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HERMITIAN ALGEBRA ON THE ELLIPSE

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To John P. D'Angelo with admiration and best wishes

Abstract. The subtle distinction between hermitian sums of squares and sums of squares, regarded as positivity certificates of a polynomial restricted to a real algebraic variety, is analyzed on the simplest, yet very relevant, example: an ellipse.

1. Introduction

The present note links two recent works [6], [7]. When dealing with the most general and abstract setting of hermitian sums of squares on real algebraic varieties we have observed that restricting the mighty real algebraic and functional analytic machinery to the case of the ellipse has much to offer. Of course, with some elementary ad-hoc arguments in place, which make the body of the following pages. For a full account of the real algebraic geometry context this essay belongs to see [9].

Let $z = (z_1, \dots, z_d) \in \mathbb{C}^d$ denote the complex variable in a d-dimensional hermitian space. We decompose $z = x + iy \in \mathbb{R}^d + i\mathbb{R}^d$ into real and imaginary part, and consider the algebras $\mathbb{C}[z], \mathbb{C}[z,\overline{z}], \mathbb{R}[x,y]$ of complex analytic polynomials, complex polynomials, respectively real polynomials. The algebra $\mathbb{C}[z,\overline{z}]$ carries an involution

$$f = \sum_{\alpha,\beta} a_{\alpha\beta} z^\alpha \overline{z}^\beta \mapsto f^* = \sum_{\alpha,\beta} \overline{a}_{\alpha\beta} z^\beta \overline{z}^\alpha,$$

so that $\mathbb{R}[x,y] = \{f \in \mathbb{C}[z,\overline{z}]; f = f^*\}$ is the ring of fixed elements.

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Let $I \subset \mathbb{R}[x,y]$ be an ideal, and let $X = V_{\mathbb{R}}(I)$ be the real zero set of I in $\mathbb{R}^{2d} = \mathbb{C}^d$. The elements of the quotient algebra $A = \mathbb{R}[x,y]/I$ can be regarded as real polynomial functions along X. We denote by ΣA^2 the convex cone of sums of squares in A. Also, we denote Σ_h to be the convex cone of sums of *hermitian squares* $|p(z)|^2$, where $p \in \mathbb{C}[z]$. On the quotient algebra A, we set $\Sigma_h A = (\Sigma_h + I)/I$. Thus, the elements of $\Sigma_h A$ are the cosets $\sum_j |p_j(z)|^2 + I$ in A, where $p_j \in \mathbb{C}[z]$ are finitely many complex analytic polynomials.

There are nontrivial examples of ideals for which $\Sigma_h A$ contains *all* strictly positive polynomials on X. The oldest and of highest impact is known as the Riesz–Fejér Theorem, which states that every nonnegative trigonometric polynomial is equal to a single hermitian square (on the trigonometric circle $\mathbb{T}$). A multi-variate analogue of it for the d-dimensional torus exists (that is $I = (|z_1|^2 - 1, \ldots, |z_d|^2 - 1)$ and $X = \mathbb{T}^d$) with a necessary stronger assumption: positive polynomials belong to $\Sigma_h A$. Along the same lines is Quillen's Theorem [8], stating that every positive polynomial on the unit sphere of $\mathbb{C}^d$ agrees with a sum of hermitian squares on the sphere, see also [2].

A rather abstract characterization of all ideals $I \subset \mathbb{C}[z,\overline{z}]$ with *Quillen's property* (that a positive polynomial on $X = V_{\mathbb{R}}(I)$ belongs to $\Sigma_h(\mathbb{R}[x,y]/I)$) was given in [6]; another proof of Quillen's theorem and a series of examples of strictly pseudoconvex, real algebraic boundaries X without Quillen's property are also contained in [6]. A filtration of obstructions, quantified by a hermitian complexity rank, was introduced in [3], ranging from Quillen's property to the existence of a complex curve in X.

The ellipses in $\mathbb{C}$ offer a surprisingly rich framework to test such novel concepts related to hermitian sums of squares. An early and very lucid work dealing with moment problems and positive polynomials on planar algebraic curves (specifically, circles or lines in general position) goes back to a 1933 note by M. G. Krein [4]. Later on, moment problems along specific curves were thoroughly investigated, see for instance [10], and in particular the Bernoulli lemniscate case discussed there.

For a better perspective on the rather lengthy preliminaries, the reader may want to have first a glance at the main results (Theorem 3.3 and Corollary 3.4), stated and proved at the end of the note.

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2. Abstract setting and general results

We start with an ideal $I \subset \mathbb{R}[x,y] \subset \mathbb{C}[z,\overline{z}]$, where $z = x + iy \in \mathbb{C}^d$. Note that the set $\Sigma_h + I$ contains $\mathbb{R}_+$ and it is a semi-ring (i.e., closed under addition and multiplication). A convex cone $C \subset A$ is called *archimedean* if for every element $f \in A$ there exists a constant $c \in \mathbb{R}$ such that $c \pm f \in C$. When applied to a semi-ring, such as $C = \Sigma_h + I$, archimedianity is equivalent to the existence of a constant $c \in \mathbb{R}$ such that $c \pm x_k, c \pm y_k \in C$ for all $1 \le k \le d$.

The importance of the archimedianity property for our note is derived from the following Representation theorem, see [5] for full details and a proof.

THEOREM 2.1. *Let $I \subset \mathbb{R}[x,y]$ be an ideal. The following are equivalent*:

(a) *The semi-ring $\Sigma_h + I$ is archimedean*;
(b) *The real zero set $V_{\mathbb{R}}(I)$ is compact and $\Sigma_h + I$ contains every $f \in A$ with $f > 0$ on $V_{\mathbb{R}}(I)$*;
(c) *The ideal I contains a polynomial of the form $\|z\|^2 + p - a$, where $p \in \Sigma_h$ and $a \geq 0$ is a real number.*

An ideal I fulfilling the conditions in the statement will be said to possess *Quillen's property*. As an application, it follows that the ideals of the sphere $(\|z\|^2 - 1)$ and the torus $(|z_1|^2 - 1, \dots, |z_d|^2 - 1)$ satisfy the above requirements. In other terms, every positive polynomial on the sphere, or on the torus, is a sum of hermitian squares restricted on the respective varieties. Due to its rather abstract form, condition (c) is not easy to check on examples. A couple of necessary conditions for Quillen's property were recently studied in [3], [7].

Following a recent trend in real algebra, often it is relevant to evaluate polynomials on elements of a noncommutative *-algebra. One classical instance is offered by the *hereditary functional calculus* introduced in operator theory by Colojoara and Foias (we refer to [7] for a reference and more details). Specifically, if $T = (T_1, \dots, T_d)$ is a d-tuple of commuting linear bounded operators acting on a Hilbert space H, and $p(z, \overline{z}) \in \mathbb{C}[z, \overline{z}]$, one defines the operator $\psi_T(p)$ by replacing $z_k, \overline{z}_k$ by $T_k, T_k^*, 1 \leq k \leq d$, respectively, and arranging all T_k^*'s to the left of T_j's, monomial by monomial. For instance, $\psi_T(z_1 \overline{z}_2 z_3) = T_2^* T_1 T_3$. Note that there is no ambiguity in the definition, thanks to the commutativity of the tuple T.

One obtains in this way a linear unital map

$$\psi_T : \mathbb{C}[z, \overline{z}] \longrightarrow L(H)$$

which is close to being multiplicative:

$$\psi_T\big(f_1(z) f_2(z, \overline{z}) f_3(\overline{z})\big) = \psi_T\big(f_3(\overline{z})\big) \psi_T\big(f_2(z, \overline{z})\big) \psi_T\big(f_1(z)\big).$$

In particular, $\psi_T(h) \geq 0$ whenever $h \in \Sigma_h$.

The tuple T is called *subnormal* if H is a closed subspace of a larger Hilbert space K and there exists a commutative tuple of normal operators $N = (N_1, \dots, N_d)$ with the property that every N_k leaves H invariant and $N_k|_H = T_k, 1 \leq k \leq d$.

Putting the hereditary functional calculus at work in conjunction with a well known subnormality criterion, one obtains the following complement to the main theorem above. Details appear in [7].

COROLLARY 2.2. *The equivalent conditions in Theorem 2.1 imply*:

(d) *Every commutative d-tuple of linear bounded operators T satisfying $\psi_T(I) = 0$ is subnormal.*

An accessible measure of the distance of a real ideal $I \subset \mathbb{R}[x,y]$ from possessing Quillen's property is offered by the *hermitian complexity* of I, introduced in [3]. More precisely, the ideal I has hermitian complexity $N \in \mathbb{N} \cup \{\infty\}$, if, roughly speaking, there are N distinct points $p_j \in \mathbb{C}^d$ with the property

$$h(p_j, \overline{p_k}) = 0, \quad h \in I, 1 \le j,k < N+1, \tag{1}$$

and N is maximal with this property. As a matter of fact, multiplicities are allowed, but we do not need here the exact definition, see for details [3].

An ideal I of hermitian complexity $N > 1$ and $V_{\mathbb{R}}(I)$ compact does not have Quillen's property for a very simple reason. Namely, assume that two distinct points $p_1, p_2 \in \mathbb{C}^d$ are subject to condition (1). Since an element $f \in I + \Sigma_h$ satisfies Cauchy–Schwarz inequality:

$$\left|f(p_1, \overline{p_2})\right|^2 \le f(p_1, \overline{p_1}) f(p_2, \overline{p_2}),$$

we infer $f(p_1, \overline{p_2}) = 0$ whenever $f(p_1, \overline{p_1}) = f(p_2, \overline{p_2}) = 0$. On the other hand, there are positive polynomials F on $V_{\mathbb{R}}(I)$ which violate the vanishing assumption (1). For instance, one can choose an element $h \in A$ satisfying $h(p_1, \overline{p_1}) = h(p_2, \overline{p_2}) = 0$ but $h(p_1, \overline{p_2}) = 1$ and choose $F = \varepsilon + h^2$ for $\varepsilon > 0$ small. Therefore, $F \notin I + \Sigma_h$.

Just for illustration, to the other end of the hermitian complexity chain lies the following result proved in [3].

THEOREM 2.3. *A principal ideal $I \subset \mathbb{R}[x,y]$ has infinite hermitian complexity if and only if $V_{\mathbb{R}}(I)$ contains a complex algebraic curve.*

Constructive methods of computing the hermitian complexity of an ideal are presented in [3].

3. The ellipse

Let $\alpha \in [0, 1/2)$ be a parameter and let $X \subset \mathbb{R}^2 = \mathbb{C}$ be the ellipse of equation

$$\phi(z, \overline{z}) = |z|^2 - \alpha z^2 - \alpha \overline{z}^2 - 1 = 0,$$

or in real coordinates

$$(1 - 2\alpha)x^2 + (1 + 2\alpha)y^2 = 1.$$

We denote by (ϕ) the principal ideal of $\mathbb{R}[x,y]$ generated by ϕ.

PROPOSITION 3.1. *Assume $\alpha \in (0, 1/2)$ and let $C > \frac{1}{1-2\alpha}$. Then $C - |z|^2$ is a positive polynomial on X, but there is no element $h \in \Sigma_h$ such that $C - |z|^2 - h \in (\phi)$.*

Proof. Clearly $|z|^2 \le \frac{1}{1-2\alpha}$ for all $z \in X$. Assume by contradiction that there is $p_1, \ldots, p_k \in \mathbb{C}[z]$ and $h \in \mathbb{C}[z, \overline{z}]$ such that

$$C - |z|^2 = \left|p_1(z)\right|^2 + \cdots + \left|p_k(z)\right|^2 + \phi(z, \overline{z}) h(z, \overline{z}).$$

The leading form of a polynomial $q(z,\overline{z}) = \sum_{a,b} c_{ab} z^a \overline{z}^b$ is $\sum_{|a|+|b|=d} c_{ab} z^a \overline{z}^b$, where d is the highest degree $|a|+|b|$ occuring in q (i.e., with $c_{ab} \neq 0$). The leading form of a product is the product of the leading forms of the factors. The leading form of a sum of hermitian squares is a scalar times $|z|^{2n}$, for some $n \geq 0$. Since $\alpha \neq 0$, the leading form of ϕ does not have this shape, and hence it cannot divide the former. Therefore, an identity as above is impossible. $\square$

On the other hand, for $\alpha = 0$ a decomposition of the form

$$C - |z|^2 = \big|p_1(z)\big|^2 + \big(|z|^2 - 1\big) h(z,\overline{z})$$

exists, in virtue of Riesz–Fejér theorem.

We turn now to the hermitian complexity degree of the ellipse, and start with analyzing whether there are pairs of points $(\lambda, \overline{\lambda}), (\mu, \overline{\mu}) \in X$ such that

$$\phi(\lambda, \overline{\mu}) = \phi(\mu, \overline{\lambda}) = 0.$$

Elementary algebra yields

$$\lambda = \alpha(\overline{\lambda} + \overline{\mu}) = \mu,$$

hence the ellipse has hermitian complexity degree equal to 1, not relevant for contradicting Quillen's property.

One step further, one can measure the distance from Quillen's property via functional calculi of linear operators. First, we treat the finite dimensional setting.

Proposition 3.2. *Let $T \in L(\mathbb{C}^N)$ satisfy the noncommutative equation of the ellipse*

$$T^*T - \alpha T^2 - \alpha T^{*2} = I. \tag{2}$$

Then T is normal: $T^*T = TT^*$.

Proof. Let $M \subset \mathbb{C}^N$ be an invariant subspace of T, so that

$$T = \begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix},$$

with respect to the decomposition $\mathbb{C}^N = M \oplus M^\perp$. Then a direct computation implies that T_{11} satisfies the same equation.

Therefore, if λ is an eigenvalue of T, then $\phi(\lambda, \overline{\lambda}) = 0$. On the other hand, if T is not normal, then there exists a two dimensional subspace $M \subset \mathbb{C}^N$ with the property that the restriction T_{11} of T to M admits a matrix decomposition of the form

$$T_{11} = \begin{pmatrix} \lambda & a \\ 0 & \mu \end{pmatrix},$$

with λ, μ in the spectrum of T and $a \neq 0$.

Equation $\phi(T_{11}, T_{11}^*) = 0$ implies
$$|\mu|^2 + |a|^2 - \alpha\mu^2 - \alpha\overline{\mu}^2 = 1,$$
whence $a = 0$, a contradiction. $\square$

And finally we consider the infinite dimensional situation.

THEOREM 3.3. *Assume $0 \le \alpha < 1/2$ and let $T \in L(H)$ be a linear bounded operator on a Hilbert space satisfying the noncommutative ellipse equation (2). Then T is subnormal.*

Proof. We identify below a scalar $\lambda \in \mathbb{C}$ with the multiple of the identity operator λI. Let $M > 1$ be a sufficiently large constant, so that the operator $M + \frac{\alpha}{M}T^2$ is invertible. Rewrite equation (2) as
$$\left(M + \frac{\alpha}{M}T^{*2}\right)\left(M + \frac{\alpha}{M}T^2\right) = M^2 - 1 + T^*T + \frac{\alpha^2}{M^2}T^{*2}T^2.$$
Denote
$$U = \sqrt{M^2-1}\left(M + \frac{\alpha}{M}T^2\right)^{-1},$$
$$V = T\left(M + \frac{\alpha}{M}T^2\right)^{-1},$$
$$W = \frac{\alpha}{M}T^2\left(M + \frac{\alpha}{M}T^2\right)^{-1},$$
remark that $[V, W] = [U, V] = [U, W] = 0$ and
$$U^*U + V^*V + W^*W = I.$$
According to Athavale's theorem [1], or Corollary 2.2 above, the triple of operators (U, V, W) is subnormal. The operator U is invertible and
$$T = \sqrt{M^2-1}U^{-1}V.$$
Therefore, T is a subnormal operator, as the result of a rational functional calculus applied to the subnormal tuple (U, V, W). $\square$

COROLLARY 3.4. *Let $\phi(x, y)$ be the equation of an ellipse E which is not a circle in $\mathbb{C} = \mathbb{R}^2$. Let $P_+(E)$ denote the set of all nonnegative polynomials on E. Then the convex cone $(\phi) + \Sigma_h$ is dense in $P_+(E)$, with respect to the finest locally convex topology of $\mathbb{R}[x, y]/(\phi)$.*

The finest locally convex topology is defined so that all linear functionals on $\mathbb{R}[x, y]/(\phi)$ are continuous.

In virtue of Haviland's theorem, the above corollary can equivalently be stated as a solution to a moment problem: *Every linear functional $L \in (\mathbb{R}[x, y]/(\phi))'$ which is nonnegative on hermitian sums of squares of polynomials is represented by a positive Borel measure supported by the ellipse E.*

Proof. Let $L \in (\mathbb{R}[x,y]/(\phi))'$ be a linear functional which is nonnegative on Σ_h, and assume $\phi(x,y) = (1-2\alpha)x^2 + (1+2\alpha)y^2 - 1, 0 < \alpha < 1/2$, as before. Then L defines a nonnegative sesquilinear form on $\mathbb{C}[z,\overline{z}]$, by

$$\langle f,g\rangle = L(f\overline{g}).$$

Note that Cauchy–Schwarz inequality

$$\left|\langle f,g\rangle\right|^2 \le \langle f,f\rangle\langle g,g\rangle, \quad f,g \in \mathbb{C}[z,\overline{z}]$$

holds, as a consequence of $L_{\Sigma_h} \ge 0$. Moreover,

$$(1-2\alpha)\|xf\|^2 + (1+2\alpha)\|yf\|^2 = \|f\|^2, \quad f \in \mathbb{C}[z,\overline{z}],$$

implying that the multiplication operators by x, y are bounded on $\mathbb{C}[z,\overline{z}]$.

In particular, the multiplication operator T by $z = x+iy$ is bounded on the subspace $\mathbb{C}[z]$, and it satisfies equation (2). According to Theorem 3.3, the operator T is subnormal, with normal extension N, the pull back via the same rational map of the minimal normal extension $(\tilde{U},\tilde{V},\tilde{W})$ of the spherical isometry (U,V,W) appearing in the proof of theorem above. Note that equation

$$\left\langle \left(\tilde{U}^*\tilde{U} + \tilde{V}^*\tilde{V} + \tilde{W}^*\tilde{W}\right)x, x\right\rangle = \|x\|^2 \tag{3}$$

is equivalent to $x \in \operatorname{Ker}(\tilde{U}^*\tilde{U} + \tilde{V}^*\tilde{V} + \tilde{W}^*\tilde{W} - I)$, thus, it defines a joint reducing subspace for the normal triple $(\tilde{U},\tilde{V},\tilde{W})$, containing the Hilbert space where (U,V,W) acts. By the minimality of the normal extension, we find that identity (3) holds everywhere. The reader can consult [1] for more details about the structure of the minimal normal extension of a spherical isometry. Consequently the normal operator N also satisfies equation $N^*N - \alpha N^2 - \alpha N^{*2} - I = 0$. Hence, the spectrum of N is contained in the curve E, and in virtue of the spectral theorem for normal operators, there exists a spectral measure Θ, supported by E, and such that

$$f\left(N,N^*\right) = \int_E f(\lambda,\overline{\lambda})\Theta(d\lambda)$$

for all polynomials $f(z,\overline{z})$. In particular, for a positive polynomial $f \in P_+(E)$ we obtain

$$\begin{aligned} L(f) &= \langle f,1\rangle = \left\langle f\left(T,T^*\right)1,1\right\rangle = \left\langle f\left(N,N^*\right)1,1\right\rangle \\ &= \int_E f(\lambda,\overline{\lambda})\left\langle\Theta(d\lambda)1,1\right\rangle \ge 0. \end{aligned}$$

□

To draw a conclusion of our computations: an ellipse which is not a circle does not posses Quillen's property, yet it has hermitian complexity one and all linear operators satisfying its associated noncommutative equation are subnormal.

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References

[1] A. Athavale, *Holomorphic kernels and commuting operators*, Trans. Amer. Math. Soc. **304** (1987), 101–110. MR 0906808

[2] D. W. Catlin and J. P. D'Angelo, *A stabilization theorem for Hermitian forms and applications to holomorphic mappings*, Math. Res. Lett. **3** (1996), 149–166. MR 1386836

[3] J. P. D'Angelo and M. Putinar, *Hermitian complexity of real polynomial ideals*, Internat. J. Math. **23** (2012), 1250026 (14 pages). MR 2925474

[4] M. G. Krein, *Über eine neue Klasse von hermitischen Formen und über eine Verallgemeinerung des trigonometrischen Momentproblems*, Izvestia Akad. Nauk. **9** (1933), 1259–1275.

[5] A. Prestel and Ch. N. Delzell, *Positive polynomials*, Springer, Berlin, 2001. MR 1829790

[6] M. Putinar and C. Scheiderer, *Sums of Hermitian squares on pseudoconvex boundaries*, Math. Res. Lett. **17** (2010), 1047–1053. MR 2729629

[7] M. Putinar and C. Scheiderer, *Quillen property of real algebraic varieties*, preprint; available at `http://arxiv.org/abs/1304.0947`.

[8] D. G. Quillen, *On the representation of Hermitian forms as sums of squares*, Invent. Math. **5** (1968), 237–242. MR 0233770

[9] C. Scheiderer, *Positivity and sums of squares: A guide to recent results*, Emerging applications of algebraic geometry, IMA Vol. Math. Appl., vol. **149**, Springer, New York, 2009, pp. 271–324. MR 2500469

[10] J. Stochel and F. H. Szafraniec, *Algebraic operators and moments on algebraic sets*, Port. Math. **51** (1994), 25–45. MR 1281954

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