Illinois Journal of Mathematics
Volume 56, Number 1, Spring 2012, Pages 7–19
S 0019-2082

SEVERAL COMPLEX VARIABLES AND CR GEOMETRY

JOHN P. D'ANGELO

Abstract. This paper discusses developments in complex analysis and CR geometry in the last forty years related to the Cauchy–Riemann equations, proper holomorphic mappings between balls, and positivity conditions in complex analysis. The paper includes anecdotes about some of the contributors to these developments.

1. Introduction

I interweave some of the developments in Complex Analysis and CR Geometry contributed by the authors and editors of this volume with some related anecdotes.

Completeness is of course impossible; my primary aim is to express thanks to all these people. I also wish to thank many other friends in complex analysis who are not explicitly mentioned here. Research mathematics is both a team effort and an individual effort. Progress occurs when many mathematicians contribute to the same area, extending and polishing each other's techniques. On occasion, individuals will introduce new ideas to a problem, thereby providing the larger community with new tools and perspectives. The editors and authors of this volume have contributed to complex analysis in both ways. I cannot possibly describe all this progress, but perhaps my comments will illuminate some of it and encourage younger researchers to continue working in complex variable theory.

This paper discusses topics from complex analysis and CR geometry in an informal fashion, interspersed with personal reminiscences. The first topic is the Cauchy–Riemann equations. I continue by considering results related to proper mappings between balls. I discuss positivity conditions and related

Received August 1, 2012; received in final form August 27, 2012.
I acknowledge support from NSF grant DMS 1066177.
2010 *Mathematics Subject Classification.* 32-02, 32T25, 32T27, 32A70, 32V99, 32W05, 32H35, 32M25, 32V15, 35N15, 12D15, 14P05, 15B57.

matters. The long reference list is in fact too short. I omitted many relevant papers and I do not include those papers from this volume I mention explicitly.

2. The Cauchy–Riemann equations

The Cauchy–Riemann equations in several complex variables define holomorphic functions: a continuously differentiable function f is holomorphic on a domain if and only if $\overline{\partial} f = 0$ there. One then considers the $\overline{\partial}$ operator on differential forms and studies the inhomogeneous equation $\overline{\partial} u = \alpha$. Here u is a form of type (p, q) and α a form of type $(p, q+1)$; these forms are allowed to have distributions as coefficients. For a solution to this system to exist, one must have $\overline{\partial} \alpha = 0$, because $\overline{\partial}^2 = 0$. One of the key approaches to the subject of complex analysis uses information about the *inhomogeneous equation* to obtain results about holomorphic functions, solutions to the *homogeneous* equation. The kind of information used arises from the point of view of partial differential equations; results about existence and regularity of solutions dominate the discussion. Decisive references to this approach include [B], [FK], [H], [S].

Perhaps the most fundamental problem in complex analysis during the twentieth century was the Levi problem. Early work showed that the boundary $b\Omega$ of a domain of holomorphy Ω in complex Euclidean space $\mathbf{C}^n$ satisfied a geometric property called pseudoconvexity; is the converse assertion true? The problem was resolved by the late 1950s using sheaf theory. In particular, three conditions on Ω are equivalent:

- Ω is a domain of holomorphy.
- Ω is pseudoconvex.
- For each $q \geq 1$, the sheaf cohomology group $H^q(\Omega, \mathcal{O})$ vanishes.

By that time, an approach (pioneered by Spencer) using the methods of partial differential equations was developing. Spencer's idea was to extend the techniques of Hodge theory to domains (or manifolds) with boundaries. He introduced the $\overline{\partial}$-Neumann problem, which was subsequently solved by Kohn [K1], [K2], [FK].

For simplicity, I will discuss the $\overline{\partial}$-Neumann problem on $(0,1)$ forms. Given a domain Ω in a complex manifold whose boundary $b\Omega$ is a smooth real manifold, introduce the (unbounded) operator

$$\square = \overline{\partial}^* \overline{\partial} + \overline{\partial}\,\overline{\partial}^*$$

on the space of square-integrable $(0,1)$ forms. With the right choice of domain, which involves boundary conditions, $\square$ is self-adjoint. For example, for $\sum a_j \, d\overline{z}^j$ to be in the domain of $\overline{\partial}$, the complex vector field $\sum a_j \frac{\partial}{\partial z^j}$ must be tangent to $b\Omega$. Thus, the very definition of $\square$ lies at the foundation of CR Geometry.

Let H denote the harmonic projector, that is, orthogonal projection onto the null space of $\Box$. Let N denote the $\overline{\partial}$-Neumann operator, the inverse of $\Box$ away from its null-space. The Hodge decomposition

$$\alpha = \Box N\alpha + H\alpha \tag{1}$$

then provides a method for finding the *Kohn solution* to the equation $\overline{\partial}u = \alpha$. For a solution to exist, it is necessary that $H\alpha = 0$ and $\overline{\partial}\alpha = 0$. In this case, it follows from (1) that

$$\alpha = \overline{\partial}\big(\overline{\partial}^* N\alpha\big). \tag{2}$$

The solution $u = \overline{\partial}^* N\alpha$ to the Cauchy–Riemann equation is the unique solution orthogonal (in L^2) to the holomorphic functions. Formula (2) also leads to the formula

$$P = I - \overline{\partial}^* N\overline{\partial}$$

for the Bergman projection operator and thus links the $\overline{\partial}$-Neumann problem with questions of smooth extension of biholomorphic maps. See [Fe] and [Be].

For a long time it had been believed that the Kohn solution was the best behaved solution in terms of regularity. Kohn proved that there always is a globally regular solution, but the particular solution $u = \overline{\partial}^* N\alpha$ does not always satisfy the global regularity property. The story is too complicated to discuss here. See [BS], [S], and their references.

3. Finite-type

The Kohn solution yields *local regularity* when so-called subelliptic estimates hold. Thus the particular solution α is smooth wherever u is smooth. Subelliptic estimates in the $\overline{\partial}$-Neumann problem go back to [K1] and [K2] where the so-called $\frac{1}{2}$ estimate holds in the strongly pseudoconvex case. Subelliptic estimates for a parameter ε with $0 < \varepsilon \leq \frac{1}{2}$ were discussed in [KN], where the connection with local regularity was established.

Kohn naturally asked for geometric conditions implying subelliptic estimates. He [K3] established the first such estimate for $\varepsilon < \frac{1}{2}$, in two complex dimensions, by introducing a finite-type condition using iterated commutators of complex vector fields. Later Kohn [K4] established such estimates using *subelliptic multipliers* and Catlin did so ([C1], [C2], [C3]) by constructing plurisubharmonic functions with large Hessians. In particular, for $(0,1)$ forms, Catlin proved that such estimates hold at a boundary point p if and only if there is a bound on the orders of contact of complex analytic varieties with the boundary hypersurface at p. Such a point is called a point of *finite-type*.

Long before his papers, Catlin told me that if I could prove that the set of points of finite-type is an open subset of the boundary, then he could prove subelliptic estimates. It took years, but we both succeeded. See [D1],

[D2], [D6] for information about points of finite-type. See [DK] for a survey on subelliptic estimates and references. More recently, Kohn's approach to subelliptic estimates from [K4] has led to the study of multiplier ideals in commutative algebra.

My own work followed the following ideas, illustrating my fondness for squared norms. I considered Taylor polynomials of a defining function for the boundary. I wrote such polynomials as differences $\|F\|^2 - \|G\|^2$ of squared norms of holomorphic polynomial mappings. I determined when such algebraic boundaries contained complex analytic varieties of positive dimension, reducing things to elementary commutative algebra. Then I made things quantitative.

The influence of Joe Kohn on complex analysis has been extraordinary. See [BCDS] both for its papers and for my feeble attempt to describe his work and influence. For years my principal mathematical goal was to tell Kohn (my advisor) something (anything at all) he didn't know about $\overline{\partial}$. I cannot resist telling the following story. When I was a graduate student I told him that I could prove some particular thing he cared about if I could show some other thing. He sarcastically reminded me of the guy who thought he had built a perpetual motion machine. The machine was all done except for getting the guy's thumb and forefinger to go back and forth forever. Five years later I did the other thing.

4. More recent stories involving the Cauchy–Riemann equations

Joe Kohn, Dave Catlin, Yum-Tong Siu, Jeff McNeal and others realized that hard analysis and commutative algebra were deeply linked by these ideas revolving around subellipticity. McNeal acted on that thought by organizing a great conference at Ohio State in 1999 and then by co-organizing with Mircea Mustata a summer program at the Park City Math Institute in 2008. This PCMI program intended to bring researchers in complex analysis and complex algebraic geometry closer. Perhaps it did so, but I am not sure. After a showing of the brilliant film "The π versus e debate," I suggested to Robert Bryant, then director of PCMI, that we hold an "Analysis versus Algebra debate." Bryant replied "it might lead to fisticuffs."

Everyone in the complex analysis side of the subject subject knows the books [FK] and [H] and the paper [Sh]. I wish to recommend two newer references, Bo Berndtsson's Park City lecture notes [B], and Emil Straube's book [S] written during his year at the Schrödinger Institute. Berndtsson's notes help bring complex analysis and geometry together. Straube's book synthesizes an unbelievable amount of hard analysis. These references give clear explanations of both background information and recent developments.

I clearly remember hearing Bo's first lecture at PCMI in 2008. I said "jättebra" (a Swedish way to say very good) to him afterwards. He asked me whether Americans actually said "giant good." I said something along the lines "only when they hear a lecture as good as yours."

I have known Emil Straube for many years and he has taught me many things. I am especially appreciative of his work with Boas [BS] on global regularity for $\overline{\partial}$, in part because they use in a meaningful way a differential form α I had discovered in my thesis (but I did little with it). Straube has also contributed significantly to the question of when the $\overline{\partial}$-Neumann problem is compact. See [CF], [FuS], [S]. He has mentored many thesis students and postdocs who have helped continue the subject.

Among Straube's postdocs, I wish to mention especially Siqi Fu. I first met Siqi when he was a graduate student at Washington University and I taught a one semester course there. Later he told me that he regarded me as his second advisor. I was touched. Among many other topics, Fu has studied whether one can hear the type of a point; in other words, to what extent does knowing the eigenvalues of the Kohn Laplacian $\Box$ determine the boundary geometry? He has also done considerable work on compactness estimates in the $\overline{\partial}$-Neumann problem and the Bergman kernel function, two topics of great interest to me. I adore the paper [BFS] on explicit computation of the Bergman kernel.

Xiaojun Huang was also a student in that class. In addition to discussing subelliptic estimates in detail, I mentioned proper mappings between balls, an area I had just begun studying and which is discussed in the next section. Huang, especially in work with Ji, has carried out some deep and difficult investigations about proper mappings. He is one of the top CR geometers of his generation.

Jeff McNeal and his former student Anne Herbig have long been good friends of mine. I will refrain from telling stories about Jeff, but not for lack of material. Both are true hard-core $\overline{\partial}$ people. As with Straube and Fu, McNeal has proved many results about two of my favorite topics, the Bergman kernel and compactness estimates. See, for example, [Mc1] and [Mc2].

Much of the work I have described presumes that the boundaries of domains are smooth. In some contexts one wishes to prove results with minimal boundary smoothness. For example, at a CR Geometry meeting in Serra Negra, Brazil in 2011, Loredana Lanzani delivered a series of nice lectures on the Bergman projection in L^p for strongly pseudoconvex domains whose boundaries are not smooth. See her article with Stein in this volume.

5. Proper mappings between balls and related developments

The work of Tanaka and Chern–Moser excited much of the mathematical community in the 1970s. Sid Webster became an expert in the Chern–Moser invariants and wrote a paper which lay the groundwork for much of my research. I vividly recall taking a long walk in Heidelberg with Webster after a meeting in Oberwolfach. I had always thought he was a Californian, but I discovered he was from Danville, Illinois (near Champaign-Urbana). Webster gave me a few glimpses of his tremendous differential-geometric insight. I still hope to follow up on some of my work by incorporating some of his ideas.

Using the Chern–Moser invariants, Webster [W] considered CR mappings from the unit sphere S^{2n-1} to S^{2n+1}. For $n \geq 3$ he showed that the only such smooth map, after composition with automorphisms in the source and target, was $z \to (z, 0)$. The proof failed when $n = 2$. Faran [Fa1], [Fa2] then solved this case; he showed that there are precisely four (equivalence classes of) CR mappings from S^3 to S^5:

$$\begin{aligned}
&(z_1, z_2) \to (z_1, z_2, 0),\\
&(z_1, z_2) \to \left(z_1, z_1 z_2, z_2^2\right),\\
&(z_1, z_2) \to \left(z_1^2, \sqrt{2} z_1 z_2, z_2^2\right),\\
&(z_1, z_2) \to \left(z_1^3, \sqrt{3} z_1 z_2, z_2^3\right).
\end{aligned}$$

Faran also showed that the analogue of Webster's result holds if $N < 2n-1$ and $f : S^{2n-1} \to S^{2N-1}$. These results got me going into my study of proper mappings between balls. One of my results described all polynomial examples in all cases. That result also led to a nice monotonicity result for the volumes of the images of these mappings.

The connection between proper mappings and CR mappings is simple to make. A holomorphic mapping $f : \Omega_1 \to \Omega_2$ between bounded domains is *proper* if the inverse image of each compact set in Ω_2 is compact in Ω_1. Assuming that the boundaries are smooth and that f has a continuous or smooth extension to the boundary, then the induced map of the boundaries is a CR mapping. Conversely, in many situations (including the ball of course) a CR mapping of the boundaries determines a holomorphic mapping of the domains. To find all rational CR mappings $\frac{p}{q}$ between spheres we must solve the equation $\|p(z)\|^2 = |q(z)|^2$ when $\|z\|^2 = 1$. My fascination with squared norms makes this problem irresistible.

I owe a great debt to Franc Forstneric, both for several of his results and for various discussions, at Oberwolfach, the Mittag-Leffler Institute, Wisconsin, and perhaps elsewhere. Forstneric [F] proved that proper maps (assumed sufficiently smooth at the boundary) between balls (with domain dimension at least 2) are rational. How much differentiability is required for the conclusion remains an open problem. He also noted that the last two of Faran's maps are group-invariant and he found some restrictions on the possible unitary groups for which invariant rational CR maps between spheres exist (spherical space form problem).

These results of Forsternic led me in many directions. For example, I asked what are all the proper rational mappings between balls, I formulated a conjecture about their degree bounds, Lichtblau and I solved the spherical space form problem in [DLi], and I discovered connections between CR Geometry and things such as the Szegö limit theorem, algebraic combinatorics, and representation theory. I initiated Dusty Grundmeier into aspects of algebraic combinatorics resulting from allowing the target to be a hyperquadric rather than a sphere. See [D7] and [G] for a glimpse of these research directions.

Peter Ebenfelt and I have organized two meetings at AIM on CR Complexity Theory, which is a natural outgrowth of these ideas. His work has helped bring back the differential geometry (connections, the Gauss map, etc.) that mine has missed. See in particular his paper with his student Son in this volume and its references. Some of his work with Baouendi, Rothschild, and Huang also informs these kinds of questions. See [BEH] and [BH]. Of course [BER] is the standard reference for CR mappings. Here I note also that my colleague Alex Tumanov [Tum] introduced *minimality*, giving a necessary and sufficient condition for holomorphic extendability of CR functions from a wedge. Alex has always been fond of complex analytic disks. His ideas and technique are on display in his volume in his paper with Sukhov and in the paper of his student Wong.

CR complexity theory includes rigidity results. In particular, Huang and Shanyu Ji (see the recent survey [HJ] among many papers) have proved many such results about proper mapping between balls. Their results and some of mine are at opposite ends of the spectrum and use different techniques. Their methods use the Chern–Moser ideas, with the unit sphere replaced by the Heisenberg group. My methods use simpler ideas, taking advantage of the symmetries provided by $\|z\|^2$ and my love of Hermitian squared norms.

Both points of view illuminate the subject of CR complexity. Recall the four examples of Faran and the earlier work of Webster. When the target dimension is small compared with the domain dimension, rational CR maps between spheres are quite restricted. Such results illustrate *rigidity.* When the target dimension rises enough, one can find rational CR maps doing almost anything one wants, illustrating *irrigidity.* Consider rational CR mappings from S^{2n-1} to S^{2N-1} and suppose $n \geq 2$. When $N < 2n-1$, the only examples are spherically equivalent to the map $z \to (z, 0)$. When $N = 2n - 1$, the first nontrivial maps appear. When $n = 2$, we obtain the four Faran maps. When $n \geq 3$ and $N = 2n - 1$, the only examples are equivalent to the Whitney map (the generalization of the second Faran map). The Whitney map is a kind of tensor product; a kind of tensor division also arises. When $N \geq 2n$, there are one-parameter families of inequivalent maps. These considerations led me to conjecturing sharp degree estimates for rational CR mappings between spheres. The conjecture states for $n \geq 3$ that a rational function from S^{2n-1} to S^{2N-1} is of degree at most $\frac{N-1}{n-1}$ and for $n = 2$ that it is of degree at most $2N - 3$. The bounds are best possible, as monomial examples exist for which equality occurs.

I want to mention the work of Lebl and Peters on degree estimates. Their paper in this volume finishes the story for (the nontrivial case of) monomial mappings. It is an inspired combination of real algebraic geometry and CR Geometry. I posed the degree-estimate problem (finding sharp bounds for the degree of a proper rational map between balls in terms of the domain and target dimensions) to them at a program for graduate students at MSRI in 2005. Jiri was just beginning his graduate work with Ebenfelt, and Han had just finished his thesis with Fornaess. They quickly became engaged by the problem, and several years later the three of us wrote a nice paper. In this volume, they have proved the conjecture (in all dimensions) in the monomial case. Han presented the paper to me earlier, wrapped up as a birthday present. Thanks!

Jiri and I wrote several papers on CR geometry during his three years as a Doob Postdoc here at Illinois. One of these shows that the Hermitian analogue of a famous result of Pfister about sums of squares fails [DL]. Lebl and my former student Lichtblau have also contributed to the study of proper mappings between balls by way of their code-writing skills. See [LL]. Bernhard Lamel was also my postdoc for one year; his work since has far transcended the ideas he and I discussed then. I have been spoiled by Bernhard and Jiri, who both came from Univ. California at San Diego. The Urbana-Champaign Sanitary District, to which I write monthly checks, is also abbreviated UCSD.

6. Positivity conditions

My study of proper mappings between balls has led to work on positivity conditions and to diverse questions of interest to other authors in this volume. Consider the following naive question. Let $f = \frac{p}{q} : \mathbf{C}^n \to \mathbf{C}^N$ be a rational mapping, reduced to lowest terms. Suppose that the image of the closed unit ball under f is contained in the open unit ball. Is it possible to add components, keeping the same denominator, and make the new map take the sphere to the sphere? The answer is yes, and the proof passes through my work (some joint with Catlin) on Hermitian analogues of Hilbert's 17-th problem. See [CD1], [CD2], [CD3], [D3], and [D4]. The following step is crucial. If a polynomial $r(z, \overline{z})$ is positive on the sphere, then it agrees with the squared norm $\|f(z)\|^2$ of a holomorphic polynomial there. This result extends as far as possible the famous Riesz–Fejer theorem from 1916 on nonnegative trig polynomials. The proof passed through compact operators and the Bergman projection. These results have interpretations as isometric embedding theorems for holomorphic bundles. See [D4] and [D5] for more information on these topics.

I mention here the various conversations I have had with Dror Varolin and Steve Bradlow. Both get (perhaps justifiably) frustrated with me for expressing my ideas in terms of polynomials rather than in terms of sections of bundles. One of the results I proved with Catlin amounts to an analytic version of the Kodaira embedding theorem. Steve persisted in asking me how the required number of tensor powers depended on the degree of the bundle. I kept on saying that it depended instead on the coefficients of the metric. I even gave examples where the manifold is complex projective space $\mathbf{CP}_1$, the degree of the bundle is 2, and the number of tensor powers needed could be arbitrarily large. Dror saved the day by giving a beautiful talk on these things in the right language.

Along the way I asked a related question about polynomials. Suppose a polynomial $r(z,\overline{z})$ is positive on the boundary of an algebraic strongly pseudoconvex domain. Must it agree with a squared norm of a polynomial map there? As noted above, Catlin and I proved this fact for the sphere. Also, by a result of Løw ([Lw]), a positive continuous function on the boundary of a strongly pseudoconvex domain does agree with a squared norm $\|f\|^2$ of a holomorphic map f. In this setting, f is defined on the pseudoconvex side and continuous on the boundary. If in addition the boundary is algebraic, and the given positive function is a polynomial, it seems natural to seek a polynomial solution.

The answer, provided by Mihai Putinar and Claus Scheiderer in [PS], is no. Their wonderful idea uses polarization (Segre sets) in a crucial way and thus connects this problem with the CR geometry of the boundary. Putinar and I have followed up by introducing and computing the *Hermitian complexity* of an ideal. We saw each other at an AIM meeting soon after his breakthrough with Scheiderer. We went into a side room of the Fry electronics warehouse and made faster progress than I ever remember making with anyone. The stunning thing is that Hermitian complexity became connected with my earliest work on the geometry of finite-type conditions. See [DP]. The paper in this volume by Putinar and Scheiderer studies these matters for an ellipse. See [Q] for a first result about Hermitian squares and see [TY] for an effective result when additional information is known.

I have always been interested in positivity conditions and Hermitian forms. Polarization enables one to treat z and $\overline{z}$ as independent variables. The simplest examples that informed my early thinking were the following two things. First is the very definition of a unitary map on $\mathbf{C}^n$. One could say either U preserves distances ($\|Uz\| = \|z\|$ for all z), or U preserves inner products ($\langle Uz, Uw\rangle = \langle z, w\rangle$ for all z, w), and one gets the same set of maps. Second was the diagonalization of a Hermitian matrix. These two simple pieces of complex linear algebra combine in one of my first results. Let $r(z,\overline{z})$ be a real-valued polynomial. Write

$$r(z,\overline{z}) = \left\|f(z)\right\|^2 - \left\|g(z)\right\|^2 \tag{3}$$

for holomorphic polynomial (vector-valued) mappings f and g. Then an irreducible complex analytic variety V lies in the zero set of r if and only if there is a unitary map U such that V is a subvariety of the variety defined by $f - Ug$.

When Dror Varolin came to Illinois, I described some of my results with Catlin mentioned above. I told him that, when a nonnegative bihomogeneous polynomial had zeroes on the sphere, it might not agree with a squared norm there. The simplest example is $(|z_1|^2 - |z_2|^2)^2$. Could he give necessary and sufficient conditions? Several years later, he gave the precise answer. See [V]. I believe that this kind of Hermitian geometry is worth considerable research effort. It ties together topics such as metric on bundles, the resolution of singularities, positivity conditions, and one of my favorite topics, the non-linear Cauchy–Schwarz inequality

$$\left|R(z,\overline{w})\right|^2 \leq R(z,\overline{z})R(w,\overline{w}). \tag{4}$$

Inequality (4) can be interpreted as a curvature condition on bundles. Taking logarithms of both sides leads to Calabi's notion (from [Cal]) of a diastatic function. It is also closely related to the following problem. Given a polynomial r in one or more real variables, is there an integer N such that r^N has all positive coefficients? In a fairly general situation, inequality (4) is equivalent to an anisotropic inequality comparing the length of the short diagonal of a parallelogram with its area. See [D3] and [DV].

7. Connections with sub-Riemannian geometry

The work on estimates for $\overline{\partial}$ seems at first glance a bit removed from some of the ideas of CR Geometry we have discussed. In fact, the ideas are closely related. My colleague Jeremy Tyson and I have noted in the survey paper [DT] that Riemann's name arises both in sub-Riemannian Geometry and CR Geometry, but for different reasons. We believe that excellent research opportunities exist in attempting to bring these subjects back together.

The unit sphere is biholomorphically equivalent to the Heisenberg group via a Cayley transformation. Stein and his school have used methods from harmonic analysis to prove $\overline{\partial}$ estimates. These methods are not limited to the sphere or even to strongly pseudoconvex domains. See for example [CNS]. The paper [RS] uses the theory of nilpotent Lie groups to give sharp subelliptic estimates in two dimensions. These papers and their references illustrate to some extent the possibilities for bringing the subjects of CR geometry and sub-Riemannian geometry back together. Iterated commutators of complex vector fields play a major role in these areas, yet the full story of finite-type brings in additional algebraic ideas. One can imagine that these algebraic ideas will someday bear on sub-Riemannian geometry.

Acknowledgments. I wish to thank AIM for the workshops on CR Complexity Theory in 2007 and 2010. Those workshops helped develop and unify some of the mathematics in this volume. I thank all the authors of the articles here. I specifically thank Steve Bradlow, Peter Ebenfelt, Jeremy Tyson, and Dror Varolin for their efforts in preparing this volume and Tom Nevins for a careful reading of this paper.

References

[BER] M. S. Baouendi, P. Ebenfelt and L. Rothschild, *Real submanifolds in complex space and their mappings*, Princeton Mathematical Series, vol. **47**, Princeton University Press, Princeton, NJ, 1999. MR 1668103

[BH] M. S. Baouendi and X. Huang, *Super-rigidity for holomorphic mappings between hyperquadrics with positive signature*, J. Differential Geom. **69** (2005), 379–398. MR 2169869

[BEH] M. S. Baouendi, P. Ebenfelt and X. Huang, *Holomorphic mappings between hyperquadrics with small signature difference*, Amer. J. Math. **133** (2011), 1633–1661. MR 2863372

[Be] S. Bell, *Biholomorphic mappings and the $\overline{\partial}$-problem*, Ann. of Math. (2) **114** (1981), 103–113. MR 0625347

[B] B. Berndtsson, *An introduction to things $\overline{\partial}$*, Analysis and algebraic geometry, IAS/Park City Mathematics Series, vol. **17**, Amer. Math. Soc., Providence, RI, 2010, pp. 7–76. MR 2743815

[BCDS] T. Bloom, D. Catlin, J. D'Angelo and Y.-T. Siu (eds.), *Modern methods in complex analysis*, Ann. of Math. Stud., Princeton Univ. Press, Princeton, NJ, 1996. MR 1369129

[BS] H. Boas and E. Straube, *Global regularity of the $\overline{\partial}$-Neumann problem: A survey of the L^2-Sobolev theory*, Several complex variables (M. Schneider and Y. T. Siu, eds.), Math. Sci. Res. Inst. Publ., vol. **37**, Cambridge Univ. Press, Cambridge, 1999, pp. 79–111. MR 1748601

[BFS] H. Boas, S. Fu and E. Straube, T*he Bergman kernel function: Explicit formulas and zeroes*, Proc. Amer. Math. Soc. **127** (1999), 805–811. MR 1469401

[Cal] E. Calabi, *Isometric imbedding of complex manifolds*, Ann. of Math. (2) **58** (1953), 1–23. MR 0057000

[C1] D. Catlin, *Necessary conditions for subellipticity of the $\overline{\partial}$-Neumann problem*, Ann. of Math. (2) **117** (1983), 147–171. MR 0683805

[C2] D. Catlin, *Boundary invariants of pseudoconvex domains*, Ann. of Math. (2) **120** (1984), 529–586. MR 0769163

[C3] D. Catlin, *Subelliptic estimates for the $\overline{\partial}$-Neumann problem on pseudoconvex domains*, Ann. of Math. (2) **126** (1987), 131–191. MR 0898054

[CD1] D. Catlin and J. D'Angelo, *A stabilization theorem for Hermitian forms and applications to holomorphic mappings*, Math. Res. Lett. **3** (1996), 149–166. MR 1386836

[CD2] D. Catlin and J. D'Angelo, *Positivity conditions for bihomogeneous polynomials*, Math. Res. Lett. **4** (1997), 1–13. MR 1470426

[CD3] D. Catlin and J. D'Angelo, *An isometric imbedding theorem for holomorphic bundles*, Math. Res. Lett. **6** (1999), 1–18. MR 1682713

[CNS] D. C. Chang, A. Nagel and E. M. Stein, *Estimates for the $\overline{\partial}$-Neumann problem in pseudoconvex domains of finite type in* $\mathbf{C}^2$, Acta Math. **169** (1992), 153–228. MR 1194003

[CF] M. Christ and S. Fu, *Compactness in the $\overline{\partial}$-Neumann problem, magnetic Schrödinger operators, and the Aharonov–Bohm effect*, Adv. Math. **197** (2005), 1–40. MR 2166176

[D1] J. P. D'Angelo, *Several complex variables and the geometry of real hypersurfaces*, CRC Press, Boca Raton, FL, 1992.

[D2] J. P. D'Angelo, *Real hypersurfaces, orders of contact, and applications*, Ann. of Math. (2) **115** (1982), 615–637. MR 0657241

[D3] J. P. D'Angelo, *Inequalities from complex analysis*, Carus Mathematical Monograph, vol. **28**, Mathematics Association of America, Washington, DC, 2002. MR 1899123

[D4] J. P. D'Angelo, *Hermitian analogues of Hilbert's 17-th problem*, Adv. Math. **226** (2011), 4607–4637. MR 2770459

[D5] J. P. D'Angelo, *Proper holomorphic mappings, positivity conditions, and isometric imbedding*, J. Korean Math. Soc. **40** (2003), 341–371. MR 1973906

[D6] J. P. D'Angelo, *A gentle introduction to points of finite type on real hypersurfaces*, Contemp. Math. **332** (2003), 19–36. MR 2016089

[D7] J. P. D'Angelo, *Invariant CR Mappings*, Complex analysis: Several complex variables and connections with PDEs and geometry, Trends in Math., Birkhäuser, Basel, 2008, pp. 95–107. MR 2885110

[DK] J. D'Angelo and J. J. Kohn, *Subelliptic estimates and finite type*, Several complex variables (M. Schneider and Y. T. Siu, eds.), Math. Sci. Res. Inst. Publ., vol. **37**, Cambridge Univ. Press, Cambridge, 1999, pp. 199–232. MR 1748604

[DL] J. D'Angelo and J. Lebl, *Pfister's theorem fails in the Hermitian case*, Proc. Amer. Math. Soc. **140** (2012), 1151–1157. MR 2869101

[DLi] J. D'Angelo and D. Lichtblau, *Spherical space forms, CR mappings, and proper maps between balls*, J. Geom. Anal. **2** (1992), 391–415. MR 1184706

[DP] J. D'Angelo and M. Putinar, *Hermitian complexity of real polynomial ideals*, Int. J. Math. **23** (2012), 1250026. MR 2925474

[DT] J. D'Angelo and J. Tyson, *An invitation to Cauchy–Riemann and sub-Riemannian geometry*, Notices Amer. Math. Soc. **57** (2010), 208–219. MR 2604488

[DV] J. D'Angelo and D. Varolin, *Positivity conditions for Hermitian symmetric functions*, Asian J. Math. **8** (2004), 215–231. MR 2129535

[Fa1] J. Faran, *Maps from the two-ball to the three-ball*, Invent. Math. **68** (1982), 441–475. MR 0669425

[Fa2] J. Faran, *The linearity of proper holomorphic maps between balls in the low codimension case*, J. Differential Geom. **24** (1986), 15–17. MR 0857373

[Fe] C. Fefferman, *The Bergman kernel and biholomorphic mappings of pseudoconvex domains*, Invent. Math. **26** (1974), 1–65. MR 0350069

[FK] G. B. Folland and J. J. Kohn, *The Neumann problem for the Cauchy–Riemann complex*, Annals of Math. Studies, vol. **75**, Princeton University Press, Princeton, 1972. MR 0461588

[F] F. Forstneric, *Extending proper holomorphic mappings of positive codimension*, Invent. Math. **95** (1989), 31–61. MR 0969413

[FuS] S. Fu and E. Straube, *Compactness in the $\overline{\partial}$-Neumann problem*, Complex Analysis and Geometry (J. D. McNeal, ed.), Ohio State Univ. Math Res. Inst. Publ., vol. **9**, de Gruyter, Berlin, 2001, pp. 141–160. MR 1912737

[G] D. Grundmeier, *Signature pairs for group-invariant Hermitian polynomials*, Internat. J. Math. **22** (2011), 311–343. MR 2782691

[H] L. Hörmander, *An introduction to complex analysis in several variables*, Van Nostrand, Princeton, NJ, 1966. MR 0203075

[HJ] X. Huang and S. Ji, *On some rigidity problems in Cauchy–Riemann analysis*, Proceedings of the international conference on complex geometry and related fields, AMS/IP Stud. Adv. Math., vol. **39**, Amer. Math. Soc., Providence, RI, 2007, pp. 89–107. MR 2338621

[K1] J. J. Kohn, *Harmonic integrals on strongly pseudo-convex manifolds, I*, Ann. of Math. (2) **78** (1963), 112–148. MR 0153030

[K2] J. J. Kohn, *Harmonic integrals on strongly pseudo-convex manifolds, II*, Ann. of Math. (2) **79** (1964), 450–472. MR 0208200

[K3] J. J. Kohn, *Boundary behavior of $\bar{\partial}$ on weakly pseudo-convex manifolds of dimension two*, J. Differential Geom. **6** (1972), 523–542. MR 0322365

[K4] J. J. Kohn, *Subellipticity of the $\bar{\partial}$-Neumann problem on pseudo-convex domains: Sufficient conditions*, Acta Math. **142** (1979), 79–122. MR 0512213

[KN] J. J. Kohn and L. Nirenberg, *Non-coercive boundary value problems*, Comm. Pure Appl. Math. **18** (1965), 443–492. MR 0181815

[LL] J. Lebl and D. Lichtblau, *Uniqueness of certain polynomials constant on a line*, Linear Algebra Appl. **433** (2010), 824–837. MR 2654111

[Lw] E. Løw, *Embeddings and proper holomorphic maps of strictly pseudoconvex domains into polydiscs and balls*, Math. Z. **190** (1985), 401–410. MR 0806898

[Mc1] J. McNeal, *Boundary behavior of the Bergman kernel function in* $\mathbf{C^2}$, Duke Math. J. **58** (1989), 499–512. MR 1016431

[Mc2] J. McNeal, *A sufficient condition for compactness of the $\overline{\partial}$-Neumann operator*, J. Funct. Anal. **195** (2002), 190–205. MR 1934357

[PS] M. Putinar and C. Scheiderer, *Sums of Hermitian squares on pseudoconvex boundaries*, Math. Res. Lett. **17** (2010), 1047–1053. MR 2729629

[Q] D. Quillen, *On the representation of Hermitian forms as sums of squares*, Invent. Math. **5** (1968), 237–242. MR 0233770

[RS] L. Rothschild and E. Stein, *Hypoelliptic differential operators and nilpotent groups*, Acta Math. **137** (1976), 247–320. MR 0436223

[Sh] M.C. Shaw, *L^2 estimates and existence theorems for the Cauchy–Riemann complex*, Invent. Math. **82** (1985), 133–150. MR 0808113

[S] E. Straube, *Lectures on the L^2-Sobolev theory of the $\overline{\partial}$-Neumann problem*, E.S.I. Lectures in Mathematics and Physics, European Mathematical Society, Zurich, Switzerland, 2011.

[Tum] A. Tumanov, *Extending CR-functions on manifolds of finite type to a wedge*, Mat. Sb. **136** (1988), 128–139. MR 0945904

[TY] W. To and S. Yeung, *Effective isometric embeddings for certain Hermitian holomorphic line bundles*, J. London Math. Soc. (2) **73** (2006), 607–624. MR 2241969

[V] D. Varolin, *Geometry of Hermitian algebraic functions: Quotients of squared norms*, Amer. J. Math. **130** (2008), 291–315. MR 2405157

[W] S. M. Webster, *On mapping an n-ball into an $(n + 1)$-ball in complex spaces*, Pacific J. Math. **81** (1979), 267–272. MR 0543749

John P. D'Angelo, Department of Mathematics, University of Illinois, 1409 W. Green St., Urbana, IL 61801, USA

E-mail address: jpda@math.uiuc.edu