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# COMPACT ACTIONS, RETRACT THEORY AND PRIME IDEALS

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Abstract. Let $N$ be a connected, simply connected, nilpotent Lie group and let $K$ be a compact subgroup of the automorphism group of $N$. We study the density of Schwartz functions in the kernels of $K$-orbits and characterize $K$-prime ideals. For this purpose a retract theory for $K$-actions has to be established.

## Introduction

Among the classical questions in harmonic analysis, we find the Fourier inversion problem, the Wiener property (see below), the characterization of maximal ideals and prime ideals in a function algebra. These questions were first solved for locally compact Abelian groups. For a non-Abelian group $G$, there exist infinite dimensional unitary irreducible representations (except in the compact case). So the Abelian Fourier transform $\hat{f}(\chi) \in \mathbb{C}$, $\chi \in \hat{G}$, has to be replaced by the family of operators $\pi(f)$, $\pi \in \hat{G}$, which act on infinite dimensional spaces. This makes the Fourier analysis questions much more difficult.

First generalizations were made for connected, simply connected, nilpotent Lie groups. For such groups $N$, the Wiener property, which says that any proper closed ideal is contained in the kernel of a unitary irreducible representation, was proved by Leptin ([9], 1976) for the algebra $L^1(N)$ and by Ludwig ([14], 1987) for the Schwartz algebra $\mathcal{S}(N)$. It is used, among others, in the proofs of the results presented in this paper.

In order to introduce the results of this paper, let us make some comments on the unitary irreducible representations and the retract problem for a connected, simply connected, nilpotent Lie group $N = \exp \mathfrak{n}$. Given $l \in \mathfrak{n}^*$,

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let $\mathfrak{p}(l)$ be a polarization for $l$ in $\mathfrak{n}$. On the subgroup $P(l) = \exp \mathfrak{p}(l)$, one defines the character $\chi_l$ by $\chi_l(x) := e^{-i\langle l, \log x\rangle}$, $x \in P(l)$. The induced representation $\pi_l := \mathrm{ind}_{P(l)}^N \chi_l$ is then irreducible, and all the unitary irreducible representations may be realized in this way. The representation $\pi_l$ acts on the function space $\mathfrak{H}_l := L^2(N/P(l), \chi_l)$ by left translation (see Section 1.2 and [4] for more details). For any Schwartz function $f \in \mathcal{S}(N)$, the operator $\pi_l(f) := \int_N f(x)\pi_l(x)\, dx$ is a kernel operator, that is, it is of the form $(\pi_l(f)\xi)(x) = \int_{N/P(l)} F(x,y)\xi(y)\, d\dot{y}$, $\xi \in \mathfrak{H}_l$, for some kernel function $F$. The pointwise Fourier inversion problem (or construction of a retract at one fixed point of the dual) is the following: Given a suitable kernel function $F$, show that there is a Schwartz function $f$ such that $\pi_l(f)$ has $F$ as an operator kernel, for fixed $l$. This was proved by Howe ([7], 1977). Of course, the kernel function $F$ depends on $l$. Hence, we may consider $F$ as a function of $l$, where $l$ runs through an appropriate submanifold of $\mathfrak{n}^*$, and we may ask whether the same retract function $f$ is valid for all $l$. The existence of retracts is a very useful tool. A big part of this paper will hence be devoted to retract problems. They are used to study questions in ideal theory of the group algebras.

The maximal ideals, the prime ideals and the kernels of the unitary irreducible representations coincide in the Abelian case. On the other hand, in the non-Abelian case, this result is not necessarily true. It remains correct for connected, simply connected, nilpotent Lie groups. Let us recall that an ideal $I$ is said to be prime, if for any two ideals $I_1$ and $I_2$ such that $I_1 * I_2 \subset I, I_1 \subset I$ or $I_2 \subset I$. For connected, simply connected, nilpotent Lie groups, J. Ludwig showed ([13], 1983) that the closed prime ideals of $L^1(N)$ coincide with the kernels of the irreducible unitary representations.

A next step consisted in introducing an exponential action on the nilpotent Lie groups. In this situation, a similar result for the characterization of the prime ideals in the set of all the ideals which are invariant under this exponential action, was established in ([15], 1998).

On the other hand, in the 80s, Poguntke studied the action of an Abelian compact group $K$ on a nilpotent Lie group ([20], [21]). He was interested in the density of Schwartz functions in the kernels of the $K$-orbits. Poguntke's result implies the characterization of the $K$-prime ideals if the compact group $K$ is Abelian. The problem for $K$ non-Abelian remained open.

In this paper, we determine the $K$-prime ideals of $\mathcal{S}(N)$ and $L^1(N)$ (with some restrictions in the case of $L^1(N)$, see below), when $K$ is a non-Abelian compact Lie subgroup of the automorphism group of $N$. After some generalities, we start by defining the space of the kernel functions. Then we construct a retract on different layers of $\mathfrak{n}^*$ (in the sense of Ludwig–Zahir, [18]) and on submanifolds contained in sections of these layers. In the fifth section, we study the orbits under the compact action and we then prove the existence of a retract for different realizations of the representations subjected to the compact action. Using these tools, we show the density of the Schwartz functions in the kernels of $K$-orbits and we characterize the corresponding $K$-prime ideals. The retract theory, which is the heart of this paper, produces a global retract theorem (Theorem 12), which is important for its own sake and which may be considered as a kind of Fourier's inversion theorem for a compact action.

The previous results are established for the layer of generic elements in the sense of Ludwig–Zahir [18], for the layer corresponding to the characters of the group and for all the layers in the case of the free nilpotent Lie groups of step 2 on 2, 3 or 4 generators. For arbitrary intermediate layers and general nilpotent Lie groups, the problem remains open.

## 1. Generalities

**1.1. Coordinates of the second kind and Haar measure.** Let $N=\exp\mathfrak{n}$ be a connected, simply connected, nilpotent Lie group. Let $\{X_1,\dots,X_n\}$ be a Jordan–Hölder basis of $\mathfrak{n}$, that is, a basis satisfying $[X_i,X_j]\in\langle X_{r+1},\dots,X_n\rangle$ where $r=\max\{i,j\}$ and where $\langle X_{r+1},\dots,X_n\rangle$ denotes the vector subspace generated by $X_{r+1},\dots,X_n$.

The elements of $N$ may be written uniquely in the form

$$x=\exp(x_1X_1)\exp(x_2X_2)\cdots\exp(x_nX_n)$$

and $x_1,\dots,x_n\in\mathbb{R}$ are called *coordinates of the second kind* of the element $x$ in the given basis. Unless otherwise stated, all our computations are done in coordinates of the second kind.

In a connected, simply connected, nilpotent Lie group, left and right *Haar measure* coincide, up to a constant, with the Lebesgue measure associated to the coordinates of the second kind.

If $\mathfrak{h}$ is a subalgebra of $\mathfrak{n}$, $\{X_1,\dots,X_r\}$ is called a *Malcev basis* of $\mathfrak{n}$ with respect to $\mathfrak{h}$ if $\mathfrak{n}=\langle X_1,\dots,X_r\rangle\oplus\mathfrak{h}$ and if, for every index $j\in\{1,\dots,r\}$, $\langle X_j,X_{j+1},\dots,X_r\rangle\oplus\mathfrak{h}$ is a subalgebra of $\mathfrak{n}$. In that case, the Lebesgue measure on $\mathbb{R}^r$ gives rise to a left invariant measure on $N/H$, with $H=\exp\mathfrak{h}$, via the formula

$$\int_{G/H} f(\dot{x})\,d\dot{x}
:=\int_{\mathbb{R}^r} f\big(\exp(x_1X_1)\exp(x_2X_2)\cdots\exp(x_rX_r)\cdot H\big)\,dx_1\,dx_2\cdots dx_r$$

for all continuous functions with compact support $f$ on $N/H$.

**1.2. Irreducible representations.** Given $l\in\mathfrak{n}^*$, let $\mathfrak{p}(l)$ be a polarization for $l$ in $\mathfrak{n}$, that is, a maximal subalgebra $\mathfrak{p}$ such that $\langle l,[\mathfrak{p},\mathfrak{p}]\rangle\equiv 0$. Let $P(l)=\exp\mathfrak{p}(l)$ be the corresponding subgroup. On $P(l)$, the character $\chi_l$ is defined by $\chi_l(x)=e^{-i\langle l,\log x\rangle}$ for all $x\in P(l)$. The *induced representation*

$\pi_l := \operatorname{ind}_{P(l)}^N \chi_l$ is defined in the following way: Let $C_c(N/P(l), \chi_l)$ be the set of all continuous functions $\xi$ from $N$ to $\mathbb{C}$, satisfying $\xi(np) = \overline{\chi_l(p)}\xi(n)$ for all $n \in N$ and all $p \in P(l)$ (*covariance condition*), with compact support mod $P(l)$. The representation space $\mathfrak{H}_l := L^2(N/P(l), \chi_l)$ is then the completion of $C_c(N/P(l), \chi_l)$ for the norm

$$\|\xi\|_2 = \left(\int_{N/P(l)} \left|\xi(\dot{x})\right|^2 d\dot{x}\right)^{\frac{1}{2}}.$$

The representation $\pi_l$ is defined on $\mathfrak{H}_l$ by

$$\left(\pi_l(x)\xi\right)(y) := \xi\left(x^{-1}y\right), \quad x, y \in N.$$

If $\mathfrak{p}(l)$ is a polarization for $l$ in $\mathfrak{n}$, then $\pi_l = \operatorname{ind}_{P(l)}^N \chi_l$ is a unitary, topologically irreducible representation of $N$. Different polarizations for a same $l$ give rise to equivalent representations; $\pi_l$ and $\pi_{l'}$ are equivalent if and only if $l$ and $l'$ belong to the same coadjoint orbit. Moreover, every unitary topologically irreducible representation is of that form, up to equivalence.

The representation $\pi_l$ gives rise to a representation of the group algebra $L^1(N)$, also denoted by $\pi_l$, defined by

$$\pi_l(f) = \int_N f(x)\pi_l(x)\, dx.$$

For any Schwartz function $f \in \mathcal{S}(N)$, the operator $\pi_l(f)$ is a kernel operator, as explained in the introduction.

**1.3. Fourier transform.** In subsequent proofs, we will use the usual Fourier transform on $\mathbb{R}$, according to the following definition:

$$\hat{f}(y) := \int_{\mathbb{R}} f(x)e^{-ixy}\, dx, \quad f \in L^1(\mathbb{R}).$$

## 2. Kernel functions

Let $N = \exp \mathfrak{n}$ be a connected, simply connected, nilpotent Lie group and $K$ be a compact subgroup of $\operatorname{Aut}(N)$. Let $\mathcal{W}$ be a smooth manifold, which will be specified later. For example, $\mathcal{W}$ may be $\mathfrak{n}^*_{\text{gen}}$ or $\mathfrak{n}^*_{\text{gen}} \cap \Sigma$, where $\mathfrak{n}^*_{\text{gen}}$ denotes the set of generic linear forms in the sense of Ludwig–Zahir [18] and $\Sigma$ denotes a Pukanszky section. But we can also choose $\mathcal{W} = K, \tilde{K}, K \cdot l_0, \tilde{K} \cdot l_0$ where $\tilde{K}$ is an open subset of $K$. Moreover, we may take $\mathcal{W} = K/K_{l_0}, \tilde{K}/K_{l_0}$ or a smooth section of $K/K_{l_0}$, resp. $\tilde{K}/K_{l_0}$ where $K_{l_0}$ denotes the stabilizer of $l_0$. We begin by giving a certain number of definitions.

### 2.1. Generalized Schwartz property.

Definition 1. Let $\mathcal{W}$ be a manifold. We say that a $C^\infty$-function $F' : \mathcal{W} \times \mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathbb{C}$ satisfies the generalized Schwartz property (GS-property) if and only if for every chart $(U, \varphi)$ of $\mathcal{W}$, for every compact subset $C_U \subset U$ and for arbitrary $A, B, C, D, E \in \mathbb{N}$, we have

$$\begin{aligned}
&\left\|F'\right\|^{C_U}_{A,B,C,D,E} \\
&\quad := \sup_{w\in\varphi(C_U);x,y\in\mathbb{R}^d} \ \sup_{|a|\le A,|b|\le B,|c|\le C,|\alpha|\le D,|\beta|\le E} \\
&\qquad \left| x^\alpha y^\beta \frac{\partial^a}{\partial w^a}\frac{\partial^b}{\partial x^b}\frac{\partial^c}{\partial y^c} F' \circ \left(\varphi^{-1} \otimes \mathrm{id} \otimes \mathrm{id}\right)(w, x_1, \dots, x_d, y_1, \dots, y_d)\right| \\
&\quad < \infty.
\end{aligned}$$

Similarly for functions $\xi : \mathcal{W} \times \mathbb{R}^d \to \mathbb{C}$.

**2.2. Kernel function spaces.** Let $\mathcal{W}$ be a manifold as before and

$$\begin{aligned} l : \mathcal{W} &\longrightarrow \mathfrak{n}^*, \\ w &\longmapsto l(w) \end{aligned}$$

be a smooth map. For example, we may choose $l(w) = w$ if $\mathcal{W} \subset \mathfrak{n}^*$ or $l(k) = k \cdot l_0$, for some fixed $l_0 \in \mathfrak{n}^*$, if $\mathcal{W} = K, \tilde{K}, \dots$.

Definition 2. Given the manifold $\mathcal{W}$ and the map $l : \mathcal{W} \longrightarrow \mathfrak{n}^*$, we define a smooth family of polarizations $(\mathfrak{p}(w))_{w\in\mathcal{W}} = (\mathfrak{p}(l(w)))_{w\in\mathcal{W}}$ in the following way:

- $\mathfrak{p}(l(w))$ is a polarization of $l(w)$ in $\mathfrak{n}$ for all $w$.
- All the polarizations $\mathfrak{p}(l(w)), w \in \mathcal{W}$, have the same fixed dimension denoted by $r$. This condition restricts the choice of $\mathcal{W}$.
- There exists $\{X_1(w), \dots, X_r(w)\}_{w\in\mathcal{W}}$ a smooth family of Malcev bases of $(\mathfrak{p}(l(w)))_w$.

Definition 3. A family of $d$ vectors $\{X_1(w), \dots, X_d(w)\}$ is said to be a smooth Malcev basis of $\mathfrak{n}$ relative to $\mathfrak{p}(l(w))$ if:

- The map $w \longmapsto X_j(w)$ is smooth for all $j$.
- $\mathfrak{n} = \bigoplus_{j=1}^d \mathbb{R}X_j(w) \oplus \mathfrak{p}(l(w))$ for all $w$.
- The space $\mathfrak{p}_j(w) := \bigoplus_{i=j}^d \mathbb{R}X_i(w) \oplus \mathfrak{p}(l(w))$ is a subalgebra of $\mathfrak{n}$ for all $j$ and for all $w$.

Given a smooth family of polarizations $(\mathfrak{p}(l(w)))_w$, we will consider the corresponding family of induced representations $(\pi_w)_{w\in\mathcal{W}}$, where

$$\pi_w := \mathrm{ind}^N_{P(l(w))}\, \chi_{l(w)},$$

with $P(l(w)) = \exp \mathfrak{p}(l(w))$. Then, for any function $f \in \mathcal{S}(N)$ and any fixed $w \in \mathcal{W}$, the operator $\pi_w(f)$ has an operator kernel $F(w; \cdot, \cdot)$ which is a function of $w$. The retract problem consists in showing, that for suitable function spaces, called kernel function spaces, there exists an $f \in \mathcal{S}(N)$ for which $\pi_w(f)$ admits as a kernel an arbitrarily chosen element of the kernel function space. These kernel function spaces are defined as follows.

DEFINITION 4. Let $\mathcal{W}$ be a manifold, $l : \mathcal{W} \to \mathfrak{n}^*$ a smooth map and $(\mathfrak{p}(l(w)))_{w\in\mathcal{W}}$ an associated smooth family of polarizations. For every compact subset $C_0$ contained in $\mathcal{W}$, we write $\mathfrak{N}_{\mathcal{W}}^{C_0}$ for the space of all complex valued smooth functions $F$ on $\mathcal{W} \times N \times N$ such that:

- The support of $F$ in $w$ is contained in $C_0$.
- $F$ satisfies the covariance condition:

$$F(w; xp, yq) = \overline{\chi_{l(w)}(p)}\chi_{l(w)}(q)F(w; x, y) \quad \forall p, q \in P\big(l(w)\big) = \exp \mathfrak{p}\big(l(w)\big).$$

- Let $\{X_1(w), \dots, X_d(w)\}$ be a smooth Malcev basis of $\mathfrak{n}$ relative to $\mathfrak{p}(l(w))$. Then, the function

$$\begin{aligned} &F'(w; x_1, \dots, x_d, y_1, \dots, y_d) \\ &\quad := F\big(w; \exp x_1X_1(w) \cdots \exp x_dX_d(w); \exp y_1X_1(w) \cdots \exp y_dX_d(w)\big) \end{aligned}$$

  verifies the GS-property.
- If $l(w)$ and $l(w')$ are in the same co-adjoint orbit, then the corresponding induced representations are equivalent. It is hence necessary to introduce a compatibility condition on the kernel function $F$ to take into account this equivalence. This condition is specific to every single choice of the manifold $\mathcal{W}$ and of the family of polarizations.

The space $\mathfrak{N}_{\mathcal{W}}^{C_0}$ will be called a space of kernel functions.

REMARK 1. The suitable compatibility condition depends on the choice of the polarizations, the definition of representations, etc. For example, if $l(w') = \mathrm{Ad}^*(m)l(w), m \in N$, and if $\mathfrak{p}(w) = \mathfrak{p}(l(w))$ is a smooth family of Vergne polarizations, the compatibility condition for kernel functions will be:

$$\begin{aligned} F\big(w'; x, y\big) &= F\big(l(w'); x, y\big) \\ &= F\big(l(w); x \cdot m, y \cdot m\big) \\ &= F(w; x \cdot m, y \cdot m) \quad \forall x, y \in N. \end{aligned}$$

In particular if $l(w) = l(w')$, then we may take $m = e$ and we have to require that

$$F\big(w'; x, y\big) = F(w; x, y) \quad \forall x, y \in N.$$

REMARK 2. We equip the space $\mathfrak{N}_{\mathcal{W}}^{C_0}$ with the topology of the semi-norms $\| \cdot \|_{A,B,C,D,E}^{C_0}$ defined in the Definition 1.

## 3. Retract theorems on the different layers

We start this section by introducing the retract problem in a few words.

It is well known that for every Schwartz function $f$ on the nilpotent Lie group $N$, the operator $\pi_l(f)$, $l \in \mathfrak{n}^*$ fixed, is completely characterized by its operator kernel

$$F(x,y) = \int_P f\left(xpy^{-1}\right)\chi_l(p)\,dp,$$

where $P = \exp \mathfrak{p}$ is a polarization of $l$, $\chi_l(p) = e^{-i\langle l, \log p\rangle}$ denotes the character of $P$ associated to $l$ and $\pi_l = \mathrm{ind}_P^N \chi_l$. The retract question is the converse problem, which means: Given a function $F$ on $N \times N$ satisfying certain hypotheses (covariance condition, Schwartz function), does there exist $f \in \mathcal{S}(N)$ such that $\pi_l(f)$ have $F(\cdot,\cdot)$ as a kernel function?

This problem has been solved by Howe [7] in 1977 for a fixed $l$ in $\mathfrak{n}^*$ and has been generalized to exponential solvable Lie groups by Ludwig [12] and Andele [1]. But these results deal only with one fixed chosen $l \in \mathfrak{n}^*$ and can therefore not be qualified as a Fourier inversion theorem. So now the question is whether these results are still true if $l$ runs through an arbitrary layer of $\mathfrak{n}^*$?

**3.1. Retract theorem on the generic layer.** For the construction of retracts, one has to stick to a precise layer of $\mathfrak{n}^*$. Such layerings of $\mathfrak{n}^*$ have first been introduced in [5] and [19]. The layering used in this paper is the one of [18]. So let $\mathfrak{n}^*_{\mathrm{gen}}$ denote the set of generic linear forms in the sense of Ludwig–Zahir [18] for a fixed Jordan–Hölder basis. Then $\mathfrak{n}^*_{\mathrm{gen}}$ is a dense open subset of $\mathfrak{n}^*$. Moreover $\mathfrak{n}^*_{\mathrm{gen}} \subset \mathfrak{n}^*_{\mathrm{puk}}$, where $\mathfrak{n}^*_{\mathrm{puk}}$ denotes the set of generic linear forms in the sense of Pukanszky [4] and $\mathfrak{n}^*_{\mathrm{gen}}$ is $N$-invariant, by [18].

Let $\Sigma$ be a corresponding Pukanszky section and $\mathfrak{p}(l)$ be the Vergne polarization for all $l \in \mathfrak{n}^*_{\mathrm{gen}} \cap \Sigma$. In this case, we consider $\mathcal{W} = \mathfrak{n}^*_{\mathrm{gen}} \cap \Sigma$ as a submanifold of $\mathfrak{n}^*$ and $\mathfrak{N}^{C_0}_{\mathfrak{n}^*_{\mathrm{gen}} \cap \Sigma}$ as the space of kernel functions defined as in Section 2.

Remark 3. As every co-adjoint orbit cuts $\Sigma$ in at most one point, there is no need for a compatibility condition.

Let $\pi_l := \mathrm{ind}_{P(l)}^N \chi_l$ be the induced representation of $N$. Then the retract problem for $\pi_l$ has been solved in the generic case [17] by the following Fourier inversion theorem.

Theorem 1. *For every $F \in \mathfrak{N}^{C_0}_{\mathfrak{n}^*_{\mathrm{gen}} \cap \Sigma}$, there exists a unique Schwartz function $f \in \mathcal{S}(N)$ such that $\pi_l(f)$ has $F(l,\cdot,\cdot)$ as an operator kernel for every $l \in \mathfrak{n}^*_{\mathrm{gen}} \cap \Sigma$ and $\pi_l(f) = 0$ if $l \notin \mathfrak{n}^*_{\mathrm{gen}}$.*

*The map $F \longmapsto f$ is continuous with respect to the appropriate function space topologies.*

**3.2. Retract theorem on the character layer.** Let

$$\mathcal{C} = \left\{ l \in \mathfrak{n}^* | \left\langle l, [\mathfrak{n}, \mathfrak{n}] \right\rangle = 0 \right\}$$

be the layer of $\mathfrak{n}^*$ corresponding to the characters of $N$. It is a vector subspace of $\mathfrak{n}^*$ and hence a submanifold of $\mathfrak{n}^*$.

The unitary irreducible representations of $N$ defined by the elements of $\mathcal{C}$ are the characters of $N$:

$$\begin{aligned} \chi_l : N &\longrightarrow \mathbb{C}, \\ x &\longmapsto \chi_l(x) = e^{-i\langle l, \log x \rangle}, \end{aligned}$$

where $l \in \mathcal{C}$.

For every compact subset $C_0$ of $\mathcal{C}$, we define

$$\mathfrak{N}_{\text{char}}^{C_0} := \{ F : \mathcal{C} \longrightarrow \mathbb{C} | F \text{ smooth and } \operatorname{supp} F \subset C_0 \}.$$

We equip $\mathfrak{N}_{\text{char}}^{C_0}$ with the topology of the semi-norms

$$\|F\|_A^{C_0} := \sup_{l \in C_0} \sup_{|a| \leq A} \left| \frac{\partial^a}{\partial l^a} F(l) \right| < \infty \quad \forall A \in \mathbb{N}.$$

REMARK 4. We do not need a compatibility condition in the definition of $\mathfrak{N}_{\text{char}}^{C_0}$ because the orbits in this case are reduced to single points.

THEOREM 2. *For every $F \in \mathfrak{N}_{\text{char}}^{C_0}$, there exists a Schwartz function $f$ such that $\widehat{f \circ \exp}(l) = \chi_l(f) = F(l)$ for all $l \in \mathcal{C}$ and $\chi_l(f) = \widehat{f \circ \exp}(l) = 0$ if $l \in \mathcal{C} \backslash C_0$.*

*There exists an algorithm to choose $f$ such that the map*

$$\begin{aligned} \mathfrak{N}_{\text{char}}^{C_0} &\longrightarrow \mathcal{S}(N), \\ F &\longmapsto R(F) := f \end{aligned}$$

*is continuous.*

*Proof.* Let $\{Y_1, \ldots, Y_n\}$ be any Jordan–Hölder basis such that

$$\mathfrak{n} = \langle Y_1, \ldots, Y_d \rangle \oplus \underbrace{\langle Y_{d+1}, \ldots, Y_n \rangle}_{\in [\mathfrak{n}, \mathfrak{n}]},$$

where $\langle Y_i, Y_{i+1}, \ldots, Y_j \rangle, i < j$, denotes the vector subspace generated by $Y_i$, $Y_{i+1}, \ldots, Y_j$. Then $F$ may be identified with a function on $\langle Y_1^*, \ldots, Y_d^* \rangle$ which we call again $F$.

Let $\varphi \in \mathcal{S}(\mathbb{R}Y_{d+1}^* \oplus \cdots \oplus \mathbb{R}Y_n^*)$ such that $\varphi(0) = 1$ and let $\tilde{F} := F \otimes \varphi$. Then $\tilde{F} \in \mathcal{S}(\mathfrak{n}^*) \equiv \mathcal{S}(\mathbb{R}^n)$ and there exists $f \in \mathcal{S}(N)$ such that $f \circ \exp \in \mathcal{S}(\mathfrak{n})$ and $\widehat{f \circ \exp} = \tilde{F} \in \mathcal{S}(\mathfrak{n}^*)$. It is sufficient to take $f \circ \exp = \mathcal{F}^{-1}(\tilde{F})$ where $\mathcal{F}^{-1}$ denotes the classical inverse Fourier transform.

If $l \in \mathcal{C}$, then $\chi_l \in \widehat{N}$ and

$$\chi_l(f \circ \exp) = \widehat{f \circ \exp}(l) = \tilde{F}(l) = \varphi(0) F(l) = F(l). \qquad \square$$

REMARK 5. The constructed retract is not unique since it depends on the choice of the function $\varphi$ and, for a fixed $\varphi$, this construction is continuous.

**3.3. Retract theorem on an intermediate layer.** In this section, we shall give a retract theorem for the free two-step nilpotent Lie group on 4 generators $F_{4,2}$.

We denote by $\mathfrak{f}_{4,2}$ the Lie algebra of $F_{4,2}$. Let $\{Y_1,\dots,Y_{10}\}$ be the Jordan–Hölder basis of $\mathfrak{f}_{4,2}$ such that

$$\begin{aligned} [Y_1,Y_2] &= Y_5, \qquad & [Y_2,Y_3] &= Y_8,\\ [Y_1,Y_3] &= Y_6, \qquad & [Y_2,Y_4] &= Y_9,\\ [Y_1,Y_4] &= Y_7, \qquad & [Y_3,Y_4] &= Y_{10}. \end{aligned}$$

Let $l=\sum_{i=1}^{10} a_iY_i^* \in \mathfrak{f}_{4,2}^*$. It is easy to compute the different layers of $\mathfrak{f}_{4,2}^*$ (generic layers in the sense of Ludwig–Zahir for any Jordan–Hölder basis obtained from the given one by a change of the order of the basis vectors $Y_1,\dots,Y_4$, character layer and intermediate layers). Following the algorithm of Ludwig–Zahir [18], one shows that the intermediate layers are given by

$$\mathcal{S}_i := \left\{ l=\sum_{i=1}^{10} a_iY_i^* \in \mathfrak{f}_{4,2}^* \,\middle|\, a_5a_{10}+a_8a_7-a_6a_9=0, a_i\neq 0 \right\} \quad \text{for } i=5,\dots,10.$$

Let us put $\mathcal{S} := \bigcup_{i=5}^{10}\mathcal{S}_i$. These intermediate layers all play the same role as is easily proven by interchanging the vectors of the Jordan–Hölder basis. Hence, it is sufficient to have a retract theorem on any one of these intermediate layers. We choose for example $\mathcal{S}_{10}$ for the proofs. The results remain valid for every $\mathcal{S}_i$ $(i=5,\dots,10)$.

One sees that for each pair of real constants $c_3$ and $c_4$,

$$\Sigma_{10} := \left\{ l=\sum_{i=1}^{10} a_iY_i^* \in \mathfrak{f}_{4,2}^* \,\middle|\, a_3=c_3; a_4=c_4 \right\}$$

is a corresponding Pukanszky section.

In this case, $\mathcal{S}_{10}\cap\Sigma_{10}$ is parameterized by $(a_1,a_2,a_6,\dots,a_{10})\in\mathbb{R}^6\times\mathbb{R}^*$ and $a_5=\frac{a_6a_9-a_7a_8}{a_{10}}$.

We consider also

$$\mathfrak{p}(l)=\mathbb{R}\left(Y_1-\frac{a_7}{a_{10}}Y_3\right)\oplus\mathbb{R}\left(Y_2-\frac{a_9}{a_{10}}Y_3\right)\oplus\mathbb{R}Y_4\oplus\cdots\oplus\mathbb{R}Y_{10}$$

the Vergne polarization obtained by the Ludwig–Zahir method [18]. Then $\{Y_3\}$ is a Malcev basis of $\mathfrak{f}_{4,2}$ relative to $\mathfrak{p}(l)$. Similarly for the other intermediate layers $\mathcal{S}_i$, $i=5,\dots,10$.

Let $C_0$ be an arbitrary compact subset contained in $\mathbb{R}^6\times\mathbb{R}^*$ and $\mathfrak{N}^{C_0}_{\mathcal{S}_{10}\cap\Sigma_{10}}$ be the corresponding space of kernel functions. Similarly for the other layers $\mathcal{S}_i$ $(5\le i\le 10)$ and the associate spaces $\mathfrak{N}^{C_0}_{\mathcal{S}_i\cap\Sigma_i}$.

REMARK 6. We do not need a compatibility condition in the definition of $\mathfrak{N}^{C_0}_{\mathcal{S}_i\cap\Sigma_i}$ since every co-adjoint orbit intersects $\Sigma_i$ in at most one point.

We denote by $\pi_l := \operatorname{ind}^{F_{4,2}}_{P(l)}\chi_l$ the unitary irreducible induced representation of $F_{4,2}$ associated to $l$ and the Vergne polarization $\mathfrak{p}(l)$. Let us compute this representation. The representation space $\mathfrak{H}_l$ may be identified with $L^2(\mathbb{R})$ by $\xi(u)\equiv\xi(\exp(uY_3))$, the function $\xi$ satisfying the covariance condition of Section 1.2. Hence,

$$
\begin{aligned}
&\big(\pi_l\big(\exp(y_1Y_1)\big)\xi\big)(u)\\
&\quad=\xi\big(\exp(-y_1Y_1)\exp(uY_3)\big)\\
&\quad=\xi\big(\exp(uY_3)\exp(-y_1Y_1-uy_1Y_6)\big)\\
&\quad=e^{-iuy_1a_6}\xi\bigg(\exp(uY_3)\exp(-y_1Y_1)\exp\bigg(y_1\bigg(Y_1-\frac{a_7}{a_{10}}Y_3\bigg)\bigg)\\
&\qquad\cdot\exp\bigg(-y_1\bigg(Y_1-\frac{a_7}{a_{10}}Y_3\bigg)\bigg)\bigg)\\
&\quad=e^{-iuy_1a_6}e^{-iy_1(a_1-\frac{c_3a_7}{a_{10}})}\\
&\qquad\cdot\xi\bigg(\exp(uY_3)\exp\bigg(-y_1\frac{a_7}{a_{10}}Y_3+\frac{1}{2}y_1^2\frac{a_7}{a_{10}}Y_6\bigg)\bigg)\\
&\quad=e^{-iy_1(ua_6+a_1-\frac{c_3a_7}{a_{10}})}e^{\frac{1}{2}iy_1^2\frac{a_7}{a_{10}}a_6}\xi\bigg(u-y_1\frac{a_7}{a_{10}}\bigg).
\end{aligned}
$$

One gets similarly:

$$
\begin{aligned}
&\big(\pi_l\big(\exp(y_2Y_2)\big)\xi\big)(u)=e^{-iy_2(ua_8+a_2-\frac{c_3a_9}{a_{10}})}e^{\frac{1}{2}iy_2^2\frac{a_8a_9}{a_{10}}}\xi\bigg(u-y_2\frac{a_9}{a_{10}}\bigg),\\
&\big(\pi_l\big(\exp(y_3Y_3)\big)\xi\big)(u)=\xi(u-y_3),\\
&\big(\pi_l\big(\exp(y_4Y_4)\big)\xi\big)(u)=e^{-iy_4(c_4-ua_{10})}\xi(u),\\
&\big(\pi_l\big(\exp(y_5Y_5)\big)\xi\big)(u)=e^{-iy_5\frac{a_6a_9-a_7a_8}{a_{10}}}\xi(u),\\
&\big(\pi_l\big(\exp(y_jY_j)\big)\xi\big)(u)=e^{-iy_ja_j}\xi(u),\quad 6\le j\le 10.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
&\big(\pi_l\big(\exp(y_1Y_1)\exp(y_2Y_2)\cdots\exp(y_{10}Y_{10})\big)\xi\big)(u)\\
&\quad=\big(\pi_l\big(\exp(y_1Y_1)\big)\big[\pi_l\big(\exp(y_2Y_2)\cdots\exp(y_{10}Y_{10})\big)\xi\big]\big)(u)\\
&\quad=\cdots\\
&\quad=e^{-iy_1(ua_6+a_1-\frac{c_3a_7}{a_{10}})}e^{-iy_2(ua_8+a_2-\frac{c_3a_9}{a_{10}})}e^{iy_1y_2\frac{a_7a_8}{a_{10}}}\\
&\qquad\cdot e^{\frac{1}{2}iy_1^2\frac{a_6a_7}{a_{10}}}e^{\frac{1}{2}iy_2^2\frac{a_8a_9}{a_{10}}}e^{-iy_4(c_4-ua_{10}+y_1a_7+y_2a_9+y_3a_{10})}\\
&\qquad\cdot e^{-iy_5\frac{a_6a_9-a_7a_8}{a_{10}}}e^{-iy_6a_6}e^{-iy_7a_7}e^{-iy_8a_8}e^{-iy_9a_9}\\
&\qquad\cdot e^{-iy_{10}a_{10}}\xi\bigg(u-y_1\frac{a_7}{a_{10}}-y_2\frac{a_9}{a_{10}}-y_3\bigg).
\end{aligned}
$$

We may then prove the following retract theorem:

THEOREM 3. *Let $i \in \{5,6,\dots,10\}$ be fixed, but arbitrary. For every $F \in \mathfrak{N}^{C_0}_{\mathcal{S}_i \cap \Sigma_i}$, there exists a Schwartz function $f$ on $F_{4,2}$ such that $\pi_l(f)$ has $F(l,\cdot,\cdot)$ as an operator kernel for all $l \in \mathcal{S}_i \cap \Sigma_i$ and $\pi_l(f) = 0$ if $l \in \mathcal{S} \backslash \mathcal{S}_i$.*

*There exists a suitable algorithm to choose $f$ such that the map*

$$\begin{aligned} \mathfrak{N}^{C_0}_{\mathcal{S}_i \cap \Sigma_i} &\longrightarrow \mathcal{S}(F_{4,2}), \\ F &\longmapsto R(F) := f \end{aligned}$$

*is continuous.*

*Proof.* The proof is done for $\mathcal{S}_{10} \cap \Sigma_{10}$. The elements of $\mathcal{S}_{10} \cap \Sigma_{10}$ are of the form

$$l = a_1 Y_1^* + a_2 Y_2^* + c_3 Y_3^* + c_4 Y_4^* + \left(\frac{a_6 a_9 - a_7 a_8}{a_{10}}\right) Y_5^* + a_6 Y_6^* + \cdots + a_{10} Y_{10}^*,$$

where $c_3, c_4$ are the constants determining the section $\Sigma_{10}$ and where $(a_1, a_2, a_6, a_7, \dots, a_{10})$ are the parameters characterizing $l$ in $\mathcal{S}_{10} \cap \Sigma_{10}$. Let $f \in \mathcal{S}(F_{4,2})$. An explicit computation of $\pi_l(f)$ gives

$$\begin{aligned} \big(\pi_l(f)\xi\big)(u) &= \int_{\mathbb{R}^{10}} f\big(\exp(y_1 Y_1) \cdots \exp(y_{10} Y_{10})\big) \\ &\qquad \cdot \big(\pi_l\big(\exp(y_1 Y_1) \cdots \exp(y_{10} Y_{10})\big)\xi\big)(u)\, dy_1 \cdots dy_{10} \\ &= \int_{\mathbb{R}^4} \hat{f}^{5,6,\dots,10}\left(y_1, y_2, y_3, y_4, \frac{a_6 a_9 - a_7 a_8}{a_{10}}, a_6, \dots, a_{10}\right) \\ &\qquad \cdot e^{-iy_1(ua_6 + a_1 - \frac{c_3 a_7}{a_{10}})} e^{-iy_2(ua_8 + a_2 - \frac{c_3 a_9}{a_{10}})} e^{iy_1 y_2 \frac{a_7 a_8}{a_{10}}} e^{\frac{1}{2} i y_1^2 \frac{a_6 a_7}{a_{10}}} \\ &\qquad \cdot e^{\frac{1}{2} i y_2^2 \frac{a_8 a_9}{a_{10}}} e^{-iy_4(c_4 - ua_{10} + y_1 a_7 + y_2 a_9 + y_3 a_{10})} \\ &\qquad \cdot \xi\left(u - y_1 \frac{a_7}{a_{10}} - y_2 \frac{a_9}{a_{10}} - y_3\right) dy_1\, dy_2\, dy_3\, dy_4, \end{aligned}$$

where $\hat{f}^{5,6,\dots,10}$ denotes the partial Fourier transform of $f$ in the last 6 variables. Changing the variable $y_3$ into a variable $w$ defined by

$$w = -y_3 + u - y_1 \frac{a_7}{a_{10}} - y_2 \frac{a_9}{a_{10}},$$

gives

$$\begin{aligned} \big(\pi_l(f)\xi\big)(u) = \int_{\mathbb{R}} \bigg[ \int_{\mathbb{R}^2} & e^{iy_1 y_2 \frac{a_7 a_8}{a_{10}}} e^{\frac{1}{2} i y_1^2 \frac{a_6 a_7}{a_{10}}} e^{\frac{1}{2} i y_2^2 \frac{a_8 a_9}{a_{10}}} \\ & \cdot \hat{f}^{4,5,\dots,10}\bigg(y_1, y_2, u - w - y_1 \frac{a_7}{a_{10}} - y_2 \frac{a_9}{a_{10}}, c_4 - w a_{10}, \\ & \frac{a_6 a_9 - a_7 a_8}{a_{10}}, a_6, \dots, a_{10}\bigg) \\ & \cdot e^{-iy_1(ua_6 + a_1 - \frac{c_3 a_7}{a_{10}})} e^{-iy_2(ua_8 + a_2 - \frac{c_3 a_9}{a_{10}})}\, dy_1\, dy_2 \bigg] \xi(w)\, dw. \end{aligned}$$

Hence the operator kernel of $\pi_l(f)$ is given by

$$\begin{aligned}&F(a_1,a_2,a_6,\dots,a_{10};u,w)\\&\quad=\hat{g}^{1,2}\left(ua_6+a_1-\frac{a_7c_3}{a_{10}},ua_8+a_2-\frac{a_9c_3}{a_{10}},u,w,a_6,\dots,a_{10}\right),\end{aligned}$$

where $\hat{g}^{1,2}$ denotes the usual partial Fourier transform of $g$ in the first two variables and where the function $g$ is obtained by

$$\begin{aligned}g(y_1,y_2,u,w,a_6,\dots,a_{10})&=e^{iy_1y_2\frac{a_7a_8}{a_{10}}}e^{\frac{1}{2}iy_1^2\frac{a_6a_7}{a_{10}}}e^{\frac{1}{2}iy_2^2\frac{a_8a_9}{a_{10}}}\\&\quad\cdot\hat{f}^{4,5,6,\dots,10}\Bigg(y_1,y_2,u-w-y_1\frac{a_7}{a_{10}}-y_2\frac{a_9}{a_{10}},\\&\qquad c_4-wa_{10},\frac{a_6a_9-a_7a_8}{a_{10}},a_6,\dots,a_{10}\Bigg).\end{aligned}$$

Conversely, given a kernel function $F$, one may compute easily the function $g$ satisfying the previous relation between $F$ and $\hat{g}^{1,2}$. Let $C_0$ be the compact subset of $\{(a_1,a_2,a_6,\dots,a_{10})\in\mathbb{R}^7|a_{10}\neq 0\}$ which contains the support in $(a_1,a_2,a_6,\dots,a_{10})$ of the considered kernel functions. Let $K_0$ be its image under the map

$$(a_1,a_2,a_6,\dots,a_{10})\mapsto\frac{a_6a_9-a_7a_8}{a_{10}}$$

and let $\varphi\in\mathcal{S}(\mathbb{R})$ be an arbitrary, but fixed Schwartz function such that $\varphi\equiv 1$ on $K_0$.

In order to deduce the function $g$ from the given $F$, let us change the variables $a_1,a_2$ into $\tilde{a}_1,\tilde{a}_2$ given by

$$\begin{cases}\tilde{a}_1=a_1+ua_6-\frac{c_3a_7}{a_{10}},\\\tilde{a}_2=a_2+ua_8-\frac{c_3a_9}{a_{10}}.\end{cases}$$

We get

$$\begin{aligned}&\hat{g}^{1,2}(\tilde{a}_1,\tilde{a}_2,u,w,a_6,\dots,a_{10})\\&\quad=F\left(\tilde{a}_1-ua_6+\frac{c_3a_7}{a_{10}},\tilde{a}_2-ua_8+\frac{c_3a_9}{a_{10}},a_6,\dots,a_{10};u,w\right).\end{aligned}$$

The function $g$ is then obtained by inverse Fourier transform in the first two variables. We hence have to produce a function $f\in\mathcal{S}(N)$ such that

$$\begin{aligned}&\hat{f}^{4,5,\dots,10}\left(y_1,y_2,u-w-y_1\frac{a_7}{a_{10}}-y_2\frac{a_9}{a_{10}},c_4-wa_{10},\frac{a_6a_9-a_7a_8}{a_{10}},a_6,\dots,a_{10}\right)\\&\quad=e^{-iy_1y_2\frac{a_7a_8}{a_{10}}}e^{-\frac{1}{2}iy_1^2\frac{a_6a_7}{a_{10}}}e^{-\frac{1}{2}iy_2^2\frac{a_8a_9}{a_{10}}}g(y_1,y_2,u,w,a_6,\dots,a_{10}).\end{aligned}$$

To do this we change the variables $u$ and $w$ into $y_3$ and $a$ by

$$\begin{cases}y_3=u-w-y_1\frac{a_7}{a_{10}}-y_2\frac{a_9}{a_{10}},\\a\ =c_4-wa_{10}\end{cases}$$

and we use the arbitrary function $\varphi$ introduced previously, to look for $f \in \mathcal{S}(N)$ such that

$$\begin{aligned}
&\hat{f}^{4,5,6,\ldots,10}(y_1, y_2, y_3, a, a_5, a_6, \ldots, a_{10}) \\
&\quad = e^{-iy_1y_2\frac{a_7a_8}{a_{10}}} e^{-\frac{1}{2}iy_1^2\frac{a_7a_6}{a_{10}}} e^{-\frac{1}{2}iy_2^2\frac{a_9a_8}{a_{10}}} \varphi(a_5) \\
&\qquad \cdot g\left(y_1, y_2, y_3 + \frac{a_7}{a_{10}}y_1 + \frac{a_9}{a_{10}}y_2 + \frac{c_4 - a}{a_{10}}, \frac{c_4 - a}{a_{10}}, a_6, \ldots, a_{10}\right).
\end{aligned}$$

Because of the restriction on the support of the kernel function $F$, the right hand side of the previous equality is a Schwartz function and $f$ may be obtained by inverse Fourier transform in the variables 4 to 10. This function $f$ may be taken as the retract of $F$, as $\pi_l(f)$ has $F(l,\cdot,\cdot)$ as an operator kernel for all $l \in \mathcal{S}_{10} \cap \Sigma_{10}$.

Let's now prove that $\pi_l(f) = 0$ for every $l \in \mathcal{S}\backslash\mathcal{S}_{10}$. Let's fix $l_0 \in \mathcal{S}\backslash\mathcal{S}_{10}$ and let's take a sequence $(l_\varepsilon)_{\varepsilon>0}$ which converges to $l_0$ in $\mathfrak{f}_{4,2}^*$ with $l_\varepsilon \in \mathcal{S}_{10}$ such that $(l_\varepsilon)_{10} = \varepsilon$.

We define $l_{\varepsilon,\Sigma_{10}}$ as follows:

$$l_{\varepsilon,\Sigma_{10}} := \mathrm{Ad}^*(F_{4,2})l_\varepsilon \cap \Sigma_{10}.$$

As $Y_{10}$ is central, $(l_{\varepsilon,\Sigma_{10}})_{10} = \varepsilon$ also. Moreover, there exists $\delta > 0$ such that for all $l = \sum_{j=1}^{10} l_j Y_j^* \in C_0$, $|\, l_{10} \,| \geq \delta > 0$ since $C_0$ is a compact subset of $\mathcal{S}_{10} \cap \Sigma_{10}$. Hence for $0 < \varepsilon < \delta$, $l_{\varepsilon,\Sigma_{10}} \notin C_0$ and $F(l_{\varepsilon,\Sigma_{10}},\cdot,\cdot) \equiv 0$ since $\mathrm{supp}_l F|_{\Sigma_{10}\times\mathbb{R}^2} \subset C_0 \times \mathbb{R}^2$. Then $\pi_{l_{\varepsilon,\Sigma_{10}}}(f) \equiv 0$ and $\pi_{l_\varepsilon}(f) \equiv 0$.

But we may of course identify the topological spaces $\mathfrak{f}_{4,2}^*/\mathrm{Ad}^*(F_{4,2})$ and $\mathrm{Prim}_* L^1(F_{4,2})$ (with the hull-kernel topology), as the group is *-regular [3]. So, as $l_\varepsilon$ converges to $l_0$, $\bigcap_{0<\varepsilon<\delta} \mathrm{Ker}(\pi_{l_\varepsilon}) \subset \mathrm{Ker}(\pi_{l_0})$. Hence, $\pi_{l_0}(f) = 0$, if $l_0 \in \mathcal{S}\backslash\mathcal{S}_{10}$. $\square$

Remark 7. 1. One is tempted to conjecture that there exists a similar retract on any layer of an arbitrary connected, simply connected, nilpotent Lie group. But this is still an open question.

2. As the rest of this paper relies on the retract results of this section, we have to limit ourselves from now on to $\mathfrak{n}_{\mathrm{gen}}^*$, to the character layer $\mathcal{C}$ or, additionally, in the case of the group $F_{2,4}$, to the intermediate layers $\mathcal{S}_i$.

## 4. Retract theorems on a submanifold contained in a section of a layer

In this section, $\mathcal{W}$ will either denote $\mathfrak{n}_{\mathrm{gen}}^* \cap \Sigma$, $\mathcal{C}$ or $\mathcal{S}_i \cap \Sigma_i$, $5 \leq i \leq 10$, in the case of the group $F_{4,2}$. Let $\mathcal{V}$ be a submanifold of $\mathcal{W}$. Then, we denote by $\mathfrak{N}_{\mathcal{V},\mathcal{W}}^C$ the corresponding space of kernel functions, if $C$ is a compact subset contained in $\mathcal{V}$.

The aim is to give a global retract theorem on the submanifold $\mathcal{V}$. To do this, we construct, locally on a chart of $\mathcal{V}$, an extension of a kernel function $F$, we apply the results of Section 3 to obtain local retracts and we use a partition of unity in order to paste together the different local retracts.

**4.1. Local retract theorem.** For each $l \in \mathcal{V}$, there exists a chart $(U, \varphi)$ of $\mathcal{V}$ in a neighborhood of $l$ such that it is the restriction to $\mathcal{V}$ of a chart $(W, \rho)$ of $\mathcal{W}$. We first assume that the fixed compact set $C$ is contained in such a chart. We then have the following theorem.

THEOREM 4. *Let $C$ be a compact subset contained in a fixed chart $(U, \varphi)$ of $\mathcal{V}$, which is a restriction to $\mathcal{V}$ of a chart of $\mathcal{W}$. For every $F \in \mathfrak{N}^C_{\mathcal{V},\mathcal{W}}$, there exists a Schwartz function $f$ on $N$, such that $\pi_l(f)$ has $F(l,\cdot,\cdot)$ as an operator kernel for all $l \in C$ and $\pi_l(f) = 0$ if $l \in \mathcal{V}\backslash C$.*

*There exists a suitable algorithm to choose $f$ such that the map*

$$F \longmapsto R(F) := f$$

*is continuous. If $\mathcal{V}$ is a submanifold of $\mathcal{W} = \mathfrak{n}^*_{\text{gen}} \cap \Sigma$, then we have moreover $\pi_l(f) = 0$ for all $l \notin \mathfrak{n}^*_{\text{gen}}$. In the case of $F_{4,2}$, we have $\pi_l(f) = 0$ for all $l \in \mathcal{S} \setminus \mathcal{S}_i$, if $\mathcal{V}$ is a submanifold of $\mathcal{W} = \mathcal{S}_i \cap \Sigma_i$.*

*Proof.* By a standard procedure, we extend $F(l,\cdot,\cdot)$ to a smooth function $F_1(l,\cdot,\cdot)$ defined on a compact subset of a chart of all of $\mathcal{W}$ and we apply the results of Section 3. $\square$

**4.2. Global retract theorem.** As before, let $\mathcal{V}$ be a submanifold of $\mathcal{W}$. We now drop the condition that the compact subset $C$ has to be contained in a fixed chart of the manifold.

THEOREM 5. *Let $C$ be any compact subset contained in $\mathcal{V}$. For every $F \in \mathfrak{N}^C_{\mathcal{V},\mathcal{W}}$, there exists a Schwartz function $f$ on $N$, such that $\pi_l(f)$ has $F(l,\cdot,\cdot)$ as an operator kernel for all $l \in C$ and $\pi_l(f) = 0$ if $l \in \mathcal{V}\backslash C$.*

*There exists a suitable algorithm to choose $f$ such that the map*

$$F \longmapsto R(F) := f$$

*is continuous. Moreover, if $\mathcal{V}$ is a submanifold of $\mathcal{W} = \mathfrak{n}^*_{\text{gen}} \cap \Sigma$, we have $\pi_l(f) = 0$ for all $l \notin \mathfrak{n}^*_{\text{gen}}$. In the case of $F_{4,2}$, we have $\pi_l(f) = 0$ for all $l \in \mathcal{S} \setminus \mathcal{S}_i$, if $\mathcal{V}$ is a submanifold of $\mathcal{W} = \mathcal{S}_i \cap \Sigma_i$.*

*Proof.* One uses a partition of unity to glue together local retracts obtained via Theorem 4. $\square$

## 5. Orbits under the compact group action

Let $N$ be a connected, simply connected, nilpotent Lie group and $K$ be a compact subgroup of the group of automorphisms of $N$, $\mathrm{Aut}(N)$, acting smoothly on $N$. We denote this action by:

$$\begin{aligned} K \times N &\longrightarrow N, \\ (k,x) &\longmapsto k \cdot x. \end{aligned}$$

This action induces naturally actions of $K$ on $\mathfrak{n}, \mathfrak{n}^*, \hat{N}, L^1(N), \mathcal{S}(N)$.

Examples.

1. Action of $\mathrm{SO}(2n)$ on the Heisenberg group $H_n$.
2. Action of $\mathrm{SO}(n)$ on $F_{n,2}$, the free nilpotent Lie group of step 2 and $n$ generators (see [2] for details on this action).

Let $\pi_l := \mathrm{ind}_{P(l)}^N \chi_l$ be the induced representation of $N$ associated to $l$, where $P(l) = \exp \mathfrak{p}(l)$ denotes the Vergne polarization of $l$. The different actions of $K$ on $\widehat{N}$, the dual of $N$, $\mathfrak{n}^*$, etc., allow us to define equivalent irreducible representations of $N$ as follows:

$$\begin{aligned} \pi_{k\cdot l} &:= \mathrm{ind}_{P(k\cdot l)}^N \chi_{k\cdot l}, \\ \tilde{\pi}_k &:= \mathrm{ind}_{k\cdot P(l)}^N \chi_{k\cdot l}, \\ {}^k\pi_l(x) &:= \pi_l\big(k^{-1}\cdot x\big), \quad x \in N. \end{aligned}$$

Here $P(l) = \exp \mathfrak{p}(l), P(k\cdot l) = \exp \mathfrak{p}(k\cdot l)$, where $\mathfrak{p}(l)$ and $\mathfrak{p}(k\cdot l)$ denote the Vergne polarizations of $l$, resp. $k\cdot l$, with respect to the given basis. Moreover, $k\cdot \mathfrak{p}(l)$ is also a polarization of $k\cdot l$, a fact which justifies the definition of $\tilde{\pi}_k$, where $k\cdot P(l) = \exp(k\cdot \mathfrak{p}(l))$.

Definition 5. For all $\pi_l \in \widehat{N}$, we define the following stabilizers:

$$\begin{aligned} K_{\pi_l} &:= \big\{k \in K | {}^k\pi_l \simeq \pi_l\big\} \quad (\simeq \text{ means: is equivalent to}) \\ &= \big\{k \in K | k\cdot l \in \mathrm{Ad}^*(N) l\big\}, \\ K_l &= \{k \in K | k\cdot l = l\}. \end{aligned}$$

These stabilizers are closed subgroups of $K$.

Remark 8. When $l_0$ defines a character, that is, when $\langle l_0, [\mathfrak{n},\mathfrak{n}]\rangle \equiv 0$, then necessarily $K_{l_0} = K_{\pi_{l_0}}$, as the co-adjoint orbit of $l_0$ reduces to $\{l_0\}$. Let us also note that

$$\mathcal{C} = \big\{l \in \mathfrak{n}^* | \big\langle l, [\mathfrak{n},\mathfrak{n}]\big\rangle \equiv 0\big\}$$

is $K$-invariant.

In general, we have $K_l \subset K_{\pi_l}$, but not necessarily $K_l = K_{\pi_l}$. Nevertheless, for technical reasons we will need $K_{l_0} = K_{\pi_{l_0}}$. Therefore, we introduce a particular linear form $l_0$, called *aligned linear form*, which satisfies $K_{l_0} = K_{\pi_{l_0}}$.

Lipsman shows in [11] the existence of such aligned linear forms. More precisely, every co-adjoint orbit in $\mathfrak{n}^*$ contains an aligned linear form $l_0$ verifying $K_{l_0} = K_{\pi_{l_0}}$ [11].

Remark 9. We can always assume that $l_0$ is aligned and generic in the sense Ludwig–Zahir (resp. aligned and in the intermediate layer $\mathcal{S}_i$, in the case of $N = F_{4,2}$) by moving on the co-adjoint orbit. This will be the case for the rest of this paper.

Let $\Sigma$ (resp. $\Sigma_i$) be the corresponding Pukanszky section of $\mathfrak{n}^*_{\text{gen}}$ (resp. $\mathcal{S}_i$) such that $l_0 \in \Sigma$ (resp. $\Sigma_i$). Let $\widetilde{\Sigma}$ denote either $\Sigma$ or $\Sigma_i$. Let $\mathcal{U}$ denote $\mathfrak{n}^*_{\text{gen}}$, resp. $\mathcal{S}_i$ in the case of the group $F_{4,2}$. As before, let $\mathcal{W} = \mathcal{U} \cap \widetilde{\Sigma}$ denote $\mathfrak{n}^*_{\text{gen}} \cap \Sigma$, resp. $\mathcal{S}_i \cap \Sigma_i$ in the case of $F_{4,2}$.

In general, $\mathcal{U}$ is not stable under the action of $K$. Therefore, we define

$$\tilde{K} := \{k \in K | k \cdot l_0 \in \mathcal{U}\}.$$

Let now

$$l_1 = \mathrm{Ad}^*(m) k_1 \cdot l_0 \in \mathrm{Ad}^*(N)\tilde{K} \cdot l_0 \cap \widetilde{\Sigma},$$

where $k_1 \in \tilde{K}$ and $m \in N$ such that $l_1 \in \widetilde{\Sigma}$.

Let $k \in \tilde{K}$ be such that $k \cdot k_1 \in \tilde{K}$. We then define $k * l_1 := (l_1)_k$ to be the unique intersection point of the co-adjoint orbit of $k \cdot l_1$ with the section $\widetilde{\Sigma}$, i.e.

$$\begin{aligned}\{(l_1)_k\} &= \mathrm{Ad}^*(N)(k \cdot l_1) \cap \widetilde{\Sigma} \\ &= \mathrm{Ad}^*(N)\big(\mathrm{Ad}^*(k \cdot m)(k k_1) \cdot l_0\big) \cap \widetilde{\Sigma} \\ &= \mathrm{Ad}^*(N)\big((k k_1) \cdot l_0\big) \cap \widetilde{\Sigma}.\end{aligned}$$

It is easy to check that, provided $k' k k_1 \in \tilde{K}$,

$$k' * (k * l_1) = (l_1)_{k'k} = \big(k'k\big) * l_1,$$

that is, we have a local group action.

In particular,

$$\{l_{0,k}\} := \{k * l_0\} = \mathrm{Ad}^*(N)(k \cdot l_0) \cap \widetilde{\Sigma}.$$

Proposition 1. *The local group action*

$$\begin{aligned}\tilde{K} &\to \mathcal{W}, \\ k &\mapsto l_{0,k} = k * l_0\end{aligned}$$

*satisfies*:

(1) *Its image* $\mathrm{Ad}^*(N)\tilde{K} \cdot l_0 \cap \widetilde{\Sigma}$ *is a submanifold of* $\widetilde{\Sigma}$ *in* $\mathcal{U}$.
(2) *There exist a smooth section* $(V, \eta)$ *of* $\tilde{K}/K_{l_0}$ *such that* $\eta(V)$ *is a submanifold of* $K$, *a relatively open subset* $\tilde{V}$ *in* $\eta(V)$ *and a smooth map*

$$\begin{aligned}\tilde{V} &\longrightarrow N, \\ k &\longmapsto m_k\end{aligned}$$

*such that* $l_{0,k} = \mathrm{Ad}^*(m_k) k \cdot l_0$.

*Proof.* As $\mathrm{Ad}^*(N)\tilde{K}\cdot l_0\cap\widetilde{\Sigma}=(\mathrm{Ad}^*(N)K\cdot l_0)\cap\mathcal{W}$ is locally closed in $\widetilde{\Sigma}$ and as we are in presence of a local group action for which the stabilizer of $l_0$ is $K_{l_0}$, the map

$$\begin{aligned}\tilde{K}/K_{l_0} &\longrightarrow \tilde{K}*l_0,\\ \dot{k} &\longmapsto l_{0,k}\end{aligned}$$

is a diffeomorphism and $\mathrm{Ad}^*(N)\tilde{K}\cdot l_0\cap\widetilde{\Sigma}$ is a submanifold of $\widetilde{\Sigma}$, by [6].

The Pukanszky orbit parametrization and the characterization of the Pukanszky sections via this parametrization, then give a system of equations for $m_k$. A smooth solution of this system exists, at least in a certain relatively open subset $\tilde{V}$ of $\eta(V)$. $\square$

REMARK 10. We have the natural local action of $\tilde{K}$ on $\mathcal{U}$ defined by

$$(k,l)\longmapsto k\cdot l.$$

Hence, for $l_0\in\mathcal{U}$ fixed, $\tilde{K}\cdot l_0$ is a submanifold of $\mathcal{U}$ [6].

## 6. $K$-retract theory for the representations induced from Vergne polarizations

In this section, we shall study the retract theory associated to the induced representations using Vergne polarizations on $K$-orbits for the different layers. We will use the notations $\widetilde{\Sigma},\mathcal{U},\mathcal{W}$ introduced in the previous section.

**6.1. On the character layer.** Recall that

$$\mathcal{C}=\{l\in\mathfrak{n}^*|\langle l,[\mathfrak{n},\mathfrak{n}]\rangle=0\}$$

is the layer of $\mathfrak{n}^*$ corresponding to the characters of $N$. It is obvious that $\mathcal{C}$ is $K$-invariant.

For $l_0\in\mathcal{C}$, $\mathcal{V}=K\cdot l_0$ is a submanifold of $\mathcal{C}$. Then, by Section 4 and Theorem 5, we have a global retract theorem for the characters $\chi_{k\cdot l_0}$ on the orbit $K\cdot l_0$. Moreover, as $K\cdot l_0$ is diffeomorphic to $K/K_{l_0}$, this gives a global retract theorem for $\chi_{k\cdot l_0}$ and kernel functions defined on $K/K_{l_0}$ (resp. on a section of $K/K_{l_0}$). Finally, we may extend this result to a retract theorem on all of $K$, if we use the following function space: Let $\mathfrak{N}_K$ be the space of all smooth functions $F:K\longrightarrow\mathbb{C}$ such that:

- F is a $\mathcal{C}^\infty$ function.
- We have the compatibility relation

$$k=k' \mod K_{l_0}\quad\Longrightarrow\quad F(k)=F(k').$$

The compatibility condition then allows us to deduce a retract theorem for $\chi_{k\cdot l_0}$ and kernel functions defined on $K$.

THEOREM 6. *For every $F \in \mathfrak{N}_K$, there exists a Schwartz function $f$ on $N$ such that $\chi_{k\cdot l_0}(f) = \widehat{f}(k \cdot l_0) = F(k)$ for all $k \in K$.*

*There exists a suitable algorithm to choose $f$ such that the map*

$$F \longmapsto R(F) := f$$

*is continuous.*

*Proof.* By Theorem 2 and the use of the compatibility condition. □

**6.2. On the sections of $\tilde{K}$-orbits.** Let $l_0 \in \mathcal{U}$ be aligned. Let $\mathcal{V} = \tilde{K} * l_0 = \mathrm{Ad}^*(N)\tilde{K} \cdot l_0 \cap \widetilde{\Sigma}$ be the submanifold of $\widetilde{\Sigma}$ obtained by the action of $\tilde{K}$. Then Theorem 5 of Section 4 is in particular true for the submanifolds $\tilde{K} * l_0 = \mathrm{Ad}^*(N)\tilde{K} \cdot l_0 \cap \widetilde{\Sigma}$ and the representations $\pi_{l_{0,k}}$.

**6.3. On the $\tilde{K}$-orbits.** The aim of this paragraph is to get out of the section $\widetilde{\Sigma}$ in order to work with the original $K$-orbit. Let us hence consider the submanifold $\mathcal{V} = \tilde{K} \cdot l_0$ of $\mathcal{U}$.

THEOREM 7. *There exist a smooth section $(V, \eta)$ of $\tilde{K}/K_{l_0}$ such that $\eta(V)$ is a submanifold of $K$ and a relatively open subset $\tilde{V}$ in $\eta(V)$, such that for every compact subset $C_0$ in $\tilde{V} \cdot l_0$, we have*:

*For every $F \in \mathfrak{N}^{C_0}_{\tilde{K}\cdot l_0, \mathcal{U}}$, there exists a Schwartz function $f$ on $N$, such that $\pi_{k\cdot l_0}(f)$ has $F(k\cdot l_0, \cdot, \cdot)$ as an operator kernel for all $k \cdot l_0 \in C_0$ and $\pi_{k\cdot l_0}(f) = 0$ if $k \cdot l_0 \in \tilde{K} \cdot l_0 \backslash C_0$. Moreover, $\pi_{k\cdot l_0}(f) = 0$ if $k \in K \setminus \tilde{K}$.*

*There exists a suitable algorithm to choose $f$ such that the map*

$$F \longmapsto R(F) := f$$

*is continuous.*

*Proof.* By Proposition 1, there exist a smooth section $(V, \eta)$ of $\tilde{K}/K_{l_0}$, a non-empty relatively open subset $\tilde{V}$ in $\eta(V)$ and a smooth map

$$\tilde{V} \longrightarrow N,$$
$$k \longmapsto m_k$$

such that $l_{0,k} = \mathrm{Ad}^*(m_k)k \cdot l_0$. We extend now $F$ to a function $F_1$ on $\mathrm{Ad}^*(N)\tilde{K} \cdot l_0 \times N \times N$ as follows:

$$F_1\big(\mathrm{Ad}^*(n)k \cdot l_0, x, y\big) := F(k \cdot l_0, x \cdot n, y \cdot n)$$

for all $n \in N$.

Then, we restrict the function $F_1$ to $(\mathrm{Ad}^*(N)\tilde{K} \cdot l_0 \cap \Sigma) \times N \times N$ in the following way:

$$F_0(l_{0,k}, x, y) = F_0\big(\mathrm{Ad}^*(m_k)k \cdot l_0, x, y\big) := F(k \cdot l_0, x \cdot m_k, y \cdot m_k).$$

Now, one checks that $F_0 \in \mathfrak{N}^{C}_{\tilde{K}*l_0, \mathcal{W}}$, where $C := \mathrm{Ad}^*(N)C_0 \cap \widetilde{\Sigma}$ is a compact subset in $\mathrm{Ad}^*(N)(\tilde{K} \cdot l_0) \cap \widetilde{\Sigma}$.

By applying Theorem 5 to the function $F_0$, one sees that there exists $f \in \mathcal{S}(N)$, such that $\pi_{l_{0,k}}(f)$ has $F_0(l_{0,k},\cdot,\cdot)$ as an operator kernel for all $k \in \tilde{K}$ and $\pi_{l_{0,k}}(f) = 0$ if $l_{0,k} \notin C$. Then, $\pi_{k\cdot l_0}(f)$ has $F(k \cdot l_0,\cdot,\cdot)$ as an operator kernel if $k \cdot l_0 \in C_0$. Moreover, $l_{0,k} = \mathrm{Ad}^*(m_k)k \cdot l_0 \notin C$ if and only if $k \cdot l_0 \notin C_0$ since every co-adjoint orbit cuts $K \cdot l_0$ in at most one point. Hence, $\pi_{k\cdot l_0}(f) = 0$ if $k \cdot l_0 \notin C_0$ and $k \in \tilde{K}$. The same is true if $k \in K \setminus \tilde{K}$, as in this case $k \cdot l_0 \notin \mathfrak{n}^*_{\mathrm{gen}}$ (resp. $k \cdot l_0 \in \mathcal{S} \setminus \mathcal{S}_i$). □

Remark 11. We get a similar retract theorem for kernel functions $F$ defined on $K$ (instead of $K \cdot l_0$), with compact support in $k$ contained in $\tilde{K}$, provided we require the compatibility condition

$$k = k' \mod K_{l_0} \quad \Longrightarrow \quad F(k,x,y) = F\big(k',x,y\big)$$

for the kernel functions $F$.

## 7. $K$-retract theory for the representations induced from translated polarizations

The purpose of this section is to give a global retract theorem for the representations induced from translated polarizations rather than Vergne polarizations, since these are more natural in the study of the compact group action.

Recall that $l_0$ is an aligned and generic linear form in the sense of Ludwig–Zahir, resp. aligned and in $\mathcal{S}_i$ (in the case of $N = F_{4,2}$) and $\mathfrak{p}(l_0)$ is the Vergne polarization of $l_0$. Then $k \cdot \mathfrak{p}(l_0)$ is a polarization for $k \cdot l_0$, for every $k$, and we define the induced representations from the translated polarizations as follows:

$$\tilde{\pi}_{0,k} := \mathrm{ind}_{k\cdot P(l_0)}^{N} \chi_{k\cdot l_0},$$

where $P(l_0) = \exp(\mathfrak{p}(l_0))$ and $k \cdot P(l_0) = \exp(k \cdot \mathfrak{p}(l_0))$. Let $\mathcal{U}$ denote $\mathfrak{n}^*_{\mathrm{gen}}$, resp. $\mathcal{S}_i$, as defined in Section 5.

### 7.1. A retract theorem for $\tilde{\pi}_{0,k}$ on a section of $\tilde{K}/K_{l_0}$.

In order to construct a global retract for $\tilde{\pi}_{0,k}$, let us recall that $\tilde{\pi}_{0,k}$ and $\pi_{k\cdot l_0}$ are unitary equivalent. We will use the results of the previous section and a result on smooth families of intertwining operators [8]. We need the following definition.

Definition 6. Let $(V,\eta)$ be a smooth section of $\tilde{K}/K_{l_0}$ such that $\eta(V)$ is a submanifold of $K$, and $V'$ a relatively open subset of $\eta(V)$. A function

$$\xi : V' \times N \to \mathbb{C}$$

is called a generalized $\mathcal{C}^\infty$-vector for the family of representations $(\pi_{k\cdot l_0})_{k\in V'}$, if

- $\xi(k, x \cdot p) = \overline{\chi_{k\cdot l_0}(p)}\xi(k,x), \forall p \in \mathfrak{p}(k \cdot l_0)$.
- The support of $\xi$ in $k$ is a compact subset of $V'$.

- For any smooth Malcev basis $\{X_1(k),\dots,X_d(k)\}$, $k\in V'$, of $\mathfrak{n}$ relative to $\mathfrak{p}(k\cdot l_0)$, the function $\xi'$ defined by
$$\xi'(k;x_1,\dots,x_d):=\xi\big(k;\exp\big(x_1X_1(k)\big)\cdots\exp\big(x_dX_d(k)\big)\big)$$
has the GS-property.

Then, for $k\in V'$ fixed, $\xi(k;\cdot)$ is a $\mathcal{C}^\infty$-vector for the representation $\pi_{k\cdot l_0}$.

We denote $P:=(P(k\cdot l_0))_{k\in V'}$ and we write $\mathcal{KS}(V',\mathcal{S}(N/P,\chi))$ for the set of all these generalized $\mathcal{C}^\infty$-vectors. We define similarly $P':=(k\cdot P(l_0))_{k\in V'}$ and $\mathcal{KS}(V',\mathcal{S}(N/P',\chi))$ associated to the family of representations $(\tilde\pi_{0,k})_{k\in V'}$.

From [8] we get the following result.

THEOREM 8. *For every smooth section $(V,\eta)$ of $\tilde K/K_{l_0}$ such that $\eta(V)$ is a submanifold of $K$, and every nonempty relatively open subset $\tilde V$ of $\eta(V)$, there exists a nonempty relatively open subset $V'$ of $\tilde V$ and a smooth family of unitary intertwining operators $U=(U_k)_{k\in V'}$ between $(\tilde\pi_{0,k})_{k\in V'}$ and $(\pi_{k\cdot l_0})_{k\in V'}$. This family of intertwining operators $U=(U_k)_{k\in V'}$,*
$$U:\mathcal{KS}\big(V',\mathcal{S}\big(N/P',\chi\big)\big)\longrightarrow\mathcal{KS}\big(V',\mathcal{S}(N/P,\chi)\big),$$
*is given by*
$$\begin{aligned}U_k\xi(k;g)&:=\frac{1}{\alpha(k)}\int_{P(k\cdot l_0)/P(k\cdot l_0)\cap k\cdot P(l_0)}\xi(k;gp)\chi_{k\cdot l_0}(p)d\dot p\\&=:\frac{1}{\alpha(k)}(T_k\xi)(k;g)\end{aligned}$$
*for all $\xi\in\mathcal{KS}(V',\mathcal{S}(N/P',\chi))$. The normalization function $\alpha$ is defined by*
$$\alpha(k)=\big\|(T_k\xi)(k;\cdot)\big\|_{L^2(N/P(k\cdot l_0),\chi_{k\cdot l_0})}:=\left(\int_{N/P(k\cdot l_0)}\big|(T_k\xi)(k;n)\big|^2\,d\dot n\right)^{\frac12}$$
*for any $\xi\in\mathcal{KS}(V',\mathcal{S}(N/P',\chi))$ such that $\xi(k,\cdot)$ has $L^2$-norm $1$ for all $k$. The function $\alpha$ is positive and smooth.*

*If $f\in\mathcal{S}(N)$, then the operator kernels $F(k,\cdot,\cdot)$ and $\tilde F(k,\cdot,\cdot)$ of $\pi_{k\cdot l_0}(f)$ and $\tilde\pi_{0,k}(f)$ are linked by the relation*
$$\begin{aligned}&F(k,x,y)\\&\quad:=(U_k\otimes\overline{U}_k)\tilde F(k,x,y)\\&\quad=\frac{1}{\alpha^2(k)}\int_{P(k\cdot l_0)/(P(k\cdot l_0)\cap k\cdot P(l_0))}\int_{P(k\cdot l_0)/(P(k\cdot l_0)\cap k\cdot P(l_0))}\tilde F(k,xp,yq)\\&\qquad\cdot\chi_{k\cdot l_0}(p)\overline{\chi_{k\cdot l_0}(q)}\,d\dot p\,d\dot q.\end{aligned}$$

In [8], these results are proven in a more general setting and with more precise indications on the choice of $V'$. In particular, they are valid for any two smooth families of polarizations associated to the same family of linear forms $(k\cdot l_0)_k$. This fact will be used in Section 7.2. In this section, we deduce the existence of a local retract for $(\tilde{\pi}_{0,k})_k$.

THEOREM 9. *There exist a smooth section* $(V,\eta)$ *of* $\tilde{K}/K_{l_0}$ *and a nonempty relatively open subset* $V'$ *of* $\eta(V)$ *such that for every compact subset* $\tilde{C}$ *of* $V'$, *we have*: *For every* $\tilde{F}\in\mathfrak{N}_{V'}^{\tilde{C}}$, *there exists a Schwartz function* $f$ *on* $N$, *such that* $\tilde{\pi}_{0,k}(f)$ *has* $\tilde{F}(k,\cdot,\cdot)$ *as an operator kernel if* $k\in\tilde{C}$ *and* $\tilde{\pi}_{0,k}(f)=0$ *if* $k\in\tilde{K}\setminus\tilde{C}\cdot K_{l_0}$. *Moreover,* $\tilde{\pi}_{0,k}(f)=0$ *if* $k\in K\setminus\tilde{K}$.

*There exists a precise algorithm to choose* $f$ *such that the map*

$$\tilde{F}\longmapsto R(\tilde{F}):=f$$

*is continuous in the given topologies.*

*Proof.* Let $(V,\eta)$ and $\tilde{V}$ be as in Theorem 7 and let $V'\subset\tilde{V}$ be such that $(U_k)_{k\in V'}$ is a smooth family of intertwining operators as given by Theorem 8. Let now $\tilde{C}$ be any compact subset of $V'$. For $\tilde{F}\in\mathfrak{N}_{V'}^{\tilde{C}}$, we define

$$\begin{aligned}
&F(k\cdot l_0,x,y)\\
&\quad:=(U_k\otimes\overline{U}_k)\tilde{F}(k,x,y)\\
&\quad=\frac{1}{\alpha^2(k)}\int_{P(k\cdot l_0)/(P(k\cdot l_0)\cap k\cdot P(l_0))}\int_{P(k\cdot l_0)/(P(k\cdot l_0)\cap k\cdot P(l_0))}\tilde{F}(k,xp,yq)\\
&\qquad\cdot\chi_{k\cdot l_0}(p)\overline{\chi_{k\cdot l_0}(q)}\,d\dot{p}\,d\dot{q}
\end{aligned}$$

for all $k\in V'$, by using Theorem 8. Then $F$ is a kernel function which belongs to $\mathfrak{N}_{\tilde{K}\cdot l_0,\mathcal{U}}^{\tilde{C}\cdot l_0}$. By Theorem 7, there exists $f\in\mathcal{S}(N)$ such that $\pi_{k\cdot l_0}(f)$ has $F(k\cdot l_0,\cdot,\cdot)$ as an operator kernel for all $k\cdot l_0\in\tilde{C}\cdot l_0$ and $\pi_{k\cdot l_0}(f)=0$ if $k\cdot l_0\in\tilde{K}\cdot l_0\setminus\tilde{C}\cdot l_0$. Moreover, $\pi_{k\cdot l_0}(f)=0$ if $k\in K\setminus\tilde{K}$.

Thanks to the smooth family of intertwining operators and the definition of $F$, we then have the same results for $\tilde{\pi}_{0,k}(f)$ and $\tilde{F}(k,\cdot,\cdot)$. The continuity of the retract is obtained by the smoothness of the intertwining operators $(U_k)_k$ and the continuity given by Theorem 7. $\square$

**7.2. A retract theorem for $\tilde{\pi}_{0,k}$ on an open subset of $K$.** By introducing a precise compatibility condition, we get a local retract on an open subset of $\tilde{K}$, instead of working only on a section of $\tilde{K}/K_{l_0}$. This will be used in Section 7.3 in the process of constructing a global retract theorem. It is done as follows.

Let $U=U\cdot K_{l_0}$ be a saturated non-empty open subset of $\tilde{K}$ and $C_0=C_0\cdot K_{l_0}$ be a saturated fixed compact subset contained in $U$. We denote by $\mathfrak{N}_K^{C_0}$ the space of kernel functions as defined in Section 2 and satisfying the following compatibility condition:

If $k = k' \bmod K_{l_0}$, there is a positive number $\alpha(k,k')$ such that we have

$$\begin{aligned}&\tilde{F}(k,x,y)\\ &\quad = \frac{1}{\alpha^2(k,k')} \int_{k\cdot P(l_0)/(k'\cdot P(l_0)\cap k\cdot P(l_0))} \int_{k\cdot P(l_0)/(k'\cdot P(l_0)\cap k\cdot P(l_0))} \tilde{F}\big(k',xp,yq\big)\\ &\qquad \cdot \chi_{k'\cdot l_0}(p)\overline{\chi_{k'\cdot l_0}(q)}\, d\dot{p}\, d\dot{q}\end{aligned}$$

for all $x,y \in N$. Here $\alpha(k,k')$ is defined as in Theorem 8, i.e. $\alpha(k,k') = \|(T_{k,k'}\xi)(\cdot)\|_{L^2(N/k\cdot P(l_0),\chi_{k\cdot l_0})}$ for any $\xi \in \mathcal{S}(N/k'\cdot P(l_0),\chi_{k'\cdot l_0})$ of $L^2$-norm 1, where

$$T_{k,k'}\xi(x) = \int_{k\cdot P(l_0)/(k'\cdot P(l_0))\cap(k\cdot P(l_0))} \xi(xp)\chi_{k'\cdot l_0}(p)\, d\dot{p}$$

and where $\mathcal{S}(N/k'\cdot P(l_0),\chi_{k'\cdot l_0})$ denotes the space of $\mathcal{C}^\infty$-vectors of the representation $\tilde{\pi}_{0,k'}$. It is easy to check that $\alpha(k,k')$ is left invariant, that is, that $\alpha(k_1k,k_1k') = \alpha(k,k')$ for all $k,k',k_1 \in K$ (see also [10]).

THEOREM 10. *There exists a nonempty open subset $U = U\cdot K_{l_0}$ in $\tilde{K}$ such that for every saturated compact subset $C_0 = C_0\cdot K_{l_0}$ in $U$, we have: For every $\tilde{F} \in \mathfrak{N}_K^{C_0}$, there exists a Schwartz function on $N$, such that $\tilde{\pi}_{0,k}(f)$ has $\tilde{F}(k,\cdot,\cdot)$ as an operator kernel for all $k \in C_0$ and $\tilde{\pi}_{0,k}(f) = 0$ if $k \in \tilde{K}\backslash C_0$. Moreover, $\tilde{\pi}_{0,k}(f) = 0$ if $k \in K\setminus\tilde{K}$.*

*There exists a precise algorithm for the construction of $f$, such that the map*

$$\tilde{F} \longmapsto R(\tilde{F}) := f$$

*is continuous in the given topologies.*

*Proof.* Let $(V,\eta)$ and $V'$ be as in Theorem 9. Let $U := V'\cdot K_{l_0}$. It then suffices to use Theorem 9 for $\tilde{F}_0 := \tilde{F}|_{V'\times N\times N}$, and the compatibility condition. ☐

Let us now translate the previous result to any point of $K$. Let $W$ be a nonempty open saturated subset of $\tilde{K}$ such that Theorem 10 holds and let $k_1 \in K$. Then, we consider the translated nonempty open subset $W_1 := k_1\cdot W$ of $K$ and take a fixed saturated compact subset $C_1 := C_1\cdot K_{l_0}$ in $W_1$. We denote by $\mathfrak{N}_K^{C_1}$ the corresponding space of kernel functions.

THEOREM 11. *Let $W$ be a nonempty open saturated (with respect to $K_{l_0}$) subset of $\tilde{K}$ such that Theorem 10 holds. For any $k_1 \in K$, let $W_1 = k_1\cdot W$ which is a nonempty subset of $K$ and let $C_1$ be any saturated compact subset contained in $W_1$. For every $\tilde{F} \in \mathfrak{N}_K^{C_1}$, there exists a Schwartz function $f_1$ on $N$, such that $\tilde{\pi}_{0,k}(f_1)$ has $\tilde{F}_1(k,\cdot,\cdot)$ as an operator kernel for all $k \in C_1$ and $\tilde{\pi}_{0,k}(f_1) = 0$ if $k \in K\backslash C_1$.*

*Moreover, there exists a precise algorithm for the construction of $f_1$ such that the map*

$$\tilde{F}_1 \longmapsto R(\tilde{F}_1) := f_1$$

*is continuous in the given topologies.*

*Proof.* Let us define

$$\tilde{F}(k,x,y) := \tilde{F}_1(k_1k, k_1x, k_1y).$$

It is clear that $\tilde{F}$ is well defined and $\operatorname{supp}\tilde{F}$ in $k$ is a compact subset contained in $C_0 = k_1^{-1}C_1$. Moreover, by a simple computation, one shows that $\tilde{F}$ verifies the covariance condition with respect to the polarization $k\cdot P(l_0)$ and also the GS-property.

In order to show that $\tilde{F}$ satisfies the compatibility condition, let us remark that if $k = k' \bmod K_{l_0}$, $\alpha(k,k')$ previously defined is invariant by translation by $k_1$. This implies that

$$\begin{aligned}
&\tilde{F}(k,x,y)\\
&\quad := \tilde{F}_1(k_1k, k_1\cdot x, k_1\cdot y)\\
&\quad = \frac{1}{\alpha^2(k_1k,k_1k')}\int_{k_1k\cdot P(l_0)/(k_1k'\cdot P(l_0)\cap k_1k\cdot P(l_0))}\int_{k_1k\cdot P(l_0)/(k_1k'\cdot P(l_0)\cap k_1k\cdot P(l_0))}\\
&\qquad \tilde{F}_1\big(k_1k', (k_1\cdot x)p, (k_1\cdot y)q\big)\chi_{k_1k'\cdot l_0}(p)\overline{\chi_{k_1k'\cdot l_0}(q)}\,d\dot{p}\,d\dot{q}\\
&\quad = \frac{1}{\alpha^2(k_1k,k_1k')}\int_{k_1k\cdot P(l_0)/(k_1k'\cdot P(l_0)\cap k_1k\cdot P(l_0))}\int_{k_1k\cdot P(l_0)/(k_1k'\cdot P(l_0)\cap k_1k\cdot P(l_0))}\\
&\qquad \tilde{F}\big(k', x(k_1^{-1}\cdot p), y(k_1^{-1}\cdot q)\big)\chi_{k'\cdot l_0}\big(k_1^{-1}\cdot p\big)\overline{\chi_{k'\cdot l_0}\big(k_1^{-1}\cdot q\big)}\,d\dot{p}\,d\dot{q}\\
&\quad = \frac{1}{\alpha^2(k,k')}\int_{k\cdot P(l_0)/(k'\cdot P(l_0)\cap k\cdot P(l_0))}\int_{k\cdot P(l_0)/(k'\cdot p(l_0)\cap k\cdot P(l_0))}\\
&\qquad \tilde{F}\big(k', xp_1, yq_1\big)\chi_{k'\cdot l_0}(p_1)\overline{\chi_{k'\cdot l_0}(q_1)}\,d\dot{p_1}\,d\dot{q_1}\quad \big(p_1 = k_1^{-1}p \text{ and } q_1 = k_1^{-1}q\big),
\end{aligned}$$

that is, $\tilde{F}$ has the correct compatibility condition. Finally, as $\tilde{F}$ satisfies the hypotheses of Theorem 10, there exists a Schwartz function $f$ in $N$ such that $\tilde{\pi}_{0,k}(f)$ has $\tilde{F}(k,\cdot,\cdot)$ as an operator kernel for all $k \in C_0$ and $\tilde{\pi}_{0,k}(f) = 0$ if $k \in K\backslash C_0$. Let us define $f_1(x) := f^{k_1^{-1}}(x) = f(k_1^{-1}\cdot x)$. It is then easy to check that $\tilde{\pi}_{0,k}(f_1)$ has $\tilde{F}_1(k,\cdot,\cdot)$ as an operator kernel for all $k \in C_1$ and $\tilde{\pi}_{0,k}(f_1) = 0$ if $k \in K\backslash C_1$.

The continuity of the retract is given by the fact that the translation is continuous and that the retract given by Theorem 10 is continuous. □

**7.3. A global retract theorem for $\tilde{\pi}_{0,k}$.** Let us denote by $\mathfrak{N}$ the space of the smooth kernel functions defined on $K \times N \times N$, satisfying the GS-property, the covariance condition with respect to the polarizations $k\cdot P(l_0)$ and the compatibility condition defined in the previous section.

THEOREM 12. *For every* $\breve{F} \in \mathfrak{N}$, *there exists a Schwartz function* $\tilde{f}$ *on* $N$ *such that* $\tilde{\pi}_{0,k}(\tilde{f})$ *has* $\breve{F}(k,\cdot,\cdot)$ *as an operator kernel for all* $k \in K$.

*Moreover, there exists for the construction of the retract a precise algorithm such that the map*

$$\breve{F} \longmapsto R(\breve{F}) := \tilde{f}$$

*is continuous in the given topologies.*

*Proof.* Let us denote by $(h_i)_{i\in I}$ a partition of unity associated to a finite number of translates of $W$ obtained by Theorem 10. As the translates of $W$ are all saturated with respect to $K_{l_0}$, we may assume that all the functions $h_i$ are constant on the classes modulo $K_{l_0}$. Then we define for all $i \in I$,

$$\tilde{F}_i(k,x,y) := h_i(k)\breve{F}(k,x,y).$$

The functions $F_i$ satisfy the hypotheses of Theorem 11. This implies that for all $i \in I$, there exists $f_i \in \mathcal{S}(N)$ such that $\tilde{\pi}_{0,k}(f_i)$ has $\tilde{F}_i(k,\cdot,\cdot)$ as an operator kernel for all $k \in K$. Since

$$\breve{F}(k,x,y) = \sum_{i\in I} h_i(k)\breve{F}(k,x,y) = \sum_{i\in I} \tilde{F}_i(k,x,y)$$

for all $k \in K$, it is sufficient to consider $\tilde{f} := \sum_{i\in I} f_i \in \mathcal{S}(N)$. Then $\pi_{0,k}(\tilde{f})$ has $\breve{F}(k,\cdot,\cdot)$ as an operator kernel for all $k \in K$.

For a fixed partition of unity of $K$, the maps $\breve{F} \longmapsto \tilde{F}_i$ are continuous. The continuity of the retract of $\tilde{F}_i$ given in Theorem 11, allows to conclude to the continuity of $\breve{F} \longmapsto R(\breve{F})$. □

This theorem is important for its own sake, but it is also useful to construct other types of retracts.

COROLLARY 1. *For every* $k_0 \in K$, *there exists a local section* $(S,\zeta)$ *of* $K/K_{l_0}$ *containing* $\dot{k}_0 = k_0 \bmod K_{l_0}$ *such that* $\zeta(S)$ *is a submanifold of* $K$. *Let* $C_0$ *be any compact subset contained in* $\zeta(S)$. *For every* $\tilde{F} \in \mathfrak{N}^{C_0}_{\zeta(S)}$, *there exists a Schwartz function* $f$ *on* $N$ *such that* $\tilde{\pi}_{0,k}(f)$ *has* $\tilde{F}(k,\cdot,\cdot)$ *as an operator kernel if* $k \in \zeta(S)$ *and* $\tilde{\pi}_{0,k}(f) = 0$ *if* $\dot{k} \notin S$.

*Moreover, there exists a precise algorithm such that the map*

$$\tilde{F} \longmapsto R(\tilde{F}) := f$$

*is continuous in the given topologies.*

## 8. $K$-retract theory for ${}^{k}\pi_{l_0}$

As in the previous sections, $l_0$ will either be aligned and generic in the sense of Ludwig–Zahir, or $l_0 \in \mathcal{S}_j$ aligned, $5 \le j \le 10$, if $N = F_{4,2}$. In this part of the paper, we shall also give a retract theorem for the representation ${}^k\pi_{l_0}$ defined as follows:

$$ {}^k\pi_{l_0}(x) := \pi_{l_0}\big(k^{-1}\cdot x\big), \quad x\in N. $$

This result will be necessary for the study of the density of Schwartz functions in the $K$-orbits and for the characterization of $K$-prime ideals in Sections 9 and 10.

It is easy to check that ${}^k\pi_{l_0}$ is unitary equivalent to $\tilde{\pi}_{0,k}$ and that the corresponding intertwining operator is given by:

$$ \begin{aligned} \mathcal{U}: \mathcal{H}_{{}^k\pi_{l_0}} &\longrightarrow \mathcal{H}_{\tilde{\pi}_{0,k}}, \\ \xi &\longmapsto \mathcal{U}\xi(t) := \xi\big(k^{-1}\cdot t\big). \end{aligned} $$

This equivalence will allow us to deduce a local retract theorem for ${}^k\pi_{l_0}$ on a section of $K/K_{l_0}$ from a local retract theorem for $\tilde{\pi}_{0,k}$ on the same section.

THEOREM 13. *For every $k_0\in K$, there exists a local section $(S,\zeta)$ of $K/K_{l_0}$ containing $\dot{k}_0 = k_0 \bmod K_{l_0}$ such that $\zeta(S)$ is a submanifold of $K$. Let $C_0$ be any compact subset contained in $\zeta(S)$. For every $\tilde{\tilde{F}}\in\mathfrak{N}^{C_0}_{\zeta(S)}$, there exists a Schwartz function $f$ on $N$ such that ${}^k\pi_{l_0}(f)$ has $\tilde{\tilde{F}}(k,\cdot,\cdot)$ as an operator kernel if $k\in\zeta(S)$ and ${}^k\pi_{l_0}(f)=0$ if $\dot{k}\notin S$.*

*Moreover, there exists a precise algorithm such that the map*

$$ \tilde{\tilde{F}} \longmapsto R(\tilde{\tilde{F}}) := f $$

*is continuous in the given topologies.*

*Proof.* Let us define $\tilde{F}$ as follows:

$$ \begin{aligned} \tilde{F}(k,x,y) &:= \mathcal{U}\otimes\mathcal{U}\tilde{\tilde{F}}(k,x,y) \\ &= \tilde{\tilde{F}}\big(k,k^{-1}\cdot x,k^{-1}\cdot y\big). \end{aligned} $$

Then $\tilde{F}$ satisfies the hypotheses of Corollary 1 and there exists $f\in\mathcal{S}(N)$ such that $\tilde{\pi}_{0,k}(f)$ has $\tilde{F}(k,\cdot,\cdot)$ as an operator kernel if $k\in\zeta(S)$ and $\tilde{\pi}_{0,k}(f)=0$ if $\dot{k}\notin S$.

So

$$ \begin{aligned} \tilde{\tilde{F}}(k,x,y) &= \tilde{F}(k,k\cdot x,k\cdot y) \\ &= \int_{k\cdot P(l_0)} f\big(kx\cdot p\cdot (ky)^{-1}\big)\chi_{k\cdot l_0}(p)\,dp \\ &= \int_{k\cdot P(l_0)} f^k\big(x\big(k^{-1}p\big)y^{-1}\big)\chi_{l_0}\big(k^{-1}p\big)\,dp \quad \big(p_1 = k^{-1}\cdot p\big) \\ &= \int_{P(l_0)} f^k\big(xp_1y^{-1}\big)\chi_{l_0}(p_1)\,dp_1 \end{aligned} $$

is the operator kernel of $\pi_{l_0}(f^k) = {}^k\pi_{l_0}(f)$, for $k \in C_0$. If $k \notin C_0 \cdot K_{l_0}$, $\tilde{\pi}_{0,k}(f) = 0$ and ${}^k\pi_{l_0}(f) = 0$ as $\tilde{\pi}_{0,k}$ and ${}^k\pi_{l_0}$ are equivalent. □

## 9. Density of Schwartz functions in the orbit

The aim of this section is to study the relationship between the kernels of the $K$-orbits $\operatorname{Ker}\Omega_l$ and $\operatorname{Ker}\Omega_l \cap \mathcal{S}(N)$. In order to have this relationship, we rely essentially on the retract theory of Section 8 and on previous results for an exponential action [15].

Let us first start by giving some definitions.

Definition 7. For all $l \in \mathfrak{n}^*$, we denote by $\Omega_l$ the $K$-orbit given by

$$\Omega_l := \{k \cdot l | k \in K\}.$$

Since ${}^k\pi_l$, $\tilde{\pi}_k = \operatorname{ind}_{k\cdot P(l)}^N \chi_{k\cdot l}$ and $\pi_{k\cdot l} = \operatorname{ind}_{P(k\cdot l)}^N \chi_{k\cdot l}$ are equivalent, we can justify the following definition of the kernel of an orbit:

Definition 8. Let $l \in \mathfrak{n}^*$. We define the kernel of the orbit $\Omega_l$ by

$$\begin{aligned}\operatorname{Ker}\Omega_l &= \left\{f \in L^1(N) | {}^k\pi_l(f) = 0 \ \forall k \in K\right\}\\ &= \left\{f \in L^1(N) | \pi_{k\cdot l}(f) = 0 \ \forall k \in K\right\}.\end{aligned}$$

We take $l_0$ an aligned and generic linear form in the sense of Ludwig–Zahir (or $l_0$ aligned and $l_0 \in \mathcal{S}_j$, $5 \le j \le 10$, in the case of $N = F_{4,2}$). Let us choose a finite family of sections $(S_i, \zeta_i)_{i\in I}$, $I$ finite, of $K/K_{l_0}$ together with finite families of open sets with compact closure $W_i \subset \overline{W}_i \subset V_i \subset S_i$, covering $K/K_{l_0}$, and a corresponding partition of unity of $\mathcal{C}^\infty$-functions $(h_i)_{i\in I}$ of $K/K_{l_0}$ such that $\operatorname{supp} h_i \subset W_i \subset \overline{W}_i$ for all $i$ and $\sum_{i\in I} h_i \equiv 1$ on $K/K_{l_0}$. We may even assume that each $W_i$ is completely contained in at least one chart of the manifold $K/K_{l_0}$. Then Theorem 13 may be applied to the sections $(S_i, \zeta_i)$.

Remark 12.

1. Every $h_i$ can be identified with a smooth function with compact support in a space $\mathbb{R}^p$ since $\operatorname{supp} h_i$ is totally contained in one chart of $K/K_{l_0}$.
2. $\zeta_i(\overline{W}_i)$ is a compact subset contained in $\zeta_i(S_i)$.

### 9.1. Relationship between $f$ and $f_i$.

Let us assume that either $l_0$ is an aligned and generic (in the sense of Ludwig–Zahir) linear form, or an aligned element of $\mathcal{S}_j$ ($5 \le j \le 10$, in the case of the group $F_{4,2}$), or defines a character. Let $f \in \mathcal{S}(N)$. We denote by $\tilde{F}(k, \cdot, \cdot)$ the kernel of the operator ${}^k\pi_{l_0}(f)$ for all $k \in K$ and $\tilde{\tilde{F}}(k, \cdot, \cdot)$ the kernel of the operator ${}^k\pi_{l_0}(f)$ for all $k \in \zeta(S)$. Let us consider

$$\tilde{F}_i(k, \cdot, \cdot) := h_i(\dot{k})\tilde{F}(k, \cdot, \cdot) \quad \forall k \in K,$$

$$\tilde{\tilde{F}}_i := \tilde{F}_i|_{\zeta_i(S_i)\times N\times N}.$$

It is then easy to see that $\tilde{\tilde{F}}_i \in \mathfrak{N}^{C_i}_{\zeta_i(S_i)}$ where $C_i := \zeta_i(\overline{W}_i)$ is a fixed compact subset contained in $\zeta_i(S_i)$. By applying Theorem 13, there exists $f_i \in \mathcal{S}(N)$ such that ${}^k\pi_{l_0}(f_i)$ has $\tilde{\tilde{F}}_i(k,\cdot,\cdot)$ as an operator kernel for every $k \in \zeta_i(S_i)$ and ${}^k\pi_{l_0}(f_i) = 0$ if $\dot{k} \notin S_i$.

Let us define

$$\tilde{f} := \sum_{i\in I} f_i \in \mathcal{S}(N).$$

Then we have the following proposition.

PROPOSITION 2. *For any* $f \in \mathcal{S}(N)$,

$$f = \tilde{f} \mod \operatorname{Ker} \Omega_{l_0} \cap \mathcal{S}(N).$$

*Proof.* The result is obtained by a simple computation, using the fact that ${}^k\pi_{l_0}(f_i) = h_i(\dot{k})({}^k\pi_{l_0}(f))$, for all $k \in K$. □

**9.2. Relationship between** $\operatorname{Ker}\Omega_{l_0}$ **and** $\operatorname{Ker}\Omega_{l_0} \cap \mathcal{S}(N)$**.** Let $\psi \in \mathcal{C}^\infty(K)$ be such that

$$\psi \geq 0, \psi(e) > 0, \quad \int_K \psi(k)\, dk = 1.$$

For every $f \in L^1(N)$, we consider

$$f^\sharp := \int_K f^k \psi(k)\, dk,$$

where $f^k(x) := f(k\cdot x)$ for all $x \in N$.

It is easy to check that the operator kernel of

$${}^k\pi_l\big(f^\sharp\big) = \int_K {}^{k''}\pi_l(f)\psi\big(k''k^{-1}\big)\, dk''$$

is given by

$$\tilde{F}^\sharp(k,\cdot,\cdot) = \int_K \tilde{F}\big(k'',\cdot,\cdot\big)\psi\big(k''k^{-1}\big)\, dk''.$$

LEMMA 1. *For every section* $(S,\zeta)$ *of* $K/K_{l_0}$, *for every compact subset of* $K$ *of the form* $\zeta(\overline{W})$, *with* $\overline{W}$ *compact and* $\overline{W} \subset V \subset \overline{V} \subset S$ *for some open subset* $V$ *of* $K/K_{l_0}$, $\zeta(\overline{W})$ *is contained in* $p^{-1}(V) = V\cdot K_{l_0} \subset p^{-1}(S) = S\cdot K_{l_0}$. *Moreover, there exists a compact neighborhood* $K_0$ *of* $e$ *in* $K$ *such that*

$$K_0^{-1}\cdot\zeta(\overline{W}) \subset p^{-1}(V) \subset p^{-1}(\overline{V}) \subset p^{-1}(S),$$

*where* $p: K \longrightarrow K/K_{l_0}$ *is the canonical projection.*

*Proof.* Obvious. □

Let us now assume that $\tilde{F}(k,\cdot,\cdot)$ is a kernel function associated to the representations ${}^k\pi_{l_0}$ and that the support of $\tilde{F}$ in $k$ is contained in $\zeta(\overline{W})\cdot K_{l_0}$. Let $K_0$ be a compact neighborhood of $e$ as in Lemma 1 and let $\psi$ be chosen as previously such that $\operatorname{supp}\psi \subset K_0$. Let us consider $\tilde{F}^\sharp$ and $\tilde{\tilde{F}}^\sharp := \tilde{F}^\sharp\,|_{\zeta(S)\times N\times N}$. Then the support of $\tilde{\tilde{F}}^\sharp$ is contained in

$$K^\sharp := \big(K_0^{-1}\cdot\zeta(\overline{W})\cdot K_{l_0}\big)\cap\zeta(S) = \big(K_0^{-1}\cdot\zeta(\overline{W})\cdot K_{l_0}\big)\cap\zeta(\overline{V})$$

which is a compact subset of $K$, contained in $\zeta(S)$. If we replace $f$, $\tilde{F}$, $W$ by $f_i$, $\tilde{F}_i$, $W_i$, $i\in I$, as in Section 9.1, we may of course choose the same $K_0$ and the same function $\psi$ for all $i\in I$. This will be done in Proposition 3.

PROPOSITION 3. *If $l_0\in\mathfrak{n}^*$ is a generic linear form in the sense of Ludwig–Zahir, or $l_0\in\mathcal{S}_j$ $(5\le j\le 10$, in the case of the group $F_{4,2})$ or if $l_0$ defines a character on $N$, we have*

$$L^1(N)*L^1(N)*\operatorname{Ker}\Omega_{l_0}*L^1(N)*L^1(N)\subset\overline{\operatorname{Ker}\Omega_{l_0}\cap\mathcal{S}\big(N\big)}^{L^1(N)}.$$

*Proof.* The proof of this proposition is an adaptation to the case of a compact action, of the corresponding proof in the case of an exponential action [15]. We will only give the parts of the proof which are really essential, which differ from the one of [15], or which show the use of the retract results. For the other parts of the proof, we refer to [15].

Let us first assume that $l_0$ is generic in the sense of Ludwig–Zahir or that $l_0\in\mathcal{S}_j$. We may of course assume that $l_0$ is aligned. By Hahn–Banach, we have to show that for every $\phi\in L^\infty(N)$ such that $\langle\phi,\operatorname{Ker}\Omega_{l_0}\cap\mathcal{S}(N)\rangle=0$, we also have

$$\big\langle\phi, L^1(N)*L^1(N)*\operatorname{Ker}\Omega_{l_0}*L^1(N)*L^1(N)\big\rangle=0.$$

Let $f\in\mathcal{S}(N)$ and $g_1,g_2,g_3,g_4\in\mathcal{S}(N)$. We apply the arguments of [15] to the functions $f_i$ given by the Section 9.1.

Let us fix $i\in I$. As in [15], we define $(g_1*g_2*f_i*g_3*g_4)^\sharp$ and we consider the corresponding operator kernel for the representations ${}^k\pi_{l_0}$, which we denote $(\tilde{G}_1\circ\tilde{G}_2\circ\tilde{\tilde{F}}_i\circ\tilde{G}_3\circ\tilde{G}_4)^\sharp$, according to the notation of Section 9.2.

One shows that $\operatorname{supp}(\tilde{G}_1\circ\tilde{G}_2\circ\tilde{\tilde{F}}_i\circ\tilde{G}_3\circ\tilde{G}_4)^\sharp\subset C_i^\sharp\times N\times N$ where $C_i^\sharp := (K_0^{-1}\cdot\zeta_i(\overline{W_i})\cdot K_{l_0})\cap\zeta_i(S_i)$ is a compact subset of $\zeta_i(S_i)$, as shown in the proof of Lemma 1.

Let $\mathcal{S}(C_i^\sharp,\mathcal{S}(\mathbb{R}^d\times\mathbb{R}^d))$ denote the set of $\mathcal{C}^\infty$ functions with support in the fixed compact set $C_i^\sharp$ and with values in the Schwartz space $\mathcal{S}(\mathbb{R}^d\times\mathbb{R}^d)$, where $d$ denotes the dimension of $\mathfrak{n}/\mathfrak{p}(l_0)$ and where $\mathfrak{p}(l_0)$ is a polarization at $l_0$.

Thanks to the covariance relation and to a fixed Malcev basis of $\mathfrak{n}$ with respect to $\mathfrak{p}(l_0)$, we may identify $(\tilde{G}_1\circ\tilde{G}_2\circ\tilde{\tilde{F}}_i\circ\tilde{G}_3\circ\tilde{G}_4)^\sharp$ with an element of $\mathcal{S}(C_i^\sharp,\mathcal{S}(\mathbb{R}^d\times\mathbb{R}^d))$.

We define a linear form $\mu_i$ on $\mathcal{S}(C_i^\sharp,\mathcal{S}(\mathbb{R}^d\times\mathbb{R}^d))$ by

$$\langle\mu_i,G\rangle:=\langle\phi,g\rangle,$$

where $g = R(G)$ is the retract of $G$ obtained by Theorem 13. It is well defined as $\langle \phi, \operatorname{Ker}\Omega_{l_0} \cap \mathcal{S}(N)\rangle = 0$.

As $C_i^\sharp$ is entirely contained in one chart of $K/K_{l_0}$, it may be identified with a compact subset of some $\mathbb{R}^p$ and $\mu_i$ may be identified with a tempered distribution. It is hence of the form

$$\begin{aligned}&\langle \mu_i, G\rangle\\ &\quad = \sum_{|\alpha|+|\beta|+|\gamma|\le M} \int_{C_i^\sharp} \tau_{\alpha,\beta,\gamma}(k,x,y)\left(\frac{\partial}{\partial k}\right)^{\alpha}\left(\frac{\partial}{\partial x}\right)^{\beta}\left(\frac{\partial}{\partial y}\right)^{\gamma} G(k,x,y)\,dk\,dx\,dy,\end{aligned}$$

where $\tau_{\alpha,\beta,\gamma}$ is a continuous function with moderate growth and where $k$ is identified with its coordinates in the given chart (see [22]). We then define a linear form $\phi^\sharp$ on $\mathcal{S}(N)$ by

$$\langle \phi^\sharp, f\rangle := \langle \phi, (g_1 * g_2 * f * g_3 * g_4)^\sharp\rangle.$$

Using the precise form of $\mu_i$, the same arguments as in Proposition 4.1 of [15] show that

$$\begin{aligned}\left|\langle \phi^\sharp, f_i\rangle\right| &\le C(i,\psi)\cdot \sup_{k\in K}\left\|{}^k\pi_{l_0}(f_i)\right\|_{\mathrm{op}}\\ &\le C(i,\psi)\cdot \sup_{k\in K}\left\|{}^k\pi_{l_0}(f)\right\|_{\mathrm{op}}\\ &\le C(i,\psi)\|f\|_1\end{aligned}$$

for some constant $C(i,\psi)$.

Let us put $C(\psi) := \sum_{i\in I} C(i,\psi)$. As $f = \tilde{f} + h$ for some $h \in \operatorname{Ker}\Omega_{l_0} \cap \mathcal{S}(N)$ and as $\langle \phi, \operatorname{Ker}\Omega_{l_0} \cap \mathcal{S}(N)\rangle \equiv 0$, it is easy to check that $\langle \phi^\sharp, h\rangle = 0$ and hence

$$\begin{aligned}\left|\langle \phi^\sharp, f\rangle\right| &= \left|\langle \phi^\sharp, \tilde{f}\rangle\right|\\ &\le \sum_{i\in I}\left|\langle \phi^\sharp, f_i\rangle\right|\\ &\le C(\psi)\cdot \sup_{k\in K}\left\|{}^k\pi_{l_0}(f)\right\|_{\mathrm{op}}\\ &\le C(\psi)\|f\|_1\end{aligned}$$

for all $f \in \mathcal{S}(N)$. So the linear form $\phi^\sharp$ may be extended to all $L^1(N)$, with the same bounds. Moreover, if $f \in \operatorname{Ker}\Omega_{l_0}$, then $\langle \phi^\sharp, f\rangle = 0$. We finish the proof as in Proposition 4.1 of [15].

If $\langle l_0, [\mathfrak{n},\mathfrak{n}]\rangle = 0$, then ${}^k\pi_{l_0} = \chi_{k\cdot l_0}$ is a character and a simplified version of the previous arguments holds. $\square$

THEOREM 14. *Let $l_0$ be a generic linear form in the sense of Ludwig–Zahir or let $l_0 \in \mathcal{S}_j$* $(5 \le j \le 10$, *in the case of the group* $F_{4,2})$, *or let $l_0$ be a linear* *form defining a character, that is,* $\langle l_0, [\mathfrak{n}, \mathfrak{n}]\rangle = 0$. *Then*

$$\operatorname{Ker}\Omega_{l_0} = \overline{\operatorname{Ker}\Omega_{l_0} \cap \mathcal{S}(N)}^{L^1(N)}.$$

*Proof.* Let $\upsilon_\iota$ be an approximate identity of $L^1(N)$. Then, for all $f \in \operatorname{Ker}\Omega_{l_0}$, we have

$$\begin{aligned}\upsilon_\iota * \upsilon_\iota * f * \upsilon_\iota * \upsilon_\iota &\in L^1(N) * L^1(N) * \operatorname{Ker}\Omega_{l_0} * L^1(N) * L^1(N)\\ &\subset \overline{\operatorname{Ker}\Omega_{l_0} \cap \mathcal{S}(N)}^{L^1(N)}.\end{aligned}$$

By taking the limit in $\iota$, the result holds. □

## 10. $K$-prime ideals

DEFINITION 9. An ideal $I$ is a $K$-prime ideal of $L^1(N)$, if $I$ is $K$-invariant and if, for all $K$-invariant ideals $I_1$, $I_2$ of $L^1(N)$,

$$I_1 * I_2 \subset I \quad \Longrightarrow \quad I_1 \subset I \quad \text{or} \quad I_2 \subset I.$$

THEOREM 15. *Let $I$ be a proper closed $K$-prime ideal of $L^1(N)$. Then there exists a $K$-orbit $\Omega_{l_0}$ in $\mathfrak{n}^*$ such that*

$$I \cap \mathcal{S}(N) = \operatorname{Ker}\Omega_{l_0} \cap \mathcal{S}(N).$$

*If $l_0$ is a generic linear form in the sense of Ludwig–Zahir (for at least one Jordan–Hölder basis), or $l_0 \in \mathcal{S}_i$ (in the case of the group $F_{4,2}$, $i = 5, \ldots, 10$) or if $l_0$ defines a character ($\langle l_0, [\mathfrak{n}, \mathfrak{n}]\rangle = 0$), we have*

$$I = \operatorname{Ker}\Omega_{l_0}.$$

*Proof.* The proof of this result is an adaptation of the proof for an exponential action [15]. As a matter of fact, $I \cap \mathcal{S}(N)$ is a $K$-prime ideal of $\mathcal{S}(N)$ which is closed in the continuous norm $\|\cdot\|_1$. By [16] which remains valid for compact actions, it has to be of the form $\operatorname{Ker}\Omega_{l_0} \cap \mathcal{S}(N)$. We use Theorem 14 to finish the proof. □

EXAMPLES. In the cases of the action of $\mathrm{SO}(2n)$ on $H_n$ (Heisenberg group), of $\mathrm{SO}(3)$ on $F_{3,2}$ (free nilpotent Lie group of step 2 on 3 generators) or of $\mathrm{SO}(4)$ on $F_{4,2}$, the $K$-prime ideals of $L^1(N)$ coincide with the kernels of $K$-orbits.

As a matter fact, in the cases of $H_n$ or $F_{3,2}$ every linear form $l_0$ is either generic in the sense of Ludwig–Zahir with respect to a suitable Jordan–Hölder basis or it defines a character. In the case of $F_{4,2}$, a Fourier inversion theorem also exists for the intermediate layers and hence the result on $K$-prime ideals remains correct.

This leads to the following conjecture:

CONJECTURE 1. *Let $N$ be a connected, simply connected, nilpotent Lie group and let $K \subset \operatorname{Aut}(N)$ be a compact subgroup of the automorphism group of $N$, acting smoothly on $N$. Then every $K$-prime ideal coincides with the kernel of a $K$-orbit.*

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