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THE INTRINSIC SQUARE FUNCTION CHARACTERIZATIONS OF WEIGHTED HARDY SPACES

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ABSTRACT. In this paper, we will study the boundedness of intrinsic square functions on the weighted Hardy spaces $H^p(w)$ for $0 < p < 1$, where w is a Muckenhoupt's weight function. We will also give some intrinsic square function characterizations of weighted Hardy spaces $H^p(w)$ for $0 < p < 1$.

1. Introduction and preliminaries

First, let's recall some standard definitions and notations. The classical A_p weight theory was first introduced by Muckenhoupt in the study of weighted L^p boundedness of Hardy–Littlewood maximal functions in [9]. Let w be a nonnegative, locally integrable function defined on $\mathbb{R}^n$, all cubes are assumed to have their sides parallel to the coordinate axes. We say that $w \in A_p$, $1 < p < \infty$, if

$$\left(\frac{1}{|Q|}\int_Q w(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{-\frac{1}{p-1}}\,dx\right)^{p-1} \le C \quad \text{for every cube } Q \subseteq \mathbb{R}^n,$$

where C is a positive constant which is independent of the choice of Q.

For the case $p = 1$, $w \in A_1$, if

$$\frac{1}{|Q|}\int_Q w(x)\,dx \le C \cdot \operatorname*{ess\,inf}_{x\in Q} w(x) \quad \text{for every cube } Q \subseteq \mathbb{R}^n.$$

For the case $p = \infty$, $w \in A_\infty$, if for any given $\varepsilon > 0$, we can find a positive number $\delta > 0$ such that if Q is a cube, E is a measurable subset of Q with $|E| < \delta|Q|$, then $\int_E w(x)\,dx < \varepsilon \int_Q w(x)\,dx$.

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It is well known that $A_\infty = \bigcup_{1<p<\infty} A_p$, namely, a nonnegative, locally integrable function $w(x)$ satisfies the condition A_∞ if and only if it satisfies the condition A_p for some $1<p<\infty$. We also know that if $w \in A_p$ with $1<p<\infty$, then $w \in A_r$ for all $r>p$, and $w \in A_q$ for some $1<q<p$. Therefore, we will use the notation $q_w \equiv \inf\{q>1 : w \in A_q\}$ to denote the critical index of w. Obviously, if $w \in A_q$, $q>1$, then we have $1 \le q_w < q$.

Given a cube Q and $\lambda>0$, λQ denotes the cube with the same center as Q whose side length is λ times that of Q. $Q = Q(x_0, r)$ denotes the cube centered at x_0 with side length r. For a weight function w and a measurable set E, we set the weighted measure $w(E) = \int_E w(x)\,dx$, and we denote the characteristic function of E by $\chi_{_E}$.

We shall need the following lemmas. For the proofs of these results, we refer the readers to [4, Chapter IV] and [5, Chapter 9].

Lemma A. *Let $w \in A_p$, $p \ge 1$. Then, for any cube Q, there exists an absolute constant $C>0$ such that*

$$w(2Q) \le C \cdot w(Q).$$

In general, for any $\lambda>1$, we have

$$w(\lambda Q) \le C \cdot \lambda^{np} w(Q),$$

where C does not depend on Q nor on λ.

Lemma B. *Let $w \in A_q$, $q>1$. Then, for all $r>0$, there exists a constant $C>0$ independent of r such that*

$$\int_{|x|\ge r} \frac{w(x)}{|x|^{nq}}\,dx \le C \cdot r^{-nq} w\big(Q(0,2r)\big).$$

Lemma C. *Let $w \in A_\infty$. For any $0<\varepsilon<1$, there exists a positive number $0<\delta<1$ such that if E is a measurable subset of a cube Q with $|E|/|Q|>\varepsilon$, then we have $w(E)/w(Q)>\delta$.*

Lemma D. *Let $w \in A_p$, $p \ge 1$. Then there exists an absolute constant $C>0$ such that*

$$C\left(\frac{|E|}{|Q|}\right)^p \le \frac{w(E)}{w(Q)},$$

for any measurable subset E of a cube Q.

Given a Muckenhoupt's weight function w on $\mathbb{R}^n$, for $0<p<\infty$, we denote by $L^p_w(\mathbb{R}^n)$ the space of all functions satisfying

$$\|f\|_{L^p_w(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} \big|f(x)\big|^p w(x)\,dx\right)^{1/p} < \infty.$$

When $q=\infty$, L^∞_w will be taken to mean L^∞, and we set $\|f\|_{L^\infty_w} = \|f\|_{L^\infty}$. As we all know, for any $0<p<\infty$, the weighted Hardy spaces $H^p_w(\mathbb{R}^n)$ can be defined in terms of maximal functions. Let φ be a function in $\mathscr{S}(\mathbb{R}^n)$ satisfying $\int_{\mathbb{R}^n} \varphi(x)\,dx = 1$. Set

$$\varphi_t(x) = t^{-n}\varphi(x/t), \quad t > 0, x \in \mathbb{R}^n.$$

We will define the maximal function $M_\varphi f(x)$ by

$$M_\varphi f(x) = \sup_{t>0} |f * \varphi_t(x)|.$$

Then $H^p_w(\mathbb{R}^n)$ consists of those tempered distributions $f \in \mathscr{S}'(\mathbb{R}^n)$ for which $M_\varphi f \in L^p_w(\mathbb{R}^n)$ with $\|f\|_{H^p_w} = \|M_\varphi f\|_{L^p_w}$. For every $1 < p < \infty$, as in the unweighted case, we have $L^p_w(\mathbb{R}^n) = H^p_w(\mathbb{R}^n)$.

The real-variable theory of weighted Hardy spaces has been studied by many authors. In 1979, Garcia-Cuerva studied the atomic decomposition and the dual spaces of H^p_w for $0 < p \le 1$. In 2002, Lee and Lin gave the molecular characterization of H^p_w for $0 < p \le 1$, they also obtained the $H^p_w(\mathbb{R})$, $\frac{1}{2} < p \le 1$ boundedness of the Hilbert transform and the $H^p_w(\mathbb{R}^n)$, $\frac{n}{n+1} < p \le 1$ boundedness of the Riesz transforms. For the results mentioned above, we refer the readers to [3], [7], [10] for further details.

In this article, we will use Garcia-Cuerva's atomic decomposition theory for weighted Hardy spaces in [3], [10]. We characterize weighted Hardy spaces in terms of atoms in the following way.

Let $0 < p \le 1 \le q \le \infty$ and $p \ne q$ such that $w \in A_q$ with critical index q_w. Set $[\,\cdot\,]$ the greatest integer function. For $s \in \mathbb{Z}_+$ satisfying $s \ge [n(q_w/p - 1)]$, a real-valued function $a(x)$ is called (p, q, s)-atom centered at x_0 with respect to w(or w-(p, q, s)-atom centered at x_0) if the following conditions are satisfied:

(a) $a \in L^q_w(\mathbb{R}^n)$ and is supported in a cube Q centered at x_0,
(b) $\|a\|_{L^q_w} \le w(Q)^{1/q-1/p}$,
(c) $\int_{\mathbb{R}^n} a(x)x^\alpha\,dx = 0$ for every multi-index α with $|\alpha| \le s$.

THEOREM E. *Let $0 < p \le 1 \le q \le \infty$ and $p \ne q$ such that $w \in A_q$ with critical index q_w. For each $f \in H^p_w(\mathbb{R}^n)$, there exist a sequence $\{a_j\}$ of w-$(p, q, [n(q_w/p - 1)])$-atoms and a sequence $\{\lambda_j\}$ of real numbers with $\sum_j |\lambda_j|^p \le C\|f\|^p_{H^p_w}$ such that $f = \sum_j \lambda_j a_j$ both in the sense of distributions and in the H^p_w norm.*

2. The intrinsic square functions and our main results

The intrinsic square functions were first introduced by Wilson in [11] and [12]; they are defined as follows. For $0 < \alpha \le 1$, let $\mathcal{C}_\alpha$ be the family of functions φ defined on $\mathbb{R}^n$ such that φ has support containing in $\{x \in \mathbb{R}^n : |x| \le 1\}$, $\int_{\mathbb{R}^n} \varphi(x)\,dx = 0$ and for all $x, x' \in \mathbb{R}^n$,

$$\left|\varphi(x) - \varphi(x')\right| \le \left|x - x'\right|^\alpha.$$

For $(y,t)\in\mathbb{R}^{n+1}_{+}=\mathbb{R}^n\times(0,\infty)$ and $f\in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, we set

$$A_\alpha(f)(y,t)=\sup_{\varphi\in\mathcal{C}_\alpha}\big|f*\varphi_t(y)\big|.$$

Then we define the intrinsic square function of f(of order α) by the formula

$$S_\alpha(f)(x)=\left(\int\int_{\Gamma(x)}\big(A_\alpha(f)(y,t)\big)^2\frac{dy\,dt}{t^{n+1}}\right)^{1/2},$$

where $\Gamma(x)$ denotes the usual cone of aperture one:

$$\Gamma(x)=\big\{(y,t)\in\mathbb{R}^{n+1}_{+}:|x-y|<t\big\}.$$

We can also define varying-aperture versions of $S_\alpha(f)$ by the formula

$$S_{\alpha,\beta}(f)(x)=\left(\int\int_{\Gamma_\beta(x)}\big(A_\alpha(f)(y,t)\big)^2\frac{dy\,dt}{t^{n+1}}\right)^{1/2},$$

where $\Gamma_\beta(x)$ is the usual cone of aperture $\beta>0$:

$$\Gamma_\beta(x)=\big\{(y,t)\in\mathbb{R}^{n+1}_{+}:|x-y|<\beta t\big\}.$$

The intrinsic Littlewood–Paley g-function (could be viewed as "zero-aperture" version of $S_\alpha(f)$) and the intrinsic g^*_λ-function (could be viewed as "infinite aperture" version of $S_\alpha(f)$) will be defined respectively, by

$$g_\alpha(f)(x)=\left(\int_0^\infty\big(A_\alpha(f)(x,t)\big)^2\frac{dt}{t}\right)^{1/2}$$

and

$$g^*_{\lambda,\alpha}(f)(x)=\left(\int\int_{\mathbb{R}^{n+1}_{+}}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\big(A_\alpha(f)(y,t)\big)^2\frac{dy\,dt}{t^{n+1}}\right)^{1/2}.$$

Similarly, we can also introduce the so-called similar-looking square functions $\tilde{S}_{(\alpha,\varepsilon)}(f)(x)$, which are defined via convolutions with kernels that have unbounded supports, more precisely, for $0<\alpha\le 1$ and $\varepsilon>0$, let $\mathcal{C}_{(\alpha,\varepsilon)}$ be the family of functions φ defined on $\mathbb{R}^n$ such that for all $x\in\mathbb{R}^n$,

$$\big|\varphi(x)\big|\le\big(1+|x|\big)^{-n-\varepsilon},$$

and for all $x,x'\in\mathbb{R}^n$,

$$\big|\varphi(x)-\varphi(x')\big|\le\big|x-x'\big|^\alpha\Big(\big(1+|x|\big)^{-n-\varepsilon}+\big(1+|x'|\big)^{-n-\varepsilon}\Big),$$

and also satisfy $\int_{\mathbb{R}^n}\varphi(x)\,dx=0$.

Let f be such that $|f(x)|(1+|x|)^{-n-\varepsilon}\in L^1(\mathbb{R}^n)$. For any $(y,t)\in\mathbb{R}^{n+1}_{+}$, set

$$\tilde{A}_{(\alpha,\varepsilon)}(f)(y,t)=\sup_{\varphi\in\mathcal{C}_{(\alpha,\varepsilon)}}\big|f*\varphi_t(y)\big|.$$

We define

$$\tilde{S}_{(\alpha,\varepsilon)}(f)(x) = \left(\int\int_{\Gamma(x)} \left(\tilde{A}_{(\alpha,\varepsilon)}(f)(y,t)\right)^2 \frac{dy\,dt}{t^{n+1}}\right)^{1/2},$$

$$\tilde{g}_{(\alpha,\varepsilon)}(f)(x) = \left(\int_0^\infty \left(\tilde{A}_{(\alpha,\varepsilon)}(f)(x,t)\right)^2 \frac{dt}{t}\right)^{1/2}$$

and

$$\tilde{g}^*_{\lambda,(\alpha,\varepsilon)}(f)(x) = \left(\int\int_{\mathbb{R}^{n+1}_+} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} \left(\tilde{A}_{(\alpha,\varepsilon)}(f)(y,t)\right)^2 \frac{dy\,dt}{t^{n+1}}\right)^{1/2}.$$

In [12], Wilson proved that the intrinsic square functions are bounded operators on the weighted Lebesgue spaces $L^p_w(\mathbb{R}^n)$ for $1<p<\infty$; namely, he showed the following result.

THEOREM F. *Let $w\in A_p$, $1<p<\infty$ and $0<\alpha\le 1$. Then there exists a positive constant $C>0$ such that*

$$\left\|S_\alpha(f)\right\|_{L^p_w} \le C\|f\|_{L^p_w}.$$

Recently, Huang and Liu [6] studied the boundedness of intrinsic square functions on the weighted Hardy spaces $H^1_w(\mathbb{R}^n)$. Moreover, they obtained the intrinsic square function characterizations of $H^1_w(\mathbb{R}^n)$.

As a continuation of their work, the purpose of this paper is to investigate the boundedness of intrinsic square functions on the weighted Hardy spaces $H^p_w(\mathbb{R}^n)$ for $0<p<1$. Furthermore, we will characterize the weighted Hardy spaces $H^p_w(\mathbb{R}^n)$ for $0<p<1$ by the intrinsic square functions including the Lusin area function, Littlewood–Paley g-function and g^*_λ-function.

In order to state our theorems, we need to introduce the Lipschitz space $\mathrm{Lip}(\alpha,1,0)$ for $0<\alpha\le 1$. Set $b_Q=\frac{1}{|Q|}\int_Q b(x)\,dx$.

$$\mathrm{Lip}(\alpha,1,0)=\left\{b\in L^1_{\mathrm{loc}}\left(\mathbb{R}^n\right) : \|b\|_{\mathrm{Lip}(\alpha,1,0)}<\infty\right\},$$

where

$$\|b\|_{\mathrm{Lip}(\alpha,1,0)} = \sup_Q \frac{1}{|Q|^{1+\alpha/n}} \int_Q \left|b(y)-b_Q\right| dy$$

and the supremum is taken over all cubes Q in $\mathbb{R}^n$.

We say that a tempered distribution f vanishes weakly at infinity, if for any $\varphi\in\mathscr{S}$, we have $f*\varphi_t(x)\to 0$ as $t\to\infty$ in the sense of distributions.

Our main results are stated as follows.

THEOREM 1. *Let $0<\alpha\le 1$, $\frac{n}{n+\alpha}<p<1$, $w\in A_{p(1+\frac{\alpha}{n})}$ and $\varepsilon>\alpha$. Suppose that $f\in(\mathrm{Lip}(\alpha,1,0))^*$, then a tempered distribution $f\in H^p_w(\mathbb{R}^n)$ if and only if $g_\alpha(f)\in L^p_w(\mathbb{R}^n)$ or $\tilde{g}_{(\alpha,\varepsilon)}(f)\in L^p_w(\mathbb{R}^n)$ and f vanishes weakly at infinity.*

THEOREM 2. *Let* $0<\alpha\le 1$, $\frac{n}{n+\alpha}<p<1$, $w\in A_{p(1+\frac{\alpha}{n})}$ *and* $\varepsilon>\alpha$. *Suppose that* $f\in(\mathrm{Lip}(\alpha,1,0))^*$, *then a tempered distribution* $f\in H^p_w(\mathbb{R}^n)$ *if and only if* $S_\alpha(f)\in L^p_w(\mathbb{R}^n)$ *or* $\tilde{S}_{(\alpha,\varepsilon)}(f)\in L^p_w(\mathbb{R}^n)$ *and* f *vanishes weakly at infinity.*

THEOREM 3. *Let* $0<\alpha\le 1$, $\frac{n}{n+\alpha}<p<1$, $w\in A_{p(1+\frac{\alpha}{n})}$, $\varepsilon>\alpha$ *and* $\lambda>\frac{3n+2\alpha}{n}$. *Suppose that* $f\in(\mathrm{Lip}(\alpha,1,0))^*$, *then a tempered distribution* $f\in H^p_w(\mathbb{R}^n)$ *if and only if* $g^*_{\lambda,\alpha}(f)\in L^p_w(\mathbb{R}^n)$ *or* $\tilde{g}^*_{\lambda,(\alpha,\varepsilon)}(f)\in L^p_w(\mathbb{R}^n)$ *and* f *vanishes weakly at infinity.*

REMARK 1. Clearly, if for every $t>0$, $\varphi_t\in\mathcal{C}_\alpha$, then we have $\varphi_t\in\mathrm{Lip}(\alpha,1,0)$. Thus, the intrinsic square functions are well defined for tempered distributions in $(\mathrm{Lip}(\alpha,1,0))^*$.

Throughout this article, we will use C to denote a positive constant, which is independent of the main parameters and not necessarily the same at each occurrence. By $A\sim B$, we mean that there exists a constant $C>1$ such that $\frac{1}{C}\le\frac{A}{B}\le C$.

3. The necessity of our conditions

In this section, we shall first prove the following lemma.

LEMMA 3.1. *Let* $0<p<1$ *and* $w\in A_\infty$. *Then for every* $f\in H^p_w(\mathbb{R}^n)$, *we have that* f *vanishes weakly at infinity.*

Proof. For any given $\varphi\in\mathscr{S}(\mathbb{R}^n)$, $\int_{\mathbb{R}^n}\varphi(x)\,dx=1$, we denote the nontangential maximal function of f by

$$M^*_\varphi(f)(x)=\sup_{|y-x|<t}\big|f*\varphi_t(y)\big|.$$

Then we have $|f*\varphi_t(x)|\le M^*_\varphi(f)(y)$ whenever $|x-y|<t$. As a consequence, we obtain the following inequality

$$\int_{|x-y|<t}\big|f*\varphi_t(x)\big|^p w(y)\,dy\le\int_{|x-y|<t}\big(M^*_\varphi(f)(y)\big)^p w(y)\,dy.$$

Hence,

$$\big|f*\varphi_t(x)\big|^p\le\frac{1}{w(Q(x,\sqrt{2}t))}\big\|M^*_\varphi(f)\big\|^p_{L^p_w}\le C\cdot\frac{1}{w(Q(x,\sqrt{2}t))}\big\|M_\varphi(f)\big\|^p_{L^p_w}.$$

It is well known that for given $w\in A_\infty$, then w satisfies the doubling condition (Lemma A). Furthermore, we can easily show that w also satisfies the reverse doubling condition; that is, for any cube Q, there exists a constant $C_1>1$ such that $w(2Q)\ge C_1w(Q)$. From this property, we can deduce $w(2^kQ)\ge C_1^k w(Q)$ by induction. Set $Q=Q(x,\sqrt{2})$. So we can get

$$\lim_{k\to\infty}\frac{1}{w(2^kQ)}=0,$$

which implies

$$\lim_{t\to\infty}\frac{1}{w(Q(x,\sqrt{2}t))}=0.$$

This completes the proof of the lemma. □

From the definitions of intrinsic square functions, we know that when $\varphi\in\mathcal{C}_\alpha$, $0<\alpha\le 1$, then there exists a positive constant c depending only on α,ε and n, such that $c\varphi\in\mathcal{C}_{(\alpha,\varepsilon)}$. Thus, we can get the pointwise inequality $S_\alpha(f)(x)\le C\tilde{S}_{(\alpha,\varepsilon)}(f)(x)$. Furthermore, in [11], the author proved that this inequality has a partial converse; that is, for every α' satisfying $0<\alpha'\le\alpha$ and $\alpha'<\varepsilon$, for all f such that $|f(x)|(1+|x|)^{-n-\varepsilon}\in L^1(\mathbb{R}^n)$, we have $\tilde{S}_{(\alpha,\varepsilon)}(f)(x)\le CS_\alpha(f)(x)$. So if we choose $\alpha'=\alpha$ and $\varepsilon>\alpha$, we obtain $S_\alpha(f)(x)\sim\tilde{S}_{(\alpha,\varepsilon)}(f)(x)$. In [11], the author also showed that the functions $S_\alpha(f)(x)$ and $g_\alpha(f)(x)$ are pointwise comparable. Meanwhile, he pointed out that by similar arguments we can show the pointwise comparability of $\tilde{S}_{(\alpha,\varepsilon)}(f)(x)$ and $\tilde{g}_{(\alpha,\varepsilon)}(f)(x)$. Therefore, in order to prove the necessity of Theorems 1 and 2, we need only to establish the following proposition.

PROPOSITION 3.2. *Let* $0<\alpha\le 1$, $\frac{n}{n+\alpha}<p<1$ *and* $w\in A_{p(1+\frac{\alpha}{n})}$. *Then for every* $f\in H^p_w(\mathbb{R}^n)$, *we have*

$$\big\|g_\alpha(f)\big\|_{L^p_w}\le C\|f\|_{H^p_w}.$$

Proof. Set $q=p(1+\frac{\alpha}{n})$. Then for $w\in A_q$, we have $[n(q_w/p-1)]=0$. By Theorem E, it suffices to show that for any w-$(p,q,0)$-atom a, there exists a constant $C>0$ independent of a such that $\|g_\alpha(a)\|_{L^p_w}\le C$.

Let a be a w-$(p,q,0)$-atom with $\operatorname{supp}a\subseteq Q=Q(x_0,r)$, and let $Q^*=2\sqrt{n}Q$. By using Hölder's inequality, Lemma A and Theorem F, we thus have

$$\begin{aligned}
(1)\qquad &\int_{Q^*}\big|g_\alpha(a)(x)\big|^p w(x)\,dx\\
&\le\Big(\int_{Q^*}\big|g_\alpha(a)(x)\big|^q w(x)\,dx\Big)^{p/q}\Big(\int_{Q^*}w(x)\,dx\Big)^{1-p/q}\\
&\le\big\|g_\alpha(a)\big\|^p_{L^q_w}w\big(Q^*\big)^{1-p/q}\\
&\le C\big\|S_\alpha(a)\big\|^p_{L^q_w}w(Q)^{1-p/q}\\
&\le C\|a\|^p_{L^q_w}w(Q)^{1-p/q}\\
&\le C.
\end{aligned}$$

Below we shall give the estimate of the integral $I=\int_{(Q^*)^c}|g_\alpha(a)(x)|^p w(x)\,dx$. For any $\varphi\in\mathcal{C}_\alpha$, by the vanishing moment condition of atom a, we get

$$\begin{aligned}
|a*\varphi_t(x)| &= \left|\int_Q \big(\varphi_t(x-y)-\varphi_t(x-x_0)\big)a(y)\,dy\right| \\
&\le \int_Q \frac{|y-x_0|^\alpha}{t^{n+\alpha}}\big|a(y)\big|\,dy \\
&\le C\cdot\frac{r^\alpha}{t^{n+\alpha}}\int_Q\big|a(y)\big|\,dy.
\end{aligned}\tag{2}$$

Denote the conjugate exponent of $q>1$ by $q'=q/(q-1)$. Hölder's inequality and the condition A_q yield

$$\begin{aligned}
\int_Q\big|a(y)\big|\,dy &\le \left(\int_Q\big|a(y)\big|^q w(y)\,dy\right)^{1/q}\left(\int_Q w(y)^{-1/(q-1)}\,dy\right)^{1/q'} \\
&\le C\|a\|_{L^q_w}\left(\frac{|Q|^q}{w(Q)}\right)^{1/q} \\
&\le C\cdot\frac{|Q|}{w(Q)^{1/p}}.
\end{aligned}\tag{3}$$

We note that $\operatorname{supp}\varphi\subseteq\{x\in\mathbb{R}^n:|x|\le 1\}$, then for any $y\in Q$, $x\in(Q^*)^c$, we have $t\ge|x-y|\ge|x-x_0|-|y-x_0|\ge\frac{|x-x_0|}{2}$. Substituting the above inequality (3) into (2), we thus obtain

$$\begin{aligned}
\big|g_\alpha(a)(x)\big|^2 &= \int_0^\infty\left(\sup_{\varphi\in\mathcal{C}_\alpha}\big|a*\varphi_t(x)\big|\right)^2\frac{dt}{t} \\
&\le C\left(\frac{|Q|}{w(Q)^{1/p}}\right)^2 r^{2\alpha}\int_{\frac{|x-x_0|}{2}}^\infty\frac{dt}{t^{2(n+\alpha)+1}} \\
&\le C\left(\frac{|Q|}{w(Q)^{1/p}}\right)^2 r^{2\alpha}\frac{1}{|x-x_0|^{2n+2\alpha}}.
\end{aligned}\tag{4}$$

It follows from the inequality (4), Lemma A and Lemma B that

$$\begin{aligned}
I &= \int_{(Q^*)^c}\big|g_\alpha(a)(x)\big|^p w(x)\,dx \\
&\le C\left(\frac{r^{n+\alpha}}{w(Q)^{1/p}}\right)^p\int_{|x-x_0|\ge\sqrt{n}r}\frac{w(x)}{|x-x_0|^{nq}}\,dx \\
&= C\left(\frac{r^{n+\alpha}}{w(Q)^{1/p}}\right)^p\int_{|y|\ge\sqrt{n}r}\frac{w_1(y)}{|y|^{nq}}\,dy \\
&\le C\left(\frac{r^{n+\alpha}}{w(Q)^{1/p}}\right)^p r^{-nq}w_1(Q_1)
\end{aligned}\tag{5}$$

$$
\begin{aligned}
&= C\left(\frac{r^{n+\alpha}}{w(Q)^{1/p}}\right)^p r^{-nq} w(Q) \\
&\le C,
\end{aligned}
$$

where $w_1(x) = w(x + x_0)$ is the translation of $w(x)$, Q_1 is a cube which is the translation of Q. It is obvious that $w_1 \in A_q$ for $w \in A_q$, $q > 1$, and $q_{w_1} = q_w$. Therefore, Proposition 3.2 is proved by combining the estimates (1) and (5). □

PROPOSITION 3.3. *Let* $0 < \alpha \le 1$, $\frac{n}{n+\alpha} < p < 1$, $w \in A_{p(1+\frac{\alpha}{n})}$ *and* $\lambda > \frac{3n+2\alpha}{n}$. *Then for every* $f \in H^p_w(\mathbb{R}^n)$, *we have*

$$\big\|g^*_{\lambda,\alpha}(f)\big\|_{L^p_w} \le C\|f\|_{H^p_w}.$$

Proof. Let $q = p(1+\frac{\alpha}{n})$. As in the proof of Proposition 3.2, we only need to show that for any w-$(p,q,0)$-atom a, there exists a constant $C > 0$ independent of a such that $\|g^*_{\lambda,\alpha}(a)\|_{L^p_w} \le C$.

Let a be a w-$(p,q,0)$-atom with $\operatorname{supp} a \subseteq Q = Q(x_0, r)$, and let $Q^*_k = 2\sqrt{n}(2^k Q)$. From the definition, we readily see that

$$
\begin{aligned}
&\big(g^*_{\lambda,\alpha}(a)(x)\big)^2 \\
&= \int\int_{\mathbb{R}^{n+1}_+} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} \big(A_\alpha(a)(y,t)\big)^2 \frac{dy\,dt}{t^{n+1}} \\
&= \int_0^\infty \int_{|x-y|<t} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} \big(A_\alpha(a)(y,t)\big)^2 \frac{dy\,dt}{t^{n+1}} \\
&\quad + \sum_{k=1}^\infty \int_0^\infty \int_{2^{k-1}t \le |x-y| < 2^k t} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} \big(A_\alpha(a)(y,t)\big)^2 \frac{dy\,dt}{t^{n+1}} \\
&\le C\left[S_\alpha(a)(x)^2 + \sum_{k=1}^\infty 2^{-k\lambda n} S_{\alpha,2^k}(a)(x)^2\right].
\end{aligned}
$$

Since $0 < p < 1$, we thus get

$$\|g^*_{\lambda,\alpha}(a)\|^p_{L^p_w} \le C\left[\big\|S_\alpha(a)\big\|^p_{L^p_w} + \sum_{k=1}^\infty 2^{-\frac{k\lambda np}{2}} \big\|S_{\alpha,2^k}(a)\big\|^p_{L^p_w}\right].$$

By Proposition 3.2, we can obtain $\|S_\alpha(a)\|_{L^p_w} \le C$. It remains to estimate $\|S_{\alpha,2^k}(a)\|_{L^p_w}$ for $k = 1, 2, \ldots$.

First, we claim that the following inequality holds.

$$\big\|S_{\alpha,2^k}(a)\big\|_{L^2_w} \le C \cdot 2^{\frac{knq}{2}} \big\|S_\alpha(a)\big\|_{L^2_w}, \quad k = 1, 2, \ldots. \tag{6}$$

In fact, by using the Fubini theorem and Lemma A, we can get

$$\begin{aligned}\big\|S_{\alpha,2^k}(a)\big\|_{L^2_w}^2 &= \int_{\mathbb{R}^n}\bigg(\int_{\mathbb{R}^{n+1}_+}\big(A_\alpha(a)(y,t)\big)^2\chi_{|x-y|<2^k t}\frac{dy\,dt}{t^{n+1}}\bigg)w(x)\,dx\\ &= \int_{\mathbb{R}^{n+1}_+}\bigg(\int_{|x-y|<2^k t} w(x)\,dx\bigg)\big(A_\alpha(a)(y,t)\big)^2\frac{dy\,dt}{t^{n+1}}\\ &\le C\cdot 2^{knq}\int_{\mathbb{R}^{n+1}_+}\bigg(\int_{|x-y|<t} w(x)\,dx\bigg)\big(A_\alpha(a)(y,t)\big)^2\frac{dy\,dt}{t^{n+1}}\\ &= C\cdot 2^{knq}\big\|S_\alpha(a)\big\|_{L^2_w}^2.\end{aligned}$$

Using Hölder's inequality, Lemma A, Theorem F and (6), we thus obtain

$$\begin{aligned}\bigg(\int_{Q_k^*}\big|S_{\alpha,2^k}(a)(x)\big|^p w(x)\,dx\bigg)^{1/p} &\le \big\|S_{\alpha,2^k}(a)\big\|_{L^2_w} w\big(Q_k^*\big)^{\frac{1}{p}-\frac{1}{2}}\\ &\le C\cdot 2^{\frac{knq}{2}}\big\|S_\alpha(a)\big\|_{L^2_w}\big(2^{knq}w(Q)\big)^{\frac{1}{p}-\frac{1}{2}}\\ &\le C\cdot 2^{\frac{knq}{p}}\|a\|_{L^2_w}\big(w(Q)\big)^{1/p-1/2}\\ &\le C\cdot 2^{\frac{knq}{p}},\end{aligned}\tag{7}$$

where we have used the fact that $w\in A_q$, $1<q<1+\frac{\alpha}{n}\le 2$, then $w\in A_2$. Below we give the estimate of the integral $J=\int_{(Q_k^*)^c}|S_{\alpha,2^k}(a)(x)|^p w(x)\,dx$.

Note that $\operatorname{supp}\varphi\subseteq\{x\in\mathbb{R}^n:|x|\le 1\}$, by a simple calculation, we know that for any $(y,t)\in\Gamma_{2^k}(x)$, $x\in(Q_k^*)^c$, then $t\ge\frac{|x-x_0|}{2^{k+1}}$. It follows from the previous estimates (2) and (3) that

$$\begin{aligned}\big|S_{\alpha,2^k}(a)(x)\big|^2 &\le C\bigg(\frac{|Q|}{w(Q)^{1/p}}\bigg)^2 r^{2\alpha}\int\int_{\Gamma_{2^k}(x)}\frac{dy\,dt}{t^{2(n+\alpha)}\cdot t^{n+1}}\\ &\le C\bigg(\frac{|Q|}{w(Q)^{1/p}}\bigg)^2 r^{2\alpha}2^{kn}\int_{\frac{|x-x_0|}{2^{k+1}}}^\infty\frac{dt}{t^{2(n+\alpha)+1}}\\ &\le C\cdot 2^{3kn+2k\alpha}\bigg(\frac{r^{n+\alpha}}{w(Q)^{1/p}}\bigg)^2\frac{1}{|x-x_0|^{2(n+\alpha)}}.\end{aligned}\tag{8}$$

Applying Lemma A, Lemma B and the above inequality (8), we have

$$\begin{aligned}J &= \int_{(Q_k^*)^c}\big|S_{\alpha,2^k}(a)(x)\big|^p w(x)\,dx\\ &\le C\cdot 2^{\frac{kp(3n+2\alpha)}{2}}\frac{r^{p(n+\alpha)}}{w(Q)}\int_{|x-x_0|\ge\sqrt{n}2^k r}\frac{w(x)}{|x-x_0|^{nq}}\,dx\\ &\le C\cdot 2^{\frac{kp(3n+2\alpha)}{2}}\frac{r^{p(n+\alpha)}}{w(Q)}\big(2^k r\big)^{-nq}\big(2^k\big)^{nq}w_1(Q_1)\\ &\le C\cdot 2^{\frac{kp(3n+2\alpha)}{2}},\end{aligned}\tag{9}$$

where the notations w_1 and Q_1 are the same as Proposition 3.2, we have $w_1(Q_1) = w(Q)$. Hence, by the estimates (7) and (9), we obtain

$$\big\|S_{\alpha,2^k}(a)\big\|^p_{L^p_w} \le C\cdot\big(2^{kp(n+\alpha)} + 2^{\frac{kp(3n+2\alpha)}{2}}\big) \le C\cdot 2^{\frac{kp(3n+2\alpha)}{2}}.$$

Therefore

$$\big\|g^*_{\lambda,\alpha}(a)\big\|^p_{L^p_w} \le C\sum_{k=1}^{\infty} 2^{-\frac{k\lambda np}{2}}\cdot 2^{\frac{kp(3n+2\alpha)}{2}} \le C,$$

where the last inequality holds since $\lambda > \frac{3n+2\alpha}{n}$. The proof of Proposition 3.3 is complete. □

Using the same arguments as above, we can also show the H^p_w–L^p_w boundedness of $\tilde{g}^*_{\lambda,(\alpha,\varepsilon)}$; that is,

$$\big\|\tilde{g}^*_{\lambda,(\alpha,\varepsilon)}(f)\big\|_{L^p_w} \le C\|f\|_{H^p_w}. \tag{10}$$

Therefore, by Lemma 3.1, Proposition 3.2, Proposition 3.3 and (10), we have proved the necessity of our conditions.

4. The sufficiency of our conditions

We shall need the following Calderón reproducing formula given in [2].

LEMMA 4.1. *Let* $\psi \in \mathscr{S}(\mathbb{R}^n)$, $\operatorname{supp}\psi \subseteq \{x\in\mathbb{R}^n : |x|\le 1\}$, $\int_{\mathbb{R}^n}\psi(x)\,dx = 0$ *and*

$$\int_0^\infty \big|\hat{\psi}(\xi t)\big|^2\frac{dt}{t} = 1 \quad \textit{whenever } \xi \ne 0.$$

Then for any $f\in\mathscr{S}'(\mathbb{R}^n)$, f *vanishes weakly at infinity, we have*

$$f(x) = \int_0^\infty\int_{\mathbb{R}^n} f*\psi_t(y)\psi_t(x-y)\frac{dy\,dt}{t}, \tag{11}$$

where the equality holds in the sense of distributions.

Suppose that ψ satisfies the conditions of Lemma 4.1. For every $f\in\mathscr{S}'(\mathbb{R}^n)$, we define the area integral of f by

$$S_\psi(f)(x) = \left(\int_{|x-y|<t}\big|f*\psi_t(y)\big|^2\frac{dy\,dt}{t^{n+1}}\right)^{1/2}.$$

We are going to prove the following result.

PROPOSITION 4.2. *Let* $0<\alpha\le 1$, $\frac{n}{n+\alpha}<p<1$ *and* $w\in A_{p(1+\frac{\alpha}{n})}$. *Then for any* $f\in\mathscr{S}'(\mathbb{R}^n)$, f *vanishes weakly at infinity, we have*

$$\|f\|_{H^p_w} \le C\big\|S_\psi(f)\big\|_{L^p_w}.$$

Proof. We follow the same constructions as in [1] and [8]. For any $k \in \mathbb{Z}$, set

$$\Omega_k = \{x \in \mathbb{R}^n : S_\psi(f)(x) > 2^k\}.$$

Let $\mathbb{D}$ denote the set formed by all dyadic cubes in $\mathbb{R}^n$ and let

$$\mathbb{D}_k = \left\{Q \in \mathbb{D} : |Q \cap \Omega_k| > \frac{|Q|}{2}, |Q \cap \Omega_{k+1}| \le \frac{|Q|}{2}\right\}.$$

Obviously, for any $Q \in \mathbb{D}$, there exists a unique $k \in \mathbb{Z}$ such that $Q \in \mathbb{D}_k$. We also denote the maximal dyadic cubes in $\mathbb{D}_k$ by Q_k^l. Set

$$\widetilde{Q} = \{(y,t) \in \mathbb{R}_+^{n+1} : y \in Q, l(Q) < t \le 2l(Q)\},$$

where $l(Q)$ denotes the side length of Q.

If we set $\widetilde{Q_k^l} = \bigcup_{Q_k^l \supseteq Q \in \mathbb{D}_k} \widetilde{Q}$, then we have $\mathbb{R}_+^{n+1} = \bigcup_k \bigcup_l \widetilde{Q_k^l}$. Hence, by the expression (11), we obtain

$$f(x) = \sum_k \sum_l \int_{\widetilde{Q_k^l}} f * \psi_t(y)\psi_t(x-y)\frac{dy\,dt}{t} = \sum_k \sum_l \lambda_{kl} a_k^l(x),$$

where

$$a_k^l(x) = \lambda_{kl}^{-1} \int_{\widetilde{Q_k^l}} f * \psi_t(y)\psi_t(x-y)\frac{dy\,dt}{t}$$

and

$$\lambda_{kl} = w(Q_k^l)^{1/p-1/2}\left(\int_{\widetilde{Q_k^l}} |f * \psi_t(y)|^2 \frac{w(Q_k^l)}{|Q_k^l|}\frac{dy\,dt}{t}\right)^{1/2}.$$

By the properties of ψ, we can easily get $\operatorname{supp} a_k^l \subseteq 5Q_k^l$, $\int_{\mathbb{R}^n} a_k^l(x)\,dx = 0$. Let $q = p(1+\frac{\alpha}{n})$, $w \in A_q$. Since

$$\|a_k^l\|_{L_w^q} = \sup_{\|b\|_{L_w^{q'}} \le 1} \left|\int_{\mathbb{R}^n} a_k^l(x)b(x)w(x)\,dx\right|.$$

Then Hölder's inequality and the definition of λ_{kl} imply

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n} a_k^l(x)b(x)w(x)\,dx\right| \\
&\le \lambda_{kl}^{-1} \int_{\widetilde{Q_k^l}} |f * \psi_t(y)||g * \psi_t(y)|\frac{dy\,dt}{t} \\
&\le \lambda_{kl}^{-1}\left(\int_{\widetilde{Q_k^l}} |f * \psi_t(y)|^2 \frac{dy\,dt}{t}\right)^{1/2}\left(\int_{\widetilde{Q_k^l}} |g * \psi_t(y)|^2 \frac{dy\,dt}{t}\right)^{1/2} \\
&\le \frac{|Q_k^l|^{1/2}}{w(Q_k^l)^{1/p}}\left(\int_{\widetilde{Q_k^l}} |g * \psi_t(y)|^2 \frac{dy\,dt}{t}\right)^{1/2},
\end{aligned}$$

where $g(x)=\chi_{5Q_k^l}(x)b(x)w(x)$. For any $(y,t)\in\widetilde{Q_k^l}$, then a direct calculation shows that

$$\big|g*\psi_t(y)\big|\le C\cdot t^{-n}\|b\|_{L^{q'}_w}w\big(Q_k^l\big)^{1/q}.$$

Hence,

$$\begin{aligned}\big\|a_k^l\big\|_{L^q_w}&\le C\cdot\frac{|Q_k^l|^{1/2}}{w(Q_k^l)^{1/p}}w\big(Q_k^l\big)^{1/q}\bigg(\int_{\widetilde{Q_k^l}}\frac{dy\,dt}{t^{2n+1}}\bigg)^{1/2}\\&\le C\cdot w\big(Q_k^l\big)^{1/q-1/p},\end{aligned}$$

where in the last inequality we have used the fact that for any $(y,t)\in\widetilde{Q_k^l}$, we have $t^n\sim|Q_k^l|$. Therefore, these functions a_k^l defined above are all w-$(p,q,0)$-atoms.

Set $\Omega_k^*=\{x\in\mathbb{R}^n: M_w(\chi_{\Omega_k})(x)>\frac{C_0}{2}\}$, where C_0 is an appropriate constant and $M_w(f)(x)=\sup_{x\in Q}\frac{1}{w(Q)}\int_Q|f(y)|w(y)\,dy$. Using the weighted weak type estimate of weighted maximal operator M_w, we have $w(\Omega_k^*)\le Cw(\Omega_k)$. Consequently

$$\int_{\Omega_k^*\backslash\Omega_{k+1}}S_\psi(f)(x)^2w(x)\,dx\le\big(2^{k+1}\big)^2w\big(\Omega_k^*\big)\le C\cdot 2^{2k}w(\Omega_k).$$

We set $E=E(y,t)=\{x\in\Omega_k^*\backslash\Omega_{k+1}:|x-y|<t\}$, then we have

$$\begin{aligned}\int_{\Omega_k^*\backslash\Omega_{k+1}}S_\psi(f)(x)^2w(x)\,dx&=\int_{\mathbb{R}^{n+1}_+}\bigg\{\int_{\mathbb{R}^n}\chi_E(x)w(x)\,dx\bigg\}\big|f*\psi_t(y)\big|^2\frac{dy\,dt}{t^{n+1}}\\&\ge\sum_{Q\in\mathbb{D}_k}\int_{\widetilde{Q}}\big|f*\psi_t(y)\big|^2w\big(E(y,t)\big)\frac{dy\,dt}{t^{n+1}}.\end{aligned}$$

We also set $\overline{\Omega}_k^*=\{x\in\mathbb{R}^n: M(\chi_{\Omega_k})(x)>\frac12\}$ and $\overline{E}=\overline{E}(y,t)=\{x\in\overline{\Omega}_k^*\backslash\Omega_{k+1}:|x-y|<t\}$, where M denotes the classical(unweighted) Hardy–Littlewood maximal operator. It is easy to check that

$$E(y,t)\supseteq\overline{E}(y,t)\quad\text{for any }(y,t)\in\widetilde{Q},Q\in\mathbb{D}_k.$$

In [1], Chang and Fefferman actually proved that $|\overline{E}(y,t)|>c|Q|$, with a positive constant c independent of Q and $(y,t)\in\widetilde{Q}$. See also [2, p. 158] for its proof. Since $w\in A_\infty$, then by Lemma C, we know that there exists a constant $0<C'<1$ such that

$$w\big(E(y,t)\big)\ge w\big(\overline{E}(y,t)\big)>C'w(Q).\tag{12}$$

Suppose that $\{Q_k^l\}$ is the family of maximal dyadic cubes containing Q which belong to $\mathbb{D}_k$. Then by Lemma D and the above inequality (12), we can get

$$2^{2k}w(\Omega_k) \geq C\sum_{Q\in\mathbb{D}_k}\int_{\widetilde{Q}}\big|f*\psi_t(y)\big|^2 w(Q)\frac{dy\,dt}{t^{n+1}} \tag{13}$$
$$\geq C\sum_{Q\in\mathbb{D}_k}\int_{\widetilde{Q}}\big|f*\psi_t(y)\big|^2 w(Q_k^l)\left(\frac{|Q|}{|Q_k^l|}\right)^q\frac{dy\,dt}{t^{n+1}}$$
$$\geq C\sum_l\int_{\widetilde{Q_k^l}}\big|f*\psi_t(y)\big|^2\frac{w(Q_k^l)}{|Q_k^l|}\cdot\frac{1}{|Q_k^l|^{\frac{\alpha}{n}}}\frac{dy\,dt}{t^{1-\alpha}}$$
$$\geq C\sum_l\int_{\widetilde{Q_k^l}}\big|f*\psi_t(y)\big|^2\frac{w(Q_k^l)}{|Q_k^l|}\frac{dy\,dt}{t},$$

where the last inequality holds since $t\sim l(Q_k^l)$. For any $l\in\mathbb{Z}_+$, since $|Q_k^l\cap\Omega_k|>\frac{|Q_k^l|}{2}$, $w\in A_\infty$, then by using Lemma C again, we have that there exists a constant $0<C''<1$ such that $w(Q_k^l\cap\Omega_k)>C''w(Q_k^l)$. Note that the maximal dyadic cubes Q_k^l are pairwise disjoint, we thus obtain

$$w(\Omega_k)\geq w\left(\left(\bigcup_l Q_k^l\right)\cap\Omega_k\right) \tag{14}$$
$$=\sum_l w\big(Q_k^l\cap\Omega_k\big)$$
$$>C''\sum_l w\big(Q_k^l\big).$$

Then it follows from Hölder's inequality, the estimates (13) and (14) that

$$\sum_k\sum_l|\lambda_{kl}|^p=\sum_k\sum_l\big(w\big(Q_k^l\big)\big)^{1-p/2}\left(\int_{\widetilde{Q_k^l}}\big|f*\psi_t(y)\big|^2\frac{w(Q_k^l)}{|Q_k^l|}\frac{dy\,dt}{t}\right)^{p/2}$$
$$\leq\sum_k\left(\sum_l w\big(Q_k^l\big)\right)^{1-p/2}\left(\sum_l\int_{\widetilde{Q_k^l}}\big|f*\psi_t(y)\big|^2\frac{w(Q_k^l)}{|Q_k^l|}\frac{dy\,dt}{t}\right)^{p/2}$$
$$\leq C\sum_k\big(w(\Omega_k)\big)^{1-p/2}\big(2^{2k}w(\Omega_k)\big)^{p/2}$$
$$\leq C\big\|S_\psi(f)\big\|_{L^p_w}^p.$$

Therefore, by using the atomic decomposition of weighted Hardy spaces, we get the desired result. □

Finally, we choose a function ψ satisfying the conditions of Lemma 4.1. Obviously, we have $\psi\in\mathcal{C}_\alpha$ for any $0<\alpha\leq 1$, which implies

$$S_\psi(f)(x)\leq S_\alpha(f)(x)\leq C\tilde{S}_{(\alpha,\varepsilon)}(f)(x)\leq C\tilde{g}^*_{\lambda,(\alpha,\varepsilon)}(f)(x). \tag{15}$$

Combining the above inequality (15) and Proposition 4.2, we have proved the sufficiency of our conditions.

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