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THE REPRODUCING KERNELS AND THE FINITE TYPE CONDITIONS

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Dedicated to Professor John D'Angelo on the occasion of his 60th birthday

Abstract. We discuss and expand upon our previous results on comparison of the Bergman and the Szegö kernels. We also study the relations between the Bergman growth exponent and the finite type conditions.

1. Introduction

The $\bar\partial$-Neumann Laplacian is a constant multiple of the usual Laplacian acting component-wise on (p,q)-forms satisfying the non-coercive $\overline{\partial}$-Neumann boundary condition. Subellipticity of the $\overline{\partial}$-Neumann Laplacian on smooth bounded strongly pseudoconvex domains in $\mathbb{C}^n$ was established by Kohn ([36], [37]). Since then, there has been a tremendous effort to generalize Kohn's result to weakly pseudoconvex domains.

Hörmander showed that a sum of squares of real vector fields is hypoelliptic provided the Lie algebra generated by these vector fields spans the whole tangent space at each point [33]. Kohn introduced a notion of finite type for a smooth bounded pseudoconvex domain in $\mathbb{C}^2$—using the defining function and the complex tangential vector field—and established subellipticity for the $\overline{\partial}$-Neumann Laplacian on such domains [38]. Several finite type notions have since been introduced for domains in higher dimensions (see [20] for a survey on these notions). In the early 1980s, D'Angelo introduced a new finite type notion which has had a profound impact on the subject. A smooth

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bounded domain Ω in $\mathbb{C}^n$ is of D'Angelo finite type if at each boundary point, the normalized order of contact of the boundary $b\Omega$ with complex analytic varieties is finite ([16], [18]). D'Angelo further established a fundamental open estimate for the type. Subsequently, Catlin showed that for a smooth bounded pseudoconvex domain Ω in $\mathbb{C}^n$, subellipticity of the $\overline{\partial}$-Neumann Laplacian is equivalent to the finite type condition introduced by D'Angelo ([7], [10]).

Reproducing kernels such as the Bergman kernel K and the Szegö kernel S have long played an important role in analytic function theory in complex analysis. In his book [47], Stein posed several questions regarding these kernels. In particular, he asked about the relationship between K and S (see [47, p. 20]). In [13], we compared the Bergman and Szegö kernels on the diagonal near the boundary using Hörmander type L^2-estimates for the $\overline{\partial}$-operator. We showed that the quotient of the Szegö and Bergman kernels for a smooth bounded pseudoconvex domain in $\mathbb{C}^n$ is bounded from above by a constant multiple of $\delta|\log\delta|^{n/a}$ for any constant $0<a<1$, where δ is the distance to the boundary. (On any compact subset, these kernels are comparable. Here and in what follows, without loss of generality, we only deal with points whose distance to the boundary is less than e^{-1}.) For a class of pseudoconvex domains that includes those of D'Angelo finite type, the quotient is also bounded from below by a constant multiple of $\delta|\log\delta|^{-a}$. Moreover, for convex domains, the quotient is bounded from above and below by a constant multiple of δ. In this largely expository paper, we elaborate and expand on these results. In particular, we show in Section 3 that this quotient is also bounded from above and below by a constant multiple of δ on a domain of the form: $\Omega_F = \{z \in \mathbb{C}^n; |F(z)|^2 < 1\}$, where $F : \mathbb{C}^n \to \mathbb{C}^m$ is a holomorphic map. Such domains were introduced by Kohn in connection with his theory of subelliptic multipliers [39]. The Bergman kernel for these domains in special cases was explicitly computed by D'Angelo [19] and others (see [5] and references therein).

Boundary behavior of the Bergman kernel has been studied extensively since the work of Hörmander [32]. The growth exponent of the Bergman kernel measures the rate of blowing-up at a boundary point. It contains information about boundary geometry of the domain. Both the growth exponent and the finite type conditions are local biholomorphic invariants. In Section 4, we discuss several relations among these quantities, as well as other quantities from algebraic geometry.

2. Preliminaries

In this section, we recall several necessary definitions. We first review the definitions of the Bergman and Szegö kernels. (We refer the reader to [47] for relevant materials.) Let Ω be a bounded domain in $\mathbb{C}^n$ and let $A^2(\Omega)$ be the space of square integrable holomorphic functions on Ω. Let $\{b_j\}_{j=1}^{\infty}$ be an orthonormal basis for $A^2(\Omega)$. The Bergman kernel $K_\Omega(z,w)$ is then given by

$$K_\Omega(z,w) = \sum_{j=1}^{\infty} b_j(z)\overline{b_j(w)}.$$

Suppose that $b\Omega$ is of class C^2. The Hardy space $H^2(\Omega)$ is then the space of holomorphic functions f on Ω such that

$$\|f\|_{b\Omega} = \limsup_{\varepsilon\to 0^+}\left(\int_{b\Omega_\varepsilon} \left|f(z)\right|^2 dS\right)^{1/2} < \infty,$$

where $b\Omega_\varepsilon$ is the level set inside Ω whose Euclidean distance to the boundary $b\Omega$ is ε. Let $\{s_j\}_{j=1}^\infty$ be an orthonormal basis for $H^2(\Omega)$. The Szegö kernel is likewise given by

$$S_\Omega(z,w) = \sum_{j=1}^{\infty} s_j(z)\overline{s_j(w)}.$$

The Bergman and Szegö kernels are the reproducing kernels for $A^2(\Omega)$ and $H^2(\Omega)$, respectively and they have the following extremal properties:

$$K_\Omega(z,z) = \sup\left\{\left|f(z)\right|^2; f \in A^2(\Omega), \|f\|_\Omega \le 1\right\} \tag{2.1}$$

and

$$S_\Omega(z,z) = \sup\left\{\left|f(z)\right|^2; f \in H^2(\Omega), \|f\|_{b\Omega} \le 1\right\}. \tag{2.2}$$

The Poisson–Szegö kernel is given by

$$\mathscr{P}_\Omega(z,w) = \frac{|S_\Omega(z,w)|^2}{S_\Omega(z,z)}, \quad z \in \Omega, w \in b\Omega.$$

Note that by the reproducing property of the Szegö kernel, we have $\int_{\partial\Omega} \mathscr{P}_\Omega(z,w)\, dS(w) = 1$.

We now recall the finite type notions introduced by D'Angelo [16] and Catlin [8]. Let $\Omega \subset \mathbb{C}^n$ be a smooth bounded domain with a defining function r such that $dr \neq 0$ on $b\Omega$. The D'Angelo 1-type at $p \in b\Omega$ is given by

$$\Delta_1(p) = \sup\left\{\frac{\nu(\varphi^*(r))}{\nu(\varphi - p)}\right\},$$

where the supremum is taken over all non-constant holomorphic maps $\varphi : \mathbb{D} \to \mathbb{C}^n$ such that $\varphi(0) = p$ and $\nu(\cdot)$ denotes the order of vanishing. The D'Angelo q-type $\Delta_q(p)$ is likewise the order of contact of q-dimensional complex varieties with $b\Omega$. More precisely,

$$\Delta_q(p) = \inf_S\left\{\Delta_1(b\Omega \cap S, p)\right\},$$

where the infimum is taken over all $(n-q+1)$-dimensional complex hyperplanes through p. By definition, $\Delta_n(p) = 1$.

Let Γ_n be the collection of n-tuples $(\mu_1, \dots, \mu_n)$ of rational numbers or ∞ such that $1 \le \mu_1 \le \mu_2 \le \cdots \le \mu_n$ and whenever μ_k is finite there exist integers $n_j \ge 0$ with $n_k > 0$ satisfying

$$\sum_{j=1}^{k} \frac{n_j}{\mu_j} = 1.$$

Elements in Γ_n are referred to as weights and they are ordered lexicographically. A weight is distinguished if there exist local holomorphic coordinates centered at p such that

$$\frac{\partial^{|\alpha|+|\beta|} r}{\partial z^\alpha \, \partial \bar{z}^\beta}(p) = 0 \quad \text{whenever} \sum_{j=1}^{n} \frac{\alpha_j + \beta_j}{\mu_j} < 1,$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\beta = (\beta_1, \dots, \beta_n)$. The Catlin multitype $\mathscr{M}(p) = (m_1, \dots, m_n)$ is then the smallest weight that dominates every distinguished weight. Evidently, $m_1 = 1$. Furthermore, we know from [8], [17] that $m_2 = \Delta_{n-1}(p)$.

Write $\Delta(p) = (\Delta_n(p), \dots, \Delta_1(p))$. Catlin [8] showed that for a smooth bounded pseudoconvex domain Ω, one always has $\mathscr{M}(p) \le \Delta(p)$. Following J. Yu ([52], [53]), we say Ω is *h-extendible* at $p \in b\Omega$ if $\mathscr{M}(p) = \Delta(p)$ and $\Delta_1(p) < \infty$, and it is h-extendible if it is h-extendible at every boundary point. (See also [25] for an equivalent concept.) The class of h-extendible domains includes smooth bounded pseudoconvex domains of finite type in $\mathbb{C}^2$ and convex domains of finite type in $\mathbb{C}^n$ (see [51]).

3. Comparison of the kernel functions

It follows from the work of Diederich and Fornæss that for any bounded pseudoconvex domain $\Omega \subset \mathbb{C}^n$ with C^2-smooth boundary, there exist a constant $a \in (0, 1]$ and a negative plurisubharmonic function φ on Ω such that

$$C_1 \delta^a(z) \le -\varphi(z) \le C_2 \delta^a(z) \tag{3.1}$$

for some positive constants C_1 and C_2, where hereafter $\delta(z)$ denotes the Euclidean distance to the boundary $b\Omega$ ([22]; see [30] for a generalization of this theorem to Lipschitz domains). The number a above is usually referred to as a Diederich–Fornæss exponent of Ω, and the supremum of all Diederich–Fornæss exponents is called the Diederich–Fornæss index of Ω. The Diederich–Fornæss index of a pseudoconvex domain can be arbitrarily small [23]. Nonetheless, the Diederich–Fornæss index of a smooth bounded pseudoconvex domain that satisfies property (P) is always one [45, Theorem 2.4]. This is true in particular for smooth bounded pseudoconvex domains of finite D'Angelo type in $\mathbb{C}^n$, as a consequence of Catlin's construction of plurisubharmonic functions with large complex hessians ([9], [10]). More recently, Fornæss and Herbig

[26] showed that a smooth bounded domain with a defining function that is plurisubharmonic on the boundary also has Diederich–Fornæss index one.

Let PSH(Ω) be the set of plurisubharmonic functions on Ω. Recall that the pluricomplex Green function of a domain Ω with a pole at $w \in \Omega$ is defined by

$$g_\Omega(z,w) = \sup\Big\{u(z); u \in \mathrm{PSH}(\Omega), u<0, \limsup_{z\to w}\big(u(z) - \log|z-w|\big) < \infty\Big\}$$

(see, e.g., [34]). It was shown by Błocki that if Ω is a bounded pseudoconvex domain in $\mathbb{C}^n$ on which there exists a negative plurisubharmonic function φ such that

$$C_1\delta^a(z) \le -\varphi(z) \le C_2\delta^a(z), \quad z \in \Omega, \tag{3.2}$$

for some positive constants C_1, C_2, and a, then for any $B>0$, there exists a positive constant C such that

$$\begin{aligned}\big\{w \in \Omega; g_\Omega(w,z) \le -B\big\} \\ \subset \Big\{C^{-1}\delta(z)\big|\log\delta(z)\big|^{-\frac{1}{a}} \le \delta(w) \le C\delta(z)\big|\log\delta(z)\big|^{\frac{n}{a}}\Big\},\end{aligned} \tag{3.3}$$

for any $z \in \Omega$ with $\delta(z) \le e^{-1}$ [2, Theorem 5.2]. The following proposition is a slight variation of Proposition 4.3 in [13].

Proposition 3.1. *Let $\Omega \subset\subset \mathbb{C}^n$ be a pseudoconvex domain with C^2-smooth boundary. Suppose that there exists a positive increasing continuous function $\beta(t)$ on $(0,\infty)$ with $\lim_{t\to 0^+}\beta(t)=0$ and positive constants B and δ_0 such that*

$$\big\{w\in\Omega; g_\Omega(w,z)\le -B\big\} \subset \big\{w\in\Omega; \delta(w)\le\beta\big(\delta(z)\big)\big\},$$

for any $z\in\Omega$ with $\delta(z)\le\delta_0$. Then there exists a constant $C>0$ such that

$$\frac{S(z,z)}{K(z,z)} \le C\beta\big(\delta(z)\big). \tag{3.4}$$

Proof. Let $z\in\Omega$. For any $f\in H^2(\Omega)$, we have

$$\int_{\{g_\Omega(\cdot,z)<-B\}} |f|^2\,dV \le \int_0^{\beta(\delta(z))} d\varepsilon \int_{b\Omega_\varepsilon} |f|^2\,dS \le C\|f\|^2_{b\Omega}\beta\big(\delta(z)\big).$$

It then follows from the extremal properties of the Bergman and Szegö kernels that

$$S(z,z) \le C\beta\big(\delta(z)\big)K_{\{g(\cdot,z)<-B\}}(z,z) \le C\beta\big(\delta(z)\big)K_\Omega(z,z).$$

Here in the last inequality, we have used [12, Lemma 4.2] or [31, Proposition 3.6]. □

By (3.3), one can choose $\beta(t)=t(-\log t)^{n/a}$ for t near 0, where a is any Diederich–Fornæss exponent of Ω. Using the facts that a can be chosen to be arbitrarily close to 1 near any boundary point ([22], Remark on p. 133) and the localization of the Bergman and Szegö kernels, we then obtain:

THEOREM 3.2. *Let $\Omega \subset\subset \mathbb{C}^n$ be a pseudoconvex domain with C^2-smooth boundary. For any $0 < a < 1$, there exists a constant $C > 0$ such that*

$$\frac{S(z,z)}{K(z,z)} \leq C\delta(z)\big|\log\big(\delta(z)\big)\big|^{n/a} \tag{3.5}$$

for any $z \in \Omega$ with $\delta(z) \leq e^{-1}$.

We refer the reader to [13, Section 4] for the detail of a proof of the above theorem. Our analysis of the lower bound for the quotient $S(z,z)/K(z,z)$ led us to a new class of domains: A bounded domain $\Omega \subset \mathbb{C}^n$ with C^2-smooth boundary is *δ-regular* if there exist a neighborhood U of $b\Omega$, a smooth bounded plurisubharmonic function φ on $U \cap \Omega$, and a defining function $\rho \in C^2$ of Ω such that

$$\partial\overline{\partial}\varphi \geq \rho^{-1}\, \partial\overline{\partial}\rho \tag{3.6}$$

on $U \cap \Omega$. Any δ-regular domain is necessarily pseudoconvex. This class of domains encompasses two classes of domains that have been widely studied in the regularity theory of the $\overline{\partial}$-Neumann problem (see [4], [20]): It is easy to see that any smooth bounded pseudoconvex domain with a defining function that is plurisubharmonic on $b\Omega$ is δ-regular, and as a simple consequence of Catlin's construction of plurisubharmonic functions with large hessians, any smooth bounded pseudoconvex domain of finite D'Angelo type is also δ-regular (see [13, Proposition 5.2]). The following proposition was proved—implicitly—in [13, Section 5], using a Hörmander type estimate due to Berndtsson [1]:

PROPOSITION 3.3. *Let $\Omega \subset\subset \mathbb{C}^n$ be a δ-regular domain with C^2-smooth boundary. Suppose that there exist a positive increasing continuous function $\alpha(t)$ on $(0,\infty)$ with $\lim_{t\to 0^+} \alpha(t) = 0$ and positive constants B and δ_0 such that*

$$\big\{w \in \Omega; g_\Omega(w,z) \leq -B\big\} \subset \big\{w \in \Omega; \delta(w) \geq \alpha\big(\delta(z)\big)\big\},$$

for any $z \in \Omega$ with $\delta(z) \leq \delta_0$. Then there exists a constant $C > 0$ such that

$$\frac{S(z,z)}{K(z,z)} \geq C\alpha\big(\delta(z)\big). \tag{3.7}$$

Using Błocki's estimate (3.3), we can choose $\alpha(t) = t(-\log t)^{-1/a}$ when t is sufficiently small, where a is any Diederich–Fornæss exponent of Ω. As a consequence, we have the following theorem.

THEOREM 3.4. *Let $\Omega \subset\subset \mathbb{C}^n$ be a δ-regular domain with C^2-smooth boundary. Let $a \in (0,1]$ be a Diederich–Fornæss exponent of Ω. Then there exists a constant $C > 0$ such that*

$$\frac{S(z,z)}{K(z,z)} \geq C\delta(z)\big|\log\big(\delta(z)\big)\big|^{-1/a}, \tag{3.8}$$

for any $z \in \Omega$ with $\delta(z) \leq e^{-1}$.

We again refer the reader to [13] for the detail of a proof. Combining Theorems 3.2 and 3.4, we then obtain the following [13] theorem.

THEOREM 3.5. *Let Ω be a smooth bounded pseudoconvex domain in $\mathbb{C}^n$. Suppose that $\partial\Omega$ is of finite D'Angelo type or has a defining function that is plurisubharmonic on $\partial\Omega$. Then for each $a \in (0,1)$, there exists a positive constant $C > 1$ such that*

$$C^{-1}\delta(z)\big|\log\delta(z)\big|^{-1/a} \leq \frac{S(z,z)}{K(z,z)} \leq C\delta(z)\big|\log\delta(z)\big|^{n/a}, \tag{3.9}$$

for any $z \in \Omega$ with $\delta(z) \leq e^{-1}$.

Note that the logarithmic terms in (3.9) correspond directly to those in Błocki's estimate (3.3). Since for convex domains, the logarithmic terms do not appear in (3.3), namely,

$$\big\{w \in \Omega; g_\Omega(w,z) \leq -B\big\} \subset \big\{C^{-1}\delta(z) \leq \delta(w) \leq C\delta(z)\big\} \tag{3.10}$$

for any $w \in \Omega$ (see [2]), we have accordingly

$$C^{-1}\delta(z) \leq \frac{S(z,z)}{K(z,z)} \leq C\delta(z). \tag{3.11}$$

It is not known whether estimates (3.10) and (3.11) hold for all smooth bounded pseudoconvex domains in $\mathbb{C}^n$. In what follows, we show that these estimates do hold for any smooth bounded pseudoconvex domain of the form:

$$\Omega_F = \big\{z \in \mathbb{C}^n;\ \big|f_1(z)\big|^2 + \cdots + \big|f_m(z)\big|^2 < 1\big\}, \tag{3.12}$$

where $F = (f_1, \ldots, f_m) : \mathbb{C}^n \to \mathbb{C}^m$ is a holomorphic map. Such domains were studied by Kohn in connection with his subelliptic multiplier theory [39, Section 7]. For brevity, we will refer to such a domain as a *Kohn special domain.*

THEOREM 3.6. *Estimates (3.10) and (3.11) hold for any Kohn special domain Ω_F.*

We will need the following simple lemma (compare [14, Proposition 2]).

LEMMA 3.7. *Let h be a holomorphic map from a domain $\Omega \subset \mathbb{C}^n$ to the unit disc $\mathbb{D}$. Then for any $z \in \Omega$,*

$$\begin{aligned}&\big\{w \in \Omega; g_\Omega(w,z) \leq -1\big\}\\ &\qquad\subset \left\{w \in \Omega; \frac{1-|h(z)|}{2+\sqrt{3}} \leq 1 - \big|h(w)\big| \leq (2+\sqrt{3})\big(1-\big|h(z)\big|\big)\right\}.\end{aligned} \tag{3.13}$$

Proof. We provide a proof for completeness. By the decreasing property of the pluri-complex Green function, we have

$$\begin{aligned}-g_\Omega(w,z) \leq -g_{\mathbb{D}}\big(h(w),h(z)\big) &= -\log\left|\frac{h(w)-h(z)}{1-\overline{h(z)}h(w)}\right|\\ &= \frac{1}{2}\log\left(1 + \frac{(1-|h(w)|^2)(1-|h(z)|^2)}{|h(w)-h(z)|^2}\right)\end{aligned}$$

$$\leq \frac{(1-|h(w)|^2)(1-|h(z)|^2)}{2|h(w)-h(z)|^2}$$
$$\leq 2\frac{(1-|h(w)|)(1-|h(z)|)}{|h(z)-h(w)|^2}.$$

When $1-|h(w)| \geq (2+\sqrt{3})(1-|h(z)|)$,

$$\big|h(w)-h(z)\big| \geq 1-\big|h(w)\big| - \big(1-\big|h(z)\big|\big) \geq (\sqrt{3}-1)\big(1-\big|h(w)\big|\big).$$

Therefore,

$$-g_\Omega(w,z) \leq (2+\sqrt{3})\frac{1-|h(z)|}{1-|h(w)|}.$$

It follows that

$$\big\{w\in\Omega; g_\Omega(w,z)\leq -1\big\} \subset \big\{w\in\Omega; 1-\big|h(w)\big| \leq (2+\sqrt{3})\big(1-\big|h(z)\big|\big)\big\}.$$

The other containment

$$\big\{w\in\Omega; g_\Omega(w,z)\leq -1\big\} \subset \left\{w\in\Omega; \frac{1-|h(z)|}{2+\sqrt{3}} \leq 1-\big|h(w)\big|\right\}$$

is proved similarly by reversing the roles of z and w. □

We now prove Theorem 3.6. Let $z\in\Omega=\Omega_F$ be sufficiently close to the boundary $b\Omega$ such that there exists a unique $\tilde{z}\in b\Omega$ with $|z-\tilde{z}|=\delta_\Omega(z)$. Note that

$$\sum_{j=1}^m \big|f_j(\zeta)\big|^2 < 1$$

if and only if

$$\sum_{j=1}^m \Big(2\mathrm{Re}\overline{f_j(\tilde{z})}\big(f_j(\zeta)-f_j(\tilde{z})\big) + \big|f_j(\zeta)-f_j(\tilde{z})\big|^2\Big) < 0.$$

Let

$$h(\zeta) = \exp\left(\sum_{j=1}^m \overline{f_j(\tilde{z})}\big(f_j(\zeta)-f_j(\tilde{z})\big)\right).$$

Then $|h(\zeta)|<1$ on Ω. Applying Lemma 3.7 to h, we then have

$$\big\{g_\Omega(\cdot,z)\leq -1\big\} \subset \big\{1-\big|h(\cdot)\big| \leq (2+\sqrt{3})\big(1-\big|h(z)\big|\big)\big\} \subset \big\{\delta_\Omega(\cdot)\leq C\delta_\Omega(z)\big\}$$

for a sufficiently large $C>0$.

To prove $\{g_\Omega(\cdot,z)\leq -1\} \subset \{C^{-1}\delta_\Omega(z)\leq\delta_\Omega(\cdot)\}$, we reverse the roles of z and w. Let $w\in\Omega$ be such that $\delta_\Omega(w)<C^{-1}\delta_\Omega(z)$ where C is a sufficiently large constant to be chosen. Let $\tilde{w}$ be the nearest point on the boundary to w. By applying Lemma 3.7 (and its proof) to

$$h(\zeta) = \exp\left(\sum_{j=1}^m \overline{f_j(\tilde{w})}\big(f_j(\zeta)-f_j(\tilde{w})\big)\right),$$

we have

$$-g_\Omega(w,z) \le (2+\sqrt{3})\frac{1-|h(w)|}{1-|h(z)|} \le C_0\frac{\delta_\Omega(w)}{\delta_\Omega(z)} < C_0/C_1 \le 1$$

when $C \ge C_0$. Therefore,

$$\{g_\Omega(\cdot,z) \le -1\} \subset \{\delta_\Omega(\cdot) \ge C^{-1}\delta_\Omega(z)\}.$$

We have now established (3.10). Since Ω_F has a plurisubharmonic defining function, it is δ-regular. By applying Propositions 3.1 and 3.3 with $\alpha(t) = \beta(t) = t$, we then obtain (3.11). This concludes the proof of Theorem 3.6.

In [47, p. 19], Stein posed another question: Does the Poisson–Szegö kernel $\mathscr{P}_\Omega$ give an approximation to the identity, in the sense that if $f(w)$ is continuous on $\partial\Omega$, then

$$u(z) = \int_{\partial\Omega} \mathscr{P}_\Omega(z,w) f(w)\, dS(w)$$

is continuous in $\overline{\Omega}$ and $u|_{\partial\Omega} = f$? It is easy to see that this is the case if for every boundary point $p \in b\Omega$ is a peak point (*i.e.*, there exists a function h holomorphic on Ω and continuous on $\overline{\Omega}$ such that $h(p) = 1$ and $|h| < 1$ on $\overline{\Omega} \setminus \{p\}$). The following proposition, an easy consequence of a theorem of Boas [3] and (a weaker version of) Theorem 3.5, answers this question of Stein affirmatively for a smooth bounded pseudoconvex domain in $\mathbb{C}^n$ of finite D'Angelo type.

PROPOSITION 3.8. *Let $\Omega \subset \mathbb{C}^n$ be a bounded pseudoconvex domain of finite D'Angelo type. Then the Poisson–Szegö kernel is an approximation to the identity.*

Proof. By [3, Corollary 5.2], we have

$$S_\Omega(z,w) \in C^\infty\big(\overline{\Omega} \times \overline{\Omega} \setminus \{(z,w) \in \partial\Omega \times \partial\Omega : z = w\}\big). \tag{3.14}$$

Let $p \in b\Omega$. For every $\varepsilon > 0$, there is $\delta > 0$ such that $|f(w) - f(p)| < \varepsilon/2$ whenever $|w - p| < \delta$. We have

$$\begin{aligned}
\big|u(z) - f(p)\big| &= \left|\int_{\partial\Omega} \mathscr{P}_\Omega(z,w)\big(f(w) - f(p)\big)\, dS(w)\right| \\
&\le \int_{\partial\Omega} \mathscr{P}_\Omega(z,w)\big|f(w) - f(p)\big|\, dS(w) \\
&= \left(\int_{\partial\Omega\cap B(p,\delta)} + \int_{\partial\Omega\setminus B(p,\delta)}\right) \mathscr{P}_\Omega(z,w)\big|f(w) - f(p)\big|\, dS(w).
\end{aligned}$$

The first integral is bounded by $\varepsilon/2$. From Theorem 3.5, we have $S(z,z) \to \infty$ as $z \to p$. Combining this with (3.14), we then obtain that the second integral is also bounded by $\varepsilon/2$ when $|z - p|$ is sufficiently small. □

Note that if $S(\cdot, z)$ is bounded for any fixed $z \in \Omega$, then

$$h(w) = \frac{S_\Omega(w,z)}{S_\Omega(z,z)} f(w) \in H^2(\Omega)$$

for any $f \in H^2(\Omega)$. Hence by the reproducing property of the Szegö kernel,

$$f(z) = \int_{\partial\Omega} S_\Omega(z,w) h(w)\, dS(w) = \int_{\partial\Omega} \mathscr{P}_\Omega(z,w) f(w)\, dS(w).$$

Therefore, it follows from the work of Boas and Straube (see [4]) that the answer to Question 2 in [47, p. 19] is also affirmative for—among other domains—smooth bounded Reinhardt domains in $\mathbb{C}^n$, smooth bounded complete Hartogs domains in $\mathbb{C}^2$, and smooth bounded pseudoconvex domains with plurisubharmonic defining functions in $\mathbb{C}^n$.

4. Bergman growth exponents

Let $\Omega \subset\subset \mathbb{C}^n$ be a domain with C^2-smooth boundary and let $p \in b\Omega$. The upper and lower growth exponents of the Bergman kernel K_Ω at p are given respectively, by

$$G_u(p;\Omega) = \inf\left\{\alpha \ge 0;\ \varlimsup_{\substack{z \to p \\ \text{n.t.}}} \delta(z)^\alpha K_\Omega(z,z) < \infty\right\} \tag{4.1}$$

and

$$G_l(p;\Omega) = \sup\left\{\alpha \ge 0;\ \varliminf_{\substack{z \to p \\ \text{n.t.}}} \delta(z)^\alpha K_\Omega(z,z) > 0\right\} \tag{4.2}$$

where $z \in \Omega$ tends to p non-tangentially. Evidently, $G_l(p;\Omega) \le G_u(p;\Omega)$. When the equality holds, these quantities are called the growth exponent of the Bergman kernel (Bergman growth exponent for abbreviation) at p and denoted by $G(p;\Omega)$.

One always has $G_u(p) \le n+1$ by comparing K_Ω with the Bergman kernel of the balls tangential to $b\Omega$ from inside. Furthermore, $G_l(p, b\Omega) \ge 2$ if and only if Ω is pseudoconvex. The necessity is a direction consequence of the solution to Levi's problem. The sufficiency follows from a theorem of Pflug [44]. In fact, by the Ohsawa–Takegoshi extension theorem [43], $K_\Omega(z,z) \ge C\delta^{-2}(z)$.

When Ω is pseudoconvex, the Bergman kernel has localization property: for any neighborhoods $V \subset\subset U$ of p, $C^{-1} K_{\Omega\cap U}(z,z) \le K_\Omega(z,z) \le K_{\Omega\cap U}(z,z)$ on $V \cap \Omega$. Therefore, the Bergman growth exponent can be considered as a local biholomorphic invariant of Ω. We refer the reader to [24], [35] for an extensive study of $G_l(p;\Omega)$.

Similarly, one can also define the upper and lower growth exponents $G_l^s(p;\Omega)$ and $G_u^s(p;\Omega)$ for the Szegö kernel S_Ω by replacing K_Ω by S_Ω in the above definitions. The following proposition is a simple consequence of Theorem 3.2 and Theorem 3.4.

PROPOSITION 4.1. (1) *If* Ω *is pseudoconvex, then* $G_u^s(p) \le G_u(p) - 1$. (2) *If* Ω *is* δ*-regular, then* $G_l(p) - 1 \le G_l^s(p) \le G_u^s(p) \le G_u(p) - 1$.

When $b\Omega$ is strongly pseudoconvex at p, it follows from the work of Hörmander [32] that $G(p;\Omega) = n+1$. Furthermore, $b\Omega$ has constant Levi-rank $k-1$ near p if and only if $G(z;\Omega) = k+1$ for all $z \in b\Omega$ near p (see [27], [28]). When Ω is pseudoconvex of finite type in $\mathbb{C}^2$, it follows from Catlin's estimates of the Bergman kernel that $G(p, b\Omega) = 2 + 2/m$ where m is the type of $b\Omega$ at p [11]. Analogous results were obtained by J. Chen [15] and McNeal [41] for convex domains of finite type in $\mathbb{C}^n$. It was shown by Boas, Straube, and Yu ([6]; see also [25]) that if Ω is a smooth bounded pseudoconvex domain that is h-extendible at p with Catlin multitype $(1, m_2, \ldots, m_n)$, then $G(p) = 2 + \sum_{j=2}^n 2/m_j$. The following theorem can be regarded as a converse to the above-mentioned theorems of Hörmander, Catlin, and Boas–Straube–Yu. Weaker results was given in [27, Theorem 3.2.1 and Theorem 3.3.1].

THEOREM 4.2. *Let* Ω *be a smooth bounded pseudoconvex domain in* $\mathbb{C}^n$. *Let* $p \in b\Omega$. *Suppose that the Catlin multitype at* p *is* $\mathscr{M}(p) = (1, m_2, \ldots, m_n)$. *If* $G_l(p) \ge 2+\varepsilon$ *for some* $\varepsilon > 0$, *then* $\sum_{j=2}^n 2/m_j \ge \varepsilon$. *In particular, the D'Angelo* $(n-1)$*-type* $\Delta_{n-1}(p) \le 2(n-1)/\varepsilon$. *Furthermore, if* $G_l(p) > n + 1/2$, *then* $b\Omega$ *is strongly pseudoconvex at* p.

Proof. Write $z = (z_1, z')$. Proving by contradiction, we assume that $\sum_{j=2}^n 2/m_j < \varepsilon$. If $m_n < \infty$, then it follows from the definition of the Catlin multitype that after a change of holomorphic coordinates on a neighborhood U of p,

$$\Omega \cap U = \{z \in U; r(z) = \operatorname{Re} z_1 + P(z') + R(z) < 0\},$$

where P is a weighted homogeneous polynomial satisfying

$$P\bigl(t^{1/m_2} z_2, \ldots, t^{1/m_n} z_n\bigr) = tP(z_2, \ldots, z_n)$$

and

$$\bigl|R(z)\bigr| \le C\left(\sum_{j=1}^n |z_j|^{m_j}\right)^{\gamma}$$

for some $\gamma > 1$. Let $z_\delta = (0, \ldots, 0, -\delta)$ and let

$$P_\delta = \bigl\{|z_j| < c\delta^{1/m_j}, 1 \le j \le n-1\bigr\} \times \bigl\{|z_n + \delta| < \delta/2\bigr\}.$$

Then $P_\delta \subset \Omega \cap U$ for sufficiently small δ and c. Therefore,

$$K_\Omega(z_\delta, z_\delta) \le K_{P_\delta}(z_\delta, z_\delta) \le C\delta^{-2-\sum_{j=2}^n 2/m_j}.$$

Thus $G_l(p) \le 2 + \sum_{j=2}^n 2/m_j < 2 + \varepsilon$, contradicting the assumption $G_l(p) \ge 2 + \varepsilon$.

If $m_n = \infty$, we may assume without loss of generality that $m_j < \infty$ for $j \le j_0$ and $m_j = \infty$ for $j \ge j_0 + 1$ for some integer $1 \le j_0 \le n-1$. It then follows that for any $M>0$, after a change of holomorphic coordinates, $b\Omega$ has a defining function r near $p=0$ of the form:

$$r(z)=\operatorname{Re} z_1+F\big(z'\big)+R(z),$$

where

$$\big|F(z)\big|\le C\left(\sum_{j=2}^{j_0}|z_j|^{m_j}+\sum_{j=j_0+1}^{n}|z_j|^M\right)$$

and

$$\big|R(z)\big|\le C\left(\sum_{j=1}^{j_0}|z_j|^{m_j}+\sum_{j=j_0+1}^{n}|z_j|^M\right)^{\gamma}$$

for some $\gamma>1$. Therefore, we have as before $P_\delta\subset\Omega\cap U$ and hence

$$G_l(p)\ge 2+\sum_{j=2}^{j_0}2/m_j+\sum_{j=j_0+1}^{n}2/M<2+\varepsilon,$$

provided M is chosen to be sufficiently large. This again leads to a contradiction with the assumption.

Since $m_j\le\Delta_{n-j+1}(p)$, $2\le j\le n$, and $m_2=\Delta_{n-1}(p)$, it follows from the inequality $\sum_{j=2}^n 2/m_j\ge\varepsilon$ that

$$\Delta_{n-1}(p)\le 2(n-1)/\varepsilon.$$

It remains to prove the last statement in the theorem. Suppose $b\Omega$ is not strongly pseudoconvex at p. Then $m_j>2$ for some $2\le j\le n$. Let j_0 be the smallest such j. Then it follows from the pseudoconvexity of $b\Omega$ that $m_{j_0}\ge 4$ (see [18, pp. 147–148]). Therefore

$$G_l(p)\le 2+\sum_{j=2}^{n}2/m_j\le 2+(j_0-2)+(n-j_0+1)/2\le n+1/2,$$

which contradicts the assumption. □

In [35], Kamimoto used a combination of singularity theory and Fourier analysis to study the Bergman kernel on domains of the form:

$$\Omega_\rho=\big\{(z,w)\in\mathbb{C}^n\times\mathbb{C}:\ \operatorname{Im} w>\rho(z)\big\}, \tag{4.3}$$

where $\rho\ge 0$ is a smooth plurisubharmonic function on $\mathbb{C}^n$ such that:

(1) $\rho(0)=0$ and $\nabla\rho(0)=0$;
(2) $\rho(z)=\rho(|z_1|,\dots,|z_n|)$;
(3) $\Delta_1(0,b\Omega_\rho)<\infty$;
(4) there exist positive constants c and ε such that $\rho(z)\ge c|z|^\varepsilon$ for sufficiently large $|z|$.

Note that the D'Angelo 1-type $\Delta_1(0)$ of $b\Omega_\rho$ at the origin is realizable by a complex line [29]. Thus, $\Delta_1(0)$ is the smallest integer m such that

$$\big|\rho(z)\big| \geq C|z|^m,$$

for all sufficiently small $|z|$.

Before we state Kamimoto's result, we first recall some definitions. Let $\mathbb{R}_+$ be the set of non-negative real numbers. Let F be a smooth real-valued function near the origin in $\mathbb{C}^n$ such that $F(0) = 0$. Let

$$F(z) = \sum_{\alpha,\beta\in\mathbb{Z}^n_+} C_{\alpha\beta} z^\alpha \bar{z}^\beta$$

be the formal power series of $F(z)$ at the origin. The Newton polyhedron $\Gamma_+(F)$ of F is the convex hull of the set

$$\bigcup \big\{\alpha + \beta + \mathbb{R}^n_+; C_{\alpha\beta} \neq 0\big\}.$$

The Newton diagram $\Gamma(F)$ is the union of the compact faces of the Newton polyhedron $\Gamma_+(F)$. If $\Gamma(F)$ intersects the real ray $t \to (t,t,\dots,t)$, $t > 0$, at $p_0 = (t_0, t_0, \dots, t_0)$ for some $t_0 > 0$, then t_0 is called the distance of the Newton diagram $\Gamma(F)$ and it is denoted by d_F. Let $\hat{m}_F$ be the number of $(n-1)$-dimensional faces of $\Gamma(F)$ containing p_0. Then the number $m_F = \max\{\hat{m}_F, n\}$ is called the multiplicity of $\Gamma(F)$.

THEOREM 4.3 (Kamimato). *Let $\Omega = \Omega_\rho$ be a domain of the form* (4.3). *Let Λ be any cone with vertex at the origin and axis the positive* $(\operatorname{Im} w)$*-axis. Let $\delta = \delta(z,w)$ be the Euclidean distance from (z,w) to $b\Omega$. Then there exist constants $C_1, C_2 > 0$ such that*

$$\frac{C_1}{\delta^{2+2/d_\rho}|\log\delta|^{m_\rho-1}} \leq K_\Omega\big((z,w),(z,w)\big) \leq \frac{C_2}{\delta^{2+2/d_\rho}|\log\delta|^{m_\rho-1}},$$

for all $(z,w)\in\Lambda$ with $|(z,w)|\leq 1$. In particular, $G(0;\Omega_\rho) = 2 + 2/d_\rho$.

EXAMPLE ([24, Lemma 6.1] and [35, Example 2.1.5]). Let a, b, c, d, and m be positive integers such that $a < b$, $d < c < m$, $2 \leq a + b < c + d$, and $b(m-c) \geq d(m-a)$. Let $\rho(z) = |z_1|^{2a}|z_2|^{2b} + |z_1|^{2c}|z_2|^{2d} + |z_1|^{2m} + |z_2|^{2m}$ and

$$\Omega = \big\{(z,w)\in\mathbb{C}^3; \operatorname{Re} w + \rho(z) < 0\big\}.$$

Write $P_1 = (0,2m)$, $P_2 = (2a,2b)$, $P_3 = (2c,2d)$, and $P_4 = (2m,0)$. Then the Newton diagram $\Gamma(\rho)$ consists of line segments $\overline{P_1P_2}$, $\overline{P_2P_3}$, and $\overline{P_3P_4}$. Thus, d_ρ is determined by the intersection of $t\to(t,t)$ with the line segment $\overline{P_2P_3}$. Hence,

$$d_\rho = \frac{2(bc-ad)}{c+b-a-d}, \qquad m_\rho = 1, \quad \text{and} \quad G(0) = 2 + \frac{2}{d_\rho} = 2 + \frac{c+b-a-d}{bc-ad}.$$

Note that the tuple of the D'Angelo types at the origin is $\Delta(0) = (1, 2(a+b), 2m)$ and the Catlin multitype is $\mathscr{M}(0) = (1, 2(a+b), 2(a+b))$. This example illustrates that for a pseudoconvex domain in $\mathbb{C}^n$, one cannot in general expect to reconstruct precise information about $\Delta_q(0)$ when $q < n-1$ from the growth exponent of the Bergman kernel as in Proposition 4.2.

Remark. (1) Let $\mathrm{BE}(n)$ be the set of all possible Bergman growth exponents for all smooth bounded pseudoconvex domains in $\mathbb{C}^n$. It follows from [11] that $\mathrm{BE}(2) = \{1 + 1/m; m \in \mathbb{N}\}$. By Theorem 4.2,

$$\mathrm{BE}(n) \cap \left(n + \frac{1}{2}, n+1\right) = \emptyset.$$

Moreover, similarly, $b\Omega$ has constant Levi-rank k, $0 \le k \le n-2$, near $p \in b\Omega$ if $G_l(z;\Omega) \in (k + 5/3, k+2]$ for all $z \in b\Omega$ near p. As a consequence,

$$\mathrm{BE}(n) \cap \left(k + \frac{5}{3}, k+2\right) = \emptyset.$$

Combining these results with the above example of Diederich–Herbort, one knows that $\mathrm{BE}(3)$ consists of all rational numbers in $[2, 2+2/3]$ and all rational numbers of the form $3 + 1/m$, $m \in \mathbb{N} \cup \{\infty\}$. More generally, one expects that for $n \ge 3$,

$$\mathrm{BE}(n) = \bigcup_{k=0}^{n-3} \left\{2 + k + r; r \in \mathbb{Q}, 0 \le r \le \frac{2}{3}\right\} \cup \left\{n + \frac{1}{m}; m \in \mathbb{N}\right\}.$$

(2) It is plausible that $G_u(p;\Omega) = G_l(p;\Omega)$ holds for all smooth bounded pseudoconvex domains in $\mathbb{C}^n$.

Recall that for a measurable function φ on a domain $U \subset \mathbb{C}^n$, the complex singularity exponent of φ at a point $p \in U$ is defined to be

$$c(p;\varphi) = \sup\big\{c \ge 0; |\varphi|^{-c} \text{ is integrable on a neighborhood of } p\big\}.$$

We refer the reader to [21] for relevant materials. The following theorem, inspired by Kamimoto's work, was obtained in [14]. It gives a partial answer to a question of D'Angelo [18, p. 259]:

Theorem 4.4 ([14]). *Let $\Omega_F = \{(z,w) \in \mathbb{C}^n \times \mathbb{C} : \operatorname{Im} w > |F(z)|^2\}$, where $F : \mathbb{C}^n \to \mathbb{C}^m$ is a holomorphic map with $F(0) = 0$. Then*

$$G(0;\Omega_F) = 2 + c\big(0; |F|^2\big).$$

The proof of the above theorem uses a version of Hironaka's theorem on resolution of singularities. We refer the reader to [14] for the detail. Only nontangential boundary behavior of the Bergman kernel was addressed in [35], [14]. Unrestricted boundary behavior remains a challenging open problem. Uniform estimates of the Bergman kernel on monomial polyhedra and other special domains were obtain by Tiao [48] and recently by Nagel and Pramanik [42].

Remark. In general, it is difficult to calculate the complex singularity exponent $c(0;|F|^2)$. Nonetheless, when $b\Omega$ is of finite D'Angelo type at 0, we have the following two estimates.

(1) It is known that $\Delta_1(0,b\Omega_F)$ is the smallest positive integer m such that $|F(z)|^2 \geq C|z|^m$ on some open neighborhood of the origin [46, Lemma I.3]. Thus in this case, $\Delta_1(0,b\Omega_F)$ is the Lelong number of the plurisubharmonic function $\log|F|^2$ at 0. Thus, the D'Angelo 1-type is connected to the complex singularity exponent via the following well-known inequality (see [21]):

$$\Delta_1(0,b\Omega) \leq c\big(0;|F|^2\big) \leq n\Delta_1(0,b\Omega_F). \tag{4.4}$$

(2) We also have

$$c\big(0;|F|^2\big) \leq \frac{1}{d_{|F|^2}}. \tag{4.5}$$

Furthermore, the equality holds if $|F|^2$, viewed as a real analytic function on $\mathbb{R}^{2n}$, is either nondegenerate in the sense of Kouchnirenko or extremely nondegenerate in the sense of Vassil'ev. Here, $d_{|F|^2}$ is the distance to the Newton diagram of $|F|^2$ in $\mathbb{R}^{2n}$. We refer the reader to [40], [49], [50] for the relevant concepts and a proof of this fact.

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