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ON THE ESTIMATION OF NONLINEAR TWISTS OF THE LIOUVILLE FUNCTION

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Dedicated in fond memory of Professor K. Ramachandra on his eightieth birthday

Abstract. We prove a nontrivial upper bound for the quantity (with $\mathbf{e}(z)=e^{2\pi i z}$),

$$\Big|\sum_{X\le n\le 2X}\lambda(n)\mathbf{e}(\alpha\sqrt{n})\Big|,$$

where α is any nonzero real number. This upper bound is an improvement of the earlier known results.

1. Introduction

In studying equi-distribution theory, zero-distribution of L-functions and so on, nonlinear exponential twists of arithmetic functions arise naturally. A more general nonlinear exponential sum is of the form

$$S(X,\alpha)=\sum_{X\le n\le 2X}a_n\mathbf{e}(\alpha\sqrt{n}),\quad 0\neq\alpha\in\mathbb{R}.$$

Here as usual $\mathbf{e}(z)=e^{2\pi i z}$. The sum $S(X,\alpha)$ was first studied by Vinogradov when $a_n=\Lambda(n)$, the von Mangoldt function (see [3], [2] and [13]). For $a_n=\Lambda(n)$ and $a_n=\mu(n)$ (μ being the Möbius function), it has been established by Iwaniec, Luo and Sarnak (see [4]) that, the sums $S(X,\alpha)$ are highly related to the L-functions of GL_2. When f is a holomorphic cusp form of even integral weight, they proved that a good upper bound for $|S(X,\alpha)|$ implies the quasi-Riemann hypothesis for $L(s,f)$ on the upper half-plane. Upper bounds for various a_n have been studied in [8] and [11].

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In this paper, we are interested on the sum $S(X,\alpha)$ when $a_n = \lambda(n)$ or $\mu(n)$ where the Liouville function $\lambda(n)$ is completely multiplicative and it is defined by $\lambda(p^k) = (-1)^k$ for prime powers and using complete multiplicativity of $\lambda(n)$ for any positive integer n with $\lambda(1) = 1$, and the Möbius function $\mu(n)$ is defined as $\mu(1) = 1, \mu(p_1 p_2 \cdots p_k) = (-1)^k$, $\mu(n) = 0$ if n is divisible by p^l with $l \geq 2$ for any prime $p \geq 2$.

As mentioned in [8], it is interesting to note that the hypothesis that for some $\theta < 1$, the bound

$$\sum_{n \leq X} \lambda(n) = O\big(X^{\theta}\big)$$

implies the quasi-Riemann hypothesis. This approach is due to Pólya. It should be noted that the Riemann hypothesis is equivalent to the assertion that the above estimate holds for every $\theta > \frac{1}{2}$. A relevant sum, namely

$$S_q(X) = \sum_{n} a_n \mathbf{e}(-2\sqrt{nq})\phi\left(\frac{n}{X}\right)$$

with $a_n \ll n^{\varepsilon}$ for any $\varepsilon > 0$ and ϕ being a smooth function compactly supported on $\mathbb{R}^+$ has been considered by Iwaniec, Luo and Sarnak in [4] and they established the bound that

$$S_q(X) \ll q^{\frac{1}{4}} X^{\frac{3}{4}+\varepsilon}, \tag{1.1}$$

under the assumption (see C.4, p. 122 of [4]) that the associated L-function $A(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ has a holomorphic continuation to $\Re s > \frac{1}{2}$ except for a possible pole of finite order at $s = 1$ and satisfies the bound

$$A(s) \ll |s|^{\varepsilon}, \quad \text{if } \Re s = \sigma$$

for any $\frac{1}{2} < \sigma < 1$ and any $\varepsilon > 0$ (the implied constant depends only on σ and ε). It should be noted that the result in (1.1) is more general. They have also dealt therein with some interesting examples and they even showed that the exponent $3/4$ in (1.1) can not be improved in certain cases (see for example C.33 of [4]).

In [11], Qingfeng Sun established unconditionally that (for any $\varepsilon > 0$ and for any $0 \neq \alpha \in \mathbb{R}$),

$$\sum_{n \sim X} \lambda(n)\mathbf{e}(\alpha\sqrt{n}) \ll X^{\frac{5}{6}} \log^{7/2} X + \left(1 + \frac{1}{|\alpha|}\right)^{1/2} X^{3/4} \log^4 X + (1+|\alpha|)^{1/2} X^{3/4} \log^{7/2} X + \left(|\alpha| + \frac{1}{|\alpha|}\right) X^{1/2+\varepsilon} \tag{1.2}$$

holds where the implied constant depends only on ε.

The aim here is to prove unconditionally Theorem 1.

THEOREM 1. *For any* $0 \neq \alpha \in \mathbb{R}$, *we have*

$$\sum_{X \le n \le 2X} \lambda(n)\mathbf{e}(\alpha\sqrt{n}) \ll_\varepsilon X^{3/4}(\log X)^{7/2}(1+|\alpha|)^{1/2} + X^{1/2}(\log X)^{7/2}\left(1+\frac{1}{|\alpha|}\right)^{1/2} + X^{\frac{1}{2}+\varepsilon}\left(|\alpha| + \frac{1}{|\alpha|}\right).$$

REMARK 1. Theorem 1 improves the bound in (1.2). This improvement essentially comes from (2.7) and using the idea of exponent pairs to estimate certain quantities (see Sections 5 and 6) in a better fashion. In addition, we need to show that with our choice of the parameters U and V, the contribution coming from the sum $S_{1,1}^{(1)}(X,\alpha)$ is controllable to at most what is claimed in the Theorem 1. This is done in Section 4. An analogous result can be proved for nonlinear exponential sums of similar type where $\lambda(n)$ is replaced by $\mu(n)$. More precisely, we also have Theorem 2.

THEOREM 2. *For any* $0 \neq \alpha \in \mathbb{R}$, *we have*

$$\sum_{X \le n \le 2X} \mu(n)\mathbf{e}(\alpha\sqrt{n}) \ll_\varepsilon X^{3/4}(\log X)^{7/2}(1+|\alpha|)^{1/2} + X^{1/2}(\log X)^{7/2}\left(1+\frac{1}{|\alpha|}\right)^{1/2} + X^{\frac{1}{2}+\varepsilon}\left(|\alpha| + \frac{1}{|\alpha|}\right).$$

REMARK 2. It should be noted that if $\alpha = -2\sqrt{q}$ with any integer $q \geq 1$ and $a_n = \lambda(n)$ or $\mu(n)$, then Theorems 1 and 2 agree with the estimate in (1.1) uniformly for $1 \le q \le c_1 X$ for some effective positive constant c_1. Conjecturally one expects that the upper bounds of Theorems 1 and 2 to hold with the exponent of X to be $\frac{1}{2}+\varepsilon$. However, it seems to be really deep and difficult to achieve.

2. Notation and preliminaries

The letters C and A (and c and a) (with or without suffixes) denote effective positive constants unless they are specified. They need not be the same at every occurrence. The notation ε always denotes any arbitrarily small positive constant. Throughout the paper, we assume $T \geq T_0$ and $X \geq X_0$, where T_0 and X_0 are large positive constants. We write $f(x) \ll g(x)$ to mean $|f(x)| < C_1 g(x)$ for $x \geq x_0$ (sometimes we denote this by the O notation also). Let $s = \sigma + it$, and $w = u + iv$. The notation $n \sim X$ means that $X \le n \le 2X$. For $n \geq 1$ an integer, let $\tau(n)$ denote the number of positive integral divisors of n.

Following Davenport (see p. 139 of [1]) and Murty and Sankaranarayanan (see Section 2 of [8]), we can describe the Vaughan's identity as follows. For any $A, B \neq 0$ and F, G, we have the formal identity

$$\frac{A}{B} = F - BGF + AG + \left(\frac{A}{B} - F\right)(1 - BG). \tag{2.1}$$

Applying this identity (2.1) with $A(s)=\zeta(2s), B(s)=\zeta(s), F(s)=\sum_{n\le U}\lambda(n)n^{-s}$ and $G(s)=\sum_{n\le V}\mu(n)n^{-s}$, then we have

$$\lambda(n)=a_1(n)+a_2(n)+a_3(n)+a_4(n), \tag{2.2}$$

where

$$a_1(n)=\begin{cases}\lambda(n) & \text{for } n\le U\\ 0 & \text{for } n>U,\end{cases} \qquad a_2(n)=-\sum_{\substack{abc=n\\ b\le V, c\le U}}\mu(b)\lambda(c), \tag{2.3}$$

$$a_3(n)=\sum_{\substack{b^2c=n\\ c\le V}}\mu(c), \qquad a_4(n)=\sum_{\substack{abc=n\\ b>V, c>U}}\mu(b)\lambda(c).$$

In (2.3), $U\ge 1$ and $V\ge 1$ are free parameters to be chosen suitably later and $\zeta(s)$ denotes the Riemann zeta-function. Note that

$$\frac{\zeta(2s)}{\zeta(s)}=\sum_{n=1}^{\infty}\frac{\lambda(n)}{n^s}, \quad \Re s>1. \tag{2.4}$$

Reduction process. Here after, our

$$S(X,\alpha):=\sum_{n\sim X}\lambda(n)\mathbf{e}(\alpha\sqrt{n}),$$

where $0\ne\alpha\in\mathbb{R}$. To simplify the arguments below, in Vaughan's identity (2.2), we make the choice of the free parameters U,V such that $U=V$ and we suppose that these parameters satisfy the condition $1\le U=V\le\frac{1}{100}X^{1/3}$ and of course $X\ge X_0$ where X_0 is sufficiently large. Then, we observe that for $n\sim X$, $a_1(n)=0$ and

$$S(X,\alpha)=S_1(X,\alpha)+S_2(X,\alpha)+S_3(X,\alpha), \tag{2.5}$$

where

$$S_1(X,\alpha)=-\sum_{n\sim X}\sum_{\substack{abc=n\\ b,c\le U}}\mu(b)\lambda(c)\mathbf{e}(\alpha\sqrt{n}),$$

$$S_2(X,\alpha)=\sum_{n\sim X}\sum_{\substack{b^2c=n\\ c\le U}}\mu(c)\mathbf{e}(\alpha\sqrt{n})$$

and

$$S_3(X,\alpha)=\sum_{n\sim X}\sum_{\substack{abc=n\\ b,c>U}}\mu(b)\lambda(c)\mathbf{e}(\alpha\sqrt{n}). \tag{2.6}$$

We notice that for $X_1\ge 0, X_2\ge 0$ and for $0\le\delta\le 1$, we have

$$\min(X_1,X_2)\le X_1^{\delta}X_2^{1-\delta}.$$

Therefore, we obtain

$$\begin{aligned}
(2.7)\qquad \left|S_2(X,\alpha)\right| &\le \sum_{b\le\sqrt{2X}} \sum_{c\le\min(U,2X/b^2)} 1\\
&\le \sum_{b\le\sqrt{2X}} \min\bigl(U,2X/b^2\bigr)\\
&\le \sum_{b\le\sqrt{2X}} U^{1/2}\frac{\sqrt{2X}}{b}\\
&\ll X^{1/2}U^{1/2}\log X\\
&\ll X^{2/3}\log X \quad \left(\text{since } 1\le U=V\le \frac{1}{100}X^{1/3}\right).
\end{aligned}$$

We follow certain arguments of Zhao (see [14]). We set $W=\sqrt{2X}$. Then, we have

$$\begin{aligned}
(2.8)\qquad S_1(X,\alpha) &= -\sum_{n\sim X}\sum_{\substack{abc=n\\ b,c\le U}} \mu(b)\lambda(c)\mathbf{e}(\alpha\sqrt{n})\\
&= -\sum_{n\sim X}\sum_{\substack{abc=n\\ b,c\le U,a\ge W}} \mu(b)\lambda(c)\mathbf{e}(\alpha\sqrt{n})\\
&\quad -\sum_{n\sim X}\sum_{\substack{abc=n\\ b,c\le U,\frac{X}{U^2}\le a<W}} \mu(b)\lambda(c)\mathbf{e}(\alpha\sqrt{n})\\
&= -S_{1,1}(X,\alpha)-S_{1,2}(X,\alpha) \quad \text{(say)}
\end{aligned}$$

and

$$\begin{aligned}
(2.9)\qquad S_3(X,\alpha) &= \sum_{n\sim X}\sum_{\substack{abc=n\\ b,c>U}} \mu(b)\lambda(c)\mathbf{e}(\alpha\sqrt{n})\\
&= \sum_{n\sim X}\sum_{\substack{abc=n\\ c>U,U<b<W}} \mu(b)\lambda(c)\mathbf{e}(\alpha\sqrt{n})\\
&\quad +\sum_{n\sim X}\sum_{\substack{abc=n\\ b\ge W,U<c\le W}} \mu(b)\lambda(c)\mathbf{e}(\alpha\sqrt{n})\\
&= S_{3,1}(X,\alpha)+S_{3,2}(X,\alpha) \quad \text{(say)}.
\end{aligned}$$

Throughout the paper, our choice for U and V are going to be $U=V=100X^{1/4}$.

Exponent pairs. We are interested on estimating the sum

$$(2.10)\qquad S:=\sum_{B\le n\le B+h} \mathbf{e}\bigl(f(n)\bigr) \quad (B\ge 1, 1<h\le B).$$

We suppose that $A \ll |f^{(1)}(x)| \ll A (A > \frac{1}{2})$. More generally, one can suppose that

$$AB^{1-r} \ll \left|f^{(r)}(x)\right| \ll AB^{1-r} \quad (r = 1, 2, 3, \ldots). \tag{2.11}$$

An exponent pair is a pair of numbers (κ, λ) with $0 \le \kappa \le \frac{1}{2} \le \lambda \le 1$ for which the estimate

$$|S| := \left|\sum_{B \le n \le B+h} \mathbf{e}\big(f(n)\big)\right| \ll A^{\kappa} B^{\lambda} \tag{2.12}$$

holds. Trivially $(0, 1)$ is an exponent pair. Using

$$\sum_{a < n \le b} \mathbf{e}\big(f(n)\big) = \sum_{\alpha - \eta < m < \beta + \eta} \int_a^b \mathbf{e}\big(f(x) - mx\big)\, dx + O\big(\log(\beta - \alpha + 2)\big) \tag{2.13}$$

and the Lemma 3.2 to estimate each integral as $\ll m_2^{-1/2}$, one obtains the bound

$$S \ll (AB)^{1/2}. \tag{2.14}$$

This means that the pair $(\frac{1}{2}, \frac{1}{2})$ is an exponent pair. We note that the set of all exponent pairs forms a convex set. We also know that (see [3]) there are at least three processes through which we can produce a lot of exponent pairs. Given exponent pairs (κ, λ) and (κ_1, λ_1), these processes are:

$$\text{Process A: } A(\kappa, \lambda) = \left(\frac{\kappa}{2\kappa + 2}, \frac{1}{2} + \frac{\lambda}{2\kappa + 2}\right),$$

$$\text{Process B: } B(\kappa, \lambda) = \left(\lambda - \frac{1}{2}, \kappa + \frac{1}{2}\right)$$

and

$$\text{Process C(t): } C(t)(\kappa, \lambda)(\kappa_1, \lambda_1) = \big(\kappa t + \kappa_1(1 - t), \lambda t + \lambda_1(1 - t)\big) \quad (0 \le t \le 1).$$

The output pairs coming from these processes are indeed exponent pairs for which we refer to [3]. Some of the exponent pairs are: $(\frac{1}{6}, \frac{2}{3}), (\frac{2}{7}, \frac{4}{7}), (\frac{5}{24}, \frac{15}{24})$, $(\frac{4}{11}, \frac{6}{11})$ etc. If $\alpha = 0.3290213568\ldots$, then Rankin showed that $(\kappa, \lambda) = (\frac{\alpha}{2} + \varepsilon, \frac{1}{2} + \frac{\alpha}{2} + \varepsilon)$ is an exponent pair such that the function $F(\kappa, \lambda) = \kappa + \lambda$ is minimal.

3. Some lemmas

LEMMA 3.1. *Let $f(x)$ be a real-valued function, differentiable on $[a, b]$. If $f'(x)$ is monotonic and $f'(x) \ge m > 0$ or $f'(x) \le -m < 0$ throughout the interval $[a, b]$, then*

$$\left|\int_a^b e^{if(x)}\, dx\right| \le \frac{4}{m}.$$

Proof. This is Lemma 4.2 of [12]. □

LEMMA 3.2. *Let $f(x)$ be a real-valued function, twice differentiable on $[a,b]$. If $f''(x) \geq \nu > 0$ or $f''(x) \leq -\nu < 0$ throughout the interval $[a,b]$, then*

$$\left| \int_a^b e^{if(x)}\,dx \right| \leq \frac{8}{\sqrt{\nu}}.$$

Proof. This is Lemma 4.4 of [12]. □

LEMMA 3.3. *Let $f(x)$ be real-valued function with $|f'(x)| \leq \theta < 1$ and $f'(x) \neq 0$ on $[a,b]$. Then,*

$$\sum_{a<n\leq 2a} \mathbf{e}\big(f(n)\big) = \int_a^{2a} \mathbf{e}\big(f(x)\big)\,dx + O\left(\frac{1}{1-\theta}\right).$$

Proof. This is Lemma 4.8 of [12] with precise O-term. For the proof, see, for instance, Theorem 7.17 of [5] or Lemma 1.2 of [3]. □

LEMMA 3.4. *Let $X,T \geq 1$. For any complex numbers a_n, we have*

$$\int_{-T}^{T} \Big| \sum_{1\leq n\leq X} a_n n^{-it} \Big|^2 dt \ll (T+X) \sum_{1\leq n\leq X} |a_n|^2.$$

Proof. This is some what a weaker version of Montgomery–Vaughan theorem (see [6] or [7] or [9] or [5]). □

4. The estimation of $S_{1,1}(X,\alpha)$

We follow some arguments of [11] closely. Recall that $W = \sqrt{2X}$. Let

$$H(d) := \sum_{\substack{bc=d\\ b,c\leq U}} \mu(b)\lambda(c). \tag{4.1}$$

Then (since $W = \sqrt{2X}$),

$$\begin{aligned} S_{1,1}(X,\alpha) &= \sum_{n\sim X} \sum_{\substack{abc=n\\ b,c\leq U, a\geq W}} \mu(b)\lambda(c)\mathbf{e}(\alpha\sqrt{n}) \\ &= \sum_{n\sim X} \sum_{\substack{ad=n\\ a\geq W}} H(d)\mathbf{e}(\alpha\sqrt{n}) \\ &= \sum_{n\sim X} \sum_{\substack{ad=n\\ a\geq W, d\leq W}} H(d)\mathbf{e}(\alpha\sqrt{n}) \\ &= \sum_{n\sim X} \sum_{\substack{ad=n\\ d\leq W}} H(d)\mathbf{e}(\alpha\sqrt{n}) - \sum_{n\sim X} \sum_{\substack{ad=n\\ a,d\leq W}} H(d)\mathbf{e}(\alpha\sqrt{n}) \\ &= S_{1,1}^{(1)}(X,\alpha) - S_{1,1}^{(2)}(X,\alpha) \quad \text{(say)}. \end{aligned} \tag{4.2}$$

Treatment of $S_{1,1}^{(1)}(X,\alpha)$. Let

$$F(u) := \sum_{n\le u}\sum_{\substack{ad=n\\ d\le W}} H(d). \tag{4.3}$$

Then

$$S_{1,1}^{(1)}(X,\alpha) = \sum_{n\sim X}\sum_{\substack{ad=n\\ d\le W}} H(d)\mathbf{e}(\alpha\sqrt{n}) = \int_X^{2X} \mathbf{e}(\alpha\sqrt{u})\,dF(u). \tag{4.4}$$

For $\Re s > 1$, consider

$$\zeta(s)\Big(\sum_{d\le W}\frac{H(d)}{d^s}\Big) = \sum_{n=1}^{\infty}\frac{b(n)}{n^s} \quad \text{where } b(n) := \sum_{\substack{ad=n\\ d\le W}} H(d). \tag{4.5}$$

Note that from (4.4), the integration variable u varies in the interval $[X, 2X]$. From Perron's formula, with $T = X$ (see Lemma 3.19 of [12], see also Corollary 2 of [10]), for any $\varepsilon > 0$, we have

$$F(u) = \sum_{n\le u} b(n) = \frac{1}{2\pi i}\int_{1+\frac{1}{\log X}-iT}^{1+\frac{1}{\log X}+iT} \zeta(s)\Big(\sum_{d\le W}\frac{H(d)}{d^s}\Big)\frac{u^s}{s}\,ds + O\big(X^{\varepsilon}\big). \tag{4.6}$$

We move the line of integration in (4.6) to $\Re s = \frac{1}{2}$. Note that $\zeta(s)$ has a simple pole at $s = 1$ and hence by Cauchy's residue theorem, we obtain (since $u \le 2X$)

$$\begin{aligned} F(u) &= u\Big(\sum_{d\le W}\frac{H(d)}{d}\Big) \\ &\quad + \frac{1}{2\pi i}\Big(\int_{\frac{1}{2}-iT}^{1+\frac{1}{\log X}-iT} + \int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} \\ &\quad + \int_{\frac{1}{2}+iT}^{1+\frac{1}{\log X}+iT}\Big)\zeta(s)\Big(\sum_{d\le W}\frac{H(d)}{d^s}\Big)\frac{u^s}{s}\,ds \\ &\quad + O\big(X^{\varepsilon}\big). \end{aligned} \tag{4.7}$$

We note that $|H(d)| \le \tau(d)$, where $\tau(d)$ denotes the number of divisors of d. Keeping in mind $T = X, W = \sqrt{2X}$, we observe that the horizontal lines contributions from (4.7) in absolute value is at most (since $u \le 2X$)

$$\begin{aligned} Q_1 &\ll \int_{1/2}^{1+\frac{1}{\log X}} \big|\zeta(\sigma \pm iT)\big|\Big(\sum_{d\le W}\frac{H(d)}{d^{\sigma}}\Big)\frac{X^{\sigma}}{T}\,d\sigma \\ &\ll \max_{\frac{1}{2}\le\sigma\le 1+\frac{1}{\log X}} \frac{T^{\frac{1}{2}(1-\sigma)}(\log T)W^{1-\sigma}(\log^2 W)X^{\sigma}}{T} \end{aligned} \tag{4.8}$$

$$\left(\text{since } \zeta(\sigma \pm iT) \ll T^{\frac{1}{2}(1-\sigma)} \log T \text{ for } \frac{1}{2} \le \sigma \le 1, |H(d)| \le \tau(d)\right)$$

$$\ll \max_{\frac{1}{2} \le \sigma \le 1 + \frac{1}{\log X}} \frac{(\sqrt{2X})^{1-\sigma} (\log^3 X) X^{\sigma + \frac{1}{2}(1-\sigma)}}{X}$$

$$(\text{since } T = X, W = \sqrt{2X})$$

$$\ll \log^3 X.$$

Note that we have used the bound

$$\sum_{n \le x} \big(\tau(n)\big)^l \ll x (\log x)^{2^l - 1}.$$

Therefore,

$$F(u) = u\Big(\sum_{d \le W} \frac{H(d)}{d}\Big) + \frac{1}{2\pi i} \int_{\frac{1}{2} - iT}^{\frac{1}{2} + iT} \zeta(s) \Big(\sum_{d \le W} \frac{H(d)}{d^s}\Big) \frac{u^s}{s}\, ds + O\big(X^{\varepsilon}\big). \tag{4.9}$$

Let $E(u) = O(u^{\varepsilon})$. Then, by partial integration, the contribution coming from the O-term in (4.9) to $S_{1,1}^{(1)}(X, \alpha)$ is

$$\int_X^{2X} \mathbf{e}(\alpha\sqrt{u})\, dE(u) \ll X^{\varepsilon}\big(1 + |\alpha|\sqrt{X}\big). \tag{4.10}$$

The contribution of the first term in (4.9) to $S_{1,1}^{(1)}(X, \alpha)$ is

$$\sum_{d \le W} \frac{H(d)}{d} \int_X^{2X} \mathbf{e}(\alpha\sqrt{u})\, du \ll \frac{\sqrt{X}}{|\alpha|} \sum_{d \le W} \frac{\tau(d)}{d} \ll \frac{\sqrt{X} \log^2 X}{|\alpha|}. \tag{4.11}$$

The contribution of the second term in (4.9) to $S_{1,1}^{(1)}(X, \alpha)$ is

$$\begin{aligned} Q_2 &:= \int_X^{2X} \mathbf{e}(\alpha\sqrt{u})\, d\Big(\frac{1}{2\pi i} \int_{\frac{1}{2} - iT}^{\frac{1}{2} + iT} \zeta(s) \Big(\sum_{d \le W} \frac{H(d)}{d^s}\Big) \frac{u^s}{s}\, ds\Big) \\ &= \frac{1}{2\pi} \int_{-T}^{T} \zeta\Big(\frac{1}{2} + it\Big) \Big(\sum_{d \le W} \frac{H(d)}{d^{\frac{1}{2} + it}}\Big) \\ &\quad \times \Big\{\int_X^{2X} u^{-\frac{1}{2} + it} \mathbf{e}(\alpha\sqrt{u})\, du\Big\}\, dt \\ &= \frac{1}{\pi} \int_{-T}^{T} \zeta\Big(\frac{1}{2} + it\Big) \Big(\sum_{d \le W} \frac{H(d)}{d^{\frac{1}{2} + it}}\Big) \\ &\quad \times \Big\{\int_{\sqrt{X}}^{\sqrt{2X}} \mathbf{e}\Big(\alpha v + \frac{t}{\pi} \log v\Big)\, dv\Big\}\, dt. \end{aligned} \tag{4.12}$$

Estimation of certain exponential integral. We need to estimate an upper bound for $|I|$ where

$$I := \int_{\sqrt{X}}^{\sqrt{2X}} \mathbf{e}\left(\alpha v + \frac{t}{\pi}\log v\right) dv.$$

Let $\mathcal{V}$ denote the interval $[\sqrt{X}, \sqrt{2X}]$ and for $v \in \mathcal{V}$, let $f(v) = \alpha v + \frac{t}{\pi}\log v$. Then

$$f'(v) = \alpha + \frac{t}{\pi v} = \frac{t + \alpha\pi v}{\pi v}, \qquad f''(v) = -\frac{t}{\pi v^2} \tag{4.13}$$

so that

$$\left|f'(v)\right| \geq \frac{\min_{v\in\mathcal{V}} |t + \alpha\pi v|}{\pi\sqrt{X}}, \qquad \left|f''(v)\right| \geq \frac{|t|}{\pi X}. \tag{4.14}$$

First of all, we notice that from Lemmas 3.1 and 3.2,

$$|I| \leq \min\left\{\frac{16\pi\sqrt{X}}{\sqrt{1+|t|}}, \frac{4\pi\sqrt{X}}{\min_{v\in\mathcal{V}}|t+\alpha\pi v|}\right\} \quad \text{if } |t| \geq 10. \tag{4.15}$$

Let $T_0 = 2\pi|\alpha|\sqrt{2X}$. If $|t| \geq T_0$, then $|t + \alpha\pi v| \geq |t| - \pi|\alpha|v \geq \frac{|t|}{2}$.

(i) Suppose that $T_0 \geq 10$. Therefore, we get

$$|I| \leq \begin{cases} \sqrt{X} & \text{if } |t| \leq 10 \text{ (trivially)},\\ \frac{16\pi\sqrt{X}}{\sqrt{1+|t|}} & \text{if } 10 \leq |t| \leq T_0 \text{ (by Lemma 3.2)},\\ \frac{4\pi\sqrt{X}}{\min_{v\in\mathcal{V}}|t+\alpha\pi v|} \leq \frac{8\pi\sqrt{X}}{|t|} & \text{if } T_0 \leq |t| \leq T \text{ (by Lemma 3.1)}. \end{cases} \tag{4.16}$$

For $\frac{1}{2} \leq \sigma < 1, |t| \leq 10$, we have

$$\begin{aligned} \left|\zeta(s) - \frac{1}{s-1}\right| &= \left|\sum_{n=1}^{\infty}\left(\frac{1}{n^s} - \int_n^{n+1}\frac{du}{u^s}\right)\right| \\ &= \left|\sum_{n=1}^{\infty}\int_n^{n+1}\left(\int_n^u \frac{dv}{v^{s+1}}(-s)\right)du\right| \\ &\leq |s|\int_1^{\infty}\frac{dv}{v^{\sigma+1}} \\ &\leq \frac{|\sigma|+|t|}{|\sigma|} \end{aligned} \tag{4.17}$$

and hence for $|t| \leq 10$, we have

$$\left|\zeta\left(\frac{1}{2} + it\right)\right| \leq 25. \tag{4.18}$$

Also,

$$\left|\sum_{d\leq W}\frac{H(d)}{d^{\frac{1}{2}+it}}\right| \leq \sum_{d\leq W}\frac{\tau(d)}{d^{\frac{1}{2}}} \ll W^{\frac{1}{2}}\log^2 W \ll X^{1/4}\log^2 X. \tag{4.19}$$

Therefore, we obtain

$$
\begin{aligned}
Q_3 &:= \frac{1}{\pi}\int_{|t|\le 10}\zeta\Big(\frac{1}{2}+it\Big)\Big(\sum_{d\le W}\frac{H(d)}{d^{\frac{1}{2}+it}}\Big)\\
&\quad\times\Big\{\int_{\sqrt{X}}^{\sqrt{2X}}\mathbf{e}\Big(\alpha v+\frac{t}{\pi}\log v\Big)dv\Big\}\,dt\\
&\ll X^{1/4}(\log^2 X)\sqrt{X}\\
&\ll X^{3/4}\log^2 X.
\end{aligned}
\tag{4.20}
$$

For $10<|t|\le T_0$, we split the interval $(10,T_0]$ into dyadic intervals of the type $(\frac{T_1}{2},T_1]$ so that there are at most $\ll \log T$ such intervals. Thus, it is enough to estimate

$$
\begin{aligned}
Q_4 &:= \int_{\frac{T_1}{2}}^{T_1}\zeta\Big(\frac{1}{2}+it\Big)\Big(\sum_{d\le W}\frac{H(d)}{d^{\frac{1}{2}+it}}\Big)\\
&\quad\times\Big\{\int_{\sqrt{X}}^{\sqrt{2X}}\mathbf{e}\Big(\alpha v+\frac{t}{\pi}\log v\Big)\,dv\Big\}\,dt\\
&\ll \frac{\sqrt{X}}{\sqrt{1+T_1}}\Big(\int_{\frac{T_1}{2}}^{T_1}\Big|\zeta\Big(\frac{1}{2}+it\Big)\Big|^2\,dt\Big)^{1/2}\\
&\quad\times\Big(\int_{\frac{T_2}{2}}^{T_2}\Big|\sum_{d\le W}\frac{H(d)}{d^{\frac{1}{2}+it}}\Big|^2\,dt\Big)^{1/2}\\
&\ll \frac{\sqrt{X}}{\sqrt{1+T_1}}\big(T_1(\log T_1)\big)^{1/2}\big((T_1+W)\log^4 W\big)^{1/2}\\
&\ll (\log X)^{5/2}\sqrt{X}(T_1+W)^{1/2}.
\end{aligned}
\tag{4.21}
$$

Note that we have used the second moment of $\zeta(s)$ on the critical line and the Lemma 3.4 in deriving the estimate (4.21) for $|Q_4|$.

Therefore, we obtain

$$
\begin{aligned}
Q_5 &:= \frac{1}{\pi}\int_{10<|t|\le T_0}\zeta\Big(\frac{1}{2}+it\Big)\Big(\sum_{d\le W}\frac{H(d)}{d^{\frac{1}{2}+it}}\Big)\\
&\quad\times\Big\{\int_{\sqrt{X}}^{\sqrt{2X}}\mathbf{e}\Big(\alpha v+\frac{t}{\pi}\log v\Big)\,dv\Big\}\,dt\\
&\ll (\log T)\max_{10\le T_1\le T_0}\Big|\int_{\frac{T_1}{2}}^{T_1}\zeta\Big(\frac{1}{2}+it\Big)\Big(\sum_{d\le W}\frac{H(d)}{d^{\frac{1}{2}+it}}\Big)\\
&\quad\times\Big\{\int_{\sqrt{X}}^{\sqrt{2X}}\mathbf{e}\Big(\alpha v+\frac{t}{\pi}\log v\Big)\,dv\Big\}\,dt\Big|
\end{aligned}
\tag{4.22}
$$

$$\ll (\log X)^{7/2}\sqrt{X}(T_0+W)^{1/2}$$
$$\ll X^{3/4}(\log X)^{7/2}(1+|\alpha|)^{1/2}$$
$$(\text{since } W=\sqrt{2X}, T_0 = 2\pi|\alpha|\sqrt{2X}).$$

For $T_0 < |t| \leq T$, we split the interval $(T_0, T]$ into dyadic intervals of the type $(\frac{T_2}{2}, T_2]$ so that there are at most $\ll \log T$ such intervals. Thus, it is enough to estimate

$$\begin{aligned}
Q_6 &:= \int_{\frac{T_2}{2}}^{T_2} \zeta\Big(\frac{1}{2}+it\Big)\Big(\sum_{d\leq W}\frac{H(d)}{d^{\frac{1}{2}+it}}\Big) \\
&\quad \times \Big\{\int_{\sqrt{X}}^{\sqrt{2X}} \mathbf{e}\Big(\alpha v + \frac{t}{\pi}\log v\Big)\, dv\Big\}\, dt \\
&\ll \frac{\sqrt{X}}{T_2}\Big(\int_{\frac{T_2}{2}}^{T_2}\Big|\zeta\Big(\frac{1}{2}+it\Big)\Big|^2 dt\Big)^{1/2} \\
&\quad \times \Big(\int_{\frac{T_2}{2}}^{T_2}\Big|\sum_{d\leq W}\frac{H(d)}{d^{\frac{1}{2}+it}}\Big|^2 dt\Big)^{1/2} \\
&\ll \frac{\sqrt{X}}{T_2}\big(T_2(\log T_2)\big)^{1/2}\big((T_2+W)\log^4 W\big)^{1/2} \\
&\ll (\log X)^{5/2}\sqrt{X}\Big(1+\Big(\frac{W}{T_2}\Big)\Big)^{1/2}.
\end{aligned} \tag{4.23}$$

Therefore, we obtain

$$\begin{aligned}
Q_7 &:= \frac{1}{\pi}\int_{T_0<|t|\leq T} \zeta\Big(\frac{1}{2}+it\Big)\Big(\sum_{d\leq W}\frac{H(d)}{d^{\frac{1}{2}+it}}\Big) \\
&\quad \times \Big\{\int_{\sqrt{X}}^{\sqrt{2X}} \mathbf{e}\Big(\alpha v + \frac{t}{\pi}\log v\Big)\, dv\Big\}\, dt \\
&\ll (\log T)\max_{T_0\leq T_2\leq T}\Big|\int_{\frac{T_2}{2}}^{T_2} \zeta\Big(\frac{1}{2}+it\Big)\Big(\sum_{d\leq W}\frac{H(d)}{d^{\frac{1}{2}+it}}\Big) \\
&\quad \times \Big\{\int_{\sqrt{X}}^{\sqrt{2X}} \mathbf{e}\Big(\alpha v + \frac{t}{\pi}\log v\Big)\, dv\Big\}\, dt\Big| \\
&\ll (\log X)^{7/2}\sqrt{X}\Big(1+\frac{W}{T_0}\Big)^{1/2} \\
&\ll X^{1/2}(\log X)^{7/2}\Big(1+\frac{1}{|\alpha|}\Big)^{1/2} \\
&(\text{since } W=\sqrt{2X}, T_0 = 2\pi|\alpha|\sqrt{2X}).
\end{aligned} \tag{4.24}$$

(ii) We suppose that $T_0 \le 10$. Then, as before, we obtain

$$Q_8 := \frac{1}{\pi}\int_{|t|\le T_0} \zeta\Big(\frac{1}{2}+it\Big)\Big(\sum_{d\le W}\frac{H(d)}{d^{\frac{1}{2}+it}}\Big) \times \Big\{\int_{\sqrt{X}}^{\sqrt{2X}} \mathbf{e}\Big(\alpha v + \frac{t}{\pi}\log v\Big)\,dv\Big\}\,dt \ll X^{3/4}\log^2 X \tag{4.25}$$

and

$$\begin{aligned} Q_9 := \frac{1}{\pi}\int_{T_0<|t|\le T} &\zeta\Big(\frac{1}{2}+it\Big)\Big(\sum_{d\le W}\frac{H(d)}{d^{\frac{1}{2}+it}}\Big) \\ &\times \Big\{\int_{\sqrt{X}}^{\sqrt{2X}} \mathbf{e}\Big(\alpha v + \frac{t}{\pi}\log v\Big)\,dv\Big\}\,dt \\ &\ll X^{1/2}\log X \max_{T_0\le T_3\le T}\frac{1}{T_3}(T_3\log T_3)^{1/2} \\ &\quad\times \big((T_3+W)\log^4 W\big)^{1/2} \\ &\ll X^{1/2}(\log X)^{7/2}\Big(1+\frac{1}{|\alpha|}\Big)^{1/2}. \end{aligned} \tag{4.26}$$

Hence from (4.10), (4.11), (4.12), (4.20), (4.22), (4.24), (4.25) and (4.26), we observe that

$$\begin{aligned} S_{1,1}^{(1)}(X,\alpha) \ll\ & \frac{\sqrt{X}\log^2 X}{|\alpha|} + X^{3/4}\log^2 X \\ &+ X^{3/4}(\log X)^{7/2}\big(1+|\alpha|\big)^{1/2} \\ &+ X^{1/2}(\log X)^{7/2}\Big(1+\frac{1}{|\alpha|}\Big)^{1/2} \\ &+ X^{\varepsilon}\big(1+|\alpha|\sqrt{X}\big) \\ \ll_\varepsilon\ & X^{\frac{1}{2}+\varepsilon}\Big(|\alpha|+\frac{1}{|\alpha|}\Big) + X^{3/4}(\log X)^{7/2}\big(1+|\alpha|\big)^{1/2} \\ &+ X^{1/2}(\log X)^{7/2}\Big(1+\frac{1}{|\alpha|}\Big)^{1/2}. \end{aligned} \tag{4.27}$$

We estimate more explicitly an upper bound than [11] for $|S_{1,1}^{(2)}(X,\alpha)|$ using exponent pairs in Section 6.

5. Better estimations of $S_{1,2}(X,\alpha), S_{3,1}(X,\alpha)$ and $S_{3,2}(X,\alpha)$

Arguing similar to Zhao (see [14]), we observe that

$$S_{1,2}(X,\alpha) = \sum_{n\sim X} \sum_{\substack{abc=n \\ b,c\le U, \frac{X}{U^2}\le a<W}} \mu(b)\lambda(c)\mathbf{e}(\alpha\sqrt{n})$$
$$= \sum_{n\sim X} \sum_{\substack{ad=n \\ \frac{X}{U^2}\le a<W}} H(d)\mathbf{e}(\alpha\sqrt{n})$$
$$= \sum_{\frac{X}{W}<d\le 2U^2} H(d) \sum_{\substack{\frac{X}{U^2}\le a<W \\ ad\sim X, a\in\mathbb{Z}}} \mathbf{e}(\alpha\sqrt{ad}). \tag{5.1}$$

It is important to note that the second sum in (5.1) exists only if the interval $[\frac{X}{U^2}, W)$ contains at least one positive integer. That is, the choice of our free parameter U must satisfy the inequality $1+\frac{X}{U^2} < W = \sqrt{2X}$. Therefore, we force our choice of U throughout the paper to satisfy $U \ge 100X^{1/4}$. Now, with this choice of U, the interval $[\frac{X}{U^2}, W)$ will contain certainly a block of consecutive positive integers. We split the summation over d and a into dyadic intervals so that we have

$$S_{1,2}(X,\alpha) \ll \log^2 X \sum_{d\sim D} \big|H(d)\big| \Big|\sum_{\substack{a\sim L \\ a\in\mathbb{Z}}} \mathbf{e}(\alpha\sqrt{ad})\Big| \tag{5.2}$$

where D and L satisfy the conditions $\frac{X}{W} < D \le 2U^2$, $\frac{X}{U^2} \le L < W$ and $DL = X$. Note that $|H(d)| \le \tau(d)$.

Estimation of $|\sum_{\substack{a\sim L \\ a\in\mathbb{Z}}} \mathbf{e}(\alpha\sqrt{ad})|$**.** Let

$$Q_{10} := \sum_{\substack{L\le a\le 2L \\ a\in\mathbb{Z}}} \mathbf{e}(\alpha\sqrt{ad}). \tag{5.3}$$

Taking $f(a) = \alpha\sqrt{ad}$, then we find that

$$f^{(r)}(a) = \frac{\alpha\sqrt{d}(\frac{1}{2})(\frac{1}{2}-1)\cdots(\frac{1}{2}-(r-1))}{a^{\frac{1}{2}+(r-1)}}.$$

We fix $A = \frac{|\alpha|\sqrt{d}}{\sqrt{L}}$. Then clearly

$$\frac{A}{2\sqrt{2}} \le \big|f^{(1)}(a)\big| \le \frac{A}{2}.$$

It should be noted that there can be some integers d in the interval $[D, 2D]$ for which $A > \frac{1}{2}$ and for the rest of the integers d in the interval $[D, 2D]$ for which $A \le \frac{1}{2}$.

(i) We consider those integers d in $[D, 2D]$ for which $A > \frac{1}{2}$.

With $B = L$, it is clear that

$$AB^{1-r} \ll_r \left|f^{(r)}(a)\right| \ll_r AB^{1-r}. \tag{5.4}$$

Therefore, by the theory of exponent pairs, we have the estimate

$$\begin{aligned} Q_{10} &:= \sum_{\substack{L \le a \le 2L \\ a \in \mathbb{Z}}} \mathbf{e}(\alpha\sqrt{ad}) \\ &\ll A^\kappa B^\lambda \\ &\ll \left(\frac{|\alpha|\sqrt{d}}{\sqrt{L}}\right)^\kappa L^\lambda, \end{aligned} \tag{5.5}$$

where this estimate (5.5) holds for any exponent pair (κ, λ).

(ii) We consider those integers d in $[D, 2D]$ for which $A \le \frac{1}{2}$.

We observe that if $f(a) = \alpha\sqrt{ad}$, then

$$\left|f'(a)\right| = \frac{|\alpha|\sqrt{d}}{2\sqrt{a}} > \frac{|\alpha|\sqrt{d}}{2\sqrt{2L}} > 0$$

for $D \le d \le 2D, L \le a \le 2L$.

Now, we use the Lemmas 3.3 and 3.1, and obtain

$$Q_{10} \ll \frac{2\sqrt{2L}}{|\alpha|\sqrt{d}} + 1. \tag{5.6}$$

Therefore, from (5.5) and (5.6), we obtain (for all d in $[D, 2D]$)

$$Q_{10} \ll \left(\frac{|\alpha|\sqrt{d}}{\sqrt{L}}\right)^\kappa L^\lambda + \frac{\sqrt{2L}}{|\alpha|\sqrt{d}} + 1. \tag{5.7}$$

Hence, we obtain

$$\begin{aligned} S_{1,2}(X, \alpha) &\ll \left(\log^2 X\right) \sum_{d \sim D} \left|H(d)\right| \left|\sum_{\substack{a \sim L \\ a \in \mathbb{Z}}} \mathbf{e}(\alpha\sqrt{ad})\right| \\ &\ll \left(\log^2 X\right) \sum_{d \sim D} \tau(d) \left\{ \left(\frac{|\alpha|\sqrt{d}}{\sqrt{L}}\right)^\kappa L^\lambda + \frac{\sqrt{2L}}{|\alpha|\sqrt{d}} + 1 \right\} \\ &\ll |\alpha|^\kappa \left(\log^2 X\right) D^{1+\frac{\kappa}{2}} (\log D) L^{\lambda - \frac{\kappa}{2}} \\ &\quad + \frac{\sqrt{2D}\sqrt{2L}(\log^2 X)(\log^2 D)}{|\alpha|} + (D \log D) \log^2 X \\ &\ll |\alpha|^\kappa \left(\log^3 X\right) (DL)^{1+\frac{\kappa}{2}} L^{\lambda-1-\kappa} \\ &\quad + \frac{X^{1/2} \log^4 X}{|\alpha|} + \frac{X \log^3 X}{L} \\ &\ll |\alpha|^\kappa \left(\log^3 X\right) X^{1+\frac{\kappa}{2}} \left(\frac{X}{U^2}\right)^{\lambda-1-\kappa} + \frac{X^{1/2} \log^4 X}{|\alpha|} \end{aligned} \tag{5.8}$$

$$
\begin{aligned}
&\quad + U^2 \log^3 X \quad \text{since } W \ge L \ge \frac{X}{U^2} \\
&\ll |\alpha|^\kappa X^{\lambda - \frac{\kappa}{2}} \big(\log^3 X\big) \frac{1}{U^{2\lambda-2-2\kappa}} \\
&\quad + \frac{X^{1/2}\log^4 X}{|\alpha|} + U^2 \log^3 X \\
&\ll |\alpha|^{1/2} X^{1/4} \big(\log^3 X\big) U^2 + \frac{X^{1/2}\log^4 X}{|\alpha|} \\
&\quad + U^2 \log^3 X \quad \text{by taking } (\kappa,\lambda) = (1/2, 1/2) \\
&\ll |\alpha|^{1/2} X^{3/4} \big(\log^3 X\big) + \frac{X^{1/2}\log^4 X}{|\alpha|} + X^{1/2}\log^3 X
\end{aligned}
$$

if we choose our $U = 100X^{1/4}$.

Estimations of $S_{3,1}(X,\alpha)$ and $S_{3,2}(X,\alpha)$. Note that $d = bc$ where b and c lie in certain intervals. To treat $S_{3,1}(X,\alpha)$, we define

$$
H_1 := \sum_{\substack{ad=n, \\ 1 \le a \le \frac{X}{U^2}}} \mu(b)\lambda(c).
$$

We note that for any given integer $n \in [X, 2X]$ and for any given integral pair (b,c) (where $U < b < W$ and $c > U$) with $d = bc$ and $ad = n$, it means that $1 \le a \le \frac{X}{U^2}$ and $U^2 < d = bc < 2X$ with $ad = n$ and such an a is uniquely determined. Therefore, $a = \frac{n}{bc}$ is a positive integer in the said interval and this means that there exists at least one integral pair (a_1, a_2) satisfying $1 \le a_1, a_2 \le \frac{X}{U^2}$ and for this pair we have $a = a_1 a_2 = \frac{n}{bc}$. Moreover, given any integral pair (b,c) lying in the said intervals with $d = bc$ and $ad = n$, the function $\mu(b)\lambda(c)$ assumes one of the values from the set $\{0, +1, -1\}$. Thus, the sum H_1 depends essentially on the factorisations of the positive integer a. This means that

$$
H_1 = H_1(a) := \sum_{\substack{ad=n, \\ 1 \le a \le \frac{X}{U^2}}} \mu(b)\lambda(c)
$$

is a function only of a. Therefore, we have

$$
S_{3,1}(X,\alpha) = \sum_{1 \le a \le \frac{X}{U^2}} H_1(a) \sum_{U^2 < d < 2X} \mathbf{e}(\alpha\sqrt{ad}).
$$

Similar reasoning holds good for $S_{3,2}(X,\alpha)$ and hence

$$
S_{3,2}(X,\alpha) = \sum_{1 \le a \le \frac{X}{WU}} H_2(a) \sum_{WU < d < 2X} \mathbf{e}(\alpha\sqrt{ad})
$$

with

$$H_2(a) := \sum_{\substack{ad=n,\\ 1\le a\le \frac{X}{WU}}} \mu(b)\lambda(c).$$

It is clear that $|H_1(a)| \le \tau(a)$ and $|H_2(a)| \le \tau(a)$. The key observation now is that the role of a and d is interchanged with a and d being in appropriate intervals. Therefore, the estimations of $S_{3,1}(X,\alpha)$ and $S_{3,2}(X,\alpha)$ are analogous to (5.8) and hence we obtain

$$\begin{aligned}(5.9)\qquad &\left|S_{1,2}(X,\alpha)\right| + \left|S_{3,1}(X,\alpha)\right| + \left|S_{3,2}(X,\alpha)\right| \\ &\ll |\alpha|^{1/2}X^{3/4}\big(\log^3 X\big) + \frac{X^{1/2}\log^4 X}{|\alpha|} + X^{1/2}\log^3 X.\end{aligned}$$

6. Better estimation of $S_{1,1}^{(2)}(X,\alpha)$ and the proof of Theorem 1

We find that

$$(6.1)\quad S_{1,1}^{(2)}(X,\alpha) = \sum_{n\sim X}\sum_{\substack{ad=n\\ a,d\le W}} H(d)\mathbf{e}(\alpha\sqrt{n}) = \sum_{d\le W} H(d) \sum_{\substack{a\le W\\ ad\sim X, a\in\mathbb{Z}}} \mathbf{e}(\alpha\sqrt{ad}).$$

As in Section 5, we split the summation over d and a into dyadic intervals and find that

$$(6.2)\qquad S_{1,1}^{(2)}(X,\alpha) \ll \big(\log^2 X\big)\sum_{d\sim\widetilde{D}} |H(d)| \Bigg|\sum_{\substack{a\sim\widetilde{L}\\ a\in\mathbb{Z}}} \mathbf{e}(\alpha\sqrt{ad})\Bigg|,$$

where $\widetilde{D}$ and $\widetilde{L}$ satisfy $\widetilde{D}\le W, \widetilde{L}\le W$ and $\widetilde{D}\widetilde{L} = X$. Now, arguments similar to Section 5 leads to

$$\begin{aligned}(6.3)\quad S_{1,1}^{(2)}(X,\alpha) &\ll \big(\log^2 X\big)\Bigg(\sum_{d\sim\widetilde{D}} |H(d)| \Bigg|\sum_{\substack{a\sim\widetilde{L}\\ a\in\mathbb{Z}}} \mathbf{e}(\alpha\sqrt{ad})\Bigg|\Bigg) \\ &\ll \big(\log^2 X\big)\sum_{d\sim\widetilde{D}} \tau(d)\left\{\left(\frac{|\alpha|\sqrt{d}}{\sqrt{\widetilde{L}}}\right)^{\kappa}\widetilde{L}^{\lambda} + \frac{\sqrt{2\widetilde{L}}}{|\alpha|\sqrt{d}} + 1\right\} \\ &\ll |\alpha|^{\kappa}\big(\log^2 X\big)\widetilde{D}^{1+\frac{\kappa}{2}}(\log\widetilde{D})\widetilde{L}^{\lambda-\frac{\kappa}{2}} \\ &\quad + \frac{\sqrt{2\widetilde{D}}\sqrt{2\widetilde{L}}(\log^2 X)(\log^2\widetilde{D})}{|\alpha|} + (\widetilde{D}\log\widetilde{D})\log^2 X \\ &\ll |\alpha|^{\kappa}\big(\log^3 X\big)W^{1+\lambda} + \frac{X^{1/2}\log^4 X}{|\alpha|} + X^{1/2}\log^3 X \\ &\ll |\alpha|^{1/2}\big(\log^3 X\big)X^{3/4} + \frac{X^{1/2}\log^4 X}{|\alpha|} + X^{1/2}\log^3 X\end{aligned}$$

by choosing the exponent pair $(\kappa, \lambda) = (1/2, 1/2)$ and since $\widetilde{D} \leq W = \sqrt{2X}$ and $\widetilde{L} \leq W = \sqrt{2X}$. Now, the Theorem 1 follows from (4.27), (6.3), (2.7) and (5.9).

7. Proof of Theorem 2

In the formal identity (2.1), we take (for $\Re s > 1$)

$$A(s) = 1, \qquad B(s) = \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \tag{7.1}$$

$$\frac{A(s)}{B(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \frac{1}{B(s)}, \qquad F(s) = \sum_{n \leq U} \frac{\mu(n)}{n^s}, \qquad G(s) = \sum_{n \leq V} \frac{\mu(n)}{n^s}.$$

The free parameters U, V are chosen in such a way to satisfy that $10 \leq U = V \leq \frac{1}{100} X^{1/3}$. In fact, our choice here too is going to be $U = V = 100 X^{1/4}$. With this setting, we observe that

$$\mu(n) = a_1(n) + a_2(n) + a_3(n) + a_4(n), \tag{7.2}$$

where

$$a_1(n) = a_3(n) = \begin{cases} \mu(n) & \text{if } n \leq U, \\ 0 & \text{otherwise,} \end{cases}$$

$$a_2(n) = - \sum_{\substack{abc=n \\ b,c \leq U}} \mu(b)\mu(c)$$

and

$$a_4(n) = \sum_{\substack{abc=n \\ b>U, c>U}} \mu(b)\mu(c). \tag{7.3}$$

We note that

$$a_1(n) = a_3(n) = 0 \quad \text{for } X \leq n \leq 2X. \tag{7.4}$$

In place of $H(d)$, we define

$$H^*(d) := \sum_{\substack{bc=d \\ b,c \leq U}} \mu(b)\mu(c), \quad \text{so that } \left| H^*(d) \right| \leq \tau(d). \tag{7.5}$$

(Analogous to $H_1(a)$ and $H_2(a)$, we can also define $H_1^*(a)$ and $H_2^*(a)$). Now one needs to treat the sums similar to $S_1(X, \alpha)$ and $S_3(X, \alpha)$, where $\lambda(c)$ is replaced by $\mu(c)$ throughout. Now the whole arguments of this paper goes through verbatim the same with these necessary changes. Thus, the proof of Theorem 2 is complete.

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