L^2-EXTENSION OF $\bar{\partial}$-CLOSED FORM

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Dedicated to John d'Angelo

ABSTRACT. Generalizing and strengthening a recent result of Koziarz, we prove a version of the Ohsawa–Takegoshi–Manivel theorem for $\bar{\partial}$-closed forms.

1. Introduction

The celebrated Ohsawa–Takegoshi–Manivel extension theorem, [7], [6] gives optimal conditions for the extension of holomorphic sections of line bundles from a divisor to the ambient space. In Manivel's article [6], it is stated that a completely parallel result holds for smooth $\bar{\partial}$-closed forms of higher degree. There is however a problem in the proof of this in [6] which is connected with the regularity of solutions of certain $\bar{\partial}$-equations with singular weights. This problem is also discussed in [4], where a strategy towards its solution is put forward.

Recently, an at least moral solution of this problem was given by Koziarz [5]. Instead of looking at the extension of individual forms, Koziarz considered the extension of cohomology classes, that is, extended closed forms up to a $\bar{\partial}$-exact error. This formulation is actually more natural than the original problem since cohomology classes have well defined restrictions on divisors, whereas $\bar{\partial}$-closed forms restrict only if a somewhat artificial condition of smoothness is imposed. Koziarz's method is inspired by work of Siu [8], and consists in representing cohomology classes by Cech cocycles. These cocycles consist of holomorphic objects for which the available machinery works better.

The purpose of this note is twofold. First, we will prove a simple proposition saying that a smooth $\bar{\partial}$-exact form on a divisor can always be extended to a closed form with arbitrary small L^2-norm in the ambient space. The converse of this statement also holds, so in fact a closed form on the divisor is exact if

and only if it can be extended to a closed form with arbitrarily small norm. This means that Koziarz's theorem on the extension up to an exact error actually gives a solution to the original problem on extension of closed forms. Second, we will give an alternative proof of Koziarz's theorem, following the method in [2]. The advantage with this alternative proof is that it gives an absolute constant for the extension, whereas in Koziarz's theorem the constant depended on the manifold and the divisor. Moreover, the curvature conditions that guarantee extendability are shown to be somewhat more liberal for forms of higher degree than for holomorphic sections. Finally, the proof exhibits the significance of extension of cohomology classes in a seemingly interesting way.

Let us comment a little bit more on this. If u is a holomorphic section of $K_\Delta + L$ over a divisor Δ, the method in [2], see also [1], consists in solving the equation
$$\bar\partial v = u \wedge [\Delta] := g.$$
The right-hand side here is not a L^2-form but a current, but nevertheless it turns out that L^2-methods can be used here. One cannot however get a solution v in L^2. If the divisor Δ is defined by a section s of some line bundle S over the ambient manifold X, the solution of the extension problem is sv, so what we want is an L^2-estimate for sv. Dually (and formally!), this corresponds to an estimate for smooth testforms α like

$$(1.1) \qquad\qquad |\langle g, \alpha\rangle|^2 \le C \int |\bar\partial^* \alpha|^2 / |s|_\psi^2$$

(where ψ is some metric on S). But this dual formulation is only formal. The fact that the weight $|s|^{-2}$ is nonintegrable causes a problem in the functional analysis involved since all smooth test forms do not have finite norm with respect to this weight. This problem can be circumvented if we instead prove a stronger estimate

$$(1.2) \qquad\qquad |\langle g, \alpha\rangle|^2 \le C \int |\bar\partial^* \alpha|^2 / |s|_\psi^r,$$

where $r < 2$. Then the functional analytic difficulty disappears and one even gets a stronger result than is asked for.

We now want to follow the same route for forms of higher degree. Both estimates (1.1) and (1.2) can then be proved in much the same manner as for holomorphic sections. As in the case of holomorphic sections, the best thing would be to use (1.2), since that is a bona fide dual formulation of the $\bar\partial$-problem. But this causes problems with regularity. One would then need to discuss regularity properties in L^2-spaces with singular weights, which leads back to the original problem with Manivel's argument. We therefore choose to work with (1.1) instead. Then the regularity problems disappear since we can go back and forth between estimates with the singular weight $|s|^{-2}$ and estimates without that weight by multiplying and dividing with s. The price we have to pay for this is that (1.1) is no longer a dual formulation

of the $\bar{\partial}$-estimate, and so not a dual formulation of the extension problem. But, miraculously, it turns out to be a dual formulation of the extension of cohomology classes, and this is what makes the scheme work.

In this paper, we will suppose all the time that X is a compact Kähler manifold. Maybe the same arguments could be pushed to noncompact situations, but the compactness assumption simplifies and makes the argument a little bit simpler than in [2].

Finally, I would like to thank the referee for a careful reading of the manuscript with many suggestions for improvement.

2. $\bar{\partial}$-exact forms

In this section, we discuss the extension of $\bar{\partial}$-exact forms. The main point is the following proposition.

PROPOSITION 2.1. *Let X be an n-dimensional compact complex manifold, and let Δ be a smooth divisor in X. Let L be a holomorphic line bundle over X. Let u be a smooth $\bar{\partial}$-closed L-valued $(0,q)$-form on Δ, $q \geq 1$. Then u is $\bar{\partial}$-exact on Δ if and only if, for any $\varepsilon > 0$, there is a $\bar{\partial}$-closed extension, U, of u to X with L^2-norm smaller than ε.*

Here L^2-norms are taken with respect to some smooth metric and some arbitrary smooth volume form. In the proof we use the next lemma.

LEMMA 2.2. *There is a sequence of cutoff-functions ρ_ε such that:*
1. *The sets where $\rho_\varepsilon(z) = 1$ are neighbourhoods of Δ shrinking to Δ, and the sets where $\rho_\varepsilon(z) = 0$ increase to $X \setminus \Delta$.*
2. *$\|\bar{\partial}\rho_\varepsilon\|$ goes to zero with ε.*

Proof. Let first the dimension be 1 and take X to be the unit disk and Δ to be the origin. The main point is that there is a complete (Kahler) metric on the punctured disk, ω, which gives $\{|z| < 1/2\}$ finite area. Indeed, the Poincare metric

$$\omega = idz \wedge d\bar{z} / (|z|^2 (\log|z|)^2)$$

has this property. Completeness implies that there is some realvalued function near the origin, ρ, such that $\rho(z)$ tends to infinity when z tends to zero and

$$i\partial\rho \wedge \bar{\partial}\rho \leq \omega.$$

Explicitly, $\rho(z) = \log\log(1/|z|)$ will do. Define functions $\chi_k(x)$ on the positive halfaxis, equal to 0 when $x < k$, to 1 when $x > k+1$, and having χ_k' bounded. Then put

$$\rho_\varepsilon = \chi_{1/\varepsilon} \circ \rho.$$

Then 1 is clear and 2 follows by dominated convergence since

$$\int_{|z|<1/2} i\partial\rho_\varepsilon \wedge \bar{\partial}\rho_\varepsilon \leq \int_{|z|<1/2} \chi_\varepsilon' \omega.$$

The general case is basically the same. We can cover Δ by a finite number of coordinate neighbourhoods, inside which Δ is defined by the equation $z_1 = 0$. Then take $\rho_\varepsilon(z_1)$ with ρ_ε defined as above and piece together with a partition of unity. $\qquad\square$

With this, we can turn to the proof of the proposition. Assume first that $u = \bar\partial v$ on Δ with v smooth. We extend v to X smoothly in an arbitrary way and let

$$U_\varepsilon = \bar\partial(\rho_\varepsilon v).$$

By the lemma, this a $\bar\partial$-closed, or even exact, extension of u with L^2-norm going to zero with ε.

For the converse, assume there are some $\bar\partial$-closed extensions, U_ε, with L^2-norms going to zero. Let $\mathcal{U}_\varepsilon$ be the harmonic representative of the cohomology class $[U_\varepsilon]$. The norms of the harmonic representatives are smaller than the norms of U_ε, so they go to zero too. Now, the space of harmonic forms is finite dimensional, so all norms are equivalent. Hence, the supnorms of $\mathcal{U}_\varepsilon$ also go to zero, so the restrictions of $\mathcal{U}_\varepsilon$ to Δ also go to zero. Since on Δ, $u - \mathcal{U}_\varepsilon$ is exact, it follows that u lies in the closure of the space of exact forms. But $\bar\partial$ has closed range on a compact manifold, so u must be exact.

3. $\bar\partial$-closed forms

In this section, we adapt the argument in [2] to forms of higher degree. We will use the residue formulation of the extension problem and the set up is as follows.

X is a compact Kähler manifold, with Kähler form ω and L is a holomorphic line bundle over X. Δ is a smooth divisor in X, given as $\Delta = s^{-1}(0)$, with s a holomorphic section of a line bundle S. Let u be a smooth L-valued $\bar\partial$-closed $(n-1, q)$-form on Δ. We want to find a smooth $L + S$-valued $\bar\partial$-closed (n, q)-form, U, on X, such that

$$(3.1) \qquad\qquad U = ds \wedge u$$

on Δ. Note that u could alternately be interpreted as a $(0, q)$-form on Δ with values in $K_\Delta + L$. By the adjunction isomorphism

$$u \mapsto ds \wedge u$$

between K_Δ and $(K_X + S)|_\Delta$ this means that we extend a $(0, q)$-form with values in

$$(3.2) \qquad\qquad F := K_X + S + L$$

to a form with values in F.

THEOREM 3.1. *Assume that ϕ is a smooth metric on L and that ψ is a smooth metric on S such that*

$$i\partial\bar\partial\phi \wedge \omega^q \geq \varepsilon i\partial\bar\partial\psi \wedge \omega^q$$

and

$$i\partial\bar{\partial}\phi \wedge \omega^q \geq 0.$$

Assume moreover the normalizing inequality

$$\log |s|^2 e^{-\psi} \leq -1/\varepsilon.$$

Let u be a smooth $\bar{\partial}$-closed $(n-1,q)$-form with values in L over Δ. Then there is a $\bar{\partial}$-closed (n,q)-form, U, with values in $S+L$ over X such that

$$U = ds \wedge u$$

on Δ and

$$\int_X |U|^2 e^{-\phi-\psi}\, dV_X \leq C_0 \int_\Delta |u|^2 e^{-\phi}\, dV_\Delta,$$

where C_0 is an absolute constant. The norms and the volume forms are defined by the Kähler form ω.

The argument starts with the observation that if U satisfies the conclusion of the theorem, and if $v := U/s$, then v has values in $K_X + L$ and solves

(3.3) $$\bar{\partial}v = \bar{\partial}(1/s) \wedge ds \wedge u = cu \wedge [\Delta],$$

where $[\Delta]$ is the current of integration on Δ and $c = (2\pi)^{-1}$. Conversely, let v solve (3.3) and assume that $U := sv$ is smooth. On Δ, we can write $U = ds \wedge \tilde{u}$ by the adjunction isomorphism. Then

$$\bar{\partial}v = \bar{\partial}(1/s) \wedge ds \wedge \tilde{u} = c\tilde{u} \wedge [\Delta].$$

Hence $\tilde{u} = u$ on Δ, so U solves the extension problem.

We now try to solve this $\bar{\partial}$-problem and start to give it a dual formulation. We will then see that the dual formulation does not at first give us a true solution to the $\bar{\partial}$-equation, and so not a solution to the extension problem. It will however give us a solution to the extension problem up to a $\bar{\partial}$-exact error term, which can then be handled by Proposition 2.1. (Going back again, if we wish, to the $\bar{\partial}$-equation, we will then after all get a solution to the $\bar{\partial}$-equation as well.) Let

$$f := u \wedge [\Delta],$$

a current with measure coefficients, concentrated on Δ and of bidegree $(n, q+1)$, with values in a certain line bundle. The proof of the next lemma will be postponed to the end of the section.

LEMMA 3.2 (The basic estimate). *Assume, in addition to the assumptions in Theorem 3.1, that*

$$\|u\|_\Delta^2 \leq 1.$$

Then, for any smooth L-valued (n,q)-form α on X

$$\left| \langle f, \alpha \rangle \right|^2 \leq C_0 \int_X \frac{|\bar{\partial}_\phi^* \alpha|^2}{|s|^2 e^{-\psi}} e^{-\phi}.$$

The norm $\|\cdot\|_\Delta$ here is the L^2-norm defined by the Kähler form ω and the metric ϕ on L.

Now consider the conjugate linear functional

$$R(\bar{\partial}^*_\phi \alpha) = \langle f, \alpha \rangle$$

defined on the space

$$E := \{\bar{\partial}^*_\phi \alpha; \alpha \text{ smooth}\}.$$

By the lemma, R is bounded by the norm

$$\int_X \frac{|\bar{\partial}^*_\phi \alpha|^2}{|s|^2 e^{-\psi}} e^{-\phi}$$

on the subspace E_0 of elements of E such that this norm is finite. Clearly, this subspace consists of forms $\bar{\partial}^*_\phi \alpha$ that vanish on Δ. By the Riesz representation theorem, there is a form w such that

$$R(\bar{\partial}^*_\phi \alpha) = \int_X \frac{w \cdot \overline{\bar{\partial}^*_\phi \alpha}}{|s|^2 e^{-\psi}} e^{-\phi}$$

for all α with $\bar{\partial}^*_\phi \alpha = 0$ on Δ. Moreover, w can be taken to satisfy

$$\int_X \frac{|w|^2}{|s|^2 e^{-\psi}} e^{-\phi} \leq C_0.$$

Substitute

$$v = w/\left(|s|^2 e^{-\psi}\right).$$

Then

(3.4)
$$\langle f, \alpha \rangle = R(\bar{\partial}^*_\phi \alpha) = \int_X v \cdot \overline{\bar{\partial}^*_\phi \alpha} e^{-\phi}$$

and

(3.5)
$$\int_X |v|^2 |s|^2 e^{-\phi-\psi} \leq C_0.$$

Notice that this *does not mean* that $\bar{\partial} v = f$ since we only know that (3.4) holds for α with $\bar{\partial}^*_\phi \alpha = 0$ on Δ.

In order to get smoothness, we now choose v with minimal norm defined in (3.5), and the first objective is to check that there is a minimizer.

LEMMA 3.3. *Assume that v_k is a sequence of forms such that*

$$\langle f, \alpha \rangle = \int_X v_k \cdot \overline{\bar{\partial}^*_\phi \alpha} e^{-\phi}$$

for all α with $\bar{\partial}^*_\phi \alpha = 0$ on the divisor. Assume also that

$$\int_X |v - v_k|^2 |s|^2 e^{-\phi-\psi} \to 0$$

for some v satisfying

$$\int_X |v|^2 |s|^2 e^{-\phi-\psi} < \infty.$$

Then

$$\langle f, \alpha \rangle = \int_X v \cdot \overline{\bar{\partial}_\phi^* \alpha} e^{-\phi}$$

for all α with $\bar{\partial}_\phi^* \alpha = 0$ on the divisor.

This means that the affine space of forms v that satisfy (3.4) is closed for the norm in (3.5), so it has an element of minimal norm. The proof of the lemma is clear since

$$\left| \int_X (v - v_k) \cdot \overline{\bar{\partial}_\phi^* \alpha} e^{-\phi-\psi} \right|^2 \leq \int_X |v - v_k|^2 |s|^2 e^{-\phi-\psi} \int_X \frac{|\bar{\partial}_\phi^* \alpha|^2}{|s|^2 e^{-\psi}} e^{-\phi}.$$

The next point is to see that if v is a minimizer, then sv is a harmonic form, hence smooth.

LEMMA 3.4. *Assume that v minimizes the norm in (3.5) among all solutions to (3.4). Then $\bar{\partial}_{\phi+\psi}^*(sv) = 0$.*

Proof. If v is a minimizer, then

$$\int_X |v|^2 |s|^2 e^{-\phi-\psi} \leq \int_X |v - \bar{\partial}u|^2 |s|^2 e^{-\phi-\psi}$$

for all smooth u. This means that

$$\int_X sv \cdot \overline{\bar{\partial}sue}^{-\phi-\psi} = 0.$$

Hence, $\bar{\partial}_{\phi+\psi}^* sv = 0$ at least outside of Δ, but sv has finite L^2-norm so a divisor is removable for this equation. (A $\bar{\partial}^*$-equation for a form is a $\bar{\partial}$-equation for $*$ of the form.) □

Finally, we have

LEMMA 3.5. *Assume v satisfies (3.4) and that the norm of v defined as in (3.5) is finite. Then*

$$\bar{\partial}(sv) = 0.$$

Proof. Take first α appearing in (3.4) to be supported outside of Δ. Then of course $\bar{\partial}_\phi^* \alpha = 0$ on Δ and $\langle f, \alpha \rangle = 0$, so

$$\int_X v \cdot \overline{\bar{\partial}_\phi^* \alpha} e^{-\phi} = 0.$$

Hence, $\bar{\partial}v = 0$ outside of Δ, so $\bar{\partial}sv = 0$ there as well. But, since sv is locally in L^2, the last equation holds across Δ too. □

All in all we have now seen that $U := sv$ is harmonic and therefore smooth, if v is the minimal solution of the dual problem. What remains is to investigate the behaviour of U on the divisor. Write $U = ds \wedge \tilde{u}$ on the divisor. Let α be a smooth L-valued $(n, q+1)$-form such that $\bar{\partial}_\phi^* \alpha = 0$ on the divisor and write

$$\alpha = \gamma_\alpha \wedge \omega^{q+1}/(q+1)!$$

for some (uniquely determined) $(n-q-1, 0)$-form γ_α. Then for any $(n, q+1)$-form g

$$\langle g, \alpha \rangle_\omega \omega^n / n! = g \wedge \bar{\gamma}_\alpha$$

(see [3] for more on this).

Hence,

$$\langle f, \alpha \rangle = \int_X f \wedge \bar{\gamma}_\alpha e^{-\phi} = \int_\Delta u \wedge \bar{\gamma}_\alpha e^{-\phi}.$$

On the other hand, by (3.4) this also equals

$$\int_X v \cdot \overline{\bar{\partial}_\phi^* \alpha} e^{-\phi} = \int_X U/s \cdot \overline{\bar{\partial}_\phi^* \alpha} e^{-\phi} = \int_X \bar{\partial}(1/s) \wedge U \wedge \bar{\gamma}_\alpha e^{-\phi} = \int_\Delta \tilde{u} \wedge \bar{\gamma}_\alpha e^{-\phi}.$$

From this, we see that

$$(3.6) \qquad\qquad \int_\Delta u \wedge \bar{\gamma}_\alpha e^{-\phi} = \int_\Delta \tilde{u} \wedge \bar{\gamma}_\alpha e^{-\phi}$$

for all α such that $\bar{\partial}_\phi^* \alpha = 0$ on Δ. This latter condition is equivalent to saying that

$$\bar{\partial}\big(\bar{\gamma}_\alpha e^{-\phi}\big) = 0.$$

Let $\bar{\gamma}_\alpha e^{-\phi} =: \chi$. This is a $(0, n-q-1)$-form with values in $-L$. Hence,

$$(3.7) \qquad\qquad \int_\Delta (u - \tilde{u}) \wedge \chi = 0$$

for all $(0, n-q-1)$-forms χ with values in $-L$ such that $\bar{\partial}\chi = 0$ on Δ. The $\bar{\partial}$ operator here is the $\bar{\partial}$ on X, but, by the next lemma, the same thing holds if only $\bar{\partial}_\Delta \chi = 0$.

LEMMA 3.6. *Let χ be a smooth—L-valued $(0,p)$-form on X such that $\bar{\partial}_\Delta \chi = 0$ on Δ. Then there is a smooth form on X, $\tilde{\chi}$ such that $\bar{\partial}_X \tilde{\chi} = 0$ on Δ and $\chi = \tilde{\chi}$ on Δ.*

Proof. Locally the divisor is given by an equation $z_1 = 0$ in some local chart. The hypothesis then means that $\bar{\partial}\chi$ is divisible by $d\bar{z}_1$. To get a local extension it therefore suffices to subtract a suitable multiple of $\bar{z}_1$, and one then obtains $\tilde{\chi}$ from a partition of unity. $\square$

It follows from the lemma that (3.8) holds for any χ on Δ such that $\bar{\partial}_\Delta \chi = 0$. But this means that the difference $u - \tilde{u}$ is $\bar{\partial}$-exact. Hence, we have proved Koziarz's theorem that u can be extended up to an exact error, and the proof of Theorem 3.1 then follows from Proposition 2.1.

All that remains is now to prove Lemma 3.2.

3.1.   Proof of the basic estimate. This follows closely the proof in [2], and the proof in the compact case is described in [3], and we refer to these notes for more details on the computations that follow.

We first write as above

$$\alpha = \gamma \wedge \omega^{q+1}/(q+1)!$$

so that γ is an L-valued $(n-q-1,0)$-form. Then define a scalar valued $(n-1,n-1)$-form

$$T_\alpha = c_q \gamma \wedge \bar{\gamma} \wedge \omega^q e^{-\phi}/q!,$$

where c_q is a unimodular constant chosen so that T_α is a positive form. We will prove the basic estimate first assuming that $\bar{\partial}\alpha = 0$. In that case, it follows from Proposition 3.4.1 in [3] that

$$(3.8) \qquad i\partial\bar{\partial}T_\alpha \geq -2\operatorname{Re}\big(\bar{\partial}\bar{\partial}_\phi^*\alpha, \alpha\big)\omega^n/n! + i\partial\bar{\partial}\phi \wedge T_\alpha.$$

Let

$$W := -\log\big(|s|^2 e^{-\psi}\big).$$

By the hypothesis in Theorem 3.1, $W \geq 1/\varepsilon$. Moreover,

$$i\partial\bar{\partial}W = i\partial\bar{\partial}\psi - c[\Delta].$$

Multiply (3.9) by W and apply Stokes' formula. This gives

$$(3.9) \qquad \int_X \big(Wi\partial\bar{\partial}\phi \wedge \omega^q - i\partial\bar{\partial}\psi \wedge \omega^q\big)/q! \wedge c_q \gamma \wedge \bar{\gamma} e^{-\phi}$$

$$+ c\int_\Delta c_q \gamma \wedge \bar{\gamma} \wedge \omega^q/q! e^{-\phi} \leq 2\operatorname{Re}\langle \bar{\partial}\bar{\partial}_\phi^*\alpha, W\alpha\rangle.$$

By the hypotheses in Theorem 3.1 the first integral in the left-hand side is nonnegative, so we get

$$c\int_\Delta c_q \gamma \wedge \bar{\gamma} \wedge \omega^q/q! e^{-\phi} \leq 2\operatorname{Re}\langle \bar{\partial}\bar{\partial}_\phi^*\alpha, W\alpha\rangle.$$

On the other hand

$$|\langle f, \alpha\rangle|^2 = \left|\int_X f \wedge \bar{\gamma} e^{-\phi}\right|^2 = \left|\int_\Delta u \wedge \bar{\gamma} e^{-\phi}\right|^2.$$

By the Cauchy inequality we get, since by assumption $\|u\|_\Delta \leq 1$ that

$$|\langle f, \alpha\rangle|^2 \leq \|\gamma\|_\Delta^2 = \int_\Delta c_q \gamma \wedge \bar{\gamma} \wedge \omega^q/q! e^{-\phi} \leq 2c^{-1}\operatorname{Re}\langle \bar{\partial}\bar{\partial}_\phi^*\alpha, W\alpha\rangle.$$

The right-hand side equals

$$2\int_X W\big|\bar{\partial}_\phi^*\alpha\big|^2 e^{-\phi} - 2\operatorname{Re}\langle \bar{\partial}W \wedge \bar{\partial}_\phi^*\alpha, \alpha\rangle.$$

The first term is obviously OK since $W \leq e^W$. For the second term, we write

$$II := \langle \bar{\partial} W \wedge \bar{\partial}_\phi^* \alpha, \alpha \rangle = \int_X \bar{\partial} W \wedge \bar{\partial}_\phi^* \alpha \wedge \bar{\gamma}_\alpha e^{-\phi}.$$

By Cauchy's inequality

$$2|II| \leq \int_X \frac{|\bar{\partial}_\phi^* \alpha|^2}{|s|^2 e^{-\psi}} e^{-\phi} + c_q \int_X e^{-W} \partial W \wedge \bar{\partial} W \wedge \gamma_\alpha \wedge \bar{\gamma}_\alpha \wedge \omega^q/q! e^{-\phi}.$$

It is only the last term that we need to worry about. Let

$$W_1 = \left(1 - e^{-W}\right).$$

Then $0 < W_1 < 1$ and

$$i\partial\bar{\partial}W_1 = -e^{-W} i\partial W \wedge \bar{\partial} W.$$

We now repeat the same argument as above, but with W replaced by W_1. The result is

$$c_q \int_X e^{-W} \partial W \wedge \bar{\partial} W \wedge \gamma_\alpha \wedge \bar{\gamma}_\alpha \wedge \omega^q/q! e^{-\phi}$$
$$\leq 2 \int_X W_1 |\bar{\partial}_\phi^* \alpha|^2 e^{-\phi} - 2 \operatorname{Re}\langle \bar{\partial} W_1 \wedge \bar{\partial}_\phi^* \alpha, \alpha \rangle.$$

The first term is controlled since $W_1 < 1$ and the second term can easily be absorbed in the left-hand side. This completes the proof of the basic estimate in case $\bar{\partial}\alpha = 0$.

The general case is easily reduced to this special case. We decompose

$$\alpha = \alpha^1 + \alpha^2,$$

where α^1 is $\bar{\partial}$-closed and α^2 is orthogonal to the space of $\bar{\partial}$-closed forms. Then in particular α^2 is orthogonal to $\bar{\partial}$-exact forms, so $\bar{\partial}_\phi^* \alpha^2 = 0$. Hence α^1 satisfies $\bar{\partial}\alpha^1 = 0$ and $\bar{\partial}_\phi^* \alpha^1 = \bar{\partial}_\phi^* \alpha$. This means, by elliptic regularity that α^1, and therefore α^2 are both smooth. Now we claim that booth sides in the basic estimate are unchanged if we replace α by α^1. Since we know the basic estimate holds for α^1 this is all we need. That the right-hand side is unchanged we have already seen. That the left-hand side is unchanged follows since f is closed and α^2 is orthogonal to closed forms. There is a minor problem here, coming from the fact that f is not an L^2-form. However, f is cohomologous to a smooth form

$$f = f_{\text{smooth}} + \bar{\partial}g$$

and this proves the claim since α^2 is smooth and satisfies $\bar{\partial}_\phi^* \alpha^2 = 0$.

References

[1] E. Amar, *Extension de fonctions holomorphes et courants*, Bull. Sci. Math. **107** (1983), 25–48. MR 0699989

[2] B. Berndtsson, *The extension theorem of Ohsawa–Takegoshi and the theorem of Donnelly–Fefferman*, Ann. Inst. Fourier (Grenoble) **46** (1996), 1083–1094. MR 1415958

[3] B. Berndtsson, *An introduction to things $\bar{\partial}$*, Analytic and algebraic geometry, IAS/Park City Math. Ser., vol. **17**, Amer. Math. Soc., Providence, RI, 2010, pp. 7–76. MR 2743815

[4] J.-P. Demailly, *On the Ohsawa–Takegoshi–Manivel L^2 extension theorem*, Complex analysis and geometry (Paris, 1997), Progr. Math., vol. **188**, Birkhäuser, Basel, 2000, pp. 47–82. MR 1782659

[5] V. Koziarz, *Extensions with estimates of cohomology classes*, Manuscripta Math. **134** (2011), 43–58. MR 2745253

[6] L. Manivel, *Un théorème de prolongement L^2 de sections holomorphes d'un fibré hermitien*, Math. Z. **212** (1993), 107–122. MR 1200166

[7] T. Ohsawa and K. Takegsohi, *On the extension of L^2-holomorphic functions*, Math. Z. **195** (1987), 197–204. MR 0892051

[8] Y.-T. Siu, *A vanishing theorem for semipositive line bundles over non-Kähler manifolds*, J. Differential Geom. **19**, 431–452. MR 0755233

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