathcal{K}$ satisfying $q_1|_{I_{\mathcal{A}} \otimes \mathcal{D}} = q_2|_{I_{\mathcal{B}} \otimes \mathcal{D}}$. Then $q_1(\mathcal{A} \otimes I_{\mathcal{D}})$ commutes with $q_1(I_{\mathcal{A}} \otimes \mathcal{D})$ in $\mathcal{K}$ and $q_2(\mathcal{B} \otimes I_{\mathcal{D}})$ commutes with $q_2(I_{\mathcal{B}} \otimes \mathcal{D})$ in $\mathcal{K}$. Let

$$\mathcal{M} = \mathcal{K} \cap \big(q_1(I_{\mathcal{A}} \otimes \mathcal{D})\big)' = \mathcal{K} \cap \big(q_2(I_{\mathcal{B}} \otimes \mathcal{D})\big)'.$$

Since $q_1(\mathcal{A} \otimes I_{\mathcal{D}})$ and $q_2(\mathcal{B} \otimes I_{\mathcal{D}})$ are both subalgebras of C*-algebra $\mathcal{M}$, there is a *-homomorphism $\widetilde{q} : \mathcal{A} \underset{\mathbb{C}}{*} \mathcal{B} \to \mathcal{M}$ by the definition of unital full free product. Moreover, the image $\widetilde{q}(\mathcal{A} \underset{\mathbb{C}}{*} \mathcal{B})$ of $\mathcal{A} \underset{\mathbb{C}}{*} \mathcal{B}$ under $\widetilde{q}$ commutes with $q_1(I_{\mathcal{A}} \otimes \mathcal{D})$ in $\mathcal{K}$. From the definition of maximal C*-norm on tensor product of two C*-algebras, there is a *-homomorphism

$$q : \big(\mathcal{A} \underset{\mathbb{C}}{*} \mathcal{B}\big) \otimes_{\max} \mathcal{D} \to \mathcal{K}.$$

such that $q \circ \pi_1 = q_1$ and $q \circ \pi_2 = q_2$. The desired conclusion now follows from the definition of full amalgamated free products of unital C*-algebras. ☐

Combining the following lemma and preceding result, we are able to obtain a result about unital full amalgamated free products of RFD C*-algebras, which can be also found in [8].

LEMMA 1 (Theorem 3.2, [8]). *Suppose $\mathcal{A}_1$ and $\mathcal{A}_2$ are unital C*-algebras. Then the unital full free product $\mathcal{A} = \mathcal{A}_1 \underset{\mathbb{C}}{*} \mathcal{A}_2$ is RFD if and only if $\mathcal{A}_1$ and $\mathcal{A}_2$ are both RFD.*

PROPOSITION 1 (Corollary 3.3, [8]). *Let $\mathcal{A}$ and $\mathcal{B}$ be unital C*-algebras. If $\mathcal{D}$ can be embedded as a unital C*-subalgebra of $\mathcal{A}$ and $\mathcal{B}$ respectively, and $\mathcal{D}$ is *-isomorphic to a full matrix algebra $\mathcal{M}_n(\mathbb{C})$ for some integer n, then the unital full amalgamated free product $\mathcal{A} \underset{\mathcal{D}}{*} \mathcal{B}$ is RFD if and only if $\mathcal{A}$ and $\mathcal{B}$ are both RFD.*

Proof. If $\mathcal{A} \underset{\mathcal{D}}{*} \mathcal{B}$ is a unital RFD algebra, then it is easy to see that $\mathcal{A}$ and $\mathcal{B}$ are both RFD. On the other hand, since $\mathcal{D}$ is *-isomorphic to a full matrix algebra, from Lemma 6.6.3 in [10], it follows that $\mathcal{A} \cong \mathcal{A}' \otimes \mathcal{D}$ and $\mathcal{B} \cong \mathcal{B}' \otimes \mathcal{D}$ where $\mathcal{A}'$ and $\mathcal{B}'$ are C*-subalgebras of $\mathcal{A}$ and $\mathcal{B}$, respectively. Therefore, $\mathcal{A}'$ and $\mathcal{B}'$ are RFD as well. Then the desired conclusion follows from Theorem 1 and Lemma 1. ☐

If a separable C*-algebra $\mathcal{A}$ can be embedded into C*-algebra

$$\prod_k \mathcal{M}_{n_k}(\mathbb{C}) \Big/ \sum_k \mathcal{M}_{n_k}(\mathbb{C})$$

for a sequence of positive integers $\{n_k\}_{k=1}^{\infty}$, then $\mathcal{A}$ is called an MF algebra. This concept was first introduced by Blackadar and Kirchberg in [2]. The class of MF algebras contains all separable RFD C*-algebras and separable quasidiagonal C*-algebras. Note that a separable C*-algebra is RFD if and only if it can be embedded into $\prod_k \mathcal{M}_{n_k}(\mathbb{C})$ for a sequence of positive integers $\{n_k\}_{k=1}^{\infty}$.

Remark 1. Since a unital full free product of quasidiagonal C*-algebras (or MF algebras) is quasidiagonal (or MF) (see [3], [9]), Proposition 1 can be stated and proved when we consider unital MF algebras or unital quasidiagonal C*-algebras.

Remark 2. Armstrong, Dykema, Exel and Li [1] showed that, for unital inclusions of C*-algebras $\mathcal{A} \supseteq \mathcal{D} \subseteq \mathcal{B}$ with $\mathcal{A}$ and $\mathcal{B}$ finite dimensional, $\mathcal{A} \underset{\mathcal{D}}{*} \mathcal{B}$ is RFD if and only if there are faithful tracial states $\tau_{\mathcal{A}}$ on $\mathcal{A}$ and $\tau_{\mathcal{B}}$ on $\mathcal{B}$ whose restrictions on $\mathcal{D}$ agree. Combining this result and the fact that each RFD C*-algebra has a faithful tracial state, it is not hard to see that $\mathcal{A} \underset{\mathcal{D}}{*} \mathcal{B}$ is RFD if and only if $\mathcal{A} \underset{\mathcal{D}}{*} \mathcal{B}$ has a faithful tracial state in this case.

The following example shows that a full amalgamated free product of two RFD (or MF, quasidiagonal) algebras may not be RFD (or MF, quasidiagonal) again, even for a unital full free product of two full matrix algebras with amalgamation over a two dimensional C*-algebra which is *-isomorphic to $\mathbb{C} \oplus \mathbb{C}$.

Example 1. Let C*-algebra $\mathcal{D} = \mathbb{C} \oplus \mathbb{C}$. Suppose that $\varphi_1 : \mathcal{D} \to \mathcal{M}_2(\mathbb{C})$ and $\varphi_2 : \mathcal{D} \to \mathcal{M}_3(\mathbb{C})$ are unital embeddings such that

$$\varphi_1(1 \oplus 0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \varphi_2(1 \oplus 0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then $\mathcal{M}_2(\mathbb{C}) \underset{\mathcal{D}}{*} \mathcal{M}_3(\mathbb{C})$ is not MF algebra (therefore it is not RFD or quasidiagonal). Actually, if we assume that $\mathcal{M}_2(\mathbb{C}) \underset{\mathcal{D}}{*} \mathcal{M}_3(\mathbb{C})$ is an MF algebra, then there exists a tracial state τ on $\mathcal{M}_2(\mathbb{C}) \underset{\mathcal{D}}{*} \mathcal{M}_3(\mathbb{C})$. So the restrictions of τ on $\mathcal{M}_2(\mathbb{C})$ and $\mathcal{M}_3(\mathbb{C})$ are the unique tracial states on $\mathcal{M}_2(\mathbb{C})$ and $\mathcal{M}_3(\mathbb{C})$,respectively. It follows that $\tau(\varphi_1(1 \oplus 0)) = \frac{1}{2} \neq \tau\ (\varphi_2(1 \oplus 0)) = \frac{1}{3}$ which contradicts to the fact that $\varphi_1(1 \oplus 0) = \varphi_2(0 \oplus 1)$ in $\mathcal{M}_2(\mathbb{C}) \underset{\mathcal{D}}{*} \mathcal{M}_3(\mathbb{C})$. Therefore, $\mathcal{M}_2(\mathbb{C}) \underset{\mathcal{D}}{*} \mathcal{M}_3(\mathbb{C})$ is not MF.

3. Full amalgamated free products of RFD C*-algebras

Throughout this section, we will only be concerned with separable C*-algebras and representations on separable Hilbert spaces. First, we will give the following well-known lemma. For completeness, we include the proof.

LEMMA 2. *Given $0<\varepsilon<1$ and $n\in\mathbb{N}$. For any two families of n pairwise orthogonal projections $\{P_1,\ldots,P_n\}$ and $\{Q_1,\ldots,Q_n\}$ in n-dimensional unital abelian C*-subalgebras $\mathcal{A}$ and $\mathcal{B}$ in $\mathcal{B}(\mathcal{H})$ with $\|P_i-Q_i\|<\frac{\varepsilon}{n+1}$ $(i=1,\ldots,n)$, there is a unitary $U\in\mathcal{B}(\mathcal{H})$ with $\|U-I\|<\varepsilon$ such that $UP_iU^*=Q_i$ for $1\le i\le n$.*

Proof. Define $X=\sum_{i=1}^{n}Q_iP_i$. Let $\delta=\frac{\varepsilon}{n+1}$. It is clear that

$$\sum_{i=1}^{n}P_i=\sum_{i=1}Q_i=I.$$

Since $\|P_i-Q_i\|<\delta$ and P_i-Q_i is self-adjoint for each i, we have that $Q_i-P_i+\delta\ge 0$. It follows that $Q_i\ge P_i-\delta$ and

$$\begin{aligned}X^*X&=\sum_{i=1}^{n}P_iQ_iP_i\ge\sum_{i=1}^{n}P_i(P_i-\delta)P_i\\&=\sum_{i=1}^{n}P_i-\sum_{i=1}^{n}\delta P_i=(1-\delta)I>0.\end{aligned}$$

Therefore, X is invertible and $\|X^*X\|\ge 1-\delta$. Assume that $X=U|X|$ is the polar decomposition of X where $|X|=(X^*X)^{\frac{1}{2}}$ and U is a partial isometry. Since X is invertible, U is a unitary. So it is not hard to see that

$$\big\||X|^{-1}-I\big\|\le\left(\frac{1}{1-\delta}\right)^{1/2}-1.$$

Meanwhile, we have $\|X^*X\|\le 1$ from the construction of X and the fact that $\{P_1,\ldots,P_n\}$ and $\{Q_1,\ldots,Q_n\}$ are two families of n pairwise orthogonal projections, respectively. Therefore, we have that

$$\begin{aligned}\|U-I\|&\le\|U-X\|+\|X-I\|\\&\le\|X\|\big\||X|^{-1}-I\big\|+\left\|\sum_{i=1}^{n}(Q_i-P_i)P_i.\right\|\\&\le\left(\left(\frac{1}{1-\delta}\right)^{1/2}-1\right)+n\delta<(n+1)\delta=\varepsilon.\end{aligned}$$

Since $X=\sum_{i=1}^{n}Q_iP_i$, it is easy to see $Q_iX=XP_i$ for $1\le i\le n$, then $P_i|X|=|X|P_i$ as well. So

$$UP_i=X|X|^{-1}P_i=XP_i|X|^{-1}=Q_iX|X|^{-1}=Q_iU.$$

Therefore, $UP_iU^*=Q_i$ for $1\le i\le n$ as desired. □

The following lemma is a useful result concerning the representations of separable C*-algebras. First, we need to recall that the rank of an operator $T\in\mathcal{B}(\mathcal{H})$, denoted by $\operatorname{rank}(T)$, is the dimension of the closure of the range of T.

LEMMA 3 (Theorem II.5.8, [7]). *Let $\mathcal{A}$ be a separable unital C^*-algebra and $\pi_i : \mathcal{A} \to \mathcal{B}(\mathcal{H}_i)$ be unital *-representations for $i = 1, 2$. Then there exists a sequence of unitaries $U_m : H_1 \to H_2$ such that $\|\pi_2(a) - U_m \pi_1(a) U_m^*\| \to 0 (m \to \infty)$ for all $a \in \mathcal{A}$ if and only if $\operatorname{rank}(\pi_1(a)) = \operatorname{rank}(\pi_2(a))$ for all $a \in \mathcal{A}$.*

DEFINITION 2. Suppose $\mathcal{H}$ is a separable Hilbert space and $F \subseteq \mathcal{H}$. For given $\varepsilon > 0$, we say that

$$\{x_1, \dots, x_n\} \subseteq_\varepsilon F$$

for $\{x_1, \dots, x_n\} \subseteq \mathcal{H}$ if there are $y_1, \dots, y_n \in F$ such that

$$\max_{1 \le i \le n} \|x_i - y_i\| \le \varepsilon.$$

The following lemma is a technical result.

LEMMA 4. *Let $\mathcal{A} \supseteq \mathcal{D} \subseteq \mathcal{B}$ be unital inclusions of separable C^*-algebras and $\mathcal{D}$ be a unital finite-dimensional Abelian C^*-algebra. Suppose $\rho_\mathcal{A} : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ and $\rho_\mathcal{B} : \mathcal{B} \to \mathcal{B}(\mathcal{H})$ are representations of $\mathcal{A}$ and $\mathcal{B}$ with $\rho_\mathcal{A}|_\mathcal{D} = \rho_\mathcal{B}|_\mathcal{D}$ on a separable Hilbert space $\mathcal{H}$, respectively. If there are two finite-dimensional subspaces F, G of $\mathcal{H}$ satisfying F is $\rho_\mathcal{A}(\mathcal{A})$ invariant and G is $\rho_\mathcal{B}(\mathcal{B})$ invariant as well as $\dim F = \dim G$, then there are a finite-dimensional subspace $\widetilde{\mathcal{H}}$ of $\mathcal{H}$ and representations $\widetilde{\rho}_\mathcal{A}$, $\widetilde{\rho}_\mathcal{B}$ of $\mathcal{A}$ and $\mathcal{B}$ on $\widetilde{\mathcal{H}}$ such that the restriction of $\widetilde{\rho}_\mathcal{A}$ on subspace F equals the restriction of $\rho_\mathcal{A}$ on F, the restriction of $\widetilde{\rho}_\mathcal{B}$ on subspace G equals the restriction of $\rho_\mathcal{B}$ on G, that is,*

$$\widetilde{\rho}_\mathcal{A}|_F = \rho_\mathcal{A}|_F, \qquad \widetilde{\rho}_\mathcal{B}|_G = \rho_\mathcal{B}|_G$$

and the restrictions of $\widetilde{\rho}_\mathcal{A}$ and $\widetilde{\rho}_\mathcal{B}$ on $\mathcal{D}$ agree, that is, $\widetilde{\rho}_\mathcal{A}|_\mathcal{D} = \widetilde{\rho}_\mathcal{A}|_\mathcal{D}$.

Proof. Suppose that $\mathcal{D} = C^*(p_1, \dots, p_t)$ where $p_1, \dots, p_t$ are orthogonal projections with $\sum_{i=1}^t p_i = I$. Let $E = F + G$. Note that E is $\rho_\mathcal{A}(\mathcal{D})$ $(= \rho_\mathcal{B}(\mathcal{D}))$ invariant. Let $d = \dim(E)$, $\widetilde{P}_i = \rho_\mathcal{A}(p_i)|_E$ and $r_i = \operatorname{rank}(\widetilde{P}_i)$. Let E' be any finite dimensional subspace of $\mathcal{H}$ that is orthogonal to E and has dimension $d' = \dim(E'_k)$ so that $d + d' = l \cdot \dim F = l \cdot \dim G$ and $\frac{\operatorname{rank}(\rho_\mathcal{A}(p_i)|_F)}{\dim(F)}(d + d') = \operatorname{rank}(\rho_\mathcal{A}(p_i)|_F) \cdot l \ge r_i$ for $i \in \{1, \dots, t\}$, $l \in \mathbb{N}$. Then we can find projections $\widetilde{Q}_1, \dots, \widetilde{Q}_t \in \mathcal{B}(E'_k)$ such that $\widetilde{Q}_1 + \cdots + \widetilde{Q}_t = I \in \mathcal{B}(E')$, and $r_i + r'_i = \operatorname{rank}(\rho_\mathcal{A}(p_i)|_F) \cdot l$ where $r'_i = \operatorname{rank}(\widetilde{Q}_i)$. Assume that $\widetilde{\mathcal{H}} = E + E'$. Since

$$\dim(\widetilde{\mathcal{H}} \ominus F) = (l - 1) \cdot \dim F$$

and

$$\begin{aligned}\operatorname{rank}\big((\widetilde{P}_i + \widetilde{Q}_i)|_{\widetilde{\mathcal{H}} \ominus F}\big) &= r_i + r'_i - \operatorname{rank}\big(\rho_\mathcal{A}(p_i)|_F\big) \\ &= \operatorname{rank}\big(\rho_\mathcal{A}(p_i)|_F\big)(l - 1).\end{aligned}$$

We can construct a representation $\rho'_\mathcal{A} : \mathcal{A} \to \mathcal{B}(\widetilde{\mathcal{H}} \ominus F)$ with $\rho'_\mathcal{A}(p_i) = (\widetilde{P}_i + \widetilde{Q}_i)|_{\widetilde{\mathcal{H}} \ominus F}$ such that $\rho'_\mathcal{A}$ is unitarily equivalent to the direct sum of $l - 1$ copies of the restriction of $\rho_\mathcal{A}$ on F, that is, $\rho_\mathcal{A}|_F$. Putting $\widetilde{\rho}_\mathcal{A}(x) = \rho_\mathcal{A}(x)|_F + \rho'_\mathcal{A}(x) \in$

$\mathcal{B}(\widetilde{\mathcal{H}})$. Then $\widetilde{\rho}_{\mathcal{A}}(p_i) = \widetilde{P}_i + \widetilde{Q}_i$. Similarly, we can construct a representation $\widetilde{\rho}_{\mathcal{B}}$ by the same way such that $\widetilde{\rho}_{\mathcal{B}}(p_i) = \widetilde{P}_i + \widetilde{Q}_i$. This implies that there are *-representations $\widetilde{\rho}_{\mathcal{A}}$ and $\widetilde{\rho}_{\mathcal{B}}$ satisfying

$$\widetilde{\rho}_{\mathcal{A}}|_F = \rho_{\mathcal{A}}|_F, \qquad \widetilde{\rho}_{\mathcal{B}}|_G = \rho_{\mathcal{B}}|_G$$

and $\widetilde{\rho}_{\mathcal{A}}|_{\mathcal{D}} = \widetilde{\rho}_{\mathcal{B}}|_{\mathcal{D}}$. □

We need one more technical result for showing our main result. Recall that a faithful representation $\pi : \mathcal{A} \to B(\mathcal{H})$ is called essential if $\pi(\mathcal{A})$ contains no nonzero finite rank operators.

Lemma 5. *Let $\mathcal{A}$ and $\mathcal{B}$ be unital separable C^*-algebras in $\prod_{n=1}^{\infty}\mathcal{M}_{k_n}(\mathbb{C})$ and $\mathcal{D}$ be a common unital C^*-subalgebra of $\mathcal{A}$ and $\mathcal{B}$ in $\prod_{n=1}^{\infty}\mathcal{M}_{k_n}(\mathbb{C})$ which is finite-dimensional and Abelian. Suppose $\Phi : \mathcal{A} \underset{\mathcal{D}}{*} \mathcal{B} \to \mathcal{B}(\mathcal{H})$ is a faithful essential representation on a separable Hilbert space $\mathcal{H}$. Then there are sequences $\{\rho_m^{\mathcal{A}}\}_{m=1}^{\infty}$ and $\{\rho_m^{\mathcal{B}}\}_{m=1}^{\infty}$ of representations of $\mathcal{A}$ and $\mathcal{B}$ on $\mathcal{H}$ such that $\rho_m^{\mathcal{B}}|_{\mathcal{D}} = \rho_m^{\mathcal{A}}|_{\mathcal{D}}$ and*

$$\left\|\rho_m^{\mathcal{A}}(a) - \Phi_{\mathcal{A}}(a)\right\| \to 0 \quad \text{for all } a \in \mathcal{A} \text{ as } m \to \infty,$$
$$\left\|\rho_m^{\mathcal{B}}(b) - \Phi_{\mathcal{B}}(b)\right\| \to 0 \quad \text{for all } b \in \mathcal{B} \text{ as } m \to \infty.$$

Moreover, for each $m \in \mathbb{N}$, we can find chains of finite-dimensional subspaces $F_1^m \subseteq F_2^m \subseteq \cdots$ and $G_1^m \subseteq G_2^m \subseteq \cdots$ of $\mathcal{H}$ with $\dim F_k^m = \dim G_k^m$ such that each F_k^m is $\rho_m^{\mathcal{A}}(\mathcal{A})$ invariant, each G_k^m is $\rho_m^{\mathcal{B}}(\mathcal{B})$ invariant and $\bigcup_{k=1}^{\infty} F_k^m$, $\bigcup_{k=1}^{\infty} G_k^m$ are both dense in $\mathcal{H}$.

Proof. Suppose $\mathcal{D} = C^*(p_1, \ldots, p_t)$ where $p_1, \ldots, p_t$ are orthogonal projections with $\sum_{i=1}^{t} p_i = I$. There are natural *-homomorphisms $\pi_n^{\mathcal{A}} : \mathcal{A} \to \mathcal{M}_{k_n}(\mathbb{C})$ and $\pi_n^{\mathcal{B}} : \mathcal{B} \to \mathcal{M}_{k_n}(\mathbb{C})$ for each $n \in \mathbb{N}$ such that the direct sums of $\{\pi_n^{\mathcal{A}}\}$ and $\{\pi_n^{\mathcal{B}}\}$ are faithful, respectively. We may assume that each $\pi_k^{\mathcal{A}}$ and $\pi_k^{\mathcal{B}}$ appear infinitely often in the lists $\{\pi_1^{\mathcal{A}}, \pi_2^{\mathcal{A}}, \ldots\}$ and $\{\pi_1^{\mathcal{B}}, \pi_2^{\mathcal{B}}, \ldots\}$, respectively so that we have an increasing sequence $N_0 = 0 < N_1 < N_2 < \cdots$ such that $\pi_k^{\mathcal{A}}$ and $\pi_k^{\mathcal{B}}$ appear at N_k position in $\{\pi_1^{\mathcal{A}}, \pi_2^{\mathcal{A}}, \ldots\}$ and $\{\pi_1^{\mathcal{B}}, \pi_2^{\mathcal{B}}, \ldots\}$, respectively. It is clear that direct sums of them are faithful essential representations, respectively. Then there are representations $\pi_{\mathcal{A}} : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ and $\pi_{\mathcal{B}} : \mathcal{B} \to \mathcal{B}(\mathcal{H})$ with a projection P_{N_k} for each $k \in \mathbb{N}$ such that P_{N_k} reduces $\pi_{\mathcal{A}}$ and $\pi_{\mathcal{B}}$, the restrictions of $\pi_{\mathcal{A}}$ and $\pi_{\mathcal{B}}$ to $(P_{N_k} - P_{N_{k-1}})(\mathcal{H})$ are unitarily equivalent to $\pi_k^{\mathcal{A}}$ and $\pi_k^{\mathcal{B}}$ respectively, and $P_{N_k} \to I$ in SOT as $k \to \infty$. Since $\pi_{\mathcal{A}}$, $\pi_{\mathcal{B}}, \Phi$ are all essential representations, we have

$$\operatorname{rank} \pi_{\mathcal{A}}(a) = \operatorname{rank} \Phi_{\mathcal{A}}(a) \quad \text{and} \quad \operatorname{rank} \pi_{\mathcal{B}}(b) = \operatorname{rank} \Phi_{\mathcal{B}}(b)$$

for each $a \in \mathcal{A}$ and $b \in \mathcal{B}$, where $\Phi_{\mathcal{A}}$ and $\Phi_{\mathcal{B}}$ are the restriction of Φ on $\mathcal{A}$ and $\mathcal{B}$, respectively. Hence, we can find sequences $\{U_m\}_{m=1}^{\infty}$ and $\{W_m\}_{m=1}^{\infty}$

of unitaries in $\mathcal{B}(\mathcal{H})$ by Lemma 3 such that, for every $a\in\mathcal{A}$ and $b\in\mathcal{B}$, we have

$$\begin{aligned}\left\|U_m\pi_{\mathcal{A}}(a)U_m^*-\Phi_{\mathcal{A}}(a)\right\|&\to 0\quad\text{as } m\to\infty,\\ \left\|W_m\pi_{\mathcal{B}}(b)W_m^*-\Phi_{\mathcal{B}}(b)\right\|&\to 0\quad\text{as } m\to\infty.\end{aligned}$$

By the fact that $\Phi_{\mathcal{A}}(p_i)=\Phi_{\mathcal{B}}(p_i)$ for every $i\in\{1,\dots,t\}$, it follows that

$$\left\|U_m\pi_{\mathcal{A}}(p_i)U_m^*-W_m\pi_{\mathcal{B}}(p_i)W_m^*\right\|\to 0\quad\text{as } m\to\infty.$$

From Lemma 2, there is $M_0\in\mathbb{N}$ such that for every $m\geq M_0$, there is a unitary V_m and ε_m satisfying $\|V_m-I\|<\varepsilon_m$, $\varepsilon_m\to 0$ $(m\to\infty)$ and

$$V_m^*W_m\pi_{\mathcal{B}}(p_i)W_m^*V_m=U_m\pi_{\mathcal{A}}(p_i)U_m^*$$

for each $i\in\{1,\dots,t\}$. Without loss of generality we can assume that, for each $m\in\mathbb{N}$, there is a V_m and ε_m such that $\|V_m-I\|<\varepsilon_m$ and

$$V_m^*W_m\pi_{\mathcal{B}}(p_i)W_m^*V_m=U_m\pi_{\mathcal{A}}(p_i)U_m^*.$$

Meanwhile, we still have

$$\left\|V_m^*W_m\pi_{\mathcal{B}}(b)W_m^*V_m-\Phi_{\mathcal{B}}(b)\right\|\to 0\quad\text{as } m\to\infty.$$

Let $\rho_m^{\mathcal{A}}(a)=U_m\pi_{\mathcal{A}}(a)U_m^*$ and $\rho_m^{\mathcal{B}}(b)=V_m^*W_m\pi_{\mathcal{B}}(b)W_m^*V_m$ for each $m\in\mathbb{N}$. It is clear that $\rho_m^{\mathcal{B}}|_{\mathcal{D}}=\rho_m^{\mathcal{A}}|_{\mathcal{D}}$ and

$$\begin{aligned}\left\|\rho_m^{\mathcal{A}}(a)-\Phi_{\mathcal{A}}(a)\right\|&\to 0\quad\text{as } m\to\infty,\\ \left\|\rho_m^{\mathcal{B}}(b)-\Phi_{\mathcal{B}}(b)\right\|&\to 0\quad\text{as } m\to\infty.\end{aligned}$$

Putting $F_k^m=U_mP_{N_k}U_m^*(\mathcal{H})$ and $G_k^m=V_m^*W_mP_{N_k}W_m^*V_m(\mathcal{H})$. Note that $\dim F_k^m=\dim G_k^m$. We also have $F_1^m\subseteq F_2^m\subseteq\cdots$ and $G_1^m\subseteq G_2^m\subseteq\cdots$ are chains of finite dimensional subspaces of $\mathcal{H}$, and each F_k^m is $\rho_m^{\mathcal{A}}(\mathcal{A})$ invariant, each G_k^m is $\rho_m^{\mathcal{B}}(\mathcal{B})$ invariant. Since $P_{N_k}\to I$ in SOT as $k\to\infty$, we have $\bigcup_{k=1}^{\infty}F_k^m$ and $\bigcup_{k=1}^{\infty}G_k^m$ are both dense in $\mathcal{H}$. This completes the proof. $\square$

From Lemmas 4 and 5, we are able to obtain a proposition below which is a key for giving our main result.

PROPOSITION 2. *Let $\mathcal{A}$ and $\mathcal{B}$ be unital separable C^*-algebras in $\prod_{n=1}^{\infty}\mathcal{M}_{k_n}(\mathbb{C})$ and $\mathcal{D}$ be a common unital C^*-subalgebra of $\mathcal{A}$ and $\mathcal{B}$ in $\prod_{n=1}^{\infty}\mathcal{M}_{k_n}(\mathbb{C})$ which is finite-dimensional and abelian. Then $\mathcal{A}\underset{\mathcal{D}}{*}\mathcal{B}$ is RFD.*

Proof. Suppose that $\mathcal{D}=C^*(p_1,\dots,p_t)$ where $p_1,\dots,p_t$ are orthogonal projections with $\sum_{i=1}^{t}p_i=I$. Let $\Phi:\mathcal{A}\underset{\mathcal{D}}{*}\mathcal{B}\to\mathcal{B}(\mathcal{H})$ be a faithful essential representation on a separable Hilbert space $\mathcal{H}$. Then by Lemma 5, there are sequences $\{\rho_m^{\mathcal{A}}\}$ and $\{\rho_m^{\mathcal{B}}\}$ of representations of $\mathcal{A}$ and $\mathcal{B}$ on $\mathcal{H}$ such that $\rho_m^{\mathcal{B}}|_{\mathcal{D}}=\rho_m^{\mathcal{A}}|_{\mathcal{D}}$ and

$$\begin{aligned}\left\|\rho_m^{\mathcal{A}}(a)-\Phi_{\mathcal{A}}(a)\right\|&\to 0\quad\text{as } m\to\infty,\\ \left\|\rho_m^{\mathcal{B}}(b)-\Phi_{\mathcal{B}}(b)\right\|&\to 0\quad\text{as } m\to\infty.\end{aligned}$$

Moreover, for each $m \in \mathbb{N}$, we can find chains of finite-dimensional subspaces $F_1^m \subseteq F_2^m \subseteq \cdots$ and $G_1^m \subseteq G_2^m \subseteq \cdots$ of $\mathcal{H}$ with $\dim F_k^m = \dim G_k^m$ such that each F_k^m is $\rho_m^{\mathcal{A}}(\mathcal{A})$ invariant, each G_k^m is $\rho_m^{\mathcal{B}}(\mathcal{B})$ invariant, and $\bigcup_{k=1}^\infty F_k^m$, $\bigcup_{k=1}^\infty G_k^m$ are both dense in $\mathcal{H}$. Then, for each $m \in \mathbb{N}$, there are sequences of representations $\{\widetilde{\rho}_{m,k}^{\mathcal{A}}\}_{k=1}^\infty$ and $\{\widetilde{\rho}_{m,k}^{\mathcal{B}}\}_{k=1}^\infty$ of $\mathcal{A}$ and $\mathcal{B}$ on a finite-dimensional Hilbert space $\mathcal{H}_{m,k}$ by Lemma 4, such that

$$\widetilde{\rho}_{m,k}^{\mathcal{A}}|_{F_k^m} = \rho_m^{\mathcal{A}}|_{F_k^m}, \qquad \widetilde{\rho}_{m,k}^{\mathcal{B}}|_{G_k^m} = \rho_m^{\mathcal{B}}|_{G_k^m}$$

and $\widetilde{\rho}_{m,k}^{\mathcal{A}}|_{\mathcal{D}} = \widetilde{\rho}_{m,k}^{\mathcal{B}}|_{\mathcal{D}}$ for each $k \in \mathbb{N}$. We first take representations $\widetilde{\rho}_{1,1}^{\mathcal{B}}$, $\widetilde{\rho}_{1,1}^{\mathcal{A}}$ of $\mathcal{A}$ and $\mathcal{B}$ on $\mathcal{H}_1^1$, respectively. Then $\widetilde{\rho}_{1,1}^{\mathcal{B}}(p_i) = \widetilde{\rho}_{1,1}^{\mathcal{A}}(p_i)$ and

$$\widetilde{\rho}_{1,1}^{\mathcal{A}}|_{F_1^1} = \rho_1^{\mathcal{A}}|_{F_1^1}, \qquad \widetilde{\rho}_{1,1}^{\mathcal{B}}|_{G_1^1} = \rho_1^{\mathcal{B}}|_{G_1^1}.$$

Using the notation in Definition 2 and the fact that $\bigcup_{k=1}^\infty F_k^m$, $\bigcup_{k=1}^\infty G_k^m$ are both dense in $\mathcal{H}$ for each m, we can find $F_{l_2}^2$ and $G_{l_2}^2$ such that

$$\begin{aligned}\left\{\eta_1^1, \ldots, \eta_{t_1}^1\right\} &\subseteq_1 G_{l_2}^2,\\ \left\{\xi_1^1, \ldots, \xi_{t_1}^1\right\} &\subseteq_1 F_{l_2}^2,\end{aligned}$$

where $\{\xi_1^1, \ldots, \xi_{t_1}^1\}$ and $\{\eta_1^1, \ldots, \eta_{t_1}^1\}$ are linear bases of F_1^1 and G_1^1 respectively. Moreover, we have representations $\widetilde{\rho}_{2,l_2}^{\mathcal{A}}, \widetilde{\rho}_{2,l_2}^{\mathcal{B}}$ of $\mathcal{A}$ and $\mathcal{B}$ on $\mathcal{H}_{l_2}^2$ such that $\widetilde{\rho}_{2,l_2}^{\mathcal{B}}(p_i) = \widetilde{\rho}_{2,l_2}^{\mathcal{A}}(p_i)$ and

$$\widetilde{\rho}_{2,l_2}^{\mathcal{A}}|_{F_{l_2}^2} = \rho_2^{\mathcal{A}}|_{F_{l_2}^2}, \qquad \widetilde{\rho}_{2,l_2}^{\mathcal{B}}|_{G_{l_2}^2} = \rho_2^{\mathcal{B}}|_{G_{l_2}^2}.$$

Sequentially, we can find $F_{l_3}^3$ and $G_{l_3}^3$ satisfying

$$\begin{aligned}\left\{\xi_1^1, \ldots, \xi_{t_1}^1, \xi_1^2, \ldots, \xi_{t_m}^2\right\} &\subseteq_{\frac{1}{2}} F_{l_3}^3,\\ \left\{\eta_1^1, \ldots, \eta_{t_1}^1, \eta_1^2, \ldots, \eta_{t_m}^2\right\} &\subseteq_{\frac{1}{2}} G_{l_3}^3,\end{aligned}$$

where $\{\xi_1^2, \ldots, \xi_{t_2}^2\}$ and $\{\eta_1^2, \ldots, \eta_{t_2}^2\}$ are linear bases of $F_{l_2}^2$ and $G_{l_2}^2$, respectively. Meanwhile, representations $\widetilde{\rho}_{3,l_3}^{\mathcal{A}}, \widetilde{\rho}_{3,l_3}^{\mathcal{B}}$ of $\mathcal{A}$ and $\mathcal{B}$ are both on $\mathcal{H}_{l_3}^3$ with $\widetilde{\rho}_{3,l_3}^{\mathcal{B}}|_{\mathcal{D}} = \widetilde{\rho}_{3,l_3}^{\mathcal{A}}|_{\mathcal{D}}$ and

$$\widetilde{\rho}_{3,l_3}^{\mathcal{A}}|_{F_{l_3}^3} = \rho_3^{\mathcal{A}}|_{F_{l_3}^3}, \qquad \widetilde{\rho}_{3,l_3}^{\mathcal{B}}|_{G_{l_3}^3} = \rho_3^{\mathcal{B}}|_{G_{l_3}^3}.$$

So from the above construction, we can find a sequence $\{\widetilde{\rho}_{m,l_m}^{\mathcal{B}}\}_{m=1}^\infty$ of representations and a sequence $\{\widetilde{\rho}_{m,l_m}^{\mathcal{A}}\}_m^\infty$ of representations satisfying $\widetilde{\rho}_{m,l_m}^{\mathcal{B}}(p_i) = \widetilde{\rho}_{m,l_m}^{\mathcal{A}}(p_i)$ for each $m \in \mathbb{N}$. We still have that $\bigcup_{m=1}^\infty F_{l_m}^m$, $\bigcup_{m=1}^\infty G_{l_m}^m$ are both dense in $\mathcal{H}$. Let $\widetilde{\rho}_{m,l_m} : \mathcal{A} \underset{\mathcal{D}}{*} \mathcal{B} \to \mathcal{B}(\mathcal{H}_{l_m}^m)$ be the *-representation such that $\widetilde{\rho}_{m,l_m}|_{\mathcal{A}} = \widetilde{\rho}_{m,l_m}^{\mathcal{A}}$ and $\widetilde{\rho}_{m,l_m}|_{\mathcal{B}} = \widetilde{\rho}_{m,l_m}^{\mathcal{B}}$. We want to show that, for a given $x \in \mathcal{A} \underset{\mathcal{D}}{*} \mathcal{B}$ and any $\varepsilon > 0$, there is $k \in \mathbb{N}$ such that

$$\left\|\widetilde{\rho}_{k,l_k}(x)\right\| \geq \|x\| - \varepsilon.$$

This will suffice to show that $\mathcal{A} \underset{\mathcal{D}}{*} \mathcal{B}$ is RFD. Write $x = w_1 + \cdots + w_M$ as the sum of finitely many words w_i in $\mathcal{A}$ and $\mathcal{B}$. Assume $\xi \in \mathcal{H}$ is a unit vector such that $\|\Phi(x)\xi\| \geq \|\xi\| - \frac{\varepsilon}{2}$. We will show that for every $i \in \{1, \dots, M\}$, there is $k(i)$ such that if $k \geq k(i)$, then

$$\big\|\widetilde{\rho}_{k,l_k}(w_i)\xi - \Phi(w_i)\xi\big\| < \varepsilon/2M.$$

Taking $k \geq \max_{1\leq i\leq M} k(i)$, this will imply $\|\widetilde{\rho}_{k,l_k}(x)\xi - \Phi(x)\xi\| < \varepsilon/2$, which will yield what we want. To show it, write

$$w_i = a_l a_{l-1} \cdots a_2 a_1$$

for some $l \in \mathbb{N}$ and $a_1, \dots, a_l \in \mathcal{A} \cup \mathcal{B}$. Let $\xi_0 = \xi$, $\xi_j = \Phi(a_j)\xi_{j-1}$ $(1 \leq j \leq l)$ and $N = \max_{1\leq j\leq l} \|a_j\|$. Choose k large enough to ensure that

$$\max\big(\operatorname{dist}\big(\xi_{j-1}, F^k_{l_k}\big), \operatorname{dist}\big(\xi_{j-1}, G^k_{l_k}\big)\big) \leq \varepsilon/\big(8lMN^{l-j}\big)$$

and

$$\big\|\Phi(a_j) - \rho^{\mathcal{A}}_k(a_j)\big\| < \frac{\varepsilon}{8lMN^{l-1}} \quad \text{if } a_j \in \mathcal{A}$$

or

$$\big\|\Phi(a_j) - \rho^{\mathcal{B}}_k(a_j)\big\| < \frac{\varepsilon}{8lMN^{l-1}} \quad \text{if } a_j \in \mathcal{B}$$

for any $j \in \{1, \dots, l\}$. Let $\eta \in \mathcal{H}$. If $a_j \in \mathcal{A}$, let $\eta_k = P_{F^k_{l_k}}(\eta) \in F^k_{l_k}$, then

$$\begin{aligned}
&\big\|\Phi(a_j)\eta - \widetilde{\rho}_{k,l_k}(a_j)\eta\big\| \\
&\quad\leq \big\|\Phi(a_j)\eta - \widetilde{\rho}_{k,l_k}(a_j)\eta_k\big\| + \big\|\widetilde{\rho}_{k,l_k}(a_j)\eta_k - \widetilde{\rho}_{k,l_k}(a_j)\eta\big\| \\
&\quad\leq \big\|\Phi(a_j)\eta - \Phi(a_j)\eta_k + \Phi(a_j)\eta_k - \widetilde{\rho}_{k,l_k}(a_j)\eta_k\big\| \\
&\qquad + \big\|\widetilde{\rho}_{k,l_k}(a_j)\eta_k - \widetilde{\rho}_{k,l_k}(a_j)\eta\big\| \\
&\quad\leq 2\|a_j\| \operatorname{dist}\big(\eta, F^k_{l_k}\big) + \big\|\Phi(a_j)\eta_k - \rho^{\mathcal{A}}_k(a_j)\eta_k\big\| \\
&\quad\leq 2\|a_j\| \operatorname{dist}\big(\eta, F^k_{l_k}\big) + \frac{\varepsilon}{4lMN^{l-1}}\|\eta_k\|.
\end{aligned}$$

Similarly, if $a_j \in \mathcal{B}$, then let $\eta_k = P_{G^k_{l_k}}(\eta) \in G^k_{l_k}$, then

$$\big\|\Phi(a_j)\eta - \widetilde{\rho}_{k,l_k}(a_j)\eta\big\| \leq 2\|a_j\| \operatorname{dist}\big(\eta, G^k_{l_k}\big) + \frac{\varepsilon}{4lMN^{l-1}}\|\eta_k\|.$$

Therefore,

$$\begin{aligned}
&\big\|\Phi(w_i)\xi - \widetilde{\rho}_{k,l_k}(w_i)\xi\big\| \\
&\quad= \big\|\Phi(a_l a_{l-1} \cdots a_2)\Phi(a_1)\xi_0 - \widetilde{\rho}_{k,l_k}(a_l a_{l-1} \cdots a_2)\widetilde{\rho}_k(a_1)\xi_0\big\| \\
&\quad\leq \big\|\Phi(a_l a_{l-1} \cdots a_2)\Phi(a_1)\xi_0 - \widetilde{\rho}_{k,l_k}(a_l a_{l-1} \cdots a_2)\Phi(a_1)\xi_0 \\
&\qquad + \widetilde{\rho}_k(a_l a_{l-1} \cdots a_2)\Phi(a_1)\xi_0 - \widetilde{\rho}_{k,l_k}(a_l a_{l-1} \cdots a_2)\widetilde{\rho}_k(a_1)\xi_0\big\| \\
&\quad\leq \big\|\Phi(a_l a_{l-1} \cdots a_2)\xi_1 - \widetilde{\rho}_{k,l_k}(a_l a_{l-1} \cdots a_2)\xi_1\big\| \\
&\qquad + \big\|\widetilde{\rho}_{k,l_k}(a_l a_{l-1} \cdots a_2)\big\|\big\|\Phi(a_1)\xi_0 - \widetilde{\rho}_{k,l_k}(a_1)\xi_0\big\|
\end{aligned}$$

$$\le \sum_{j=1}^{l-1} \big\|\widetilde{\rho}_{k,l_k}(a_l \cdots a_{j+1})\big\| \big\|\Phi(a_j)\xi_{j-1} - \widetilde{\rho}_{k,l_k}(a_j)\xi_{j-1}\big\|$$
$$< \sum_{j=1}^{l-1} N^{l-j+1} \cdot 2N$$
$$\times \left(\max\big(\operatorname{dist}\big(\xi_{j-1}, F^k_{l_k}\big), \operatorname{dist}\big(\xi_{j-1}, G^k_{l_k}\big)\big) + \frac{\varepsilon}{4lMN^l} N^{j-1}\|\xi_0\|\right)$$
$$= \frac{\varepsilon}{2M}.$$

It follows that $\mathcal{A} \underset{\mathcal{D}}{*} \mathcal{B}$ is RFD. □

The following lemma can be found in [4]. Combining previous lemmas and proposition as well as the lemma below, we will be ready to state and prove our main result.

LEMMA 6 (Lemma 2.2, [4]). *Let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-algebras having $\mathcal{D}$ embedded as a unital C^*-subalgebra of each of them. Let*

$$\mathcal{C} = \mathcal{A} \underset{\mathcal{D}}{*} \mathcal{B}$$

be the full amalgamated free product of $\mathcal{A}$ and $\mathcal{B}$ over $\mathcal{D}$. If there is a projection $p \in \mathcal{D}$ and there are partial isometries $v_1, \ldots, v_n \in \mathcal{D}$ such that $v_i^* v_i \le p$ and $\sum_{i=1}^n v_i v_i^* = 1 - p$, then

$$p\mathcal{C}p \cong (p\mathcal{A}p) \underset{p\mathcal{D}p}{*} (p\mathcal{B}p).$$

THEOREM 2. *Let $\mathcal{A}$, $\mathcal{B}$ be separable unital C^*-algebras and $\mathcal{D}$ be a finite-dimensional C^*-algebra. Suppose $\psi_{\mathcal{A}} : \mathcal{D} \to \mathcal{A}$ and $\psi_{\mathcal{B}} : \mathcal{D} \to \mathcal{B}$ are unital embeddings. Then $\mathcal{A} \underset{\mathcal{D}}{*} \mathcal{B}$ is RFD if and only if there are unital embeddings $q_1 : \mathcal{A} \to \prod_{n=1}^\infty \mathcal{M}_{k_n}(\mathbb{C})$ and $q_2 : \mathcal{B} \to \prod_{n=1}^\infty \mathcal{M}_{k_n}(\mathbb{C})$ for a sequence $\{k_n\}_{n=1}^\infty$ of integers such that the following diagram commutes*

$$\begin{array}{ccc} \mathcal{D} & \overset{\psi_{\mathcal{A}}}{\hookrightarrow} & \mathcal{A} \\ {\scriptstyle \psi_{\mathcal{B}}}\downarrow & & \downarrow{\scriptstyle q_1} \\ \mathcal{B} & \overset{q_2}{\hookrightarrow} & \prod_{m=1}^\infty \mathcal{M}_{k_m}(\mathbb{C}). \end{array}$$

Proof. If $\mathcal{A} \underset{\mathcal{D}}{*} \mathcal{B}$ is RFD, then there is a unital embedding

$$\Phi : \mathcal{A} \underset{\mathcal{D}}{*} \mathcal{B} \to \prod_{n=1}^\infty \mathcal{M}_{k_n}(\mathbb{C})$$

for a sequence $\{k_n\}_{n=1}^\infty$ of integers. Let q_1 and q_2 be the restrictions of Φ on $\mathcal{A}$ and $\mathcal{B}$ respectively. Then the above diagram is commutative. Conversely, we may assume that $\mathcal{A}$, $\mathcal{B}$ are unital subalgebras of $\prod_{n=1}^{\infty}\mathcal{M}_{k_n}(\mathbb{C})$ for a sequence $\{k_n\}_{n=1}^{\infty}$ of integers and $\mathcal{A}\supseteq\mathcal{D}\subseteq\mathcal{B}$ are unital inclusions of C*-algebras. Since $\mathcal{D}$ is a finite-dimensional C*-subalgebra, we can find a projection $p\in\mathcal{D}$ and partial isometries $v_1,\ldots,v_n\in\mathcal{D}$ such that $v_i^*v_i\leq p$ and $\sum_{i=1}^{n}v_iv_i^*=1-p$. Therefore, for showing $\mathcal{A}\underset{\mathcal{D}}{*}\mathcal{B}$ is RFD, it is sufficient to show that $P\mathcal{A}P\underset{P\mathcal{D}P}{*}P\mathcal{B}P$ is RFD by Lemma 6 and Lemma 2.1 in [4]. Since $P\mathcal{D}P$ is a finite-dimensional abelian C*-algebra. Then the desired result follows from Proposition 2. □

Corollary 1. *Let $\mathcal{A}\supseteq\mathcal{D}\subseteq\mathcal{B}$ be unital C*-inclusions of C*-algebras in the C*-algebra $\prod_{n=1}^{\infty}\mathcal{M}_{k_n}(\mathbb{C})$ and $\mathcal{D}$ is a unital finite-dimensional C*-subalgebra. Then $\mathcal{A}\underset{\mathcal{D}}{*}\mathcal{B}$ is RFD.*

Corollary 2. *Suppose that $\mathcal{A}$ is a separable unital RFD C*-algebra and $\mathcal{D}$ is a unital finite-dimensional C*-subalgebra of $\mathcal{A}$. Then $\mathcal{A}\underset{\mathcal{D}}{*}\mathcal{A}$ is RFD.*

Corollary 3. *For unital C*-inclusions $\mathcal{D}\subseteq\mathcal{B}\subseteq\mathcal{A}$ and $\mathcal{D}\subseteq\mathcal{C}\subseteq\mathcal{A}$, if $\mathcal{A}$ is a separable unital RFD algebra and $\mathcal{D}$ is finite-dimensional, then $\mathcal{B}\underset{\mathcal{D}}{*}\mathcal{C}$ is RFD.*

Example 2. Let $\mathcal{M}_k(\mathbb{C})\supseteq\mathcal{D}\subseteq\mathcal{M}_l(\mathbb{C})$ be unital inclusions of unital C*-algebras. If $\mathrm{tr}_k|_{\mathcal{D}}=\mathrm{tr}_l|_{\mathcal{D}}$ where tr_k and tr_l are tracial states on $\mathcal{M}_k(\mathbb{C})$ and $\mathcal{M}_l(\mathbb{C})$ respectively, then there exists an integer n and there are two unital embeddings $q_1:\mathcal{M}_k(\mathbb{C})\to\mathcal{M}_n(\mathbb{C})$ and $q_2:\mathcal{M}_l(\mathbb{C})\to\mathcal{M}_n(\mathbb{C})$ such that $q_1|_{\mathcal{D}}=q_2|_{\mathcal{D}}$. It implies that there is a commutative diagram which is same as the one in Theorem 2. Therefore, $\mathcal{M}_k(\mathbb{C})\underset{\mathcal{D}}{*}\mathcal{M}_l(\mathbb{C})$ is RFD. In fact, this result has been proved in [4].

Remark 3. From the previous example and the fact that every MF algebra has a tracial state, it is not hard to see that $\mathcal{M}_k(\mathbb{C})\underset{\mathcal{D}}{*}\mathcal{M}_l(\mathbb{C})$ is RFD if and only if $\mathcal{M}_k(\mathbb{C})\underset{\mathcal{D}}{*}\mathcal{M}_l(\mathbb{C})$ is an MF algebra.

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