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# THE EXACTNESS OF CERTAIN RANDOMIZED $C^*$-ALGEBRAS

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ABSTRACT. We construct a non-atomic strong operator topology-dense probability measure on the set of unitary operators acting on a separable Hilbert space, such that the $C^*$-algebra generated by $n \geq 3$ independently chosen random unitaries is almost surely non-exact.

## 1. Introduction

Recall that a $C^*$-algebra $E$ is called *exact* if for every short exact sequence $0 \to A \to B \to C \to 0$ of $C^*$-algebras $A, B$ and $C$ we have a short exact sequence

$$0 \to E \otimes_{\min} A \to E \otimes_{\min} B \to E \otimes_{\min} C \to 0.$$

Many constructions of $C^*$-algebras are given by a couple or countable many generators satisfying certain "ideal" relations. Examples we have in mind are the group $C^*$-algebra of a discrete group, or the Cuntz algebra $\mathcal{O}_n$. Then often these $C^*$-algebras satisfy certain nice properties, for example the property to be exact. But what happens if one little bit disturbs the "perfect" relations of such constructions. What, if the relations of the generators of the $C^*$-algebra are "unperfect" and "non-constructed"? Or, what happens if we choose the relations by random? Is then the resulting $C^*$-algebra exact? Almost never, or almost sure?

In this paper, we give a partial answer to this question as follows ($H$ is always a separable Hilbert space, and $\mathcal{U}(B(H))$ denotes the unitary group of $B(H)$). Recall that the distribution $\mu$ of a random element $x : \Omega \to X$, where $X$ is a measurable space and $(\Omega, \mathbb{P})$ is the underlying probability space, is

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the probability measure $\mu$ on $X$ given by $\mu(A) = \mathbb{P}(x^{-1}(A)) = \mathbb{P}(x \in A)$ for measurable subsets $A \subseteq X$.

THEOREM 1.1. *There exits a non-atomic strong operator topology-dense probability measure $\mathbb{P}_{\mathcal{U}}$ on $\mathcal{U}(B(H))$, such that independent $\mathbb{P}_{\mathcal{U}}$-distributed random variables $U_1, \dots, U_n$ generate almost surely a non-exact $C^*$-algebra $C^*(U_1, \dots, U_n)$ when $n \geq 3$.*

We point out that we use a classical "commutative" probability space, but the random objects are non-commutative algebras. On the other hand, in non-commutative or quantum probability theory (recent samples are for example [4], [6], [7]), the probability space itself is non-commutative, but the random objects could be interpreted as real or complex valued in their primary intention. It is worth to mention that algebraical random objects were also considered in other categories, for example random groups by Gromov in [1]. Haagerup and Thorbjørnsen show in [2, Corollary 8.4] that there exists a probability measure $P_{\text{free}}$ on the unitary group of the ultraproduct $\prod_{n \in \mathbb{N}} M_n / \sum_{n \in \mathbb{N}} M_n$ such that $r$ independent $P_{\text{free}}$-distributed random unitaries generate almost surely the reduced $C^*$-algebra $C^*_\lambda(\mathbb{F}_r)$ of the free groups $\mathbb{F}_r$. In this sense, our theorem might be regarded as the full $C^*$-algebraic counterpart of the result of Haagerup and Thorbjørnsen: perhaps $C^*(U_1, \dots, U_n)$ of Theorem 1.1 is almost surely the full $C^*$-algebra $C^*(\mathbb{F}_r)$, see Remark 4.4 why one may conjecture this. There is even some overlapping operator space theoretical technique of proof, compare [2, Section 2] with Sections 3 and 4.

This paper is organized as follows. In Section 2, we introduce a family of probability measures on $B(H)$. A random $C^*$-algebra is then a $C^*$-algebra $C^*(x_1, \dots, x_n)$ which is generated by $n$ (usually independent and identically distributed) random elements $x_1, \dots, x_n$ in $B(H)$. In Section 3, we recall the local theory of operator spaces (completely bounded Banach–Mazur distance) investigated by Pisier, and we state exactness criteria for exact $C^*$-algebras due to him ([8], [10]). The proof of Theorem 1.1 is heavily relying on these results. We highlight that it is possible and also promising to ask whether a random $C^*$-algebra is an exact $C^*$-algebra (to be precise, it is possible to ask for the probability that the $C^*$-algebra is exact).

In Section 4, we introduce the notion of widely spread isometries $(S_1, \dots, S_n)$. We show that in that case the $C^*$-algebra $C^*(S_1, \dots, S_n)$ is not exact (here we use the theory in [8]), provided $n \geq 3$. In Section 5, we use this result to prove Theorem 1.1.

## 2. Probability measures on $B(H)$ and random $C^*$-algebras

Let $H$ be a separable Hilbert space. Since $B(H)$ is non-separable, there is some radius $r > 0$ such that $B(H)$ contains uncountably many disjoint balls $B_i$ with radius $r$. Hence, if we have a probability measure $\mathbb{P}$ on $B(H)$, then $\mathbb{P}(B_i)$ is non-zero only for countably many indices $i$'s. In this respect, a probability measure on $B(H)$ is always unsatisfying. However, this drawback can be compensated to some extend by just requiring $\mathbb{P}(X) > 0$ for all open strong operator topology neighborhoods $X$ in $B(H)$. In this case we say that $\mathbb{P}$ is *strong operator topology-dense.*

If one asks for translation invariant measures the answers are well known. There does not exist a translation invariant measure which generalizes the Lebesgue measure to infinite dimensional spaces. Also, there does not exist a translation invariant measure on the not locally compact unitary group $\mathcal{U}(B(H))$. Anyway, there exist more or less interesting probability measures on $B(H)$.

A simple approach is to choose a discrete dense set $D$ in $\mathbb{K}$, the compact operators on $B(H)$, and endow $B(H)$ with a measure $P_D$ that is atomic on $D$ and vanishing outside of $D$. Then $P_D$ is a strong operator topology-dense measure, however, trivially a $C^*$-algebra generated by $P_D$-distributed random elements is a subalgebra of $\mathbb{K}$, and thus nuclear and exact. This example hints that the strong operator topology-density, though a natural property to require, is not a too strong indicator for the "quality" of a measure. Another more natural approach would be to consider the random operator $X = (\alpha_{i,j}T_{i,j})_{i,j\in\mathbb{N}}$ where $T$ is a Hilbert–Schmidt operator with matrix representation $T = (T_{i,j})_{i,j\in\mathbb{N}}$, and where $(\alpha_{i,j})_{i,j\in\mathbb{N}}$ is a family of independent $N(0,1)$ normal distributed random variables. However, $X$ is almost surely Hilbert–Schmidt and hence once again compact.

We regard $H$ and $B(H)$, respectively, as measurable spaces by endowing them with the Borel structure induced by the norm topology on $H$ and $B(H)$, respectively. In this respect, the following lemma is useful (and its proof is straightforward).

LEMMA 2.1. *Let $H$ be separable. The Borel structures induced by the norm, the strong operator, and the weak operator topology on $B(H)$ all coincide.*

We now introduce the type of probability measures we will use in the proof of Theorem 1.1. At first we need a probability measure on a separable Hilbert space $H$ with normal base $(e_i)_{i\geq 1}$. A natural candidate is the Wiener measure on $C([0,1])$, which we can extend to $L^2([0,1])$ (the probability measure on $L^2([0,1]) \setminus C([0,1])$ is set to zero). Another construction is the probability measure on $H$ associated to the random element

$$x = \sum_{k=1}^{\infty} a_k(\alpha_k + i\beta_k)e_k \in \ell^2(\mathbb{N}),$$

where the $\alpha_k$ and $\beta_k$ are independent $N(0,1)$ normal distributed random variables. Here $(a_k)$ is an element in $\ell^2(\mathbb{N})$. The series $x$ then converges a.s. in $H$ (since $\mathbb{E}\|x\|^2 = \mathbb{E}\sum \|a_k(\alpha_k + i\beta_k)e_k\|^2 < \infty$).

We say that a random element $x \in H$ is *norm-dense*, if $\mathbb{P}(x \in B) > 0$ for all non-empty open balls $B$ in $H$. We say that $x$ is *non-degenerate*, if $\mathbb{P}(x \in L) = 0$

for all finite dimensional subspaces $L \subseteq H$. Notice, that both above measures on $H$ are norm-dense and non-degenerate.

To construct a random element in $\mathcal{U}(B(H))$, we start with a sequence $(x_1, x_2, x_3, \dots)$ of independent non-degenerate norm-dense random elements $x_i \in H$. Then we form a normal basis $(y_1, y_2, y_3, \dots)$ of the Hilbert space $\overline{\text{lin}}\{x_1, x_2, x_3, \dots\}$ by the well-known procedure of Gram–Schmidt, that is, we define

$$y_n = \left( x_n - \sum_{i=1}^{n-1} \langle x_n, y_i \rangle y_i \right) \Bigg/ \left\| x_n - \sum_{i=1}^{n-1} \langle x_n, y_i \rangle y_i \right\|.$$

Observe that $y_n$ is a measurable random element in $H$, and $y_n$ is a.s. well defined, since the probability that we have a division by zero in $y_n$ is

$$\int \cdots \int \int 1_{\{x_n \in \text{lin}(x_1, \dots, x_{n-1})\}} \, dx_n \, dx_{n-1} \cdots dx_1 = 0$$

by the non-degenerateness of $x_n$. Then we introduce a measurable (because the map $(x_1, x_2, \dots) \mapsto y_n$ is measurable) random operator $U \in B(H)$ by $U(e_n) = y_n$ for all $n \geq 1$. In fact, $U$ is a.s. an isometry with range

$$\text{Im}(U) = \overline{\text{lin}}\{y_1, y_2, y_3, \dots\} = \overline{\text{lin}}\{x_1, x_2, x_3, \dots\}.$$

We can achieve that $U$ is a.s. a unitary operator as follows.

LEMMA 2.2. *$U$ is almost surely a unitary operator if the sequence $(x_i)$ is independent identically distributed.*

*Proof.* Given a non-empty open ball $B$ in $H$ we have infinitely many chances $i \in \mathbb{N}$ such that the event $x_i \in B$ happens, and therefore $\mathbb{P}(\exists i \geq 1 : x_i \in B) = 1$. Let $D$ be a countable dense subset of $H$, and $B(x, r)$ be the open ball in $H$ with center $x$ and radius $r$. Then

$$\mathbb{P}\big((x_i) \text{ is dense in H}\big) = \mathbb{P}\big(\forall k \geq 1, \forall x \in D, \exists i \geq 1 : x_i \in B(x, 1/k)\big) = 1.$$

Hence, $U(H) = H$ a.s. □

To obtain a random element $X$ in $B(H)$ one may take two independent random elements $U, V \in \mathcal{U}(B(H))$, and two independent normal distributed real valued random variables $\alpha, \beta$, and set $X = \alpha(U + U^*) + i\beta(V + V^*)$. This approach seems natural, because on the other hand any $X \in B(H)$ permits such a representation (by spectral calculus; this is well known).

A random $C^*$-algebra $A$ is then a $C^*$-algebra $A = C^*(X_1, \dots, X_n)$ or $A = C^*(X_1, X_2, \dots)$ which is generated in $B(H)$ by a finite or infinite sequence of random elements $X_i \in B(H)$. Notice, however, that $A$ is not a random element in the usual sense in the set of separable $C^*$-algebras since we do not ask for measurability.

## 3. Pisier's local theory of exactness

Most of this section can be found well presented in [10]. Let $E, F$ be finite dimensional operator spaces which are isomorphic as vector spaces. Then the completely bounded Banach–Mazur distance is the number

$$d_{\mathrm{cb}}(E,F) = \inf\left\{\|u\|_{\mathrm{cb}}\left\|u^{-1}\right\|_{\mathrm{cb}} \middle| u : E \to F \text{ linear isomorphism}\right\}.$$

Notice that $d_{\mathrm{cb}}(E,F) \geq 1$, and $d_{\mathrm{cb}}(E,F) = 1$ if and only if $E$ and $F$ are completely isometric. The word "distance" for $d_{\mathrm{cb}}$ is justified by the following fact: if we consider the set $OS_n$ of $n$-dimensional operator spaces where completely isometric operator spaces are identified, then $\delta_{\mathrm{cb}}(E,F) = \log d_{\mathrm{cb}}(E,F)$ defines a metric on $OS_n$. We will always regard linear subspaces of $B(H)$ as operator subspaces of $B(H)$.

LEMMA 3.1. *Let $X$ be a normed linear space* (*endowed with the Borel structure of the norm topology*) *and $n \geq 1$. Then the linear dimension function* $\dim : X^n \to \mathbb{N}_0 : \dim(x_1, \ldots, x_n) = \dim(\mathrm{lin}(x_1, \ldots, x_n))$ *is measurable.*

*Proof.* The set

$$Y = Y_{i,j_1,\ldots,j_k} = \left\{(x_1, \ldots, x_n) \in X^n | x_i \in \mathrm{lin}(x_{j_1}, \ldots, x_{j_k})\right\}$$

is measurable for fixed $1 \leq i, j_1, \ldots, j_k \leq n$. Indeed, $Y$ can be expressed as $Y = p(f^{-1}(0))$ where $f : \mathbb{C}^k \times X^n \to X$ is given by $f(\lambda, x) = x_i - \sum_{r=1}^k \lambda_r x_{j_r}$, and $p : \mathbb{C}^n \times X^n \to X^n$ is the canonical projection. Now the set $A_m = \{x \in X^n | \dim(x) \leq m\}$ can be described as a finite Boolean expression of such sets $Y_{i,j_1,\ldots,j_k}$. ☐

LEMMA 3.2. *If $x_1, \ldots, x_n, y_1, \ldots, y_n \in B(H)$, $E_x = \mathrm{lin}(x_1, \ldots, x_n)$ and $E_y = \mathrm{lin}(y_1, \ldots, y_n)$, then $d_{\mathrm{cb}}(E_x, E_y)$* (*there where it is defined*) *is continuous in $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ w.r.t. the norm in $B(H)^{2n}$.*

*If we put $d_{\mathrm{cb}}(E_x, E_y) = \infty$ for $\dim(E_x) \neq \dim(E_y)$, then $(x, y) \mapsto d_{\mathrm{cb}}(E_x, E_y)$ is a measurable function everywhere on $B(H)^{2n}$.*

*Proof.* The first claim follows by an application of [10, Lemma 2.13.2]. The second claim follows from this and the fact that the set $\{(x, y) \in B(H)^{2n} | \dim(E_x) = \dim(E_y)\}$ is measurable by Lemma 3.1. ☐

Let $\mathbb{K}$ be the set of compact operators of $B(H)$. Let $X$ be any operator space. Then the completely bounded Banach–Mazur distance of $X$ to the compact operators is ([8])

$$d_{S\mathbb{K}}(X) = \sup_{E \subseteq X} \inf_{F \subseteq \mathbb{K}} d_{\mathrm{cb}}(E,F),$$

where the supremum is taken over all finite dimensional subspaces $E \subseteq X$, and the infimum is taken over all finite dimensional subspaces $F \subseteq \mathbb{K}$ such that $E$ and $F$ are isomorphic as linear spaces. Notice that $d_{S\mathbb{K}}(X) = d_{S\mathbb{K}}(\overline{X})$ for not necessarily norm closed operator spaces $X$ by Lemma 3.2.

We have the following local characterization of exact $C^*$-algebras.

THEOREM 3.3 ([8]). *A $C^*$-algebra $A$ is exact if and only if $d_{S\mathbb{K}}(A) = 1$.*

Even more, the following remarkable theorem shows that under some circumstances a small particular subspace $E$ of a $C^*$-algebra $A$ "decides" whether $A$ is exact or not.

THEOREM 3.4 ([10], Theorem 17.9). *Let $V$ be a set of unitary operators in $B(H)$ containing* $1$. *Let $A$ be the $C^*$-algebra generated by $V$. Then $A$ is exact if and only if $d_{S\mathbb{K}}(\mathrm{lin}(V)) = 1$.*

If $A = C^*(X_1, X_2, \dots)$ is a random $C^*$-algebra generated by a finite or infinite sequence $X = (X_1, X_2, \dots)$ of random elements $X_i \in B(H)$, then we will check that $d_{S\mathbb{K}}(A)$ is measurable. In particular, we can ask for the probability that $A$ is exact since we have

$$\mathbb{P}(A \text{ is exact}) = \mathbb{P}\big(d_{S\mathbb{K}}(A) = 1\big).$$

LEMMA 3.5. *$d_{S\mathbb{K}}(A)$ is measurable.*

*Proof.* Let $W$ be the countable set of all $*$-polynomials in the variables $x_1, x_2, x_3, \dots$ with scalar coefficients in $\mathbb{Q} + i\mathbb{Q} \subseteq \mathbb{C}$. Let $D$ be a countable dense subset of $\mathbb{K}$. Then we have

$$d_{S\mathbb{K}}(A) = \sup_{n \geq 1} \sup_{f \in W^n} \inf_{y \in D^n} d_{\mathrm{cb}}\big(\mathrm{lin}\big(f_1(X), \dots, f_n(X)\big), \mathrm{lin}(y_1, \dots, y_n)\big).$$

Hence, $d_{S\mathbb{K}}(A)$ is measurable since the function

$$X \mapsto d_{\mathrm{cb}}\big(\mathrm{lin}\big(f_1(X), \dots, f_n(X)\big), \mathrm{lin}(y_1, \dots, y_n)\big)$$

is measurable by Lemma 3.2. □

The following theorem can be used to show that the full group $C^*$-algebra $C^*(\mathbb{F}_n)$ of the free group $\mathbb{F}_n$ with $n$ generators is not exact. Let $U_1, \dots, U_n$ be the canonical generators of $C^*(\mathbb{F}_n)$, and let $E_U^n = \mathrm{lin}(U_1, \dots, U_n)$.

THEOREM 3.6 ([8]). *$d_{S\mathbb{K}}(E_U^n) \geq n(2\sqrt{n-1})^{-1}$. In particular, $d_{S\mathbb{K}}(E_U^n) > 1$ for $n \geq 3$.*

Hence, if $n \geq 3$ and $A$ is any $C^*$-algebra which contains a completely isometric copy of $E_U^n$, then $A$ is not exact, since $d_{S\mathbb{K}}(A) \geq d_{S\mathbb{K}}(E_U^n) > 1$. In particular, this is obviously true for $A = C^*(\mathbb{F}_n)$.

THEOREM 3.7 ([3]). *The metric space $OS_n = (OS_n, \delta_{\mathrm{cb}})$ is not separable for $n \geq 3$.*

Let $OS_n(\mathbb{K})$ be the subset of $OS_n$ which consists of all $n$-dimensional suboperator spaces of $\mathbb{K}$. Then $OS_n(\mathbb{K})$ is separable (cf. Lemma 3.2). Let $D$ be a dense subset of $OS_n(\mathbb{K})$. Let $B(F, r)$ be the ball in $OS_n$ with center $F$ and radius $r$. Now let $A = C^*(X_1, \dots, X_n)$ be a random $C^*$-algebra and

$E=\mathrm{lin}(X_1,\dots,X_n)$. Since $OS_n$ contains uncountable many disjoint balls with radius $r$ (for suitable small $r$) by Theorem 3.7, is it then really likely that

$$E\in\bigcup_{F\in D}B(F,r)?$$

We ask this, because otherwise $\log d_{S\mathbb{K}}(E)\geq r$, or $d_{S\mathbb{K}}(E)>1$, and $A$ would not be exact. However, the answer of this question does not seem so easy as it looks at first glance. We namely have the following: Let $d_k(E,F)$ denote the Banach–Mazur distance of two $n$-dimensional operator spaces $E$ and $F$ on the $k$th matrix level. Then the distance of any $E$ to the compact operators $\mathbb{K}$ on the $k$th matrix level vanishes, that is, we have $\inf_{F\subseteq\mathbb{K}}d_k(E,F)=1$ ([10, Lemma 21.9]). So one really needs to know the operator space structure of $E$ *thoroughly*, and one has to use the $d_{\mathrm{cb}}$-distance to answer the above question.

## 4. Widely spread isometries

Let $H$ be a separable Hilbert space with normal base $(e_1,e_2,e_3,\dots)$. Let $U_1,\dots,U_n$ be the canonical generators of $C^*(\mathbb{F}_n)$ of the free group $\mathbb{F}_n$ with $n$ free generators, and let $E_U^n=\mathrm{lin}(U_1,\dots,U_n)$.

The goal of the following definition is Lemma 4.3.

Definition 4.1. A tuple $(S_1,\dots,S_n)$ of isometries $S_i\in B(H)$ is called *widely spread* if for all $k\geq 1$, all isometries $T_1,\dots,T_n\in B(H)$, and all $\varepsilon>0$ there exists an isometry $V\in B(H)$ such that

$$\bigl|\langle S_aVe_i,S_bVe_j\rangle-\langle T_ae_i,T_be_j\rangle\bigr|\leq\varepsilon$$

for all $1\leq a,b\leq n$ and $1\leq i,j\leq k$.

Notice that the definition is independent of the normal base $(e_i)$. Further, the values of $T_a$ and $V$ are just relevant on the vectors $e_1,\dots,e_k$, and we thus have the following fact.

Lemma 4.2. *There exists a countable set $\mathcal{S}$ of unitaries which is dense w.r.t. the strong operator topology in the set of all isometries. Hence, in the last definition, it is sufficient to require that the $T_1,\dots,T_n$ are elements of $\mathcal{S}$.*

*Proof.* For each $n$ let $D_n$ be a countable dense subset of $\{(Se_1,\dots,Se_n)\in H^n|S \text{ isometry}\}$. Choose any unitary $U_{n,x}$ $(x\in D_n)$ such that $U_{n,x}(e_k)=x_k$ for all $1\leq k\leq n$. Then the family $\mathcal{S}=(U_{n,x})_{n\geq 1,x\in D_n}$ satisfies the claim. □

Lemma 4.3. *Let $S_1,\dots,S_n$ be widely spread isometries and $F=\mathrm{lin}(S_1,\dots,S_n)$. Then the linear map $\phi:E_U^n\to F$, such that $\phi(U_i)=S_i$, is completely isometric.*

*Proof.* Assume that $C^*(\mathbb{F}_n)$ is represented on $H$. Let $P_k$ be the orthogonal projection onto $\mathrm{lin}(e_1,\dots,e_k)$. Let $\varepsilon>0$ and $\alpha_1,\dots,\alpha_n\in M_m(\mathbb{C})$. For some large $N \geq 1$, and any unitaries $\tilde{S}_1, \ldots, \tilde{S}_n \in B(H)$ satisfying $S_i P_N = \tilde{S}_i P_N$ $(1 \leq i \leq N)$ we obtain

$$\Big\| \sum_{i=1}^n \alpha_i \otimes S_i \Big\|_{M_m \otimes B(H)} \leq \Big\| \sum_i \alpha_i \otimes \tilde{S}_i P_N \Big\| + \varepsilon$$
$$\leq \Big\| \sum_i \alpha_i \otimes U_i \Big\| + \varepsilon.$$

(For the last inequality, notice that the canonical $*$-homomorphism $C^*(U_1, \ldots, U_n) \to C^*(\tilde{S}_1, \ldots, \tilde{S}_n)$ is a complete contraction.) For the reverse estimation, choose $N \geq 1$ and $\xi \in \ell^2_m \otimes \mathrm{lin}(e_1, \ldots, e_N)$ with $\|\xi\| \leq 1$ such that

$$\Big\| \sum_{i=1}^n \alpha_i \otimes U_i \Big\|_{M_m \otimes B(H)} \leq \Big\| \Big( \sum_i \alpha_i \otimes U_i P_N \Big) \xi \Big\|_{\ell^2_m \otimes H} + \varepsilon.$$

An easy calculation shows that there exists a continuous function $f : \mathbb{C}^{n^2 N^2} \to \mathbb{R}$ such that (using $\|\eta\| = \sqrt{\langle \eta, \eta \rangle}$ in a Hilbert space)

$$\Big\| \Big( \sum_{i=1}^n \alpha_i \otimes x_i P_N \Big) \xi \Big\|_{\ell^2_m \otimes H}$$
$$= f\Big( \big( (\langle x_a e_s, x_b e_t \rangle)_{a,b=1}^n \big)_{s,t=1}^N \Big)$$

for all $x_1, \ldots, x_n$ in $B(H)$, and where $((\langle x_a e_s, x_b e_t \rangle)_{a,b=1}^n)_{s,t=1}^N \in \mathbb{C}^{n^2 N^2}$. Since $(S_1, \ldots, S_n)$ is widely spread, we can choose some small $\delta > 0$ and some isometry $V \in B(H)$ such that

$$\big| \langle S_a V e_s, S_b V e_t \rangle - \langle U_a e_s, U_b e_t \rangle \big| \leq \delta$$

for all $1 \leq a, b \leq n$ and $1 \leq s, t \leq N$, and such that, by the continuity of $f$, we have

$$\Big\| \sum_i \alpha_i \otimes U_i \Big\| - \varepsilon \leq \Big\| \Big( \sum_i \alpha_i \otimes U_i P_N \Big) \xi \Big\|$$
$$= f\Big( \big( (\langle U_a e_s, U_b e_t \rangle)_{a,b} \big)_{s,t} \Big)$$
$$\leq f\Big( \big( (\langle S_a V e_s, S_b V e_t \rangle)_{a,b} \big)_{s,t} \Big) + \varepsilon$$
$$= \Big\| \Big( \sum_i \alpha_i \otimes S_i V P_N \Big) \xi \Big\| + \varepsilon$$
$$\leq \Big\| \sum_i \alpha_i \otimes S_i \Big\| + \varepsilon.$$

Since $\varepsilon > 0$ and $\alpha_i \in M_m$ were arbitrary, $\phi$ is completely isometric. $\square$

By the last proof, we see that $\phi$ is also a complete isometry if the condition stated in Definition 4.1 only holds for $U_1, \dots, U_n$ rather than for all isometries $T_1, \dots, T_n$.

REMARK 4.4. If $S_1, \dots, S_n$ are unitaries and the unitization

$$\phi^+ : \operatorname{lin}(1, U_1, \dots, U_n) \to \operatorname{lin}(1, S_1, \dots, S_n)$$

of $\phi$ of Lemma 4.3 is a complete isometry, then $\phi^+$ extends to a $*$-isomorphism $\Phi : C^*(\mathbb{F}_n) \to C^*(S_1, \dots, S_n)$ by [9, Proposition 6] (or see [10, Proposition 13.6]). However, $\Phi$ need not be an isomorphism in general. If $T_1, \dots, T_n$ are widely spread isometries, $S$ is an isometry and $U$ is a unitary, then $ST_1U, \dots, ST_nU$ are also widely spread isometries. Taking here widely spread unitaries $T_1, \dots, T_n$, $S = T_1^*$, and $U = 1$, yields an example of widely spread unitaries $1, T_1^*T_2, \dots, T_1^*T_n$ whose associated unitization $\phi^+$ is obviously not a complete isometry, and so $\Phi$ is not an isomorphism.

THEOREM 4.5. *Let $S_1, \dots, S_n$ be widely spread isometries in $B(H)$, and let $n \geq 3$. Then $C^*(S_1, \dots, S_n)$ is not exact.*

*Proof.* Let $F = \operatorname{lin}(S_1, \dots, S_n)$ and $A = C^*(S_1, \dots, S_n)$. By Lemma 4.3 and Theorem 3.6, we have $d_{S\mathbb{K}}(A) \geq d_{S\mathbb{K}}(F) = d_{S\mathbb{K}}(E_U^n) > 1$. Hence, the claim follows from Theorem 3.3. $\square$

The integer $n \geq 3$ is really sharp here. By what we have remarked above, there exist widely spread unitaries $u_1u_1^*, u_2u_1^*$, say, but $C^*(u_2u_1^*)$ is not exact.

The following Lemma 4.6 follows also immediately from Theorem 5.4 (the proof of Theorem 5.4 does not depend on this lemma), but the lemma is also a corollary of Theorems 3.3 and 3.6, and we will give a short proof.

LEMMA 4.6. *Let $n \geq 3$. Then there exists a strong operator topology-dense subset $D$ of $\mathcal{U}(B(H))^n$ such that $A = C^*(V_1, \dots, V_n)$ is not exact for all $(V_1, \dots, V_n) \in D$.*

*Proof.* Let $P_m$ be the orthogonal projection onto $\operatorname{lin}(e_1, \dots, e_m)$. For $k \geq 1$ and $(u_1, \dots, u_n) \in \mathcal{U}(B(H))^n$ consider the linear space $L = \operatorname{lin}\{u_a e_i | 1 \leq a \leq n, 1 \leq i \leq k\}$ with dimension $r = \dim(L)$. Choose isometries $S$ and $T$, respectively, with range $I - P_r$ and $L^\perp$, respectively. Then choose any unitaries $V_1, \dots, V_n$ such that

$$V_a P_k = u_a P_k, \qquad (V_a P_r)H = L, \qquad V_a(I - P_r) = TU_aS^*.$$

Hence, the proofs of Lemma 4.3 and Theorem 4.5 show that $\operatorname{lin}(V_1, \dots, V_n)$ is completely isometric to $E_U^n$, and $A$ is not exact. $\square$

By choosing $D$ countably in Lemma 4.6, we can choose an atomic strong operator topology-dense probability measure on $\mathcal{U}(B(H))^n$ such that $C^*(X_1, \dots, X_n)$ is almost surely non-exact for random elements $(X_1, \dots, X_n) \in \mathcal{U}(B(H))^n$. By contrast, notice that in Theorem 1.1 the $X_i$ are chosen *independently identically distributed.*

## 5. A probability measure resulting in non-exactness

The aim of this section is the proof of Theorem 1.1. To this end, we will start with three simple lemmas which give rough uniform estimates on the output of the Gram–Schmidt process depending on the input, in particular when the input is already almost a normalized orthogonal sequence.

Let $(x_1, x_2, x_3, \ldots)$ be a sequence in $H$, and let $(y_1, y_2, y_3, \ldots)$ be the normalized orthogonal sequence in $H$ by applying the Gram–Schmidt process to $(x_1, x_2, x_3, \ldots)$. Then $(y_1, y_2, y_3, \ldots)$ can be described by

$$(y_1, y_2, y_3, \ldots) = \big(f(P_0, x_1), f(P_1, x_2), f(P_2, x_3), \ldots\big),$$

where $P_i \in B(H)$ is the orthogonal projection onto $\operatorname{lin}(x_1, \ldots, x_i)$, and

$$f(P, x) = (x - Px)/\|x - Px\|$$

for all projections $P \in B(H)$ and $x \in H$ (as far as $\|x - Px\| \neq 0$).

Lemma 5.1. *Let $P_0 \in B(H)$ be a projection, $0 < \varepsilon \leq 1/4$, and $x_1, \ldots, x_n \in H$ such that*

$$\big|\|x_i\| - 1\big| \leq \varepsilon, \qquad \big\|P_0(x_i)\big\| \leq \varepsilon, \qquad \big|\langle x_i, x_j\rangle\big| \leq \varepsilon$$

*for all $1 \leq i \neq j \leq n$. Let $P_i$ be the orthogonal projection onto the Hilbert space spanned by the image space of $P_0$ and the vectors $x_1, \ldots, x_i$. Then $\|P_{i-1} x_i\| \leq 8^i \varepsilon$.*

*Proof.* Let $\tilde{x}_1 = f(P_0, x_1)$. Then for the projection $P_1$ we have $P_1(x_i) = P_0(x_i) + \langle x_i, \tilde{x}_1\rangle \tilde{x}_1$ for $2 \leq i \leq n$. We thus get

$$\begin{aligned}
\big\|P_1(x_i)\big\| &\leq \big\|P_0(x_i)\big\| + \frac{|\langle x_i, x_1 - P_0(x_1)\rangle|}{1 - 2\varepsilon} \\
&\leq 2\big(\big\|P_0(x_i)\big\| + \big|\langle x_i, x_1\rangle\big| + \big|\langle x_i, P_0(x_1)\rangle\big|\big) \\
&\leq 8\varepsilon.
\end{aligned}$$

In the same way we proceed by induction. □

Lemma 5.2. *Let $P \in B(H)$ be a projection, let $e, x \in H$ and $0 < \varepsilon \leq 1/4$, such that $\|e\| = 1$, $\|x - e\| \leq \varepsilon$ and $\|Px\| \leq \varepsilon$. Then $\|f(P, x) - e\| \leq 11\varepsilon$.*

*Proof.* The estimation is straightforward. □

Lemma 5.3. *Let $P_0 \in B(H)$ be a projection and $0 < \varepsilon \leq 1/4$. Let $e_1, \ldots, e_n \in H$ be orthogonal normalized vectors. Let $x_1, \ldots, x_n \in H$ such that*

$$\|x_i - e_i\| \leq \varepsilon, \qquad \big\|P_0(x_i)\big\| \leq \varepsilon \quad (i = 1, \ldots, n).$$

*Let $P_i$ be the orthogonal projection onto the Hilbert space spanned by the image space of $P_0$ and the vectors $x_1, \ldots, x_i$. Then*

$$\big\|f(P_{i-1}, x_i) - e_i\big\| \leq 33 \cdot 8^n \varepsilon \quad (i = 1, \ldots, n).$$

*Proof.* For $i \neq j$, we have $|\langle x_i, x_j\rangle| = |\langle x_i, x_j\rangle - \langle e_i, e_j\rangle| \leq 3\varepsilon$. Hence, $\|P_{i-1}(x_i)\| \leq 8^n \cdot 3\varepsilon$ by Lemma 5.1, and consequently $\|f(P_{i-1}, x_i) - e_i\| \leq 11 \cdot 8^n \cdot 3\varepsilon$ by Lemma 5.2. ☐

THEOREM 5.4. *There exists a non-atomic strong operator topology-dense probability measure $\mathbb{P}_{\mathcal{U}}$ on $\mathcal{U}(B(H))$, such that independent $\mathbb{P}_{\mathcal{U}}$-distributed random elements $U_1, \dots, U_n$ are almost surely widely spread for $n \geq 1$.*

*In particular, $C^*(U_1, \dots, U_n)$ is almost surely not exact when $n \geq 3$.*

*Proof. Step 1.* In the first step, we construct the probability measure $\mathbb{P}_{\mathcal{U}}$. Let $H = \ell^2(\mathbb{Z})$, and $\sigma \in B(\ell^2(\mathbb{Z}))$ be the shift operator. We choose independent non-degenerate norm-dense random variables $x_1, x_2, x_3, \dots$ and $w_1, w_2, w_3, \dots$ in $\ell^2(\mathbb{Z})$. We require that the sequence $(w_i)$ is identically distributed. For a sequence of integers $k_i$, which we will specify below, we put

$$\begin{aligned}
y_1 &= \sigma^{k_1}(x_1),\\
y_2 &= w_1,\\
(y_3, y_4) &= \big(\sigma^{k_2}(x_3), \sigma^{k_2}(x_4)\big),\\
y_5 &= w_2,\\
(y_6, y_7, y_8) &= \big(\sigma^{k_3}(x_6), \sigma^{k_3}(x_7), \sigma^{k_3}(x_8)\big),\\
y_9 &= w_3,\\
\cdots &= \cdots,\\
(y_{s_n}, y_{s_n+1}, \dots, y_{s_n+n-1}) &= \big(\sigma^{k_n}(x_{s_n}), \sigma^{k_n}(x_{s_n+1}), \dots, \sigma^{k_n}(x_{s_n+n-1})\big),\\
y_{s_n+n} &= w_n,\\
\cdots &= \cdots,
\end{aligned}$$

where $(s_1, s_2, s_3, \dots) = (1, 3, 6, 10, \dots)$. For the distribution of the sequence $(x_i)$, we require that

$$x_{s_n+i-1} \overset{d}{=} x_{s_m+i-1} \quad \forall 1 \leq n < m \ \forall i = 1, \dots, n.$$

Let $P_i$ be the orthogonal projection onto $\operatorname{lin}(y_1, y_2, \dots, y_i)$. Since for fixed $i$ and any (independent) random element $z \in \ell^2(\mathbb{Z})$ we have $\|P_i\sigma^k(z)\| \to 0$ $(k \to \infty)$ a.s., we also have convergence of this sequence in probability ([5, Lemma 3.2]). That means that for all $n \geq 1$

$$\sup_{i=1,\dots,n} \big\|P_{s_n-1}\sigma^k(x_{s_n+i-1})\big\| \xrightarrow{\mathbb{P}} 0 \quad (k \to \infty)$$

in probability. We define the $k_i$ inductively as follows. Let $n \geq 1$, and assume that we have already defined $k_1, \dots, k_{n-1}$. Then for $\varepsilon_n = 1/n$ we choose $k_n \in \mathbb{N}$ such that

$$\mathbb{P}\Big(\sup_{i=1,\dots,n} \big\|P_{s_n-1}\sigma^{k_n}(x_{s_n+i-1})\big\| \leq \varepsilon_n\Big) \geq 1 - \varepsilon_n.$$

The basic idea behind is that this means

$$\mathbb{P}\big(\big\|P_{s_n-1}(y_{s_n+i-1})\big\| \le \varepsilon_n \ \forall i = 1,\dots,n\big) \ge 1-\varepsilon_n.$$

(Roughly speaking, the idea is that when we orthogonalize the sequence $(y_1, y_2, y_3, \dots)$ by Gram–Schmidt to obtain a normal basis $(z_1, z_2, z_3, \dots)$, then $z_{s_n}, \dots, z_{s_n+n-1}$ is with high probability almost independent from the predecessor sequence $(z_1, \dots, z_{s_n-1})$. Thus, the sequence $(z_{s_n}, \dots, z_{s_n+n-1})$ is a new chance to obtain desired values in $H$.)

We fix a normal base $(e_1, e_2, e_3, \dots)$ in $\ell^2(\mathbb{Z})$. Let $(z_i)$ be the orthogonal normalized sequence obtained by applying the Gram–Schmidt procedure to $(y_i)$. Then we define a random operator $U$ by

$$U(e_i) = z_i = f(P_{i-1}, y_i) \quad \forall i \ge 1.$$

Since the sequence $(w_i)$ is i.i.d., the set $\{w_1, w_2, \dots\}$ is a.s. a dense subset of $H$ by the proof of Lemma 2.2. Hence $U$ is a.s. a unitary operator since the set $\{w_1, w_2, \dots\}$ lies in the image of $U$ by construction.

Let $V$ be a unitary operator, $n \ge 1$ and $\varepsilon > 0$. Then the probability that $\sup_{i=1,\dots,n} \|y_i - Ve_i\| \le \varepsilon$ is positive. Applying Lemma 5.3 ($P_0 = 0$), we obtain that the probability of $\sup_{i=1,\dots,n} \|Ue_i - Ve_i\| \le 33 \cdot 8^n \varepsilon$ is positive. Hence, the random element $U$ is strong operator topology-dense in $\mathcal{U}(B(H))$.

*Step 2.* Let $U^1, \dots, U^m$ be independent $\mathbb{P}_{\mathcal{U}}$-distributed random elements. That is, for $1 \le a \le m$ we choose independent random sequences $(x_1^a, x_2^a, x_3^a, \dots)$ and $(w_1^a, w_2^a, w_3^a, \dots)$, respectively, which are distributed as $(x_i)$ and $(w_i)$, respectively, construct $(y_i^a)$ from $(x_i^a)$ and $(w_i^a)$ like $(y_i)$ is constructed from $(x_i)$ and $(w_i)$ (for one common sequence $(k_i)$), and set

$$U^a(e_i) = z_i^a := f\big(P_{i-1}^a, y_i^a\big) \quad (i \in \mathbb{N}),$$

where $P_i^a$ is the projection onto $\operatorname{lin}(y_1^a, \dots, y_i^a)$.

Now we fix any $n \ge 1$, any isometries $T^1, \dots, T^m \in B(\ell^2(\mathbb{Z}))$, and any $0 < \varepsilon \le 1/4$. For all $v \ge n$ set

$$A_v^a = \big\{\omega \in \Omega \big| \big\|x_{s_v+i-1}^a(\omega) - T^a(e_i)\big\| \le \varepsilon \ \forall i = 1, \dots, n\big\}.$$

Notice that $\mathbb{P}(A_n^a) = \mathbb{P}(A_v^a)$ for $v \ge n$, since $x_{s_n+i-1}^a \stackrel{d}{=} x_{s_v+i-1}^a$ as required. Set

$$B_v^a = \big\{\omega \in \Omega \big| \big\|P_{s_v-1}^a\big(y_{s_v+i-1}^a(\omega)\big)\big\| \le \varepsilon_v \ \forall i = 1, \dots, n\big\}.$$

By an above inequality, we have $\mathbb{P}(B_v^a) \to 1$ if $v$ tends to infinity. Let

$$C_v^a = A_v^a \cap B_v^a \quad (v \ge n).$$

Choose $v_0 \ge n$ such that $\varepsilon_{v_0} \le \varepsilon$. Then we have

$$\begin{aligned} \big\|y_{s_v+i-1}^a(\omega) - \sigma^{k_v}\big(T^a(e_i)\big)\big\| &\le \varepsilon, \\ \big\|P_{s_v-1}^a\big(y_{s_v+i-1}^a(\omega)\big)\big\| &\le \varepsilon \end{aligned}$$

for all $i = 1, \dots, n$, $v \geq v_0$ and $\omega \in C_v^a$. By applying Lemma 5.3, we obtain

$$\left\| z^a_{s_v+i-1}(\omega) - \sigma^{k_v}\left(T^a(e_i)\right)\right\| \leq 33 \cdot 8^n \varepsilon$$

for all $i = 1, \dots, n$, $v \geq v_0$ and $\omega \in C_v^a$. We let

$$C_v = C_v^1 \cap \dots \cap C_v^m,$$

and we similarly define $A_v$ and $B_v$. Let $V$ be the isometry $V(e_i) = e_{i+1}$. We then obtain

$$\begin{aligned} (1) \qquad & \left|\left\langle U^a(\omega)V^{s_v-1}e_i, U^b(\omega)V^{s_v-1}e_j\right\rangle - \left\langle T^a e_i, T^b e_j\right\rangle\right| \\ &= \left|\left\langle U^a(\omega)e_{s_v+i-1}, U^b(\omega)e_{s_v+j-1}\right\rangle - \left\langle T^a e_i, T^b e_j\right\rangle\right| \\ &= \left|\left\langle z^a_{s_v+i-1}(\omega), z^b_{s_v+j-1}(\omega)\right\rangle - \left\langle \sigma^{k_v}T^a e_i, \sigma^{k_v}T^b e_j\right\rangle\right| \\ &\leq 3 \cdot 33 \cdot 8^n \varepsilon \end{aligned}$$

for all $1 \leq a, b \leq m$ and $1 \leq i, j \leq n$, $v \geq v_0$ and $\omega \in C_v$.

We next choose a sequence $v_1, v_2, v_3, \dots$ of integers inductively as follows. Let $v_0 < v_1 < \dots < v_{r-1}$ be already chosen. Let $D_s$ be the event

$$D_s = (\Omega - C_{v_1}) \cap (\Omega - C_{v_2}) \cap \dots \cap (\Omega - C_{v_s}) \quad \forall s \geq 1.$$

Since $\mathbb{P}(B_v) \to 1$ for $v \to \infty$, we have

$$\left|\mathbb{P}(B_{v_r} \cap A_{v_r} | D_{r-1}) - \mathbb{P}(A_{v_r} | D_{r-1})\right| \leq \mathbb{P}(A_n)/2$$

for some $v_r > v_{r-1}$. Observe that the event $A_{v_r}$ (which only depends on $x^a_{s_{v_r}}, \dots, x^a_{s_{v_r}+n-1}$) is independent from the event $D_{r-1}$ (which only depends on $x_1^a, \dots, x^a_{s_{v_r}-1}$ and $(w_i^a)$). We hence obtain

$$\left|\mathbb{P}(C_{v_r} | D_{r-1}) - \mathbb{P}(A_{v_r})\right| \leq \mathbb{P}(A_n)/2.$$

Since $\mathbb{P}(A_{v_r}) = \mathbb{P}(A_n)$, this yields

$$\mathbb{P}(\Omega - C_{v_r} | D_{r-1}) \leq 1 - \mathbb{P}(A_n)/2 =: \delta < 1.$$

Thus we have

$$\mathbb{P}(D_r) = \mathbb{P}(D_{r-1})\mathbb{P}(\Omega - C_{v_r} | D_{r-1}) \leq \mathbb{P}(D_{r-1})\delta.$$

By induction, we obtain $\mathbb{P}(D_r) \leq \mathbb{P}(D_1)\delta^{r-1} \to 0$ for $r \to \infty$. Hence, $\mathbb{P}(\bigcup_{v=v_0}^\infty C_v) = 1$, which means that the above estimate (1) holds a.s.

If we vary over all $n \geq 1$, all $\varepsilon = 1/k$, and all $T^1, \dots, T^m \in \mathcal{S}$ for the countable set $\mathcal{S}$ of Lemma 4.2, then we have proved that $(U^1, \dots, U^m)$ is a.s. widely spread. It now follows from Theorem 4.5 that $C^*(U^1, \dots, U^m)$ is a.s. not exact for $m \geq 3$. □

REMARK 5.5. We remark that in the proof of the last theorem we may also remove the sequence $(w_i)$ and set $y_1 = \sigma^{k_1}(x_1)$, $(y_2, y_3) = (\sigma^{k_2}(x_2), \sigma^{k_2}(x_3))$, and $(y_4, y_5, y_6) = (\sigma^{k_3}(x_4), \sigma^{k_3}(x_5), \sigma^{k_3}(x_6))$, and so on. But then the random element $U$ may no longer be a unitary, but just a random *isometry.* All other claims of the theorem, however, remain valid. It is an open question whether we would also get a.s. widely spread isometries if we used the sequence $y_i = \sigma^i(x_i)$ ($(x_i)$ i.i.d.) in the last proof. For the sequence $y_i = x_i$ ($(x_i)$ i.i.d.) we are less confident (mainly due to numerical experiments). However, there exists a rearrangement $f$ of $\mathbb{N}$ such that the random measure associated to the sequence $y_i = \sigma^{f_i}(x_i)$ ($(x_i)$ i.i.d.) yields widely spread isometries. Indeed, start with $f_1 = 1$, choose $k_2$ "large enough" (similarly as in the last proof) and set $(f_2, f_3) = (k_2, k_2 + 1)$, then fill the gap $(2, \ldots, k_2 - 1)$ in the image of $f$ by setting $(f_4, \ldots, f_{4+(k_2-1)-2}) = (2, \ldots, k_2 - 1)$, then, once again, choose $k_3$ large enough and set $(f_{k_2+2}, f_{k_2+3}, f_{k_2+4}) = (k_3, k_3 + 1, k_3 + 2)$, then once again fill the gap $(k_2 + 2, \ldots, k_3 - 1)$ in the image of $f$ by letting $(f_{k_2+5}, \ldots, f_{k_2+5+(k_2+2)-(k_3-1)}) = (k_2 + 2, \ldots, k_3 - 1)$, and so on.

Let $U_i$ be the random unitaries of the last theorem, and fix two random unitaries $V$ and $W$. Then $VU_1W, \ldots, VU_nW$ is another example of a.s. widely spread unitaries. It is unclear whether the random $C^*$-algebra constructed in Theorem 5.4 is almost surely canonically isomorphic to $C^*(\mathbb{F}_n)$, see Remark 4.4 why one might conjecture this.

## References

[1] M. Gromov, *Random walk in random groups*, Geom. Funct. Anal. **13** (2003), 73–146. MR 1978492

[2] U. Haagerup and S. Thorbjørnsen, *A new application of random matrices:* $\mathrm{Ext}(C^*_{\mathrm{red}}(F_2))$ *is not a group*, Ann. of Math. **162** (2005), 711–775. MR 2183281

[3] M. Junge and G. Pisier, *Bilinear forms on exact operator spaces and* $B(H) \otimes B(H)$, Geom. Funct. Anal. **5** (1995), 329–363. MR 1334870

[4] M. Junge and Q. Xu, *On the best constants in some non-commutative martingale inequalities*, Bull. Lond. Math. Soc. **37** (2005), 243–253. MR 2119024

[5] O. Kallenberg, *Foundations of modern probability*, Probability and Its Applications (New York), Springer-Verlag, New York, 1997. MR 1464694

[6] T. Kusalik, J. A. Mingo and Roland Speicher, *Orthogonal polynomials and fluctuations of random matrices*, J. Reine Angew. Math. **604** (2007), 1–46. MR 2320312

[7] A. Nica, D. Shlyakhtenko and R. Speicher, *R-cyclic families of matrices in free probability*, J. Funct. Anal. **188** (2002), 227–271. MR 1878637

[8] G. Pisier, *Exact operator spaces*, Astérisque **232** (1995), 159–186. MR 1372532

[9] G. Pisier, *A simple proof of a theorem of Kirchberg and related results on* $C^*$*-norms*, J. Operator Theory **35** (1996), 317–335. MR 1401692

[10] G. Pisier, *Introduction to operator space theory*, London Mathematical Society Lecture Note Series, vol. 294, Cambridge University Press, Cambridge, 2003. MR 2006539

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