

POLYNOMIALS CONSTANT ON A HYPERPLANE AND CR MAPS OF SPHERES

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Dedicated to Professor John P. D'Angelo on the occasion of his 60th birthday

ABSTRACT. We prove a sharp degree bound for polynomials constant on a hyperplane with a fixed number of nonnegative distinct monomials. This bound was conjectured by John P. D'Angelo, proved in two dimensions by D'Angelo, Kos and Riehl and in three dimensions by the authors. The current work builds upon these results to settle the conjecture in all dimensions. We also give a complete description of all polynomials in dimensions 4 and higher for which the sharp bound is obtained. The results prove the sharp degree bounds for monomial CR mappings of spheres in all dimensions.

1. Introduction

We are interested in the degree and the number of monomials for polynomials with nonnegative coefficients that are constant on the hypersurface

$$x_1 + \cdots + x_n = 1.$$

We denote by $N(p)$ the number of nonzero coefficients of a polynomial p . Our main result is the following.

THEOREM 1.1. *Let $n \geq 2$ and let p be a polynomial with non-negative coefficients of degree d in n dimensions such that $p(x) = 1$ whenever $x_1 + \cdots +$*

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$x_n = 1$. Then the following inequality holds and is sharp

$$(1) \quad d \leq \begin{cases} 2N(p) - 3 & \text{if } n = 2, \\ \frac{N(p)-1}{n-1} & \text{if } n \geq 3. \end{cases}$$

For $n \geq 3$, sharp means that for every n and every d there exist examples for which equality holds. When $n = 2$ equality can only hold for odd degree, for even degree there exist examples so that the right hand side is exactly one larger.

Inequality (1) was conjectured by D'Angelo [3]. The 2-dimensional case was proved by D'Angelo, Kos and Riehl in [3]. The 3-dimensional case was proved by the authors in [10]. In this article we use ideas of the 2-dimensional proof for an induction argument that uses the 3-dimensional result as a base to obtain the result in dimensions 4 and higher.

In dimensions 4 and higher, we also give a complete description of all polynomials for which inequality (1) is an equality. We note that this description does not hold in dimensions 2 and 3.

It is instructive to note that no degree bound can hold when $n = 1$; the polynomial $p(x) = x^d$ gives 1 whenever $x = 1$, but $N(p) = 1$ while d can be arbitrarily large.

We end the introduction with a brief description of the history of the problem. The motivation of the problem comes from the study of proper holomorphic maps between balls in different dimensions. We denote by \mathbb{B}^n the unit ball in n complex dimensions. Faran [5] showed that a proper holomorphic map from \mathbb{B}^2 to \mathbb{B}^3 that is sufficiently smooth on the boundary is spherically equivalent to a monomial map of degree at most 3. It was shown by Forstnerič [6] that any proper holomorphic map from \mathbb{B}^n to \mathbb{B}^N , with $n \geq 2$, is rational if the map is sufficiently smooth up to the boundary, and for fixed dimensions n and N the degree is bounded from above by a constant depending only on n and N .

The bound on the degree in terms of n and N has been studied by various authors. As mentioned above, Forstnerič [6] proved that a bound exists. The best currently known bound in the general rational case was proved by Meylan [11] in $n = 2$ and extended to $n \geq 3$ by D'Angelo and Lebl [4]. D'Angelo has conjectured that the best possible bound for $n = 2$ is given by $d \leq 2N - 3$, and for $n \geq 3$ it is given by $d \leq \frac{N-1}{n-1}$. The reader will notice that these are exactly the same inequalities as those in Theorem 1.1. When N is small compared to n , then the conjectured bound is known to hold in the general rational case. When $n = 2$ and $N = 3$ it follows from Faran's work [5] that $d \leq 3$. When $n \geq 3$ and $N \leq 3n - 4$, then $d \leq 2$ by the work of Huang and Ji [7] and Huang, Ji and Xu [8].

Suppose that the proper holomorphic map $f : \mathbb{B}^n \rightarrow \mathbb{B}^N$ is a monomial map, that is, every coordinate function is a monomial. Of course in this case the map automatically extends to the boundary, and the properness of the

map means exactly that

$$(2) \quad \|f(z_1, \dots, z_n)\|^2 = 1 \quad \text{when } |z_1|^2 + \dots + |z_n|^2 = 1,$$

which we can write as

$$(3) \quad |f_1(z)|^2 + \dots + |f_N(z)|^2 = 1 \quad \text{when } |z_1|^2 + \dots + |z_n|^2 = 1.$$

Since every coordinate function f_j is a monomial, we get that $|f_1(z)|^2 + \dots + |f_N(z)|^2$ is a real polynomial p in the variables x_1, \dots, x_n , where $x_j = |z_j|^2$. The polynomial p has nonnegative coefficients and (at most) N distinct nonzero coefficients. Theorem 1.1 therefore implies the following result.

COROLLARY 1.2. *Let $n, N \in \mathbb{N}$, with $n \geq 2$ and suppose that $f : \mathbb{B}^n \rightarrow \mathbb{B}^N$ is a monomial map that is proper and of degree d . Then the following inequality holds and is sharp*

$$(4) \quad d \leq \begin{cases} 2N - 3 & \text{if } n = 2, \\ \frac{N-1}{n-1} & \text{if } n \geq 3. \end{cases}$$

We note again that this result was proved by D’Angelo, Kos and Riehl in [3] for $n = 2$ and by the authors in [10] for $n = 3$.

While monomial mappings may seem like a special case, we would like to note that all known examples of proper rational maps between balls are homotopic to a monomial map. Furthermore, the first author proved in [9] that all degree two proper maps between balls are equivalent to monomial maps.

2. Whitney polynomials

Denote by \mathcal{H} the set of polynomials p with nonnegative coefficients that satisfy $p = 1$ on the hyperplane $x_1 + \dots + x_n = 1$. We start by giving a method of constructing examples in \mathcal{H} . From now on we write $s = x_1 + \dots + x_n$. Of course s itself lies in \mathcal{H} . Let $p \in \mathcal{H}$ be a polynomial of degree d , and suppose that m is a monomial with a positive coefficient such that $p - m$ still has only non-negative coefficients. Then we can replace m by s times m , in other words,

$$(5) \quad p - m + s \cdot m \in \mathcal{H}.$$

Note that if m has degree d , and the coefficient of m is as large as possible, then $p - m + sm$ has degree $d + 1$, and has exactly $n - 1$ nonzero coefficients more than p . So starting with s , we can repeatedly replace one of the terms of maximal degree with s times that term, to obtain many examples of any degree for which $N = n + (d - 1)(n - 1)$, or $N = d(n - 1) + 1$. For example when $n = 3$ (calling the variables x, y , and z), we could obtain a degree 3 polynomial with 7 monomials by the following procedure:

$$(6) \quad \begin{aligned} x + y + z(x + y + z) &= x + y + xz + yz + z^2, \\ x + y + xz(x + y + z) + yz + z^2 &= x + y + x^2z + xyz + xz^2 + yz + z^2. \end{aligned}$$

We call polynomials obtained by following this procedure *sharp generalized Whitney polynomials*, and note that

$$(7) \quad d = \frac{N-1}{n-1}.$$

As we will see, this is the best possible bound for $n \geq 3$, and in fact, for $n \geq 4$ these are the only polynomials for which there can be equality.

THEOREM 2.1. *Let $n \geq 4$ and let $p \in \mathcal{H}$. Then (7) holds if and only if p is a sharp generalized Whitney polynomial.*

The following example shows that Theorem 2.1 does not hold for $n = 3$.

$$(8) \quad p(x, y, z) = x^3 + 3x^2z + 3xz^2 + z^3 + 3xy + 3yz + y^3.$$

This polynomial can be obtained from the polynomial $F = x^3 + 3xy + y^3$ by substituting $x + z$ for x . Since $F = 1$ when $x + y = 1$ one immediately sees that $p = 1$ when $x + y + z = 1$. The polynomial p has degree 3, has 7 nonzero coefficients and 3 variables, and $\frac{7-1}{3-1} = 3$. It is clear that p is not a sharp generalized Whitney polynomial, as the highest degree terms are not of the form $s \cdot m$.

In dimension 2 the equality $d = N - 1$ is not optimal, in fact N can be decreased by roughly a factor 2. The map $F = x^3 + 3xy + y^3$ is an example where $d = 2N - 3$. It was shown by D'Angelo [1] (see also [2]) that such examples occur for any odd degree.

The sharp polynomials in $n = 2$ constructed by D'Angelo happen to be group invariant, see [2]. Sharp polynomials that are group invariant also occur in $n = 3$, for example (8) above. However, as a consequence of Theorem 2.1 no sharp polynomials are group invariant when $n \geq 4$, as sharp generalized Whitney polynomials are not group invariant.

3. Newton diagrams

As before, let $s = x_1 + \dots + x_n$ and let $p \in \mathcal{H}$. As $p - 1$ is divisible by $s - 1$, we can define the polynomial

$$(9) \quad q = \frac{p-1}{s-1}.$$

In general, we work with q rather than with p . While the polynomial q can have negative coefficients, the fact that the coefficients of p are all nonnegative puts serious restrictions on q , which we discuss in this section.

DEFINITION 3.1. For $\alpha \in \mathbb{Z}^n$, we write $|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n$. Let $p - 1 = (s - 1)q$ as above, where p is of degree d . We define the function $D: \mathbb{Z}^n \rightarrow \{0, P, N\}$ as follows. Write q in multi-index notation

$$(10) \quad q(x) = \sum_{\alpha} c_{\alpha} x^{\alpha}.$$

If c_α does not appear in the expansion, we let $c_\alpha = 0$.

$$(11) \quad D(\alpha) := \begin{cases} P & \text{if } c_\alpha > 0, \\ 0 & \text{if } c_\alpha = 0, \\ N & \text{if } c_\alpha < 0. \end{cases}$$

We call D the *Newton diagram* of q , and we say that D is the Newton diagram corresponding to p .

We call the $\alpha \in \mathbb{Z}^n$ *points* of D , and we call α a 0-point if $D(\alpha) = 0$, a P -point if $D(\alpha) = P$ and an N -point if $D(\alpha) = N$. We say that the monomial x^α is the monomial associated to $\alpha \in \mathbb{Z}^n$ and vice-versa. We often identify points of \mathbb{Z}^n with the associated monomials.

Let $\widehat{K} \subset \mathbb{Z}^n$ is the smallest set such that $D^{-1}(\{P, N\}) \subset \widehat{K}$, that is \widehat{K} contains all the points where D is nonzero, and such that for some $a = (a_1, \dots, a_n) \in \mathbb{Z}^n$ and some $k \in \mathbb{N}$ we have

$$(12) \quad \widehat{K} = \{x \in \mathbb{Z}^n \mid x_j \geq a_j \text{ for all } j \text{ and } |x| \leq k\}.$$

We define the *size* of D as $k - |a| + 1$.

Therefore, the Newton diagram of q is an n -dimensional array of P 's, N 's, and 0's, one for each coefficient of q . For convenience, we include negative powers in the Newton diagram, even though $D(\alpha) = 0$ any time α is not in the positive quadrant. We note that if $p - 1$ has a nonzero constant term then the size of D is equal to the degree of p .

We generally ignore points $\alpha \in \mathbb{Z}^n$ where $D(\alpha) = 0$. We give a graphical representation of D by drawing a lattice, and then drawing the values of D in the lattice as circles or spheres. For convenience, when drawing the $n = 2$ lattice, we put $(0, 0)$ at the origin, and then let $(0, 1)$ be directed at angle $\frac{\pi}{3}$ and $(1, 0)$ at angle $\frac{2\pi}{3}$. Similarly, we depict the diagram for $n = 3$.

In the figures, we do not draw the circles and spheres corresponding to the zero coefficients. See Figure 1 for sample diagrams.

We adopt the following terminology from [3] and [10].

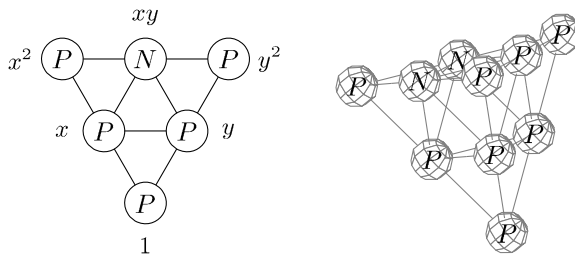


FIGURE 1. Diagrams corresponding to the polynomial $p(x, y) = x^3 + 3xy + y^3$ on the left, and the polynomial from (8) on the right.

DEFINITION 3.2. For a Newton diagram D and $\alpha \in \mathbb{Z}^n$ we define

$$(13) \quad \text{down}_k(\alpha) = \text{down}_k(\alpha_1, \dots, \alpha_n) := (\alpha_1, \dots, \alpha_{k-1}, \alpha_k - 1, \alpha_{k+1}, \dots, \alpha_n)$$

to be the function that subtracts 1 from the k th element of α .

We call $\alpha \in \mathbb{Z}^n$ a *sink*, if

$$(14) \quad \begin{aligned} D(\alpha) &= N \text{ or } 0 \quad \text{and} \\ D(\text{down}_k(\alpha)) &= P \text{ or } 0 \quad \text{for all } k = 1, \dots, n, \end{aligned}$$

and at least one of $D(\alpha)$ or $D(\text{down}_k(\alpha))$ is nonzero.

Similarly we call $\alpha \in \mathbb{Z}^n$ a *source*, if

$$(15) \quad \begin{aligned} D(\alpha) &= P \text{ or } 0 \quad \text{and} \\ D(\text{down}_k(\alpha)) &= N \text{ or } 0 \quad \text{for all } k = 1, \dots, n, \end{aligned}$$

and at least one of $D(\alpha)$ or $D(\text{down}_k(\alpha))$ is nonzero.

When we want to say that α is a sink or a source, but we do not necessarily need to differentiate, we say α is a *node*. Define

$$(16) \quad \#(D) := \# \text{ of nodes in } D.$$

See Figure 2 for a sample two dimensional diagram with sinks and sources marked. In the diagram sinks and sources are marked with a triangle, with i for sink and o for source. The vertices of the triangle mark the points that affect the sink or source. The top vertex marks the α and the bottom two vertices of the triangle mark $\text{down}_1(\alpha)$ and $\text{down}_2(\alpha)$.

Note that the monomials $x^\alpha, x^{\text{down}_1(\alpha)}, \dots, x^{\text{down}_n(\alpha)}$ in q are those monomials that when multiplied by $(s - 1)$ are the ones that contribute to the monomial x^α in $p - 1 = q(s - 1)$. Therefore, if α is a sink, then x^α in p must have a positive coefficient. Similarly, if α is a source then x^α in p must have a negative coefficient. The following proposition is then immediate.

PROPOSITION 3.3. *Let p be a polynomial such that $p - 1 = (s - 1)q$. Let D be the corresponding Newton diagram. If $\alpha \in \mathbb{Z}^n$ is a sink, then the monomial*

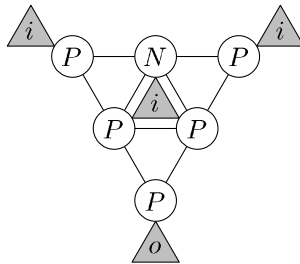


FIGURE 2. The two-dimensional diagram corresponding to $p = x_1^3 + 3x_1x_2 + x_2^3$ with sinks and sources marked respectively i and o .

x^α has a nonzero positive coefficient in the polynomial $p - 1$. If $\alpha \in Z^n$ is a source then the monomial x^α has a nonzero negative coefficient in $p - 1$. In particular,

$$(17) \quad \#(D) \leq N(p - 1) \quad \text{which implies } \#(D) - 1 \leq N(p).$$

Furthermore, if $p \in \mathcal{H}$ (that is p has all positive coefficients) and p is nonconstant, then D has a unique source at $\alpha = (0, \dots, 0)$, and otherwise D has only sinks.

In [10], we have further symmetrized the problem to avoid the change in signs for sinks and sources. In most of the proofs in this paper the symmetric version is not necessary and may in fact obscure the main argument. Certain parts of the argument, however, will require the symmetric version in two dimensions, and we therefore make the following definition.

DEFINITION 3.4. Fix d . Let

$$(18) \quad H_d = \{(a, b, c) \in \mathbb{Z}^3 : a + b + c = d, \text{ and } a, b, c \geq 0\}.$$

A 2-dimensional symmetric diagram is a function

$$(19) \quad D : H_d \rightarrow \{P, N, 0\}.$$

We call $(a, b, c) \in \mathbb{Z}^3$ a symmetric node or just node if D is nonzero on at least one of $(a - 1, b, c)$, $(a, b - 1, c)$, and $(a, b, c - 1)$, and D does not take both the values P and N on $(a - 1, b, c)$, $(a, b - 1, c)$, and $(a, b, c - 1)$. In other words, (a, b, c) is a node if the direct image

$$(20) \quad D(\{(a - 1, b, c), (a, b - 1, c), (a, b, c - 1)\}) = \{0, P\} \text{ or } \{0, N\}.$$

As before let $\#(D)$ denote the number of nodes in D .

PROPOSITION 3.5. Let $Q(x_1, x_2, x_3)$ be a homogeneous polynomial:

$$(21) \quad Q(x_1, x_2, x_3) = \sum_{|\alpha|=d} C_\alpha x^\alpha$$

and define

$$(22) \quad D(\alpha) := \begin{cases} P & \text{if } C_\alpha > 0, \\ 0 & \text{if } C_\alpha = 0, \\ N & \text{if } C_\alpha < 0. \end{cases}$$

Then D is a 2-dimensional symmetric diagram and

$$(23) \quad P(x_1, x_2, x_3) = (x_1 + x_2 + x_3)Q(x_1, x_2, x_3)$$

has at least $\#(D)$ distinct monomials.

If we start with a Q as above and look at $q(x_1, x_2) = Q(x_1, x_2, -1)$, then $(x_1 + x_2 + x_3)$ becomes $(x_1 + x_2 - 1)$ and we are in the nonsymmetric setup. We obtain a (nonsymmetric) 2-dimensional diagram by mapping H_d into \mathbb{Z}^2

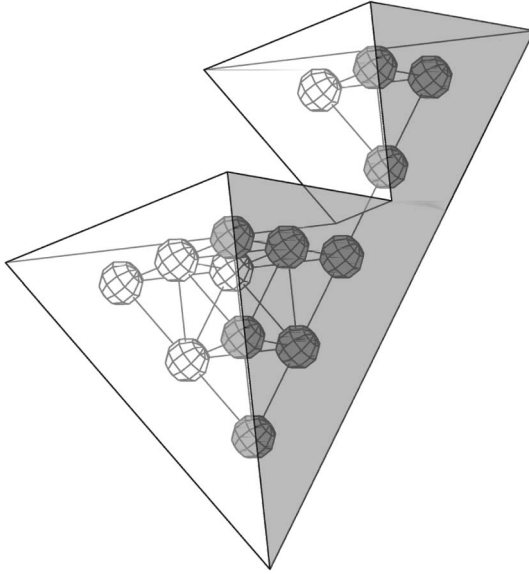


FIGURE 3. Treating a diagram as a solid for $n = 3$. A vertical face and its corresponding points are shaded.

and flipping signs for points corresponding to odd powers of x_3 . Sinks and sources correspond to the symmetric nodes in the symmetric diagram.

The arguments in [10] relied heavily on a geometric interpretation of the Newton diagram of q . Our terminology treats the Newton diagram as a solid object formed by simplices at each of $\alpha \in \mathbb{Z}^n$ with $D(\alpha) \neq 0$. The definitions of corners, edges and faces are made with this picture in mind. See Figure 3 for the geometric intuition.

Our naming convention changes slightly from [10] so that the naming is logical and consistent for diagrams of all dimensions.

DEFINITION 3.6. An $\alpha \in \mathbb{Z}^n$ is called a *top-corner* of a Newton diagram D if $D(\alpha) = 0$ and if there exists exactly one $k \in [1, n]$ with $D(\text{down}_k(\alpha)) \neq 0$. An $\alpha \in \mathbb{Z}^n$ is called a *bottom-corner* of a Newton diagram D if $D(\alpha) \neq 0$ and if $D(\text{down}_k(\alpha)) = 0$ for all $k \in [1, n]$.

An $\alpha \in \mathbb{Z}^n$ is said to lie on a *horizontal edge* of D if $D(\alpha) = 0$ and $D(\text{down}_k(\alpha)) \neq 0$ for exactly two distinct $k \in [1, n]$.

An $\alpha \in \mathbb{Z}^n$ is said to lie on a *vertical edge* of D if $D(\alpha) \neq 0$ and $D(\text{down}_k(\alpha)) \neq 0$ for exactly one $k \in [1, n]$.

Finally, an $\alpha \in \mathbb{Z}^n$ is said to lie in the *interior of a vertical face* of D if $D(\alpha) \neq 0$ and $D(\text{down}_j(\alpha)) \neq 0$ for exactly two distinct $k \in [1, n]$, and α lies

in the interior of a *horizontal face* of D if $D(\alpha) = 0$ and $D(\text{down}_k(\alpha)) \neq 0$ for exactly three distinct $k \in [1, n]$.

If a node $\alpha \in \mathbb{Z}^n$ lies on a corner or edge or in the interior of a vertical or horizontal face, then we call α respectively a corner-, edge- or facial-node. We often distinguish between top-corner nodes and bottom-corner nodes, and sometimes between vertical-edge nodes and horizontal-edge nodes.

Two points α and β are said to be *adjacent* if there is a k such that $\alpha_j = \beta_j$ for all $j \neq k$ and $\alpha_k - \beta_k = \pm 1$. The subset $S \subset \mathbb{Z}^n$ is said to be *connected* if for each $\alpha, \beta \in S$ there is a path from α to β along adjacent elements of S . A vertical (2-dimensional) face of D is a maximal connected subset of the elements $\alpha \in \mathbb{Z}^n$ that are either corners, lie on edges or lie in a vertical face, and that all lie in the same 2-dimensional plane in \mathbb{Z}^n defined by $n - 2$ equations of the form $\alpha_j = c_j$. Similarly, a horizontal face of D is a maximal connected subset of the elements $\alpha \in \mathbb{Z}^n$ that are either corners, lie on a horizontal edge or in a horizontal face, and that all lie in the same 2-dimensional plane in \mathbb{Z}^n defined by $|\alpha| = d$ and $n - 3$ equations of the form $\alpha_j = c_j$. For a vertical face, there are exactly two parameters α_k and α_m that can vary. We say that the vertical face corresponds to the variables x_k and x_m .

An important step in the proof of the $n = 3$ case of Theorem 1.1, proved in [10], was to count the number of nodes that lie in a 2-dimensional face, and then add the numbers for all the faces. A difficulty that arises when counting nodes in this manner is that the edge-nodes and corner-nodes are counted multiple times, hence the distinction between these nodes and facial nodes, which lie on only one face and are therefore only counted once. A similar idea comes up in the proof for $n \geq 4$, which we present in the next two sections.

4. Views and sides

In a 2-dimensional diagram, we say that α and β are in the same *row* if $|\alpha| = |\beta|$, that is if they correspond to monomials of the same degree.

DEFINITION 4.1. We define a *simple diagram* to be a 2-dimensional diagram with bottom nodes only in its lowest nonzero row. The *height* of a simple diagram is the number of nonzero rows.

In all arguments below, we simply ignore the bottom nodes of simple diagrams.

For each pair (x_k, x_m) , where $k \neq m$, in an n -dimensional diagram D of size d , let us consider the corresponding vertical faces and call this the *side* of the diagram. For each side we can find a finite set of simple diagrams $\mathcal{F} = (F_1, F_2, \dots, F_s)$ such that the top and side edges of each F_j correspond to top and side edges of the diagram D . Furthermore, the heights of F_i sum up to d . Each edge node, top-corner node, and facial node of F_i corresponds to a node

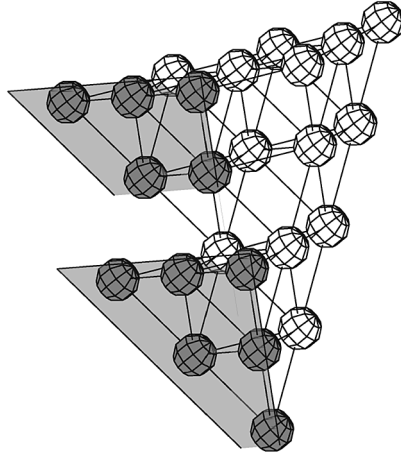


FIGURE 4. Two simple diagrams F_1 and F_2 as faces corresponding to an edge in a three-dimensional diagram.

in D . See Figure 4 for an example when $n = 3$. To find the (F_1, \dots, F_s) , start at the unique bottom node of D . Then find the face F_1 containing this bottom node corresponding to the edge (x_k, x_m) . Suppose that F_1 has height h_1 . At the row directly above the top row of F_1 (row $h_1 + 1$ in D), find a nonzero point with the smallest degree in the $x_1, \dots, \hat{x}_k, \dots, \hat{x}_m, \dots, x_n$ variables (where the $\hat{}$ means we are excluding that variable). Find the corresponding face and consider only that part of the face from row $h_1 + 1$ upwards and mark the corresponding simple diagram F_2 . Continue until we reach the top row of D . Depending on the points chosen it is possible to get a different set of simple diagrams corresponding to the edge (x_k, x_m) . However, it suffices to fix one set of simple diagrams corresponding to each edge.

DEFINITION 4.2. We call $\mathcal{F} = (F_1, \dots, F_s)$ a *complete set of simple diagrams corresponding to the edge (x_k, x_m)* .

LEMMA 4.3. *Let D be a simple diagram of height d . Let f be the number of facial nodes, e the number of edge nodes, and c the number of corner nodes (excluding bottom nodes). Then*

$$(24) \quad 2f + e + c \geq d + 1.$$

Proof. We claim that we can “fill” the diagram without increasing the sum $2f + e + c$, starting with the zeroes on the lowest nonzero row, and working our way up to the highest nonzero row of the diagram. By a filled diagram, we mean that the nonzero points in the lowest row are connected, and that up to the highest row of the diagram, a nonzero point has only nonzero points above

it. Once the diagram is filled, the claim follows from the following argument, which was also used in the proof of Lemma 4.2 in [10].

We refer to the horizontal rows of our filled simple diagram by rows 1 through d , starting with the lowest row and ending with the highest. For each row we count the number of sign changes in the row as we move from one end to the other. We denote this number by s_j , the number of sign changes, where j refers to the number of the row. For example, if the lowest row consists of only P -points then $s_1 = 0$.

Now consider rows j and $j + 1$, and note that if s_j differs from s_{j+1} then there must be at least one facial or edge node consisting of points in these two rows. In fact, note that 2 times the number of facial nodes plus the number of edge nodes that occur here must be at least as large as $|s_{j+1} - s_j|$. Hence the total number of nodes (counting facial nodes for 2) in the diagram without counting the edge and corner nodes on the top row must be at least $|s_d - s_1|$. The number of edge or corner nodes on the top row equals exactly $l_d + 1 - s_d$, where l_j denotes the length of the j th row. Hence, the total number of nodes is at least

$$(25) \quad l_d + 1 - s_d + |s_d - s_1| \geq l_d + 1 - s_d \geq l_d + 1 - (l_1 - 1) \geq d + 1,$$

which gives the desired inequality.

In order to prove our claim that we can always fill the diagram without increasing the number of nodes, we need to consider several cases. We start with the lowest nonzero row. Suppose the P - and N -points on this row do not form a connected subset of \mathbb{Z}^n . By the definition of a simple diagram there are no higher bottom-nodes, which means that there must exist a sequence of 0-points on the lowest row that is enclosed on either side by nonzero points. We fill this sequence with alternating P 's and N 's. Of course such a filling can be done in two distinct ways, and it turns out that for at least one way the sum $2f + e + c$ does not increase. To see this fact, note that the only way the count can be increased is when an edge-node above one of the most outer 0-points is changed into an facial node. However, if this can happen then we can choose what sign we change the corresponding 0-point to, such that the edge-node disappears. In this case the $(2f + e + c)$ -count decreases by 1 on that outer 0-point, and as it can increase by at most 1 on the other side the sum $2f + e + c$ does not increase. See Figure 5.

The argument for higher rows is similar but slightly different cases need to be considered. We consider the lowest row for which there exist zeroes that lie above nonzero points. If there are no nonzero points on this row then we are done filling, so let us suppose that there exist nonzero points in this row. Consider a 0-point on this row that lies above at least one nonzero point, and that is adjacent to a nonzero point. We first consider the case when there are nonzero points on both sides of this 0-point. In this case we again have that by assigning an N or P to this 0-point, we can only increase the count if an

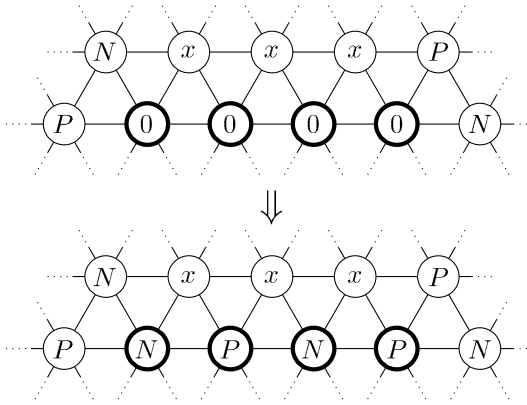


FIGURE 5. Filling in the lowest row by a correctly chosen sequence of P 's and N 's. The exact values of the points marked with x 's is not relevant.

edge-node turns into an facial node. However, each time this can happen, the edge node disappears if we assign the other sign to the 0-point. By choosing the sign for which the number of edge-nodes that changes into a facial node is minimal, the $(2f + e + c)$ -count does not increase.

The last case we need to consider is when the 0-point lies next to exactly one nonzero point. By changing this 0-point into an N - or P -point we may not only increase the $2f + e + c$ count by changing an edge-node into a facial node, but we also create a new corner-node which increases the count by 1. If the number of edge-nodes that can change to a facial node is odd (in which case it necessarily is 1), then we can always choose the sign so that more edge nodes disappear than there are changed into facial nodes and we are done. So let us assume that the number of edge nodes that can change into facial nodes is even, either 0 or 2. In this case, we fill a horizontal list of 0-nodes, starting with the given 0-node and ending with the nearest 0-node that is either adjacent to a nonzero node, or that is part of a node itself. We fill this list with alternating P - and N -points, see Figure 6 for an example. Note that in this example we cannot fill fewer 0-points without increasing the number of nodes.

When the nearest 0-point is adjacent to a nonzero node we no longer create a new corner-node and we again choose the sign so that at least as many edge-nodes are removed as there are edge-nodes changed into facial nodes and we are done. In the case when the nearest 0-point is not adjacent to a nonzero point but is a node itself, then by filling that point we can change the $(2f + e + c)$ -count by exactly 1. By our assumption, the number of nodes that is changed for the 0-point that we started with was even, so the total

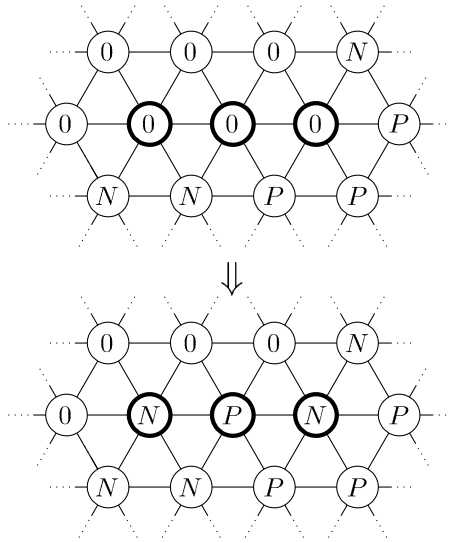


FIGURE 6. Filling in the rightmost three 0-points by alternating P - and N -points.

is odd. Therefore we can always choose the sign of the alternating list such that more edge- and corner-nodes disappear than edge-nodes are changed into facial nodes. The gain obtained from the new corner-node can therefore be countered by choosing the proper sign. This completes the argument. \square

DEFINITION 4.4. Let D be an n -dimensional diagram. For each $\alpha \in \mathbb{Z}^{n-1}$ define

$$(26) \quad \gamma_{k,m}(\alpha, a, b) := (\alpha_1, \dots, \alpha_{k-1}, a, \alpha_{k+1}, \dots, \alpha_{m-1}, b, \alpha_m, \alpha_{m+1}, \dots, \alpha_{n-1}) \in \mathbb{Z}^n,$$

where $1 \leq k \leq n - 1$, $1 \leq m \leq n$, and $k \neq m$. That is, we replace the k th element with a and we prepend b before the m th element. If $m = n$ then we append b onto the end. For example, $\gamma_{2,3}((0, 2), a, b) = (0, a, b)$.

Let

$$(27) \quad \gamma_{k,m}(\alpha) := \gamma_{k,m}(\alpha, a, b),$$

where $a + b = \alpha_k$, and a is the smallest integer such that $D(\gamma_{k,m}(\alpha, a, b))$ is nonzero.

Define $D' = V(D, k, m)$ as an $(n - 1)$ -dimensional diagram by setting

$$(28) \quad D'(\alpha) := D(\gamma_{k,m}(\alpha)).$$

We say that a node $\alpha \in \mathbb{Z}^{n-1}$ of D' corresponds to a node β of D if $\beta = \gamma_{k,m}(\alpha)$.

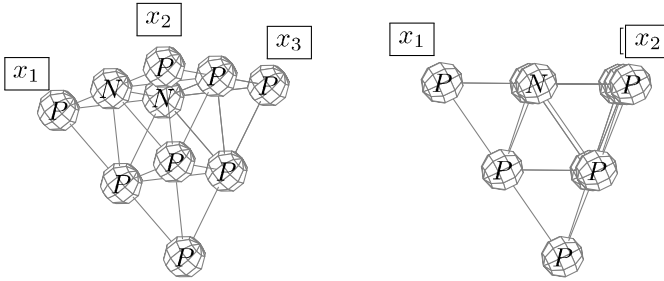


FIGURE 7. Example “view” of a diagram. On the left is a 3-dimensional diagram D with directions labeled by the corresponding variables. On the right we have the 2-dimensional $V(D, 2, 3)$.

Intuitively, $V(D, k, m)$ is the “view” along the edge looking from x_k towards x_m . See Figure 7 for an example in $n = 3$.

It is not hard to see that if $\alpha \in \mathbb{Z}^{n-1}$ is a node of $V(D, k, m)$ then it always corresponds to a node $\gamma_{k,m}(\alpha)$ of D . Thus, we have the rather simple estimate

$$(29) \quad \#(V(D, k, m)) \leq \#(D).$$

In fact, if α is a sink (resp. source), then $\gamma_{k,m}(\alpha)$ is a sink (resp. source). Hence, we also have the following proposition.

PROPOSITION 4.5. *If D has a unique source at the origin $\alpha = (0, \dots, 0)$ and all other nodes are sinks, then $V(D, k, m)$ has a unique source at the origin and all other nodes are sinks.*

When diagrams have only a single source at the origin, they have a special form. We define two conditions that we need in the proof.

DEFINITION 4.6. A set $K \subset \mathbb{Z}^2$ has *left overhang* if there exists a point $(a, b) \in K$, $(a, b) \neq (0, 0)$ such that $(a, b - 1) \notin K$ and $(a - 1, b) \notin K$ for all $y \geq b$.

A point $(a, b) \in K$ is *right overhang* if it is a left overhang after swapping variables. We say simply that (a, b) is an *overhang* if it is a left or a right overhang.

Let $K \subset \mathbb{Z}^n$ be the support of a diagram D , that is let $K = D^{-1}(\{P, N\})$. Define a 2-dimensional projection

$$(30) \quad \pi(K, k, m) := \{(a, b) \in \mathbb{Z}^2 : \text{there exists } \alpha \in K \text{ with } \alpha_k = a \text{ and } \alpha_m = b\}.$$

We say that D has *no overhang* if $\pi(K, k, m)$ has no overhang for every $k, m = 1, \dots, n$, $k \neq m$, where K is the support of D .

The above definition of no overhang agrees precisely with the definition from [10]. See Figure 8 for an example.

DEFINITION 4.7. Let D be an n -dimensional diagram ($n \geq 2$). A set $E \subset D^{-1}(\{P, N\}) \subset \mathbb{Z}^n$ (subset of the support of D) is an *outside vertical edge* if there exists $\alpha \in \mathbb{Z}^n$ and an integer k such that

$$(31) \quad E = \{ \text{down}_k(\alpha), \text{down}_k(\text{down}_k(\alpha)), \dots, \text{down}_k(\text{down}_k(\dots \text{down}_k(\alpha) \dots)) \},$$

and such that if $D(\alpha) = 0$, $D(\text{down}_j(\alpha)) = 0$ for all $j \neq k$, and $D(\text{down}_j(\beta)) = 0$ for all $j \neq k$ and all $\beta \in E$. See Figure 9.

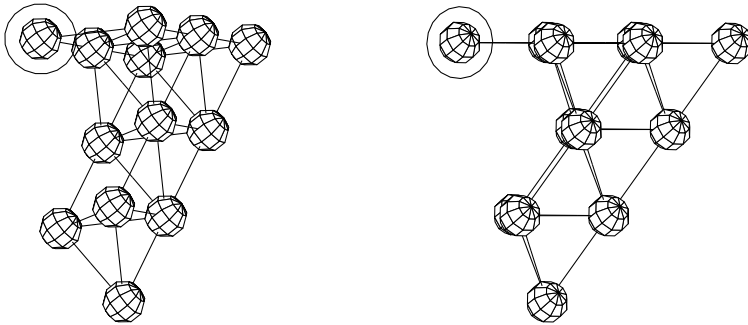


FIGURE 8. Diagram showing an example of overhang in $n = 3$.

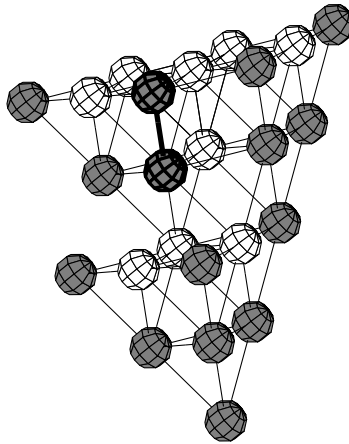


FIGURE 9. Diagram with outside edges marked by dark balls. One specific outside edge is marked with thick lines.

If $\beta \in E$ and $D(\text{down}_k(\beta)) = 0$, then we say β is a *bottom node* (it is in fact a node).

We say that a diagram D has *no outside vertical edge nodes* if for every outside vertical edge E , no element $\beta \in E$ is a node, except possibly the bottom node if E contains one.

PROPOSITION 4.8. *If D is an n -dimensional ($n \geq 2$) diagram with a unique source at the origin (all other nodes are sinks) then D has no overhang and no outside vertical edge nodes.*

Proof. First, suppose that D has an outside vertical edge node, then it has to be a sink at $\beta \in E$ for some outside vertical edge E . Then $D(\beta) = N$ as $D(\beta)$ cannot be 0. If α and k are as in the definition of the edge E , then α must be a node, and by our hypothesis on D it has to be a sink. As $D(\alpha) = 0$, then $D(\text{down}_k(\alpha)) = P$, but then there must be some source on the edge between α and β contradicting the hypothesis on D .

Suppose that there is an overhang. We note that the 2-dimensional projection of the support in the definition of no overhang is the support of some diagram

$$(32) \quad D' = V(\cdots V(V(D, k_1, k_2), k_3, k_4), \dots, k_{m-1}, k_m)$$

for some sequence of integers k_1, \dots, k_m taking views along distinct edges until we arrive at a 2-dimensional diagram D' that has an overhang at some $(a, b) \in \mathbb{Z}^2$. By definition of overhang, (a, b) is the bottom node of an outside vertical edge while not being the origin. The outside vertical edge must consist of P 's only otherwise we would obtain a source in the diagram, but if $D((a, b)) = P$, then (a, b) is a source. We have a contradiction as (a, b) was not the origin. \square

The definition of $V(D, k, m)$ can be used even for a symmetric 2-dimensional diagram. The result is a 1-dimensional symmetric diagram, that is, simply a single row of P 's, N 's, and 0's. A node on such a 1-dimensional diagram is then simply two points next to each other that are of the configuration $(0, N)$, $(N, 0)$, (N, N) , $(0, P)$, $(P, 0)$, (P, P) . In other words, a node is two points next to each other not in the configuration $(0, 0)$, (P, N) or (N, P) .

PROPOSITION 4.9. *Let D be a 2-dimensional symmetric diagram. Either*

- (i) $\#(D) = 3$, in which case D has one nonzero point, or
- (ii) $\#(D) > 3$, in which case size of D is greater than one, and there exists a view $V(D, k, m)$ (where $k, m = 1, 2, 3$), such that

$$(33) \quad \#(V(D, k, m)) \leq \#(D) - 2.$$

Proof. In [10], the authors have shown that if $\#(D) = 3$, then D is of size one, that is D is a single point.

In case $\#(D) > 3$, first suppose that the support of every $V(D, k, m)$ is connected, that is, there is exactly one node of the form $(0, *)$ and exactly one node of the form $(*, 0)$, where $*$ is P or N . If every $V(D, k, m)$ has only

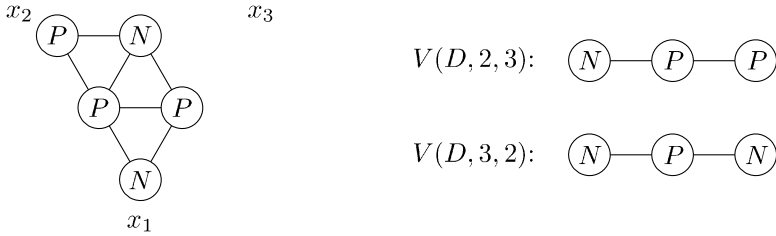


FIGURE 10. A 2-dimensional symmetric diagram with directions labeled by the corresponding variables. Note that $V(D, 2, 3)$ contains a node of the form (P, P) that does not correspond to any node (gets hidden) in $V(D, 3, 2)$.

those two nodes, then $\#(V(D, k, m)) \leq \#(D) - 2$ for every $V(D, k, m)$. Thus, suppose that $V(D, k, m)$ is such that there is a node α in D that corresponds to a node of the form (P, P) or a node (N, N) in $V(D, k, m)$. Then α does not correspond to any node in $V(D, m, k)$, see Figure 10. It is easy to see that at least one node of the form $(0, *)$ or $(*, 0)$ gets hidden by $V(D, m, k)$. Thus, $\#(V(D, m, k)) \leq \#(D) - 2$ for every $V(D, m, k)$.

If the support of some $V(D, k, m)$ is not connected, the proposition follows easily by noting that there are two nodes in D corresponding to nodes of the form $(*, 0)$ or $(0, *)$ in $V(D, k, m)$ that get hidden. \square

5. Proof of the main result

The main step towards the proof of Theorem 1.1 is taken in the following lemma. By a *top horizontal face*, we mean a horizontal face of the diagram corresponding to the highest degree.

LEMMA 5.1. *Let D be an n -dimensional, $n \geq 4$, diagram of size d with a unique source at the origin. Then there exist integers k, m ($k \neq m$, and $1 \leq k, m \leq n$) such that*

$$(34) \quad \#(V(D, k, m)) \leq \#(D) - d.$$

Furthermore, if any complete set of simple diagrams corresponding to any edge contains a facial node, or if any top horizontal face has more than three nodes, then we can choose k and m such that

$$(35) \quad \#(V(D, k, m)) < \#(D) - d.$$

Proof. For each edge, we look at all possible corresponding complete sets of simple diagrams, and count the number of facial nodes in these complete sets. We select an edge (x_k, x_m) that has a corresponding complete set of simple diagrams \mathcal{F} with the minimal number of facial nodes.

In the view $V(D, k, m)$, we note that no facial node on \mathcal{F} corresponds to a node in $V(D, k, m)$. Similarly, no edge node corresponds to a node in

$V(D, k, m)$ since only outside vertical edge nodes could possibly be seen in $V(D, k, m)$ and no such nodes exist by Proposition 4.8. Let F_1, \dots, F_s be the simple diagrams contained in \mathcal{F} as described above. If d_j is the height of F_j then $\sum d_j = d$. A corner node in F_j that is not a top-corner of maximal degree (maximal degree in F_j) is not visible in $V(D, k, m)$. Were such a node visible in $V(D, k, m)$, it would be an outside vertical edge node and $V(D, k, m)$ has no such nodes by Proposition 4.8. Of the top-corner nodes of maximal degree, we can “see” exactly one of them (the one corresponding to the x_k corner), hence one top-corner node is visible.

Let $e_j, c_j,$ and f_j denote the edge, corner and facial nodes in F_j (not counting any bottom corner nodes). First, suppose that there are no facial nodes in any complete set of simple diagrams corresponding to any edge. Then by Lemma 4.3, we see that there must be at least $e_j + c_j \geq d_j + 1$ nodes in F_j and at most one of them is visible. Thus,

$$(36) \quad \sum_{j=1}^s (e_j + c_j) \geq \sum_{j=1}^s (d_j + 1) = d + s.$$

Hence, there are at least $d + s$ nodes in \mathcal{F} and of those only s are visible in $V(D, k, m)$. Thus, d nodes in D do not correspond to any node in $V(D, k, m)$ and (34) holds.

Still assume that there are no facial nodes in any complete set of simple diagrams corresponding to any edge. Suppose that there is some top horizontal face with more than 3 nodes. Without loss of generality, let this be the horizontal face (or faces) corresponding to the vertices $\{x_1, x_2, x_3\}$. Then there is at least one view $V(D, 1, 2), V(D, 1, 3)$ or $V(D, 2, 3)$ that must hide at least two nodes by Proposition 4.9. But in the above estimate we only counted one hidden node in the top degree. Hence for that view, at least $d + 1$ nodes are hidden and so (35) holds.

Now suppose that there are facial nodes in \mathcal{F} . There are at least two other edges with one vertex being x_m . Let \mathcal{F}' and \mathcal{F}'' be the corresponding complete sets of simple diagrams. As \mathcal{F} had the least number of facial nodes, then \mathcal{F}' and \mathcal{F}'' have at least as many facial nodes as \mathcal{F} .

We observe that as the view goes in the direction from x_k to x_m , any facial nodes in \mathcal{F}' and \mathcal{F}'' do not correspond to nodes in $V(D, k, m)$. Hence for each facial node in \mathcal{F} , two other facial nodes are hidden, one in \mathcal{F}' and one in \mathcal{F}'' . Therefore, by the above calculation, for each F_j there are at least

$$(37) \quad 3f_j + e_j + c_j - 1$$

nodes hidden in $V(D, k, m)$. As $2f_j + e_j + c_j \geq d + 1$, then if $f_j > 0$ for any j , more than d nodes are hidden in $V(D, k, m)$ and (35) holds. \square

Lemma 5.1 is the induction step in the proof of Theorem 1.1. The basis for this induction is the following result, which was proved by the authors in [10].

THEOREM 5.2 (see [10]). *Let D be a 3-dimensional diagram of size d with a unique source at the origin. Then*

$$(38) \quad 2d + 2 \leq \#(D).$$

THEOREM 5.3. *Let D be an n -dimensional, $n \geq 4$, diagram of size d with a unique source at the origin. Then*

$$(39) \quad (n - 1)d + 2 \leq \#(D).$$

Proof. The proof follows by induction on n . We know the result holds for $n = 3$ by Theorem 5.2. Let $n \geq 4$ and suppose that the result holds for dimension $n - 1$. By Lemma 5.1, we can find a view V with $\#(V) \leq \#(D) - d$. As V is $(n - 1)$ -dimensional we have that

$$(40) \quad (n - 2)d + 2 \leq \#(V)$$

and inequality (39) follows. □

THEOREM 5.4. *Let D be an n -dimensional, $n \geq 4$, diagram of size d with a unique source at the origin. Suppose that D minimizes $\#(D)$ for the given degree d , that is $(n - 1)d + 2 = \#(D)$. Then D contains precisely one point in each degree.*

It is not hard to see that if a diagram D contains precisely one point in each degree such that for each degree precisely one of the possible nodes is cancelled by the point in the higher degree (a diagram corresponding to a sharp generalized Whitney polynomial), then we see that $\#(D) = (n - 1)d + 2$.

Proof of Theorem 5.4. If there were more than 3 nodes on any top horizontal face, then we could apply Lemma 5.1 to find a view V with $\#(V) < (n - 1)d + 2 - d = (n - 2)d + 2$, which would contradict the bound of Theorems 5.2 (if $n = 4$) and 5.3 (if $n > 4$).

Thus, all top horizontal faces have 3 nodes. By the 2-dimensional bounds, this means that the top horizontal faces must all consist of a single point, therefore the top degree part of D consists of isolated points. By “isolated” we mean that the corresponding horizontal faces (each consisting of a single point) are disjoint. It is not hard to see that removing any of those points would get a diagram D' with $n - 1$ fewer nodes (only one node can be created and n nodes would be removed). The diagram D' would still have a unique source at the origin. As D minimizes $\#(D)$ for the given degree, then the degree of D' must be one less. That is, there could have been at most one point in the top degree part of D . The proof follows by induction on the degree. □

As we have mentioned above, Proposition 3.3, a term in $p \in \mathcal{H}$ corresponds to a sink in the corresponding diagram, and in fact $\#(D) - 1 \leq N(p)$. Theorem 5.3 thus proves the main estimate in Theorem 1.1. Theorem 5.4 implies that a $p \in \mathcal{H}$ that satisfies equality in the bound, must be such that $q = \frac{p-1}{s-1}$ has exactly one nonzero term of each degree. It follows immediately that such a p must be a sharp generalized Whitney polynomial, proving Theorem 2.1.

REMARK 5.5. We conjecture that the same degree bound holds for all n without requiring that p has positive coefficients, under the weaker indecomposability condition discussed in [10]. This conjecture was proved for $n = 2$ in [10]. In the above the key assumptions that make the proof work are no overhang, which was required in [10] to prove the bound for $n = 3$, and in the present proof we required no outside vertical edge nodes for $n \geq 4$. Both assumptions are automatically satisfied for a diagram with a unique source at the origin. We note that apart from these assumptions we did not differentiate between sinks and sources.

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