THE BERGMAN PROJECTION IN L^p FOR DOMAINS WITH MINIMAL SMOOTHNESS

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Dedicated to John P. D'Angelo, on the occasion of his 60th birthday

ABSTRACT. Let $D \subset \mathbb{C}^n$ be a bounded, strongly Levi-pseudoconvex domain with minimally smooth boundary. We prove $L^p(D)$-regularity for the Bergman projection B, and for the operator $|B|$ whose kernel is the absolute value of the Bergman kernel with p in the range $(1, +\infty)$. As an application, we show that the space of holomorphic functions in a neighborhood of $\overline{D}$ is dense in $\vartheta L^p(D)$.

1. Introduction

This is the first in a series of papers dealing with the L^p-theory of reproducing operators such as the Cauchy–Fantappié integrals and the Szegő and Bergman projections for domains in $\mathbb{C}^n$ whose boundaries have minimal smoothness. In the present paper, we study the Bergman projection on domains in $\mathbb{C}^n$ that are strongly pseudoconvex and have C^2 boundaries. In succeeding papers, we will establish analogous L^p results for the Cauchy–Leray integral on strongly $\mathbb{C}$-linearly convex domains with $C^{1,1}$ boundaries, and for the Szegő projection, as well as certain holomorphic Cauchy–Fantappié integrals, on strongly pseudoconvex domains with C^2 boundaries.

In considering each of these reproducing operators and their integral kernels, we may divide them into several classes. First, the "universal" Bochner–Martinelli kernel, which is unique among all these in that it is essentially

independent of the domain. However, its serious drawback is that it does not produce holomorphic functions in general (because it is not holomorphic in the parameter) and so plays no role below. Next, there are the "canonical" kernels that are determined by the domain in question, but do not depend on any particular choice of defining function for the domain. Among these are the Cauchy–Leray, Szegő, and Bergman kernels. Finally, there are various kinds of non-canonical kernels, such as the kernels of the operators B_ε used below, that also depend on the choice of defining function for the domain.

Regularity properties of the Bergman projection and in particular its L^p boundedness have been the object of considerable interest for more that 40 years. When the boundary of the domain D is sufficiently smooth, decisive results were obtained in the following settings:

(a) When D is strongly pseudoconvex [F], [PS].
(b) When $D \subset \mathbb{C}^2$ and its boundary is of finite type [M], [NRSW].
(c) When $D \subset \mathbb{C}^n$ is convex and its boundary is of finite type [M1], [MS].
(d) When $D \subset \mathbb{C}^n$ is of finite type and its Levi form is diagonalizable [CD].

Related results were obtained in [B], [B1], [BaLa], [BB], [BoLo], [EL], [H], [KP] and [Z].

Turning to the problem of what might be the suitable condition of minimal smoothness for a domain D, we see a clear distinction between the case of $\mathbb{C}$ (one complex variable) and $\mathbb{C}^n, n \geq 2$. In the former case, all the Cauchy–Fantappié kernels reduce to the familiar Cauchy kernel (which is, of course, holomorphic in the parameter) and the natural limit of regularity of the boundary involves one derivative, if only because of the necessity, for the Cauchy integral and the Szegő projection, that the boundary be rectifiable. Indeed deep results of this kind—"near C^1" for example, Lipschitz boundaries—have been obtained by a number of mathematicians for the Cauchy integral and the Szegő and Bergman projections. (Recent results are in [LS], along with a review of the extensive literature.) However when $n \geq 2$, pseudoconvexity must intervene, and because of this the natural lower limit of regularity should by necessity involve two orders of differentiability. Now if we want the generality of considering all domains that are of class C^2, we then have essentially the following choice: either allow all (weakly) pseudoconvex domains, or restrict attention to strongly pseudoconvex domains. This is because to consider an intermediate class of domains (say those of "finite-type") would in effect require differentiability of higher order, related to the type in question. With these limitations in mind, we state our main result.

THEOREM. *Let D be a bounded domain in $\mathbb{C}^n$ which has a C^2 boundary that is strongly pseudoconvex. Then the Bergman projection B of $L^2(D)$ to the Bergman Space $\vartheta L^2(D)$ is bounded on $L^p(D)$, for $1 < p < \infty$.*

The main difference when dealing with the situation when D is only of class C^2 compared with the cases of the more regular domains treated in (a)–(d) is

that in each of these cases known formulas for the Bergman kernel, or at least size estimates, played a decisive role. In our general situation, such estimates are unavailable and we must proceed by a different analysis.

Our starting point is the idea used in [KS] to study the Szegő projection which was adapted by [L] and then [R] for the Bergman projection, all these when the domains are sufficiently smooth. In terms of the Bergman operator B, this proceeded by constructing another (non-orthogonal) projection B_1 via the Cauchy–Fantappiè formalism, corrected by the solution of a $\bar{\partial}$-problem. Then a simple formula connected B with B_1 and B_1^*, and the problem was resolved because $B_1 - B_1^*$ was "smoothing" (or sufficiently small). In our general situation this regularity or smallness is not possible, but what works instead can be described imprecisely as follows: One constructs an appropriate family $\{B_\varepsilon\}_{\varepsilon>0}$ of non-orthogonal projections. While this family does not approximate B (in fact the norms of the B_ε are in general unbounded as $\varepsilon \to 0$), suitable truncations of the B_ε approximate in a specific sense the "essential part" of B. This is made precise in Lemma 5.1.

There are two further results worth mentioning that follow from our analysis. First, not only is the operator B bounded on L^p, but so is the operator $|B|$ whose kernel is the absolute value of the Bergman kernel. This is in sharp contrast with the Cauchy–Leray integral and the Szegő projection, because such operators are non-absolutely convergent singular integrals, and this makes their treatment more intricate than that of B. Secondly, we have the following approximation property: The linear space of functions holomorphic in a neighborhood of $\overline{D}$ is dense in $\vartheta L^p(D)$, $1 < p < \infty$.

2. The Levi polynomial and its variants

2.1. Preliminaries. D is a bounded domain in $\mathbb{C}^n$ which is of class C^2 and is strongly pseudoconvex. Thus, there is a defining function $\rho \in C^2(\mathbb{C}^n)$ for D, with $D = \{\rho < 0\}$ and $|\nabla \rho| > 0$ on bD, with ρ strictly pluri-subharmonic. Define $P_w(z)$, the Levi polynomial at w by

$$P_w(z) = \sum_j \frac{\partial \rho(w)}{\partial w_j}(z_j - w_j) + \frac{1}{2}\sum_{j,k}\frac{\partial^2 \rho(w)}{\partial w_j\,\partial w_k}(z_j - w_j)(z_k - w_k),$$

which is a quadratic (holomorphic) polynomial in z. A basic property of P_w is that

(2.1) $$\rho(w) + 2\operatorname{Re}P_w(z) + L_w(z - w) = \mathcal{T}_w(z)$$

is the second-order Taylor expansion of $\rho(z)$ about w. (See, e.g., [K].) Here

$$L_w(z - w) = \sum_{j,k}\frac{\partial^2 \rho(w)}{\partial w_j\,\partial \overline{w}_k}(z_j - w_j)(\overline{z}_k - \overline{w}_k)$$

is the Levi form and

$$L_w(z - w) \geq c|z - w|^2, \quad c > 0,$$

by the strict pluri-subharmonicity we assumed. So we can write the above as

(2.2)
$$2\operatorname{Re}\bigl(-\rho(w) - P_w(z)\bigr)$$
$$= -\rho(z) - \rho(w) + L_w(z - w) + o\bigl(|z - w|^2\bigr)$$

as $|z - w| \to 0$. Now to extend the above when $z - w$ is not small, we define the function $g(w, z)$ by

(2.3)
$$g(w, z) = -P_w(z)\chi + |z - w|^2(1 - \chi) - \rho(w).$$

Here $\chi = \chi(|z-w|^2)$ is a C^∞ function which is 1 when $|z-w| \leq \frac{\mu}{2}$ and vanishes when $|z - w| \geq \mu$. We take μ to be a small constant, fixed so that (2.2) and the strict positivity of L_w guarantee that we have

(2.4)
$$2\operatorname{Re} g(w, z) \geq \begin{cases} -\rho(w) - \rho(z) + c|w - z|^2, & \text{if } |w - z| \leq \mu, \\ c, & \text{if } |w - z| \geq \mu \end{cases}$$

for some constant $c > 0$.

Now g, and its variants g_ε defined below, will be our basic tools.

Note $g(w, z)$ is holomorphic (or C^∞) in z, hence smooth in that variable, but is only continuous in w, since we have only assumed ρ is of class C^2. Because of this we introduce for each $\varepsilon > 0$ an $n \times n$ matrix

$$\tau^\varepsilon(w) = \bigl(\tau^\varepsilon_{j,k}(w)\bigr)$$

of functions that are each smooth (precisely: C^2 in $\overline{D}$), so that

$$\sup_{w \in \overline{D}} \left| \frac{\partial^2 \rho(w)}{\partial w_j \, \partial w_k} - \tau^\varepsilon_{j,k}(w) \right| \leq \varepsilon \quad \text{for all } 1 \leq j, k \leq n.$$

We define accordingly

(2.5)
$$P^\varepsilon_w(z) = \sum_j \frac{\partial \rho(w)}{\partial w_j}(z_j - w_j) + \frac{1}{2} \sum_{j,k} \tau^\varepsilon_{j,k}(w)(z_j - w_j)(z_k - w_k)$$

and

(2.6)
$$g_\varepsilon(w, z) = -P^\varepsilon_w(z)\chi + |z - w|^2(1 - \chi) - \rho(w).$$

Note that since

(2.7)
$$\bigl| g(w, z) - g_\varepsilon(w, z) \bigr| \leq c\varepsilon |w - z|^2$$

once we have chosen ε sufficiently small, we can assert that

(2.8)
$$2\operatorname{Re} g_\varepsilon(w, z) \geq \begin{cases} -\rho(w) - \rho(z) + c|w - z|^2, & \text{if } |w - z| \leq \mu, \\ c, & \text{if } |w - z| \geq \mu \end{cases}$$

(after having decreased the sizes of μ and c, if necessary).

2.2. Size estimates of the g_ε. There are two properties of the g_ε that are needed below: size and symmetry estimates. The first is contained in the following. We use the notation

$$\langle a, b \rangle = \sum_j a_j b_j,$$

where $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ are vectors in $\mathbb{C}^n$.

PROPOSITION 2.1. *If z and w are in $\overline{D}$ and ε is sufficiently small, then:*
(a) $|g(w, z)| \approx |\rho(w)| + |\rho(z)| + |\operatorname{Im}\langle \partial\rho(w), w - z \rangle| + |w - z|^2$.
(b) $|g_\varepsilon(w, z)| \approx |g(w, z)|$.

Here (and below) the constants implicit in the equivalences $\approx$ are independent of ε.

Proof of Proposition 2.1. Note that $|\rho(z)| = -\rho(z)$ and $|\rho(0)| = -\rho(w)$, since z and w are in $\overline{D}$. Next, when $|w - z| \geq \mu/2$, the assertion (a) is immediate from (2.4). We turn to the case $|w - z| \leq \mu/2$.

We first observe that (2.2) and (2.3) show that

(2.9) $$\big|\operatorname{Re} g(w, z)\big| \approx |\rho(z)| + |\rho(w)| + |w - z|^2.$$

Also by (2.3)

$$\operatorname{Im} g(w, z) = -\operatorname{Im} P_w(z)$$

and by the definition of $P_w(z)$ this yields

(2.10) $$\big|\operatorname{Im}\big(g(w, z) - \langle \partial\rho(w), w - z \rangle\big)\big| \leq c|z - w|^2.$$

Hence

$$\big|\operatorname{Im} g(w, z)\big| + c|z - w|^2 \geq \big|\operatorname{Im}\langle \partial\rho(w), w - z \rangle\big|,$$

which gives

$$\big|g(w, z)\big| \gtrsim \big|\rho(z)\big| + \big|\rho(w)\big| + \big|\operatorname{Im}\langle \partial\rho(w), w - z \rangle\big| + |w - z|^2.$$

To see the reverse, note that by (2.10)

$$\big|\operatorname{Im} g(w, z)\big| \leq \big|\operatorname{Im}\langle \partial\rho(w), w - z \rangle\big| + c|w - z|^2,$$

and combining this with (2.9) gives

$$\big|g(w, z)\big| \lesssim \big|\rho(z)\big| + \big|\rho(w)\big| + \big|\operatorname{Im}\langle \partial\rho(w), w - z \rangle\big| + |w - z|^2.$$

Hence, (a) is fully established. The conclusion (b) follows immediately from (a) and (2.7). □

From now on, all our statements will be restricted to the ε that are small enough so that both (2.8) and the above conclusion holds.

2.3. Symmetries of the g_ε. One can observe that by Proposition 2.1 we have the symmetries

(2.11) $\big|g(w, z)\big| \approx \big|g(z, w)\big| \quad \text{and} \quad \big|g_\varepsilon(w, z)\big| \approx \big|g_\varepsilon(z, w)\big|.$

In fact since ρ is of class C^2,

$$\langle \partial\rho(w), z - w \rangle - \langle \partial\rho(z), z - w \rangle = O\big(|w - z|^2\big),$$

thus

$$\big|\mathrm{Im}\langle \partial\rho(w), w - z \rangle\big| + |w - z|^2 \approx \big|\mathrm{Im}\langle \partial\rho(z), w - z \rangle\big| + |w - z|^2$$

proving by (a) of Proposition 2.1, the asserted symmetry of $|g(w, z)|$ and hence of $|g_\varepsilon(w, z)|$.

However for what follows below a more refined version of this symmetry is crucial.

PROPOSITION 2.2. *For any $\varepsilon > 0$, there is $\delta = \delta_\varepsilon > 0$, so that*

$$\big|g_\varepsilon(w, z) - \overline{g}_\varepsilon(z, w)\big| \leq c\varepsilon|z - w|^2, \quad \text{if } |z - w| < \delta_\varepsilon.$$

The inequality above takes into account the modules of continuity of the second derivatives of ρ. In this connection, we define

$$\omega_{j,k}(\delta) = \sup_{|z-w|\leq\delta} \left(\left| \frac{\partial^2\rho(z)}{\partial z_j\,\partial z_k} - \frac{\partial^2\rho(w)}{\partial w_j\,\partial w_k} \right| + \left| \frac{\partial^2\rho(z)}{\partial z_j\,\partial\overline{z}_k} - \frac{\partial^2\rho(w)}{\partial w_j\,\partial\overline{w}_k} \right| \right)$$

taken over all $z, w \in \overline{D}$. We set

$$\omega(\delta) = \sum_{j,k} \omega_{j,k}(\delta).$$

Then in view of the uniform continuity of the second derivatives of ρ, the function $\omega(\delta)$ decreases to 0 as $\delta \to 0$.

We observe next, with $\mathcal{T}_w(z)$ denoting the Taylor polynomial of order 2 of $\rho(z)$ centered at w, see (2.1), that then

(2.12) $\big|\rho(z) - \mathcal{T}_w(z)\big| \leq c\omega\big(|z - w|\big)|z - w|^2, \quad \text{if } |w - z| < \delta.$

In fact, for any C^2 function f on $\mathbb{R}$ we have the identity

$$f(t) - f(0) - tf'(0) - \frac{t^2}{2}f''(0) = \int_0^t (t - s)\big(f''(s) - f''(0)\big)\,ds;$$

also the integral is majorized by

$$\frac{1}{2}t^2 \sup_{0\leq s\leq t} \big(t\big|f''(s) - f''(0)\big|\big).$$

Applying this to $f(t) = \rho(w + t(z - w))$, then yields the inequality (2.12), when we take $t = 1$.

To prove Proposition 2.2, note by (2.6) that if we take $\delta \le \mu/2$, so that $|z - w| \le \mu/2$, by (2.5) we are reduced to considering

$$(2.13) \qquad \rho(z) - \rho(w) + \langle \partial\rho(w), w - z \rangle + \langle \overline{\partial}\rho(z), \overline{w} - \overline{z} \rangle$$
$$+ \frac{1}{2}\langle \overline{\tau}^\varepsilon(z); \overline{w} - \overline{z}, \overline{w} - \overline{z} \rangle - \frac{1}{2}\langle \tau^\varepsilon(w); w - z, w - z \rangle.$$

Here we are using the shorthand

$$\langle M; a, b \rangle = \sum_{j,k} M_{j,k} a_j b_k$$

if $M = (M_{j,k})$ is an $n \times n$ matrix and $a = (a_1, \ldots, a_n), b = (b_1, \ldots, b_n)$ are vectors. Next, we write the Taylor expansion of order 1 of $\partial\rho(w)/\partial w_j$ centered at z, and use the same argument that led to (2.12). We thus see that

$$\frac{\partial\rho(w)}{\partial w_j} = \frac{\partial\rho(z)}{\partial z_j} + \sum_k \left(\frac{\partial^2\rho(z)}{\partial z_j\, \partial z_k}(w_k - z_k) + \frac{\partial^2\rho(z)}{\partial z_j\, \partial\overline{z}_k}(\overline{w}_k - \overline{z}_k) \right) + R_1,$$

where $|R_1| \le c\omega(|w - z|)|w - z|$. If we insert this for $\partial\rho(w)$ in (2.13), we obtain

$$g_\varepsilon(w, z) - \overline{g}_\varepsilon(z, w)$$
$$= \rho(z) - \rho(w) + \langle \partial\rho(z), w - z \rangle + \langle \overline{\partial}\rho(z), \overline{w} - \overline{z} \rangle$$
$$+ \frac{1}{2}\left\langle \frac{\partial^2\rho(z)}{\partial z\, \partial z}; w - z, w - z \right\rangle + \frac{1}{2}\left\langle \frac{\partial^2\rho(z)}{\partial\overline{z}\, \partial\overline{z}}; \overline{w} - \overline{z}, \overline{w} - \overline{z} \right\rangle$$
$$+ \left\langle \frac{\partial^2\rho(z)}{\partial z\, \partial\overline{z}}; w - z, \overline{w} - \overline{z} \right\rangle + O(|w - z|R_1) + R_2,$$

where

$$R_2 = \frac{1}{2}\left\langle \frac{\partial^2\rho(z)}{\partial z\, \partial z} - \tau^\varepsilon(w); w - z, w - z \right\rangle$$
$$+ \frac{1}{2}\left\langle \overline{\tau}^\varepsilon(z) - \frac{\partial\rho(z)}{\partial\overline{z}\, \partial\overline{z}}; \overline{w} - \overline{z}, \overline{w} - \overline{z} \right\rangle.$$

So this can be written as

$$g_\varepsilon(w, z) - \overline{g}_\varepsilon(z, w) = \rho(z) - \mathcal{T}_w(z) + O(|w - z|R_1) + R_2.$$

Now recall (2.12), and by the same token, $|R_1| \le c\omega(|z - w|)|z - w|$, while clearly $|R_2| \le c\varepsilon|z - w|^2$. Thus we get the inequality $|g_\varepsilon(w, z) - \overline{g}_\varepsilon(z, w)| \le c\varepsilon|z - w|^2$ for $|z - w| \le \delta$, as soon as we choose $\delta \le \delta_\varepsilon$, with $\omega(\delta_\varepsilon) \le \varepsilon$. Proposition 2.2 is therefore proved.

We should note that in this section we have not used the C^2 regularity of the $(\tau_{j,k}^\varepsilon(w))$; this will be needed in the next section.

3. An "approximation" for the Bergman projection

The first step in the study of the Bergman projection B for the domain D is to form a family $\{B_\varepsilon\}_{\varepsilon>0}$ of operators that are each (non-orthogonal) projections from $L^2(D)$ to the Bergman space $\vartheta L^2(D)$, that is the subspace of $L^2(D)$ consisting of holomorphic functions. While these operators do not converge to the Bergman projection B as $\varepsilon \to 0$ (in fact their norms are in general unbounded as $\varepsilon \to 0$), they nevertheless play a crucial role in the analysis of B. The operator B_ε will be given as a sum

$$B_\varepsilon = B_\varepsilon^1 + B_\varepsilon^2.$$

The first, B_ε^1, is an adaptation of a Cauchy–Fantappié integral constructed explicitly using the function $g_\varepsilon(w,z)$ of the previous section. The second, B_ε^2, is a correction obtained by solving a $\overline{\partial}$-problem in a domain that strictly contains D. Here we follow the approach of [KS], [L], and [R], which as it stands, works only in the case of smooth domains (say of class C^3). Another significant difference is that now we need a family $\{B_\varepsilon\}_{\varepsilon>0}$, as opposed to a single such operator, and that the properties of B_ε as $\varepsilon \to 0$ are now indispensable. We turn first to B_ε^1.

3.1. The Cauchy–Fantappié part. Keeping in mind the $\{g_\varepsilon(w,z)\}_{\varepsilon>0}$ given by (2.6), we define the $(1,0)$-forms η_ε by

$$(3.1) \qquad \eta_\varepsilon(w,z) = \chi\left(\sum_j \frac{\partial \rho(w)}{\partial w_j}\, dw_j - \frac{1}{2}\sum_{j,k} \tau_{j,k}^\varepsilon(w)(w_k - z_k)\, dw_j\right)$$
$$+ (1-\chi)\sum_j (\overline{w}_j - \overline{z}_j)\, dw_j,$$

so that

$$(3.2) \qquad g_\varepsilon(w,z) = \langle \eta_\varepsilon(w,z), w - z\rangle - \rho(w).$$

Now with the above, the definition of B_ε^1 will be taken to be

$$(3.3) \qquad B_\varepsilon^1(f)(z) = \frac{1}{(2\pi i)^n} \int_D f(w)(\overline{\partial}_w G_\varepsilon)^n(w,z), \quad z \in D,$$

where we have set

$$(3.4) \qquad G_\varepsilon(w,z) = \frac{\eta_\varepsilon(w,z)}{g_\varepsilon(w,z)}.$$

From inequality (2.8), it follows that for any $z_0 \in D$ there is an open ball $\mathcal{B}_{z_0}$ centered at z_0, and a neighborhood $\mathcal{N}_{z_0}$ of bD, so that

$$g_\varepsilon(w,z) + \rho(w) \neq 0, \quad \text{if } z \in \mathcal{B}_{z_0},\ w \in \mathcal{N}_{z_0}.$$

Thus, if we set

$$\widehat{G}_\varepsilon(w,z) = \frac{\eta_\varepsilon(w,z)}{g_\varepsilon(w,z) + \rho(w)} = \frac{\eta_\varepsilon(w,z)}{\langle \eta_\varepsilon(w,z), w - z\rangle}$$

then

$$(3.5) \qquad \langle \widehat{G}_\varepsilon(w,z), w-z \rangle = 1, \quad \text{for } z \in \mathcal{B}_{z_0},\ w \in \mathcal{N}_{z_0}.$$

A $(1,0)$-form $\mathcal{G}(w,z)$ is a *generating form over* z_0 if it is of class C^2 in the variable w and satisfies (3.5), and it is a *generating form over* D if it is generating over z for any $z \in D$. One has the following lemma.

LEMMA 3.1. *Suppose that* $\mathcal{G}$ *is a generating form over* $z \in D$. *Then*

$$(3.6) \qquad f(z) = \frac{1}{(2\pi i)^n} \int_{bD} f(w) \big(\mathcal{G} \wedge (\overline{\partial}_w \mathcal{G})^{n-1} \big)(w,z)$$

for any f *that is holomorphic in* D *and continuous in* $\overline{D}$.

Here we have set $(\overline{\partial}_w \mathcal{G})^{n-1} = \overline{\partial}_w \mathcal{G} \wedge \overline{\partial}_w \mathcal{G} \wedge \cdots \wedge \overline{\partial}_w \mathcal{G}$, with the wedge product taken $n-1$ times. The expression

$$\frac{1}{(2\pi i)^n} \mathcal{G} \wedge (\overline{\partial}_w \mathcal{G})^{n-1}$$

is called the *Cauchy–Fantappié form of order* 0 *generated by* $\mathcal{G}$.

Starting with (3.6) we might hope to use Stokes' theorem to show that

$$(3.7) \qquad B_\varepsilon^1(f)(z) = \frac{1}{(2\pi i)^n} \int_{bD} f(w) \big(G_\varepsilon \wedge (\overline{\partial}_w G_\varepsilon)^{n-1} \big)(w,z), \quad z \in D$$

and hence that

$$(3.8) \qquad f(z) = B_\varepsilon^1(f)(z), \quad z \in D$$

at least for holomorphic f that are of class $C^1(\overline{D})$. The problem is that G_ε is not a generating form (it is not of class $C^2(\overline{D})$, and it does not satisfy (3.5)), nor do we want to restrict ourselves to the f that are in $C^1(\overline{D})$, but rather to the holomorphic f that are in (say) $L^1(D)$. We get around these obstacles in two stages. First, establishing (3.8) for f in $C^1(\overline{D})$, and then relaxing this requirement on f. Our result is as follows.

PROPOSITION 3.1. *Suppose* f *is holomorphic in* D *and belongs to* $L^1(D)$. *Then, the identity* (3.8) *holds in* D, *with* B_ε^1 *given by* (3.3).

The first step is to approximate the defining function ρ (which is of class C^2) by a sequence $\{\rho^r\}$ of functions each of class C^3, so that the ρ^r and their derivatives of order ≤ 2 converge uniformly to ρ. Recalling the definition of η_ε in (3.1) we set

$$(3.9) \qquad \eta_\varepsilon^r(w,z) = \chi \left(\partial \rho^r(w) - \frac{1}{2} \sum_{j,k} \tau_{j,k}^\varepsilon(w)(w_k - z_k)\, dw_j \right)$$

$$+ (1-\chi) \sum_j (\overline{w}_j - \overline{z}_j)\, dw_j, \quad \text{and}$$

$$g_\varepsilon^r(w,z) = \langle \eta_\varepsilon^r(w,z), w-z \rangle - \rho(w).$$

We note that for $z \in D$, the $\eta_\varepsilon^r(w,z) \to \eta_\varepsilon(w,z)$ and $g_\varepsilon^r(w,z) \to g_\varepsilon(w,z)$ as $r \to \infty$, uniformly for $w \in \overline{D}$ together with their first derivatives in w. Moreover by (2.8), we have $g_\varepsilon^r(w,z) \neq 0$ when $w \in bD$, for sufficiently large r. We then set

$$(3.10) \qquad G_\varepsilon^r(w,z) = \frac{\eta_\varepsilon^r(w,z)}{g_\varepsilon^r(w,z)} \quad \text{and} \quad G_\varepsilon(w,z) = \frac{\eta_\varepsilon(w,z)}{g_\varepsilon(w,z)}$$

and we see that the $(1,0)$-forms G_ε^r converge to G_ε uniformly in $\overline{D}$, together with their first derivatives. Finally, each G_ε^r has continuous derivatives of order ≤ 2 for $w \in \overline{D}$ (because ρ^r was of class C^3 and τ^ε of class C^2).

Now assume f is holomorphic in D and belongs to $C^1(\overline{D})$. Then $\overline{\partial}_w(fG_\varepsilon^r \wedge (\overline{\partial}_w G_\varepsilon^r)^{n-1}) = f\overline{\partial}_w(G_\varepsilon^r \wedge (\overline{\partial}_w G_\varepsilon^r)^{n-1}) = f(\overline{\partial}_w G_\varepsilon^r)^n$, since $\overline{\partial}_w(\overline{\partial}_w G_\varepsilon^r)^{n-1} = 0$. And since $fG_\varepsilon^r \wedge (\overline{\partial}_w G_\varepsilon^r)^{n-1}$ is an $(n, n-1)$-form,

$$\overline{\partial}_w\big(fG_\varepsilon^r \wedge (\overline{\partial}_w G_\varepsilon^r)^{n-1}\big) = d_w\big(fG_\varepsilon^r \wedge (\overline{\partial}_w G_\varepsilon^r)^{n-1}\big).$$

So Stokes' theorem gives

$$(3.11) \qquad \int_D f(w)(\overline{\partial}_w G_\varepsilon^r)^n = \int_{bD} f(w)G_\varepsilon^r \wedge (\overline{\partial}_w G_\varepsilon^r)^{n-1}.$$

One notes that when $w \in bD$, we have $G_\varepsilon^r = \widehat{G}_\varepsilon^r$ where we have defined

$$(3.12) \qquad \widehat{G}_\varepsilon^r(w,z) = \frac{\eta_\varepsilon^r(w,z)}{g_\varepsilon^r(w,z) + \rho(w)} = \frac{\eta_\varepsilon^r(w,z)}{\langle \eta_\varepsilon^r(w,z), w - z \rangle}.$$

Also one checks that

$$G_\varepsilon^r \wedge (\overline{\partial}_w G_\varepsilon^r)^{n-1} = \widehat{G}_\varepsilon^r \wedge (\overline{\partial}_w \widehat{G}_\varepsilon^r)^{n-1} \quad \text{when } w \in bD.$$

However now, $\widehat{G}_\varepsilon^r$ is a generating form over D (it is of class $C^2(D)$ in the variable w, and it satisfies (3.5) for each $z \in D$), thus by Lemmas 3.1 and (3.11),

$$f(z) = \frac{1}{(2\pi i)^n} \int_D f(w)(\overline{\partial}_w G_\varepsilon^r)^n, \quad z \in D$$

and a passage to the limit, $r \to \infty$, then yields the representation (3.8), under the restriction that f is holomorphic and is in $C^1(\overline{D})$.

To relax this requirement on f we use what we have established, but instead for domains that are interior to our domain D, proceeding as follows: for any positive λ, define D_λ by

$$D_\lambda = \big\{w : \rho(w) + \lambda < 0\big\}.$$

Thus, D_λ arises by replacing the defining function ρ by $\rho_\lambda = \rho + \lambda$. Note that $\overline{D}_\lambda \subset D_0$ and more generally $\overline{D}_{\lambda'} \subset D_\lambda$ if $\lambda' > \lambda$.

Returning to g_ε and η_ε if we define

$$(3.13) \qquad \widetilde{G}_\varepsilon^\lambda(w,z) = \frac{\eta_\varepsilon(w,z)}{g_\varepsilon(w,z) - \lambda}$$

then whenever f is holomorphic in D, we have by what has already been proved,

$$f(z) = \frac{1}{(2\pi i)^n} \int_{D_\lambda} f(w)(\overline{\partial}_w \widetilde{G}_e^\lambda)^{n-1}, \quad \text{whenever } z \in D_\lambda,$$

because f is automatically in $C^1(\overline{D}_\lambda)$.

However it is also clear that for fixed $z \in D$ and for fixed $\lambda(z)$ chosen so that $z \in D_\lambda$ for all $0 < \lambda < \lambda(z)$, that

$$\widetilde{G}_\varepsilon^\lambda \to G_\varepsilon, \quad \text{as } \lambda \to 0, \text{ uniformly for } w \in D_{\lambda(z)},$$

with a corresponding convergence of derivatives of order one. However if $z \in D$,

$$
\begin{aligned}
(2\pi i)^n f(z) &= \int_{D_\lambda} f(w)(\overline{\partial}_w \widetilde{G}_\varepsilon^\lambda)^n \\
&= \int_D f(w)(\overline{\partial}_w G_\varepsilon)^n + \int_{D_\lambda} f(w)\big[(\overline{\partial}_w \widetilde{G}_\varepsilon^\lambda)^n - (\overline{\partial}_w G_\varepsilon)^n\big] \\
&\quad + \int_{D\setminus D_\lambda} f(\overline{\partial}_w G_\varepsilon)^n.
\end{aligned}
$$

Now the second integral on the right tends to zero because of the nature of the convergence of $\overline{\partial}_w \widetilde{G}_\varepsilon^\lambda$ to $\overline{\partial}_w G_\varepsilon$, as $\lambda \to 0$; and the third integral tends to zero in view of the assumption that f is integrable on D. Proposition 3.1 is therefore proved.

3.2. The correction. While the operator B_ε^1 satisfies the reproducing property (3.8), the function $B_\varepsilon^1(f)$ is not necessarily holomorphic for general f that are not holomorphic. This is because $G_\varepsilon(w,z) = \eta_\varepsilon(w,z)/g_\varepsilon(w,z)$ is holomorphic in z only for z near w (see (3.1) and (3.2)). We now proceed to correct B_ε^1 to overcome this difficulty. We write

$$(3.14) \qquad\qquad B_\varepsilon^1(w,z) = (\overline{\partial}_w G_\varepsilon)^n(w,z),$$

so that

$$B_\varepsilon^1(f)(z) = \frac{1}{(2\pi i)^n} \int_D f(w) B_\varepsilon^1(w,z).$$

PROPOSITION 3.2. *There is an (n,n) form (in w), $B_\varepsilon^2(w,z)$, depending on z, that is continuous for $(w,z) \in \overline{D} \times \overline{D}$, so that if*

$$B_\varepsilon(f)(z) = \frac{1}{(2\pi i)^n} \int_D f(w)\big(B_\varepsilon^1(w,z) + B_\varepsilon^2(w,z)\big), \quad then:$$

(a) *$B_\varepsilon(f)(z)$ is holomorphic for $z \in D$, for each $f \in L^1(D)$.*
(b) *In addition, if f is also holomorphic in D then $B_\varepsilon(f)(z) = f(z)$, for $z \in D$.*

The proof proceeds by solving a $\overline{\partial}$-problem for a domain Ω, strictly containing D, which has a smooth (C^∞) boundary and is strongly pseudoconvex. Here the focus will be on the z-variable, with $w \in \overline{D}$ fixed (a reversal of the attention paid to w and z above).

We define the "parabolic" region

$$\mathcal{P}_w = \left\{ z : \rho(z) + \rho(w) < \frac{c}{2}|z-w|^2 \right\}, \quad w \in \overline{D}.$$

Here c is the constant appearing in (2.4). We note that $\mathcal{P}_w \supset D$, and in fact $\mathcal{P}_w \supset \overline{D}$ for $w \in D$; moreover when $w \in bD$, the exterior of $\mathcal{P}_w$ intersects $\overline{D}$ only at w. We also set $\mathcal{B}_w$ to be the open ball centered at w, $\{z : |z-w| < \mu/2\}$ where μ is the constant which occurs in the inequality (2.4). We will use the notation $D_\lambda = \{z : \rho(z) < -\lambda\}$ which appeared above, but now for λ negative, with $\lambda = -\lambda_0$, and $\lambda_0 > 0$. We have

$$(3.15) \qquad \mathcal{P}_w \cup \mathcal{B}_w \supset D_{-\lambda_0}, \quad \text{where } \lambda_0 = c\mu^2/8.$$

The proof of (3.15) divides into two cases: when $|z-w| \geq \mu/2$ and when $|z-w| < \mu/2$. In the first case, if $z \in D_{-\lambda_0}$ then $\rho(z) < \lambda_0$, and since $\rho(w) \leq 0$, it follows from our choice of λ_0 that $z \in \mathcal{P}_w$. In the second case, z is automatically in $\mathcal{B}_w$, and hence (3.15) is proved. It is equally easy to see that we also have

$$(3.16) \qquad \mathcal{P}_w \cap \mathcal{B}_w \neq \emptyset, \quad w \in \overline{D}.$$

Next, by approximating ρ, the defining function of D, by an appropriate C^∞ function which is close to ρ and its derivatives of order not exceeding two, we can find $\widetilde{\rho}$, so that $\rho + \lambda_0/2 < \widetilde{\rho} < \rho + \lambda_0$, and so that the domain $\Omega = \{z : \widetilde{\rho}(z) < 0\}$ has a boundary that is C^∞ and strongly pseudoconvex. Note that

$$(3.17) \qquad \overline{D} \subset D_{-\lambda_0/2} \subset \Omega \subset D_{-\lambda_0}$$

and so we have

$$\overline{D} \subset \Omega, \quad \text{and} \quad \overline{\Omega} \subset \mathcal{P}_w \cup \mathcal{B}_w, \quad \text{for every } w \in \overline{D}.$$

We now set up our $\overline{\partial}$-problem on Ω. For each $w \in \overline{D}$ we denote by $F(w,z) = F_\varepsilon(w,z)$ the following double form, which is of type $(0,1)$ in z, and of type (n,n) in w

$$(3.18) \qquad F(w,z) = \begin{cases} -\overline{\partial}_z(B_\varepsilon^1(w,z)), & \text{if } z \in \mathcal{P}_w, \\ 0, & \text{if } z \in \mathcal{B}_w. \end{cases}$$

In fact, note by (3.1) and (3.2), that $g_\varepsilon(w,z), \eta_\varepsilon(w,z)$ are holomorphic in z for $z \in \mathcal{B}_w$ and by (2.8), $g_\varepsilon(w,z)$ is non-vanishing in $\mathcal{P}_w \cap \mathcal{B}_w$. Thus $\overline{\partial}_z(B_\varepsilon^1(w,z)) = 0$ in $\mathcal{P}_w \cap \mathcal{B}_w$ and so $F(w,z)$ is defined consistently in $\mathcal{P}_w \cup \mathcal{B}_w$. It is also clear from this and from (3.16) that $F(z,w)$ is C^∞ for $z \in \mathcal{P}_w \cup \mathcal{B}_w$,

and as such it depends continuously on $w \in \overline{D}$. Moreover $\overline{\partial}_z F(w, z) = 0$, for $z \in \mathcal{P}_w \cup \mathcal{B}_w, w \in \overline{D}$.

Now let $S = S_z$ be the solving operator, giving the normal solution of the problem $\overline{\partial} u = \alpha$ in Ω, via the $\overline{\partial}$-Neumann problem, so that $u = S(\alpha)$ satisfies the above whenever α is a $(0,1)$-form with $\overline{\partial}\alpha = 0$. We set

$$(3.19) \qquad B_\varepsilon^2(w, z) = S_z\big(F(w, \cdot)\big).$$

Then by the regularity properties of S for which (see, e.g., [CS, Chapters 4 and 5], or [FK]), we have that $B_\varepsilon^2(w, z)$ is $C^\infty(\overline{\Omega})$, as a function of z, and continuous for $w \in \overline{\Omega}$. Moreover $\overline{\partial}_z(B_\varepsilon^2(w, z)) = -\overline{\partial}_z B_\varepsilon^1(w, z)$, for $z \in D$, so $\overline{\partial}_z(B_\varepsilon(w, z)) = 0$ for $z \in D$. So conclusion (a) is proved. To establish conclusion (b) it suffices to see that

$$(3.20) \qquad \int_D f(w) F(w, z) = 0 \quad \text{for } z \in \overline{\Omega},$$

whenever $f \in \vartheta L^1(D)$. In fact if (3.20) holds, then

$$\int_D f(w) B_\varepsilon^2(w, z) = \int_D f(w) S_z\big(F(w, z)\big) = S_z\left(\int_D f(w) F(w, z)\right) = 0$$

thus (b) is a consequence of (3.20). Note that we have

$$\int_D f(w) F(w, z) = -\int_D f(w) \overline{\partial}_z B_\varepsilon^1(w, z) = 0 \quad \text{for } z \in D,$$

but this does not suffice to give (3.20) (for $z \in \overline{\Omega}$). To prove (3.20), we need to use Stokes' theorem, and as in the proof of Proposition 3.1, we assume initially that $f \in C^1(\overline{D})$, besides being holomorphic.

Now set $G_\varepsilon^r = \eta_\varepsilon^r / g_\varepsilon^r$ as in the proof of Proposition 3.1, with $g_\varepsilon^r(w, z) = \langle \eta_\varepsilon^r(w, z), w - z \rangle - \rho(w)$, see (3.9)–(3.10). Also, set

$$F^r(w, z) = \begin{cases} -\overline{\partial}_z \overline{\partial}_w (G_\varepsilon^r \wedge (\overline{\partial}_w G_\varepsilon^r)^{n-1}), & \text{if } z \in \mathcal{P}_w, \\ 0, & \text{if } z \in \mathcal{B}_w. \end{cases}$$

Now

$$\int_D f(w) F^r(w, z) = \int_{D \setminus \mathcal{B}_w} f(w) F^r(w, z) = \int_{D \setminus \mathcal{B}_z'} f(w) F^r(w, z),$$

where $\mathcal{B}_z' = \{w : |w - z| < \mu/2\}$, since $w \in \mathcal{B}_z'$ exactly when $z \in \mathcal{B}_w$.

However the boundary of $D \setminus \mathcal{B}_z'$ consists of two parts, denoted I and II below, where

$$I = bD \cap \big(\mathcal{B}_z'\big)^c \quad \text{and} \quad II = D \cap b\mathcal{B}_z'.$$

Now

$$\overline{\partial}_w \big(f(w)\big(G_r^r \wedge \overline{\partial}_w G_\varepsilon^r\big)^{n-1}\big) = d_w \big(f(w) G_\varepsilon^r \wedge \big(\overline{\partial}_w G_\varepsilon^r\big)^{n-1}\big)$$

since f is holomorphic and $G_\varepsilon^r \wedge (\overline{\partial}_w G_\varepsilon^r)^{n-1}$ is an $(n, n-1)$-form, so Stokes' theorem gives us that

$$\int_{D \setminus \mathcal{B}'_z} f(w) F^r(w, z) = \left(\int_I + \int_{II} \right) f(w) \overline{\partial}_z \big(G_\varepsilon^r \wedge \overline{\partial}_w G_\varepsilon^r \big)^{n-1}.$$

But the integral over II vanishes since G_ε^r is holomorphic in z, for $z \in \mathcal{B}_w$, that is $w \in \mathcal{B}'_z$. To treat the integral over I we involve the modification $\widehat{G}_\varepsilon^r$ that was defined in (3.12). Then since $I \subset bD$, the same argument as before shows

$$\int_I = \int_I f(w) \overline{\partial}_z \big(\widehat{G}_\varepsilon^r \wedge (\overline{\partial}_w \widehat{G}_\varepsilon^r)^{n-1} \big).$$

However, $\widehat{G}_\varepsilon^r$ is a generating form over D so the following identity holds for it.

LEMMA 3.2 ([R, Lemma IV.3.5]).

(3.21) $$\overline{\partial}_z \big(\widehat{G}_\varepsilon^r \wedge (\overline{\partial}_w \widehat{G}_\varepsilon^r)^{n-1} \big) = d_w \big(H(w, z) \big),$$

where H is the double form of type $(0, 1)$ in z, and of type $(n, n-2)$ in w

$$H(w, z) = -(n-1) \widehat{G}_\varepsilon^r \wedge \big(\overline{\partial}_w \widehat{G}_\varepsilon^r \big)^{n-2} \wedge \overline{\partial}_z \widehat{G}_\varepsilon^r.$$

Using (3.21), we find

$$f(w) \overline{\partial}_z \big(\widehat{G}_\varepsilon^r \wedge (\overline{\partial}_w \widehat{G}_\varepsilon^r)^{n-1} \big) = \frac{1}{1-n} d_w \big(f(w) H(w, z) \big),$$

since f is holomorphic. So by Stokes' theorem

$$\int_I f(w) \overline{\partial}_z \big(\widehat{G}_\varepsilon^r \wedge (\overline{\partial}_w \widehat{G}_\varepsilon^r)^{n-1} \big) = \frac{1}{1-n} \int_{bI} f(w) H(w, z) = 0,$$

since

$$\overline{\partial}_z \widehat{G}_\varepsilon^r(w, z) = 0 \quad \text{for any } w \in bI \subset b\mathcal{B}'_z.$$

As a result,

$$\int_D f(w) F^r(w, z) = 0 \quad \text{for } z \in \overline{\Omega},$$

and a limit as $r \to \infty$ gives (3.20) in the case when $f \in \vartheta(D) \cap C^1(\overline{D})$.

To drop the assumption that f is in $C^1(\overline{D})$ we argue as in the proof of Proposition 3.1, by replacing the domain D with defining function ρ, by the proper sub-domains D_λ, with defining function $\rho + \lambda$ for λ positive. With $\widetilde{G}_\varepsilon^\lambda$ defined as in (3.13), we now set

$$\tilde{F}^\lambda(w, z) = \begin{cases} -\overline{\partial}_z \overline{\partial}_w (\widetilde{G}_\varepsilon^\lambda \wedge (\overline{\partial}_w \widetilde{G}_\varepsilon^\lambda)^{n-1}), & \text{if } z \in \mathcal{P}_w, \\ 0, & \text{if } z \in \mathcal{B}_w. \end{cases}$$

Then as before,

$$\int_{D_\lambda} f(w) \tilde{F}^\lambda(w, z) = 0 \quad \text{for } \lambda < \lambda(z),$$

if is holomorphic in D. A passage to the limit, $\lambda \to 0$, then gives (3.20) and the proof of Proposition 3.2 is complete.

For later applications, it will be useful to note that

$$B_\varepsilon(w, z) = B_\varepsilon^1(w, z) + B_\varepsilon^2(w, z)$$

is holomorphic for z in a neighborhood of $\overline{D}$, when every (fixed) w is inside D. To make this precise, recall the family of domains $D_\lambda = \{\rho < -\lambda\}$ used above, and the positive number λ_0 appearing in (3.15).

COROLLARY 1. *Suppose $0 < \lambda < \lambda_0/2$, and $w \in D_\lambda$. Then $B_\varepsilon(w, z)$ extends to a holomorphic function for $z \in D_{-\lambda}$.*

Proof. Let $F(w, z)$ be as in (3.18) and Ω be as in (3.17). Note that by (3.19) we have

$$\overline{\partial}_z B_\varepsilon^2(w, z) = F(w, z) \quad \text{for } z \in \Omega,$$

and

$$F(w, z) = -\overline{\partial}_z B_\varepsilon^1(w, z) \quad \text{for } z \in \mathcal{P}_w,$$

by the definition of $F(w, z)$. Hence, $B_\varepsilon(w, z)$ extends to a holomorphic function for $z \in \Omega \cap \mathcal{P}_w$. However when $w \in D_\lambda$, we have that $\mathcal{P}_w \supset D_{-\lambda}$ (because $w \in D_\lambda$ and $z \in D_{-\lambda}$ imply that $\rho(z) + \rho(w) < 0 \leq c/2|z - w|^2$). Also $\Omega \supset D_{-\lambda_0/2}$ by our construction, see (3.17). Thus, $\Omega \cap \mathcal{P}_w \supset (D_{-\lambda} \cap D_{-\lambda_0/2}) = D_{-\lambda}$ for $\lambda < \lambda_0/2$, and $B_\varepsilon(w, z)$ is holomorphic in $D_{-\lambda}$. $\square$

4. An L^p estimate

We prove a basic L^p inequality needed in what follows. We deal with a comparison operator Γ defined by

$$\Gamma(f)(z) = \int_D |g(w, z)|^{-n-1} f(w) \, dV(w), \quad z \in D,$$

where dV is the Euclidean volume element in $\mathbb{C}^n$. We also consider the operator Γ_ε defined similarly, with g replaced by g_ε.

PROPOSITION 4.1. *For $1 < p < \infty$, we have*

$$\left\| \Gamma(f) \right\|_{L^p(D)} \leq c_p \|f\|_{L^p(D)}$$

and also,

$$\left\| \Gamma_\varepsilon(f) \right\|_{L^p(D)} \leq c_p \|f\|_{L^p(D)}$$

with a bound c_p independent of ε.

Here $\|f\|_{L^p(D)}$ denotes the norm of f in the Lebesgue space $L^p(D)$ with respect to the Euclidean measure in $\mathbb{R}^{2n}$. It is clear that by Proposition 2.1, the result for Γ_ε is immediately reducible to that for Γ, which in turn is a consequence of the following.

LEMMA 4.1. *For each $\alpha, 0 < \alpha < 1$*

$$(4.1) \qquad \int_D \bigl|g(w,z)\bigr|^{-n-1}\bigl|\rho(w)\bigr|^{-\alpha}\, dV(w) \le c_\alpha \bigl|\rho(z)\bigr|^{-\alpha}.$$

Once (4.1) is proved, the symmetry $|g(w,z)| \approx |g(z,w)|$, see (2.11), shows that the analogous inequality

$$(4.2) \qquad \int_D \bigl|g(w,z)\bigr|^{-n-1}\bigl|\rho(z)\bigr|^{-\alpha}\, dV(z) \le c'_\alpha \bigl|\rho(w)\bigr|^{-\alpha}$$

also holds. The fact that (4.1) and (4.2) imply Proposition 4.1 is a consequence of standard arguments, see, for example, [MS]. Indeed, for fixed $z \in D$ we write

$$\bigl|g(w,z)\bigr|^{-n-1}\bigl|f(w)\bigr| = F_1(w) \cdot F_2(w)$$

with

$$F_1 = \bigl|g(w,z)\bigr|^{-(n+1)/p}\bigl|f(w)\bigr|\bigl|\rho(w)\bigr|^{\beta/q}; \qquad F_2 = \bigl|g(w,z)\bigr|^{-(n+1)/q}\bigl|\rho(w)\bigr|^{-\beta/q},$$

where q is the dual exponent to p, $1/p + 1/q = 1$ (β will be chosen later). Then by Hölder's inequality

$$\bigl|\Gamma(f)(z)\bigr|^p \le \left(\int |F_1| \cdot |F_2| \right)^p \le \int |F_1|^p \cdot \left(\int |F_2|^q \right)^{p/q}.$$

Now

$$\int \bigl|F_2(w)\bigr|^q\, dV(w) = \int \bigl|g(w,z)\bigr|^{-n+1}\bigl|\rho(w)\bigr|^{-\beta}\, dV(w) \le c_\beta\bigl|\rho(z)\bigr|^{-\beta}$$

by (4.1), if we choose $\beta \in (0,1)$. Thus,

$$\bigl|\Gamma(f)(z)\bigr|^p \le c\left(\int \bigl|g(w,z)\bigr|^{-n-1}\bigl|f(w)\bigr|^p\bigl|\rho(w)\bigr|^{\beta p/q}\, dV(w) \right)\bigl|\rho(z)\bigr|^{-\beta p/q}.$$

If we integrate this in z, interchange the order of integration and use (4.2) with $\alpha = \beta/q$ (having chosen β sufficiently small so that $0 < \beta p/q < 1$ in addition to $0 < \beta < 1$), then the result is

$$\int_D \bigl|\Gamma(f)(z)\bigr|^p\, dV(z) \le C_p \int_D \bigl|f(w)\bigr|^p\, dV(w),$$

proving the proposition.

We turn to the proof of Lemma 4.1. We consider first the heart of the matter, where $z \in D$ is sufficiently close to the boundary, and the region of integration in (4.1) is limited to those w sufficiently close to z. That is, for a small positive constant c_1, we show first that

$$(4.3) \quad \int_{|w-z|<c_1} \bigl|g(w,z)\bigr|^{-n-1}\bigl|\rho(w)\bigr|^{-\alpha}\, dV(w) \le c\bigl|\rho(z)\bigr|^{-\alpha}, \quad \text{for } \bigl|\rho(z)\bigr| < c_1.$$

Now if z is sufficiently close to bD, then the normal projection $\pi(z)$ of z to the boundary is well defined. Hence by a translation and unitary linear transformation of $\mathbb{C}^n$, we can find a new coordinate system $(z_1, z_2, \ldots, z_n)$

centered at $\pi(z)$, that is $\pi(z) = (0, \ldots, 0)$, so that if $z_n = x_n + iy_n$, the points $(z_1, \ldots, z_{n-1}, x_n)$ are in the (real) tangent plane at $\pi(z) = (0, \ldots, 0)$; moreover our initial z is given as $(0, \ldots, 0, iy_n)$ in this coordinate system. It follows that

$$y_n \approx |\rho(z)|; \qquad |z - \pi(z)| \approx |\rho(z)|.$$

In addition

$$\frac{\partial \rho}{\partial z_j}(\pi(z)) = 0, \quad 1 \le j \le n-1, \qquad \text{while } \frac{\partial \rho}{\partial y_n}(\pi(z)) \neq 0,$$

and

$$\left(\frac{\partial \rho}{\partial z_n}\right)(\pi(z)) = -\frac{i}{2}\left(\frac{\partial \rho}{\partial y_n}\right)(\pi(z)).$$

Thus if $w = (w_1, \ldots, w_n)$, with $w_n = u_n + iv_n$, then

$$\left|\text{Im}\langle \partial\rho(\pi(z)), w - z\rangle\right| \approx |u_n - x_n| = |u_n|.$$

Now by Proposition 2.1 and the symmetry (2.11), we have that

$$|g(w, z)| \approx |\rho(w)| + |\rho(z)| + |\text{Im}\langle \partial\rho(z), w - z\rangle| + |w - z|^2,$$

and by the above,

$$\text{Im}\langle \partial\rho(z), w - z\rangle = \text{Im}\langle \partial\rho(\pi(z)), w - z\rangle + O(|\rho(z)||w - z|).$$

Combining this with the above and the fact that $|w - z| < c_1$ with c_1 small, we obtain

$$\left|\rho(z)\right| + \left|\text{Im}\langle \partial\rho(z), w - z\rangle\right| \gtrsim |\rho(z)| + |u_n|,$$

and so

$$(4.4) \qquad |g(w, z)| \gtrsim |\rho(z)| + |\rho(w)| + |u_n| + |w'|^2,$$

where $z' = (z_1, \ldots, z_{n-1}) = (0, \ldots, 0)$, and $w' = (w_1, \ldots, w_{n-1})$.

We now introduce a further coordinate system about $\pi(z) = (0, \ldots, 0)$ via a C^2 change of variables so that these new coordinates $(\zeta_1, \zeta_2, \ldots, \zeta_n)$ are related to the previous coordinates by

$$\zeta_j = z_j, \quad 1 \le j \le n-1; \qquad \zeta_n = x_n - i\rho(z).$$

Thus because of (4.4), if we write $-\rho(z) = t$ and $-\rho(w) = s$, then to prove (4.3) it suffices to see that

$$\int_{s \in \mathbb{R}^+} \int_{u_n \in \mathbb{R}} \int_{w' \in \mathbb{C}^{n-1}} \frac{s^{-\alpha}}{(t + s + |u_n| + |w'|^2)^{n+1}}\, ds\, du_n\, dV(w') \le ct^{-\alpha}$$

for any $t > 0$, and $0 < \alpha < 1$. To prove this inequality, note that by rescaling by the mapping

$$w' \to \delta^2 w'; \qquad u_n \to \delta u_n; \qquad s \to \delta s,$$

we are reduced to checking that

$$(4.5) \qquad \int_{\mathbb{R}^+ \times \mathbb{R} \times \mathbb{C}^{n-1}} \frac{s^{-\alpha}}{(1 + s + |u_n| + |w'|^2)^{n+1}}\, ds\, du_n\, dV(w') < \infty.$$

Now recall that for $0 < \alpha < 1$ we have

$$\int_0^\infty \frac{s^{-\alpha}}{(s+X)^{n+1}} \, ds = c_\alpha X^{-n-\alpha}, \quad \text{whenever } X > 0$$

(the restriction on α guarantees the convergence of the integral). Applying this fact to $X = 1 + |u_n| + |w'|^2$ we see that the integral (4.5) is further reduced to a multiple of

$$\int_{\mathbb{R} \times \mathbb{C}^{n-1}} \frac{1}{(1 + |u_n| + |w'|^2)^{n+\alpha}} \, du_n \, dV(w'),$$

and the finiteness of the latter is easily seen for $\alpha > 0$. This proves (4.3).

What remains is the situation where either $|\rho(z)| \geq c_1$ or $|z - w| \geq c_1$ (or both). In any case we note from Proposition 2.1 that $|g(w, z)| \geq c' > 0$ and hence it suffices to see that

$$(4.6) \qquad \int_D |\rho(w)|^{-\alpha} \, dV(w) < \infty, \quad \text{whenever } \alpha < 1.$$

To verify (4.6) cover the boundary of D by finitely many small balls, so that in each ball we can introduce a coordinate system as above. Then the finiteness of the part of the integral (4.6) taken over each such ball is easily reduced the fact that

$$\int_0^1 s^{-\alpha} \, ds < \infty, \quad \text{when } \alpha < 1.$$

This concludes the proof of Lemma 4.1 and hence also of Proposition 4.1.

We now apply this proposition to obtain a preliminary estimate for the operators $\{B_\varepsilon^1\}_{\varepsilon > 0}$. We rewrite $B_\varepsilon^1(w, z)$, the kernel of B_ε^1, as

$$(4.7) \qquad B_\varepsilon^1(w, z) = \frac{N_\varepsilon(w, z)}{g_\varepsilon(w, z)^{n+1}},$$

where $N_\varepsilon(w, z)$ can be computed from (3.14) and (3.4), which give

$$(4.8) \qquad N_\varepsilon(w, z) = -\left((\overline{\partial}_w \eta_\varepsilon)^{n-1} \wedge \overline{\partial}_w g_\varepsilon \wedge \eta_\varepsilon + g_\varepsilon (\overline{\partial}_w \eta_\varepsilon)^n\right).$$

Here and in the sequel, we use the notation O_ε to indicate a form (or a function) whose coefficients are bounded by $c_\varepsilon |w - z|$. Now looking back at the definition of g_ε and η_ε, see (2.6) and (3.1), we see that

$$\overline{\partial}_w g_\varepsilon(w, z) = -\overline{\partial}\rho(w) + O_\varepsilon(|w - z|),$$
$$\eta_\varepsilon(w, z) = \partial\rho(w) + O_\varepsilon(|w - z|),$$
$$\overline{\partial}_w \eta_\varepsilon(w, z) = \overline{\partial}\partial\rho(w) + O_\varepsilon(|w - z|).$$

Inserting these expressions in (4.8) yields

$$(4.9) \qquad N_\varepsilon(w, z) = \left(\overline{\partial}\rho \wedge \partial\rho \wedge (\overline{\partial}\partial\rho)^{n-1}\right)(w) + O_\varepsilon(|w - z|).$$

Since these forms have degree $2n$ in w, and the Euclidean volume element in $\mathbb{R}^{2n}$ may be expressed as

$$dV(w) = \prod_{j=1}^{n} \frac{i}{2} dw_j \wedge d\overline{w}_j,$$

we have

(4.10) $$\left(\overline{\partial}\rho \wedge \partial\rho \wedge (\overline{\partial}\partial\rho)^{n-1}\right)(w) = \mathcal{K}_0(w) \, dV(w),$$

where $\mathcal{K}_0(w)$ is a continuous function on $\overline{D}$. We also have

(4.11) $$\left|O_\varepsilon\left(|w-z|\right)\right| \le c_\varepsilon |w-z|,$$

however, the bound c_ε may not remain bounded as $\varepsilon \to 0$ because it depends on the first derivatives of τ^ε. Nevertheless, we do have the following corollary.

COROLLARY 2. *For each $\varepsilon > 0$ and $1 < p < \infty$, the operator B_ε is bounded on $L^p(D)$.*

By Proposition 3.2, $B_\varepsilon = B_\varepsilon^1 + B_\varepsilon^2$, and the operator B_ε^2 has a kernel which is bounded on $\overline{D} \times \overline{D}$, hence it gives rise to bounded operator on L^p, for all $1 \le p \le \infty$. For the operator B_ε^1, we apply (4.7) through (4.11) to see that

$$\left|B_\varepsilon^1(f)(z)\right| \le (c_0 + c_\varepsilon)\Gamma_\varepsilon\left(|f|\right)(z).$$

Now Proposition 4.1 grants the boundedness of B_ε^1 on L^p for $1 < p < \infty$, proving the corollary.

5. The approximate "symmetry" of B_ε

Our goal here is to understand the degree to which the operator B_ε is symmetric. Indeed, if it had exactly that property, that is, $B_\varepsilon^* = B_\varepsilon$, then in view of properties (a) and (b) in Proposition 3.2, B_ε would necessarily be identical with the Bergman projection. While this is not the case, the facts concerning the approximate symmetry of B_ε are centered in the following key lemma. If

$$(f, g) = \int_D f(w)\overline{g}(w) \, dV(w)$$

is the inner product on $L^2(D)$, and T is a bounded operator on $L^2(D)$, we denote by T^* its adjoint with respect to this inner product; it satisfies

$$\left(T^*f, g\right) = (f, Tg).$$

LEMMA 5.1. *For each $\varepsilon > 0$, we can decompose $B_\varepsilon - B_\varepsilon^*$ as*

(5.1) $$B_\varepsilon - B_\varepsilon^* = A_\varepsilon + C_\varepsilon,$$

where:

(a) $\|A_\varepsilon\|_{L^p \to L^p} \le \varepsilon c_p$, *for $1 < p < \infty$.*

(b) *Each C_ε has a kernel which is continuous on $\overline{D} \times \overline{D}$, and hence C_ε maps $L^1(D)$ to $C(\overline{D})$.*

In general, the norm of the operator C_ε may increase as $\varepsilon \to 0$.

Proof of Lemma 5.1. To construct the operators A_ε and C_ε we need to break up B_ε into a part whose kernel is supported near the "diagonal" of $D \times D$, and a complementary part supported away from this set. For this purpose, fix a continuous function φ on $\mathbb{R}^+$, so that $\varphi(t) = 1$, when $0 \le t \le \frac{1}{2}$ and $\varphi(t) = 1$ for $t > 1$, and for each $r > 0$ set

$$\varphi_r(w, z) = \varphi\left(\frac{|\rho(z)| + |\rho(w)| + |z - w|}{r} \right).$$

We write

$$D^r(f)(z) = \frac{1}{(2\pi i)^n} \int_D \varphi_r(w, z) f(w) B_\varepsilon^1(w, z),$$

$$E^r(f)(z) = \frac{1}{(2\pi i)^n} \int_D \left(1 - \varphi_r(w, z)\right) f(w) B_\varepsilon^1(w, z)$$

so that $B_\varepsilon^1 = D^r + E^r$. The parameter r will be chosen in terms of ε momentarily. We then define A_ε and C_ε by

$$A_\varepsilon = D^r - \left(D^r\right)^*$$

and

$$C_\varepsilon = E^r - \left(E^r\right)^* + B_\varepsilon^2 - \left(B_\varepsilon^2\right)^*.$$

With these definitions, the identity (5.1) clearly holds, since $B_\varepsilon = B_\varepsilon^1 + B_\varepsilon^2$. To proceed further, we recall that if T is a bounded operator on $L^2(D)$ given with a kernel K,

$$T(f)(z) = \int_D K(w, z) f(w) \, dV(w),$$

with say $|K(w, z)| \lesssim |g(w, z)|^{-n-1}$, then its adjoint T^* is given by the kernel $\overline{K}(z, w)$ in place of $K(w, z)$. In view of the representation (4.7), the kernel of D^r is

$$\frac{(\mathcal{K}_0(w) + O_\varepsilon(|w - z|))\varphi_r(w, z)}{(2\pi i)^n (g_\varepsilon(w, z))^{n+1}},$$

and so to study $D^r - (D^r)^*$ we must estimate

$$\frac{1}{(g_\varepsilon(w, z))^{n+1}} - \frac{1}{(\overline{g}_\varepsilon z, w))^{n+1}}$$

and

(5.2) $$\frac{\mathcal{K}_0(w)}{(2\pi i)^n} - \overline{\left(\frac{\mathcal{K}_0(z)}{(2\pi i)^n}\right)} + \frac{O_\varepsilon(|w - z|)}{(2\pi i)^n} - \overline{\left(\frac{O_\varepsilon(|w - z|)}{(2\pi i)^n}\right)}$$

on the support of φ_r. (Note that $\varphi_r(w, z)$ is obviously symmetric in w and z). Now

$$\left(g_\varepsilon(w, z)\right)^{-n-1} - \left(\overline{g}_\varepsilon(z, w)\right)^{-n-1} = \frac{(\overline{g}_\varepsilon(z, w))^{n+1} - (g_\varepsilon(w, z))^{n+1}}{(g_\varepsilon(w, z)\overline{g}_\varepsilon(z, w))^{n+1}}.$$

Taking into account the fact that $|g_\varepsilon(w, z)| \approx |g_\varepsilon(z, w)|$, we see that the above is majorized by a multiple of

$$\frac{|g_\varepsilon(w, z) - \overline{g}_\varepsilon(z, w)|}{|g_\varepsilon(w, z)|^{n+2}}.$$

In turn, by Propositions 2.1 and 2.2, this is majorized by a multiple of

$$\frac{\varepsilon|z - w|^2}{|g_\varepsilon(w, z)|^{n+2}} \lesssim \frac{\varepsilon}{|g_\varepsilon(w, z)|^{n+1}},$$

whenever $|z - w| < \delta_\varepsilon$. So if we choose r so that $r \leq \delta_\varepsilon$, then

$$(5.3) \qquad \left| \frac{1}{(g_\varepsilon(w, z))^{n+1}} - \frac{1}{\overline{g}_\varepsilon(z, w)^{n+1}} \right| \leq \frac{c\varepsilon}{|g_\varepsilon(w, z)|^{n+1}}$$

for (w, z) in the support of φ_r, because then $|z - w| < r$.

Next, we examine the numerator (5.2). To this end, we first note that

$$O_\varepsilon\big(|w - z|\big) = O_\varepsilon(r)$$

for (w, z) in the support of φ_r, because then $|z - w| < r$. To estimate the difference $\mathcal{K}_0(w)/(2\pi i)^n - \mathcal{K}_0(z)/(2\pi i)^n$, we invoke the identities (4.9) through (4.10) along with the computation [R, Exercise VII.E.7.2], and obtain

$$\frac{\mathcal{K}_0(w)}{(2\pi i)^n} = \frac{n!}{\pi^n} |\nabla\rho(w)|^2 \det\mathcal{L}_w,$$

where $\det\mathcal{L}_w$ is the determinant of the Levi form at the point w (for the domain $\{\zeta : \rho(\zeta) < \rho(w)\}$). This shows that $\mathcal{K}_0(w)/(2\pi i)^n$ is a *real*-valued continuous function on $\overline{D}$. By its uniform continuity, there is a $\delta' = \delta'_\varepsilon$, so that

$$\left| \frac{\mathcal{K}_0(w)}{(2\pi i)^n} - \frac{\mathcal{K}_0(z)}{(2\pi i)^n} \right| < \varepsilon \quad \text{if } |z - w| < \delta'_\varepsilon.$$

Finally, suppose that in the above inequalities we write

$$O_\varepsilon(r) \leq r A_\varepsilon$$

for an appropriate bound A_ε (depending on ε). We can then choose r in terms of ε by taking $r = \min(\delta_\varepsilon, \delta'_\varepsilon, \varepsilon/A_\varepsilon)$. This guarantees that

$$(5.4) \qquad \left| \frac{\mathcal{K}_0(w) + O_\varepsilon(|w - z|)}{(2\pi i)^n} - \overline{\left(\frac{\mathcal{K}_0(z) + O_\varepsilon(|w - z|)}{(2\pi i)^n} \right)} \right| \leq c\varepsilon$$

on the support of φ_r. Combining (5.4) with (5.3) then shows that

$$(5.5) \quad \left| \frac{\mathcal{K}_0(w) + O_\varepsilon(|w-z|)}{(2\pi i)^n (g_\varepsilon(w,z))^{n+1}} - \overline{\left(\frac{\mathcal{K}_0(z) + O_\varepsilon(|w-z|)}{(2\pi i)^n (g_\varepsilon(z,w))^{n+1}} \right)} \right|$$

$$\leq \frac{c\varepsilon}{|g_\varepsilon(w,z)|^{n+1}}.$$

So by Proposition 4.1, we conclude that

$$\|A_\varepsilon\|_{L^p \to L^p} = \left\| D^r - \left(D^r \right)^* \right\|_{L^p \to L^p} \leq \varepsilon c_p,$$

which is conclusion (a) of this lemma.

The second conclusion is immediate. Indeed since $|\rho(z)| + |\rho(w)| + |z - w| \geq r/2$ on the support of $1 - \varphi_r$, we have by Proposition 2.1 that $|g_\varepsilon(w,z)| \gtrsim r^2$ there, and hence the kernel of B_ε^1 is a continuous function of (w, z) there, see (4.7).

Thus the kernels of E^r and $(E^r)^*$ are continuous on $\overline{D} \times \overline{D}$, and by Proposition 3.2, the same is true for B_ε^2 and $(B_\varepsilon^2)^*$. Since $C_\varepsilon = E^r - (E^r)^* + B_\varepsilon^2 - (B_\varepsilon^2)^*$, the operator C_ε has a continuous kernel on $\overline{D} \times \overline{D}$. From this, it is also evident that C_ε maps $L^1(D)$ to $C(\overline{D})$, proving the lemma. $\square$

6. The main theorems

Let B be the Bergman projection for the domain D. The operator B is the orthogonal projection of $L^2(D)$ to the Bergman space $\vartheta L^2(D)$, the subspace of holomorphic functions in $L^2(D)$. Then as is well known,

$$(6.1) \quad B(f)(z) = \int_D B(w,z) f(w) \, dV(w), \quad f \in L^2(D), \ z \in D,$$

where $B(w, z)$ is the Bergman kernel, which satisfies $B(z, w) = \overline{B}(w, z)$.

THEOREM 6.1. *Suppose the domain is of class C^2 and strongly pseudoconvex. Then $f \mapsto B(f)$ extends to a bounded mapping of $L^p(D)$ to itself, for each p, $1 < p < \infty$.*

As is well known, for each $z \in D$, the function $B(w, z)$ belongs to $L^2(D)$, see [K]. Hence, $B(f)(z)$ is well-defined by (6.1) when $f \in L^p(D)$ for $p \geq 2$. When $p \leq 2$, the operator (6.1) is initially defined only on $L^2(D)$, a dense subspace of $L^p(D)$. In either case, the thrust of Theorem 6.1 is the inequality

$$\left\| B(f) \right\|_{L^p} \leq c_p \|f\|_{L^p}$$

for f that belongs to $L^2(D)$.

There is also a stronger version that involves the operator whose kernel is the absolute value of the Bergman kernel. Define $|B|$ to be the operator

$$|B|(f)(z) = \int_D \left| B(z, w) \right| f(w) \, dV(w)$$

(defined initially on $L^2(D)$).

THEOREM 6.2. *Under the assumptions of Theorem* 6.1, *we have that* $f \mapsto |B|(f)$ *is also bounded on* $L^p(D)$, $1 < p < \infty$.

To prove Theorem 6.1, we start with the following identities that hold on $L^2(D)$. First $BB_\varepsilon = B_\varepsilon$, which follows from (a) of Proposition 3.2 (together with Corollary 2, for $p = 2$); also $B_\varepsilon B = B$, which is a consequence of (b) of Proposition 3.2. Taking adjoints of the second of these identities and using that $B^* = B$ immediately yields

$$B\bigl(I - \bigl(B_\varepsilon^* - B_\varepsilon\bigr)\bigr) = B_\varepsilon.$$

It then follows from Lemma 5.1 that

$$B(1 + A_\varepsilon) = B_\varepsilon - BC_\varepsilon.$$

Now for fixed p, $1 < p < \infty$, choose $\varepsilon = \varepsilon(p)$ so small that according to Lemma 5.1 we have that $\|A_\varepsilon\|_{L^p \to L^p} < 1$. Then $I + A_\varepsilon$ is invertible as an operator on L^p, using a Neumann series. Writing the inverse as $(I + A_\varepsilon)^{-1}$ gives us

$$(6.2) \qquad B = (B_\varepsilon - BC_\varepsilon)(I + A_\varepsilon)^{-1}.$$

Since we know that B_ε is bounded on L^p (see Corollary 2), it suffices to note that BC_ε is bounded on L^p. We observe this first for $p \leq 2$ as follows. The fact that the kernel of C_ε is continuous on $\overline{D} \times \overline{D}$ gives that C_ε maps L^1 to L^∞, and hence L^p to L^2. Thus BC_ε maps L^p to L^2, and since $p \leq 2$, we have BC_ε maps L^p to L^p. This proves that the boundedness of B on L^p, when $p \leq 2$.

To obtain the result for $p \geq 2$, we may use the (self) duality of B to reduce to the case $p \leq 2$. Alternatively we may retrace the steps leading to (6.2) to obtain

$$(6.3) \qquad (I - A_\varepsilon)B = B_\varepsilon^* + C_\varepsilon^* B.$$

Then, after inverting $(I - A_\varepsilon)$, it suffices to see that $C_\varepsilon^* B$ is bounded on L^p. For this one notes that B maps L^p to L^2 (since now $L^p \subset L^2$), and C_ε^* maps L^2 to L^p (since it maps L^1 to L^∞). The proof of Theorem 6.1 is now complete.

To prove Theorem 6.2, we need to manipulate operators with positive kernels. To do so rigorously, we formulate the following definition. Suppose T is a bounded linear operator on L^p. We say that T has a *positive majorant* $\widehat{T}$, if $\widehat{T}$ is bounded linear operator on L^p that satisfies

$$(6.4) \qquad \begin{cases} \widehat{T}(f) \geq 0 & \text{if } f \geq 0, \text{ and} \\ \bigl|T(f)(z)\bigr| \leq \widehat{T}(|f|)(z), & \text{for a.e. } z. \end{cases}$$

Observe that if T_1 and T_2 have positive majorants $\widehat{T}_1$ and $\widehat{T}_2$ respectively, then

- $\widehat{T}_1 + \widehat{T}_2$ and $\widehat{T}_1 \circ \widehat{T}_2$ are positive majorants for $T_1 + T_2$ and $T_1 \circ T_2$, respectively.

Also suppose T_n have positive majorants $\widehat{T}_n$ and $T_n \to T$, while $\widehat{T}_n \to S$, as $n \to \infty$ strongly in L^p, then

- S is a positive majorant of T.

As a result, if T has a positive majorant $\widehat{T}$ and $\|\widehat{T}\|_{L^p \to L^p} < 1$ (hence $\|T\|_{L^p \to L^p} < 1$), then

- $(I - \widehat{T})^{-1}$ is a positive majorant of $(I - T)^{-1}$.

Indeed, by the above, $\sum_{n=0}^{N} \widehat{T}^n$ is a positive majorant of $\sum_{n=0}^{N} T^n$, and the assertion follows by a passage to the limit as $N \to \infty$.

Let us now consider the case $p \geq 2$. We will show first that the Bergman projection has a positive majorant, as an operator on L^p. The proof of Corollary 2 shows that B_ε and B_ε^* both have positive majorants (these can be taken to be $c_\varepsilon' \Gamma_\varepsilon$, for suitable c_ε'). Moreover this and the proof of (a) of Lemma 5.1 also shows that the operator A_ε also has a positive majorant of the form $c_\varepsilon \Gamma_\varepsilon$. If we set $T = A_\varepsilon$, this means that there is a positive majorant $\widehat{T}$ for T, with $\|\widehat{T}\|_{L^p \to L^p} < 1$, if ε is sufficiently small. The same is true for $T = (I - A_\varepsilon)^{-1}$.

Thus by (6.3), B will have a positive majorant as a result of the following simple lemma applied to $T_0 = C_\varepsilon^* B$, which (by Lemma 5.1) is bounded from $L^p(D)$ to $C(\overline{D})$ for all $p \geq 2$.

LEMMA 6.1. *Suppose T_0 is a bounded linear mapping from $L^p(D)$ to $C(\overline{D})$. Then as a linear mapping from $L^p(D)$ to $L^p(D)$, T_0 has a positive majorant.*

Proof. For each $z \in D$, consider the linear functional $f \mapsto T_0(f)(z)$. Then by assumption, $|T_0(f)(z)| \leq c\|f\|_{L^p}$, and so there is a $\psi = \psi_z \in L^q(D)$ with $1/p + 1/q = 1$, with $\|\psi_z\|_{L^q} \leq c$, such that

$$T_0 f(z) = \int_D \psi_z(w) f(w) \, dV(w).$$

Now define $\widehat{T}_0$ by

$$\widehat{T}_0 f(z) = \int_D |\psi_z(w)| f(w) \, dV(w),$$

so

$$\|\widehat{T}_0 f\|_{L^p} \leq c' \|\widehat{T}_0 f\|_{L^\infty} \leq c'c \|f\|_{L^p},$$

by Hölder inequality. The lemma is therefore proved. $\qquad\square$

As a result B, the Bergman projection, has a positive majorant $\widehat{B}$. This means that

(6.5) $$\left| \int_D B(w,z) f(w) \, dV(w) \right| \leq \widehat{B}(|f|)(z) \quad \text{a.e. } z \in D.$$

To pass from (6.5) to the operator with kernel $|B(w,z)|$ requires a further step, because (6.4) is asserted only for a.e. $z \in D$, and the exceptional set may depend on f. To get around this difficulty we modify the majorant $\widehat{B}$,

replacing it by another majorant $B^{\#}$ for which the analogue (6.4) holds for *all* $z \in D$.

In fact, define the *averaging operator* $\mathcal{M}$, by

$$\mathcal{M}(F)(z) = \frac{1}{V(\mathcal{B}_z)} \int_{\mathcal{B}_z} F(w)\, dV(w), \quad z \in D,$$

where $\mathcal{B}_z$ is the ball centered at z of radius equal to $1/2$ the distance of z from bD. Note that $\mathcal{M}(F)(z)$ is defined for all $z \in D$, and has the following basic properties:

(i) $\mathcal{M}(Bf)(z) = B(f)(z)$ for all $z \in D$ (by the mean value property applied to $B(f)$, since this is holomorphic).

(ii) If $|f(w)| \leq |g(w)|$ a.e. w, then $\mathcal{M}(|f|)(z) \leq \mathcal{M}(|g|)(z)$ for all z.

(iii) $|\mathcal{M}(f)(z)| \leq \mathcal{M}(|f|)(z)$.

Now define $B^{\#}$ by

$$B^{\#}(f)(z) = \mathcal{M}(\widehat{B}(f))(z), \quad z \in D.$$

Then (6.4) and the basic properties listed above imply that

(6.6) $$\left| \int_D B(w,z) f(w)\, dV(w) \right| \leq B^{\#}(|f|)(z) \quad \text{for all } z \in D.$$

Moreover by the maximal theorem $f \to B^{\#}(f)$ is also bounded on L^p, by virtue of the L^p boundedness of $\widehat{B}$. Finally, for fixed z apply (6.5) to $f(w)$ replaced by

$$h(w) = \begin{cases} f(w) \frac{|B(w,z)|}{B(w,z)}, & \text{if } B(w,z) \neq 0, \\ 0, & \text{if } B(w,z) = 0. \end{cases}$$

Since the Bergman kernel, $B(w,z)$ is anti-holomorphic in $w \in D$, then for any fixed $z \in D$ it will vanish only on a zero-measure subset of D. This means that

$$|h(w)| = |f(w)| \quad \text{a.e. } w \in D,$$

and so

$$B^{\#}(|h|)(z) = B^{\#}(|f|)(z) \quad \text{for all } z \in D.$$

Also note that

$$B(h)(z) = |B|(f)(z) \quad \text{for all } z \in D,$$

and it follows from (6.6) that

$$|B(f)(z)| \leq B^{\#}(|f|)(z) \quad \text{for all } z \in D.$$

From these (and the fact that $B^{\#}$ is a positive majorant for B everywhere in D), we obtain

$$\left| |B|(f)(z) \right| \leq B^{\#}(|f|)(z) \quad \text{for all } z \in D.$$

This shows that $f \to |B|(f)$ is bounded on $L^p(D)$ when $p \geq 2$. Since the operator $|B|$ with kernel $|B(w,z)|$ is obviously self-dual, the usual duality

shows that it extends to a bounded operator on L^p, for $1 < p \le 2$. Theorem 6.2 is therefore proved.

7. Concluding remarks

7.1. Density in $\vartheta L^p(D)$.

PROPOSITION 7.1. *The collection of functions that are each holomorphic in some neighborhood of $\overline{D}$ is dense in $\vartheta L^p(D)$, $1 < p < \infty$.*

In fact suppose that $f \in \vartheta L^p(D)$. Then we know that $f = B_\varepsilon(f)$ (see Proposition 3.2). Now let

$$f_n = \begin{cases} f(w), & \text{for } w \in D_{1/n} = \{\rho < -1/n\}, \\ 0, & \text{for } w \in D \setminus D_{1/n}, \end{cases}$$

and set $F_n = B_\varepsilon(f_n)$. Then clearly $\|f_n - f\|_{L^p(D)} \to 0$ as $n \to \infty$, which implies that

$$\|F_n - f\|_{L^p(D)} = \left\| B_\varepsilon(f_n - f) \right\|_{L^p(D)} \le c \|f_n - f\|_{L^p(D)} \to 0$$

by Corollary 2. However if $1/n < \lambda_0/2$, where λ_0 is as in (3.15), then Corollary 1 shows that F_n is holomorphic in $D_{-1/n} \supset \overline{D}$, which proves Proposition 7.1.

7.2. The Bergman projection B^σ.

Suppose σ is a positive function on D. Define the inner product $(\cdot, \cdot)_\sigma$ by

$$(F, G)_\sigma = \int_D F(w) \overline{G}(w) \sigma(w) \, dV(w)$$

and write $L_\sigma^2(D)$ for the corresponding L^2 space with norm

$$\|F\|_{L_\sigma^2} = (F, F)_\sigma^{1/2}.$$

Let B^σ be the Bergman projection corresponding to L_σ^2, that is, the orthogonal projection via $(\cdot, \cdot)_\sigma$ of L_σ^2 to the weighted Bergman space ϑL_σ^2. Note that $B = B^\sigma$ if $\sigma \equiv 1$.

We may ask whether B^σ (like B) is bounded on L^p (or L_σ^p), for a given σ. (This question was raised by Polam Yung.) We consider here only the first case of interest, when σ is continuous in $\overline{D}$ and nowhere zero. Then we can assert

- B^σ is bounded on L_σ^p, for $1 < p < \infty$.

Under our assumptions on σ the norm on L_σ^p is clearly equivalent with the norm on L^p. Thus we can also assert

- B^σ is bounded on L^p, for $1 < p < \infty$.
- Similar results hold for the operator $|B^\sigma|$.

Note however that these results are not direct consequences of Theorem 6.1 since there is no simple relation expressing B^σ in terms of B. These assertions can, however, be established by reprising the proofs of Theorems 6.1 and 6.2. One starts by proving an analogue of Lemma 5.1, which is based on the following observation. Suppose that $\widetilde{B}_\varepsilon^\sigma$ is defined to be the adjoint of B_ε with respect to $(\cdot, \cdot)_\sigma$. Then $\widetilde{B}_\varepsilon^\sigma = \sigma^{-1} \cdot B_\varepsilon^* \cdot \sigma$. Further details are left to the interested reader.

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