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WHITNEY'S EMBEDDING THEOREM FOR PSEUDO-HOLOMORPHIC DISCS

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ABSTRACT. We prove an analog of Whitney's embedding theorem for pseudo-holomorphic discs.

1. Introduction

Thom's transversality theorem and Whitney's approximation theorem are useful tools in analysis and geometry. They are proved by local perturbations and use cut-off functions to obtain a global result. Since there is no holomorphic cut-off functions in complex analytic category, we might need a global perturbation in order to prove the corresponding results. Kaliman and Zaidenberg [3] proved the jet transversality theorem for any holomorphic mapping from a Stein manifold to a complex manifold if the domain of the initial map is shrinked. Forstnerič [4] proved the a similar result without shrinking the domain, but the target space is required to be either subelliptic or satisfies the Oka property. In almost complex category, Sukhov and Tumanov [1] proved the Thom's transversality theorem and Whitney's immersion theorem for pseudo-holomorphic discs. In this paper, we show Whitney's embedding theorem holds for pseudo-holomorphic discs.

THEOREM 1.1. *Let (M,J) be a C^∞-smooth almost complex manifold with $\dim_{\mathbb{C}} M > 2$, fix $m \geq 0$ and let $f_0 : \mathbb{D} \to M$ be a pseudo-holomorphic disc of class $C^m(\overline{\mathbb{D}})$. Then there exists a pseudo-holomorphic embedding $f : \mathbb{D} \to M$ arbitrarily close to f_0 in $C^m(\overline{\mathbb{D}})$.*

2. Pseudo-holomorphic discs and Cauchy–Green integral

We first recall some basic results in several complex variables. Let $\mathbb{D}$ be the unit disc in $\mathbb{C}$ and let J_{st} be the standard almost complex structure of $\mathbb{C}^n$.

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Suppose (M,J) is an almost complex manifold, a smooth map $f:\mathbb{D}\to M$ is called a *pseudo-holomorphic disc* if

$$df\circ J_{\mathrm{st}}=J\circ df.$$

Fix a positive integer k and a real number $0<\alpha<1$. We denote by $\mathcal{C}^{\alpha}(\overline{\mathbb{D}})$ the space of functions on $\overline{\mathbb{D}}$ satisfying the Hölder condition with Hölder index α. We denote

$$\mathcal{C}^{k,\alpha}(\overline{\mathbb{D}})=\left\{f\in C^{k}(\overline{\mathbb{D}})|\text{the } k\text{th partial derivatives of } f \text{ is in } \mathcal{C}^{\alpha}(\overline{\mathbb{D}})\right\}.$$

The main tool that we will use in this paper is the Cauchy–Green integral

$$Tu(\zeta)=\frac{1}{2\pi i}\int_{\mathbb{D}}\frac{u(\omega)\,d\omega\wedge d\overline{\omega}}{\omega-\zeta}.$$

The following properties of the Cauchy–Green integral can be found in [2]:

COROLLARY 2.1.

(1) *Let $p>2$. If $f\in L^{p}(\mathbb{D})$, then $\partial_{\overline{\zeta}}Tf=f$ in the sense of distribution, where $\zeta\in\mathbb{D}$.*
(2) *Let $k\geq 0$ be an integer and $0<\alpha<1$, then $T:\mathcal{C}^{k,\alpha}(\overline{\mathbb{D}})\to\mathcal{C}^{k+1,\alpha}(\overline{\mathbb{D}})$ is bounded.*

3. Surjectivity of the jet map

Let B_1,B_2 be $n\times n$ matrix functions on $\mathbb{D}$ of class $L^{p}(\mathbb{D})$. Consider the solution of the equation

$$u_{\overline{\zeta}}=B_1u+B_2\overline{u}. \tag{1}$$

Fix $\tau\in\mathbb{D}$, we define an operator

$$Pu=u-T(B_1u+B_2\overline{u})+T(B_1u+B_2\overline{u})(\tau).$$

By using a similar proof in Theorem 3.1 of [1], with changing the evaluation point from the origin of $\mathbb{D}$ to τ, we have the following:

THEOREM 3.1. *Let B_1,B_2 be $n\times n$ matrices in $L^{p}(\mathbb{D}),p>2$ and $\tau\in\mathbb{D}$.*

(1) *Let $w_1,\dots,w_d$ form a basis of $kerP$ over $\mathbb{R}$ and let $r>2p(p-2)^{-1}$, then there exists holomorphic polynomial vectors $p_1,\dots,p_d$ with $p_1(\tau)=\cdots=p_d(\tau)=0$ such that the operator $\widetilde{P}:L^{r}(\mathbb{D})\to L^{r}(\mathbb{D})$ defined by*

$$\widetilde{P}u=Pu+\sum_{j=1}^{d}\big(\mathrm{Re}(u,w_j)\big)p_j$$

has trivial kernel. The polynomials p_j can be chosen to be arbitrarily small. (Here $(\cdot,\cdot)$ means the inner product for $u=(u_1,\dots,u_n),v=(v_1,\dots,v_n)$

$$(u,v)=\sum_{j=1}^{n}\frac{i}{2}\int_{\mathbb{D}}u_j\overline{v_j}\,d\zeta\wedge\overline{d\zeta}.)$$

(2) *If* $B_1, B_2 \in \mathcal{C}^{k,\alpha}(\mathbb{D})$, $0<\alpha<1$ *(resp.* $W^{k,p}(\mathbb{D})$*) then* $\widetilde{P}$ *is an invertible bounded operator in* $\mathcal{C}^{k+1,\alpha}(\mathbb{D})$ *(resp.* $W^{k+1,p}(\mathbb{D})$*). The function* $\phi = \widetilde{P}u$ *is holomorphic if and only if* u *satisfies equation* (1).

Now we are going to prove the existence of solution of equation (1) with prescribed value at two given points.

THEOREM 3.2. *Let* B_1, B_2 *be* $n \times n$ *matrix functions on* $\mathbb{D}$ *of class* $\mathcal{C}^{k-1,\alpha}(\mathbb{D})$, $k \geq 1, 0<\alpha<1$. *Then for all* $\zeta_1, \zeta_2 \in \mathbb{D}$, $a_1, a_2 \in \mathbb{C}^n$, *there exists a solution* u *of equation* (1) *such that* $u \in \mathcal{C}^{k,\alpha}(\mathbb{D})$ *and*

$$u(\zeta_1) = a_1, \qquad u(\zeta_2) = a_2.$$

Proof. By the Theorem 5.1 in [1], there exists $u_1 \in \mathcal{C}^{k,\alpha}(\mathbb{D})$ such that $u_1(\zeta_1) = a_1$. Let $u(\zeta) = u_1(\zeta) + (\zeta - \zeta_1)w(\zeta)$ where w has to be determined. If we require $B_1 u + B_2 \overline{u} = u_{\overline{\zeta}}$ and $u(\zeta_2) = a_2$, then w has to satisfy

$$B_1(\zeta)w(\zeta) + B_2(\zeta)\frac{\overline{\zeta - \zeta_1}}{\zeta - \zeta_1}\overline{w(\zeta)} = w_{\overline{\zeta}}(\zeta) \tag{2}$$

and

$$w(\zeta_2) = \frac{a_2 - u_1(\zeta_2)}{\zeta_2 - \zeta_1} \triangleq \phi \quad \text{which is a constant vector.}$$

Note that $\widehat{B_2(\zeta)} = B_2(\zeta)\frac{\overline{\zeta-\zeta_1}}{\zeta-\zeta_1} \in L^p(\mathbb{D})$ for all $2<p<\infty$, therefore we can apply Theorem 3.1 with $\tau = \zeta_2$ and $B_2 = \widehat{B_2}$. Set $w = \widetilde{P}^{-1}(\phi)$, then we have $w \in W^{1,p}(\mathbb{D})$ for all $2<p<\infty$, hence $w \in \mathcal{C}^{\alpha}(\mathbb{D})$, therefore $\phi = \widetilde{P}w(\zeta_2) = Pw(\zeta_2) = w(\zeta_2)$, and $\widetilde{P}w = \phi$ implies $w_{\overline{\zeta}} = B_1 w + \widehat{B_2}\overline{w}$.

It remains to show that u is in the class $\mathcal{C}^{k,\alpha}(\mathbb{D})$. We first choose $0<r<1$ so that $\mathbb{D}_r$, the disc centered at the origin with radius r, satisfying $\mathbb{D}_r \subset \mathbb{D}$ and $\zeta_1 \in \mathbb{D}_r$. Note that the coefficient matrices $B_1, \widehat{B_2}$ is in $C^{k-1,\alpha}(\mathbb{D} \setminus \mathbb{D}_r)$ and it is proved that $w \in \mathcal{C}^{\alpha}(\mathbb{D})$, so we can apply bootstrapping to the equation $\widetilde{P}w = \phi$ to conclude that w is in $\mathcal{C}^{k,\alpha}(\mathbb{D} \setminus \mathbb{D}_r)$.

Note that the function $v(\zeta) = (\zeta - \zeta_1)w(\zeta)$ satisfies equation (1), then we have

$$v(z) = T(B_1 v + B_2 \overline{v})(z) + \psi(z) \tag{3}$$

for some vector $\psi(z)$ holomorphic on $\mathbb{D}$ and in $\mathcal{C}^{\alpha}(\mathbb{D})$. We claim that $\psi \in \mathcal{C}^{k,\alpha}(\partial\mathbb{D})$: write $T(B_1 v + B_2 \overline{v})(z)$ as

$$\frac{1}{2\pi i}\int_{\mathbb{D}_r} \frac{B_1 v + B_2 \overline{v}}{\zeta - z}\, d\zeta \wedge d\overline{\zeta} + \frac{1}{2\pi i}\int_{\mathbb{D}\setminus\mathbb{D}_r} \frac{B_1 v + B_2 \overline{v}}{\zeta - z}\, d\zeta \wedge d\overline{\zeta},$$

for which we understand it as an integration applied to each entry of the vector function. Now the first term is holomorphic on $\partial\mathbb{D}$ since the function $\frac{1}{\zeta - z}$ is holomorphic in z on $\partial\mathbb{D}$ whenever ζ is in $\mathbb{D}_r$, hence it is in $\mathcal{C}^{k,\alpha}(\partial\mathbb{D})$. The second term is in $\mathcal{C}^{k,\alpha}(\mathbb{D}\setminus\mathbb{D}_r)$ since $v\in\mathcal{C}^{k,\alpha}(\mathbb{D}\setminus\mathbb{D}_r)$ and $B_1, B_2\in\mathcal{C}^{k-1,\alpha}(\mathbb{D}\setminus\mathbb{D}_r)$. Combining with the fact that $v\in\mathcal{C}^{k,\alpha}(\mathbb{D}\setminus\mathbb{D}_r)$, we have $\psi\in\mathcal{C}^{k,\alpha}(\partial\mathbb{D})$.

Now ψ is holomorphic in $\mathbb{D}$ and in $\mathcal{C}^{k,\alpha}(\partial\mathbb{D})$, hence by the regularity of Laplace operator we have $\psi\in\mathcal{C}^{k,\alpha}(\mathbb{D})$ (see, for example, [5]). By using $v\in\mathcal{C}^{\alpha}(\mathbb{D})$, we can apply bootstrapping to the equation

$$v = T(B_1 v + B_2\overline{v}) + \psi$$

to conclude that v, and hence u, is in fact in $\mathcal{C}^{k,\alpha}(\mathbb{D})$. □

4. Approximation by embedding pseudo-holomorphic discs

Given $f_0:\mathbb{D}\to M$ a pseudo-holomorphic disc of class $C^m(\overline{\mathbb{D}})$, we first use Theorem 1.1 in [1] to approximate f_0 by a $C^\infty(\overline{\mathbb{D}})$ pseudo-holomorphic immersion $f_1:\mathbb{D}\to M$, so that f_1 is close to f_0 in $C^m(\overline{\mathbb{D}})$.

For $0<r<1$, the function $z\mapsto f_{2,r}(z) = f_1(rz)$ is arbitrarily close to f_1 in $C^m(\overline{\mathbb{D}})$ as r approaches 1: By Nash's embedding theorem, we can assume $M=\mathbb{C}^N$ for some N sufficiently large, then the result can be easily proved by the fact that the i-derivative $f^{(i)}$ is uniformly continuous on $\overline{\mathbb{D}}$.

For all $\varepsilon>0$, choose r_0 so that f_{2,r_0} is ε-close to f_1 in $C^m(\overline{\mathbb{D}})$, for simplicity denote f_{2,r_0} by f_2. Note that f_2 is an immersed pseudo-holomorphic disc.

LEMMA 4.1. *Let*

$$R=\inf\bigl\{|c_1-c_2| : c_1, c_2\in\overline{\mathbb{D}}, c_1\neq c_2, f_2(c_1)=f_2(c_2)\bigr\},$$

then $R>0$.

Proof. Suppose $R=0$, then there exists two sequences c_1^n, c_2^n in $\overline{\mathbb{D}}$ such that $|c_1^n - c_2^n|\to 0$ as $n\to\infty$, which means $c_j^n\to c_0\in\overline{\mathbb{D}}$ for each $j=1,2$. Note that $r_0c_0\in\mathbb{D}$ and f_1 is an immersion at r_0c_0 implies f_1 is injective on some neighborhood $N\subset\mathbb{D}$ of r_0c_0. However for n large enough, all $r_0c_1^n, r_0c_2^n$ belong to N and we have $f_1(r_0c_1^n)=f_2(c_1^n)=f_2(c_2^n)=f_1(r_0c_2^n)$, a contradiction. Therefore, $R>0$. □

Define

$$U=\left\{(\zeta_1,\zeta_2)\in\overline{\mathbb{D}}\times\overline{\mathbb{D}} : |\zeta_1-\zeta_2|<\frac{R}{2}\right\}.$$

Let $\mathcal{V}^{k,\alpha}$ be the space of all solutions of equation (1) in the class $\mathcal{C}^{k,\alpha}(\mathbb{D})$. For any function $f:X\to Y$, let $E_{\zeta_1,\zeta_2}f = Ef(\zeta_1,\zeta_2)$ be the 0-jet of $f\times f$ at $\zeta_1,\zeta_2\in X$ defined by

$$E_{\zeta_1,\zeta_2}f=\bigl(\zeta_1, f(\zeta_1), \zeta_2, f(\zeta_2)\bigr).$$

By using the compactness argument as in Proposition 5.2 of [1], we have the following corollary.

COROLLARY 4.2. *For $B_1, B_2 \in \mathcal{C}^{k-1,\alpha}(\mathbb{D})$ $(k \geq 1, 0 < \alpha < 1)$, there exists a subspace $V \subset \mathcal{V}^{k,\alpha}$ with $\dim V < \infty$, such that for all $(\zeta_1, \zeta_2) \in (\overline{\mathbb{D}} \times \overline{\mathbb{D}}) \setminus U$, the mapping $u \mapsto (u(\zeta_1), u(\zeta_2))$ is surjective from V onto $\mathbb{C}^{2n}$.*

Before proving Theorem 1.1, we need some results from [1]. Assume for the moment that $f_2 : \mathbb{D} \to M$ is a pseudo-holomorphic disc of class $\mathcal{C}^{k,\alpha}(\mathbb{D})$ where k is large and $0 < \alpha < 1$, the particular values of k and α are unimportant. For every point $p \in M$ there exists a chart $\psi : U \subset M \to \mathbb{C}^n$ such that $p \in U, \psi(p) = 0$, and for the push-forward $\psi_* J = d\psi \circ J \circ d\psi^{-1}$ we have $\psi_* J(0) = J_{\mathrm{st}}$. It is proved in [1] that it is possible to choose such chart ψ^ζ for each $p = f_2(\zeta)$ and ψ^ζ $\mathcal{C}^{k,\alpha}$-smoothly depends on $\zeta \in \overline{\mathbb{D}}$. Furthermore, a map $f : \mathbb{D} \to M$ close to f_2 is in $\mathcal{C}^{k,\alpha}$ if and only if the map $\zeta \mapsto g(\zeta) \triangleq \psi^{r\zeta}(f(\zeta)) \in \mathbb{C}^n$ satisfies an equation of the form

$$g_{\overline{\zeta}} = A(\zeta, g)\overline{g}_{\overline{\zeta}} + b(\zeta, g), \tag{4}$$

where b is some smooth matrix function and $A(\zeta, \cdot)$ is the complex matrix of $\psi_*^{r\zeta} J$.

Let g_0 be a solution of (4) in $\mathcal{C}^{k,\alpha}(\mathbb{D})$ and put $A_0(\zeta) = A(\zeta, g_0(\zeta))$. For any other solution g of (4) in the same class, put $h = g - A_0\overline{g}$, so h and g is in one-to-one correspondence. By a direct computation equation, (4) can be written as

$$h_{\overline{\zeta}} = K_0\overline{h}_{\overline{\zeta}} + K_1 h + K_2\overline{h} + q \tag{5}$$

for some matrix functions K_0, K_1, K_2, q. Put $h_0 = g_0 - A_0\overline{g}_0$. Equation (5) is equivalent to

$$h = T_0(K_0\overline{h}_{\overline{\zeta}} + K_1 h + K_2\overline{h} + q) + \phi + \phi_0,$$

where ϕ is holomorphic, ϕ_0 is a fixed holomorphic function such that (5) holds with $h = h_0$ and $\phi = 0$.

Consider the C^∞ map

$$h \mapsto F_0(h) = \phi = h - T_0(K_0\overline{h}_{\overline{\zeta}} + K_1 h + K_2\overline{h} + q) - \phi_0,$$

and let $P = F_0'(h_0)$. By Theorem 3.1, we can modify F_0 to get a new function $F : h \mapsto F(h)$ where

$$F(h) = h - T_0(K_0\overline{h}_{\overline{\zeta}} + K_1 h + K_2\overline{h} + q) - \phi_0 + \sum_{j=1}^{d} \mathrm{Re}(w_j, h - h_0)p_j.$$

Note that the Fréchet derivative of F at h_0 has the form

$$F'(h_0)u = u - T_0(B_1 u + B_2\overline{u}) + \sum_{j=1}^{d} \mathrm{Re}(w_j, u)p_j, \quad B_1, B_2 \in \mathcal{C}^{k-1,\alpha}(\mathbb{D}).$$

In fact $F'(h_0)$ is an isomorphism, hence by the inverse function theorem F gives a one-to-one correspondence between all solutions of (5) close to h_0 and all holomorphic functions ϕ close to 0 in $\mathcal{C}^{k,\alpha}(\mathbb{D})$.

As a conclusion we have a one-to-one correspondence $f \leftrightarrow h$ between pseudo-holomorphic disc f close to f_2 and solutions h of the equation (5) close to h_0.

Proof of Theorem 1.1. Let $J_1 = \{$0-jet of $f \times f : (\overline{\mathbb{D}} \times \overline{\mathbb{D}}) \setminus U \to M \times M | f$ is J-holo$\}$, $J_2 = \{$0-jet of $h \times h : (\overline{\mathbb{D}} \times \overline{\mathbb{D}}) \setminus U \to \mathbb{C}^{2n} | h$ satisfies (5)$\}$

The correspondence $f \leftrightarrow h$ gives rise to a diffeomorphism

$$\Psi : W \to \widetilde{W}$$

defined in the neighborhood $W \supset Ef_2((\overline{\mathbb{D}} \times \overline{\mathbb{D}}) \setminus U)$ in J_1, $\widetilde{W} = \Psi(W)$ in J_2. Let $\Delta = \{(\zeta_1, x, \zeta_2, x) \in \overline{\mathbb{D}} \times M \times \overline{\mathbb{D}} \times M | (\zeta_1, \zeta_2) \in (\overline{\mathbb{D}} \times \overline{\mathbb{D}}) \setminus U, x \in M\}$ be a subset of J_1, then Ef is transversal to Δ if and only if the corresponding Eh is transversal to $\widetilde{\Delta} = \Psi(\Delta)$. (Here, we only consider the case $\Delta \cap W \neq \emptyset$ and denote $\Delta \cap W$ by Δ again, otherwise $Ef \pitchfork \Delta$ for small perturbation of f_2.)

Let $u_1, \dots, u_N$ be the basis of $V \subset \mathcal{V}^{k,\alpha}$, the space of all solutions of $u_{\overline{\zeta}} = B_1 u + B_2 \overline{u}$ in the class $\mathcal{C}^{k,\alpha}(\mathbb{D})$. Let $\phi_j = F'(h_0)(u_j)$ and $\phi_s = \sum_l s_l \phi_l$ for $s = (s_1, \dots, s_N) \in \mathbb{R}^N$. The map

$$\begin{aligned} \Phi : \big((\overline{\mathbb{D}} \times \overline{\mathbb{D}}) \setminus U\big) \times \mathbb{R}^N &\to J_2, \\ (\zeta_1, \zeta_2, s) &\to EF^{-1}(\phi_s)(\zeta_1, \zeta_2) \end{aligned}$$

is defined for small $s \in \mathbb{R}^N$. By Proposition 4.2, the mapping is a submersion for $s = 0, (\zeta_1, \zeta_2) \in (\overline{\mathbb{D}} \times \overline{\mathbb{D}}) \setminus U$, by shrinking the domain of s in $\mathbb{R}^N$, we see that Φ is a submersion. Hence by parametric transversality, there is $s \in \mathbb{R}^N$ arbitrarily close to 0 such that $Eh = {}^{(2)}jF^{-1}(\phi_s)$ is transversal to $\widetilde{\Delta}$, hence the corresponding Ef is transversal to Δ.

Now Ef is transversal to Δ implies $Ef((\overline{\mathbb{D}} \times \overline{\mathbb{D}}) \setminus U) \cap \Delta = \emptyset$. To see this, first observe that $\dim_{\mathbb{R}}((\overline{\mathbb{D}} \times \overline{\mathbb{D}}) \setminus U) = 4$, so

$$\dim_{\mathbb{R}} Ef\big((\overline{\mathbb{D}} \times \overline{\mathbb{D}}) \setminus U\big) = 4, \qquad \dim_{\mathbb{R}}\big(\big((\overline{\mathbb{D}} \times \overline{\mathbb{D}}) \setminus U\big) \times M\big) = 4 + 2n$$

and

$$\dim_{\mathbb{R}}(\overline{\mathbb{D}} \times M \times \overline{\mathbb{D}} \times M) = 4 + 4n.$$

Therefore, if $Ef((\overline{\mathbb{D}} \times \overline{\mathbb{D}}) \setminus U) \cap \Delta$ is nonempty, it will imply $4 + 4 + 2n \geq 4 + 4n$ which gives $\dim_{\mathbb{C}} M = n \leq 2$, a contradiction.

Finally, we need to check that f is injective on $\overline{\mathbb{D}}$, suppose there exists $\zeta_1 \neq \zeta_2$ in $\overline{\mathbb{D}}$ with $f(\zeta_1) = f(\zeta_2)$, then $Ef((\overline{\mathbb{D}} \times \overline{\mathbb{D}}) \setminus U) \cap \Delta = \emptyset$ implies $(\zeta_1, \zeta_2) \in U$, which means $|\zeta_1 - \zeta_2| < \frac{R}{2}$, but this contradicts the definition of R. Hence, the immersed pseudo-holomorphic disc f is injective. Because $\overline{\mathbb{D}}$ is compact, f is in fact an embedding. Therefore, we have found an embedded pseudo-holomorphic disc f $C^\infty(\overline{\mathbb{D}})$ arbitrarily close to f_2 and hence to f_0 in $C^m(\overline{\mathbb{D}})$. The proof of Theorem 1.1 is complete. $\square$

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References

[1] A. Sukhov and A. Tumanov, *Deformations and transversality of pseudo-holomorphic discs*, J. Anal. Math. **116** (2012), 1–16. MR 2892615

[2] I. N. Vekua, *Generalized analytic functions*, Pergamon, London, 1962. MR 0150320

[3] S. Kaliman and M. Zaidenberg, *A transverality theorem for holomorphic mappings and stability of Eisenman–Kobayashi measures*, Trans. Amer. Math. Soc. **348** (1996), 661–672. MR 1321580

[4] F. Forstnerič, *Holomorphic flexibility properties of complex manifolds*, Amer. J. Math. **128** (2006), 239–270. MR 2197073

[5] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, Classics in Mathematics, Springer, Berlin, 2001. Reprint of the 1998 edition. MR 1814364

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