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ON THE CLASSIFICATION OF DEGREE 1 ELLIPTIC THREEFOLDS WITH CONSTANT j-INVARIANT

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Abstract. We describe the possible Mordell–Weil groups for degree 1 elliptic threefold with rational base and constant j-invariant. Moreover, we classify all such elliptic threefolds if the j-invariant is nonzero. We can use this classification to describe a class of singular hypersurfaces in $\mathbf{P}(2,3,1,1,1)$ that admit no variation of Hodge structure (Remark 9.3).

1. Introduction

In this paper, we work over the field $\mathbf{C}$ of complex numbers. Let $\pi : X \to B$ be an elliptic threefold with a (fixed) section $\sigma_0 : B \to X$, such that B is a rational surface. Assume that X is not birationally equivalent to a product $E \times B$, with E an elliptic curve.

Fix a Weierstrass equation for the generic fiber of π. As explained in Section 2, this establishes a degree $6k$ hypersurface $Y \subset \mathbf{P}(2k, 3k, 1, 1, 1)$ that is birational to X and such that the fibration π is birationally equivalent to projection from $(1 : 1 : 0 : 0 : 0)$ onto a plane.

This integer k is not unique. We call the minimal possible k for which such an Y exists the *degree* of $\pi : X \to B$. One can easily show that if X is a rational threefold then the degree equals 1 or 2, and that if X is Calabi–Yau then the degree equals 3.

For a general point $p \in B$, we can calculate the j-invariant of the elliptic curve $\pi^{-1}(p)$. This yields a rational function $j(\pi) : B \dashrightarrow \mathbf{P}^1$.

In this paper, we study elliptic threefolds of degree 1 with rational base and constant j-invariant. We would like to classify all such possible threefolds. The two invariants that interest us are the configuration of singular fibers of π and the structure of the Mordell–Weil group $\mathrm{MW}(\pi)$, the group of rational

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sections of π. The actual classification we are aiming at in this paper is a classification of possible singular loci of irreducible and reduced degree 6 threefolds Y in $\mathbf{P}(2,3,1,1,1)$ together with the possibilities for $\mathrm{MW}(\pi)$. In [8] it is explained how to obtain an elliptic threefold X from Y.

One way of constructing elliptic threefolds is taking a cone Y over an elliptic surface $S \subset \mathbf{P}(2,3,1,1) \subset \mathbf{P}(2,3,1,1,1)$. The Mordell–Weil group and the configuration of singular fibers can be obtained from S. All possible degree 6 surfaces in $\mathbf{P}(2,3,1,1)$, that correspond to elliptic surfaces, have already been classified by Oguiso and Shioda [9]. We refer to such Y as 'obtained by the cone construction'. We exclude such Y from our classification. One can show that Y is a cone over an elliptic surface if and only if the discriminant curve is a union of lines through one point.

We split our considerations in three cases, namely the general one $j(\pi) \neq 0, 1728$, and two special cases $j(\pi) = 1728$ and $j(\pi) = 0$.

The case $j(\pi) \neq 0, 1728$ is the easiest one. In this case, it is well known that Y is given by

$$y^2 = x^3 + AP^2x + BP^3$$

with $A, B \in \mathbf{C}$ and $P \in \mathbf{C}[z_0, z_1, z_2]_2$, i.e., P is homogeneous of degree 2. Our assumptions on Y imply that $P = 0$ is a smooth conic. It turns out that in this case $\mathrm{MW}(\pi) \cong (\mathbf{Z}/2\mathbf{Z})^2$.

The exceptional cases $j(\pi) = 0, 1728$ are more interesting. In these cases, one has an equation of the form

$$y^2 = x^3 + R, \quad \text{resp.} \quad y^2 = x^3 + Qx$$

with $Q \in \mathbf{C}[z_0, z_1, z_2]_4$ and $R \in \mathbf{C}[z_0, z_1, z_2]_6$.

To calculate the group $\mathrm{MW}(\pi)$, we use the results of [5]. It turns out that $\mathrm{MW}(\pi)$ is determined by the type of singularities and the configuration of singular points of $Q = 0$, resp., $R = 0$.

More precisely, the main result of [5] states that $\mathrm{MW}(\pi)$ is isomorphic to the group of Weil divisors on Y modulo the Cartier divisors on Y. In our case, this can be reformulated as

$$\operatorname{rank} \mathrm{MW}(\pi) = h^4(Y)_{\mathrm{prim}} = \dim \operatorname{coker}\left(F^2H^4(\mathbf{P} \setminus Y, \mathbf{C}) \to \bigoplus_{p \in \mathcal{P}} H^4_p(Y, \mathbf{C})\right),$$

where $\mathcal{P}$ consists of the points $\{x = y = Q = 0\}_{\mathrm{sing}}$, respectively, $\{x = y = R = 0\}_{\mathrm{sing}}$.

The Poincaré residue map yields a natural surjection from $\mathbf{C}[z_0, z_1, z_2]_2 x \oplus \mathbf{C}[z_0, z_1, z_2]_4$ onto $F^2H^4(\mathbf{P} \setminus Y, \mathbf{C})$. To determine $H^4_p(Y, \mathbf{C})$, we use three methods. Let $p \in \mathcal{P}$.

(1) If (Y, p) is an isolated singularity and is semi-weighted homogeneous, then there is a method of Dimca to compute an explicit basis for $H^4_p(Y, \mathbf{C})$, together with the Hodge filtration.

(2) If (Y,p) is not weighted homogeneous, but (Y,p) is isolated, then there is a classical method of Brieskorn [1] to calculate $H^4_p(Y)$. This method does not produce the Hodge filtration, and in the weighted homogeneous case it is more complicated than Dimca's method.

This method is implemented in the computer algebra package Singular. Since this case is rather exceptional, we preferred to calculate $H^4_p(Y,\mathbf{C})$ by using Singular. Hence, several of the results in the sequel are only valid up to the correct implementation of Brieskorn's method in Singular.

(3) If (Y,p) is a non-isolated singularity, but is weighted homogeneous, then the transversal type is an ADE-surface singularity. To calculate $H^4_p(Y,\mathbf{C})$, we apply a generalization of Dimca's method, due to Hulek and the author [5].

We list now the possible groups.

THEOREM 1.1. *Suppose* $Y\subset\mathbf{P}(2,3,1,1,1)$ *is a degree* 6 *hypersurface, corresponding to an elliptic threefold* $\pi: X\to B$, *not obtained by the cone construction and not birational to a product* $E\times B$. *Then* $\mathrm{MW}(\pi)$ *is one of the following*

- $(\mathbf{Z}/2\mathbf{Z})^2$ *if* $j(\pi)\neq 0, 1728$.
- $(\mathbf{Z}/2\mathbf{Z}), (\mathbf{Z}/2\mathbf{Z})^2$ *or* $(\mathbf{Z}/2\mathbf{Z})\times\mathbf{Z}^2$ *if* $j(\pi)=1728$.
- $0,\mathbf{Z}/3\mathbf{Z},(\mathbf{Z}/2\mathbf{Z})^2,\mathbf{Z}^2,\mathbf{Z}^4,\mathbf{Z}^6$ *if* $j(\pi)=0$.

In the case $j(\pi)=1728$, we get a complete classification.

THEOREM 1.2. *Suppose* Y *satisfies the conditions of the previous theorem, and suppose that* $j(\pi)=1728$.

We have that $\mathrm{MW}(\pi)\cong(\mathbf{Z}/2\mathbf{Z})^2$ *if and only if* $Q=0$ *defines a double conic and* $\mathrm{MW}(\pi)\cong\mathbf{Z}/2\mathbf{Z}\times\mathbf{Z}^2$ *if and only if* $Q=0$ *is the unique quartic with two* A_3 *singularities.*

For $j(\pi)=0$, the number of cases to consider is quite large. One should apply the following program:

(1) Determine all possible types of singularities of sextic curves. This is done in [6].
(2) For each type of singularity, determine $H^4_p(Y)$.
(3) Determine which combinations of singularities are possible on a sextic curve. Here one might restrict oneself to combinations of singularities that yield nontrivial $H^4_p(Y)$.
(4) For each configuration, study the relation between $h^4(Y)$ and the position of the singularities.

The second point is completely done in this paper, except for six types of singularities that are both not weighted homogeneous and not isolated. The number of cases to consider at the third and fourth point is quite big. We restrict ourselves to the case where the sextic is non-reduced, and the case where the sextic has ordinary cusps. (It turns out that if the sextic has a node at p then $H^4_p(Y)$ vanishes, for this reason we study sextic with cusps.)

The curves with only cusps as singularities yield examples for the groups $0, \mathbf{Z}^2, \mathbf{Z}^4$ and $\mathbf{Z}^6$. One can show that $\mathrm{MW}(\pi) = (\mathbf{Z}/2\mathbf{Z})^2$ if and only if R defines a triple conic, and $\mathrm{MW}(\pi) = \mathbf{Z}/3\mathbf{Z}$ if and only if R defines a double cubic. This suffices to provide examples for each of the groups mentioned in Theorem 1.1.

As pointed out to the author by several persons there is a different approach possible. A degree 1 elliptic threefold X with base $\mathbf{P}^2$ defined over $\mathbf{C}$ can also be considered as a rational elliptic surface S over $\mathbf{C}(t)$. The possible Mordell–Weil groups for rational elliptic surfaces over algebraically closed fields have been determined by Oguiso and Shioda [9], hence the possible Mordell–Weil groups for degree 1 elliptic threefolds correspond to the to the possible subgroups of the Mordell–Weil group of $S/\overline{\mathbf{C}(t)}$ fixed by the Galois group $\mathrm{Gal}(\overline{\mathbf{C}(t)}/\mathbf{C}(t))$. In this way, one can obtain Theorem 1.1 with a little effort (one needs to exclude the groups $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}^4$, $(\mathbf{Z}/2\mathbf{Z})^2 \times \mathbf{Z}^2$ in the case $j(\pi) = 1728$ and the group $\mathbf{Z}^8$ in the case $j(\pi) = 0$). Theorem 1.2 and our results in the case $j(\pi) = 0$ are harder to obtain with this method. That is, one needs to relate the singularities of the discriminant curve with the Galois representation on the Mordell–Weil group. The main obstruction to this approach is that the Oguiso–Shioda classification gives estimates for the degree of the generators of the Mordell–Weil group, but no explicit formulae in terms of the coefficient of the Weierstrass equation. This final obstruction is severe, that is, in many cases the Galois group is not solvable and hence we have cannot give closed formulae for the generators. For this reason, we take a more geometric approach.

The organization of this paper is as follows: In Section 2, we recall several standard facts concerning elliptic threefolds. In particular we construct our model Y. In Section 3, we limit the possibilities for the group $\mathrm{MW}(\pi)$. This is done by studying the behavior of $\mathrm{MW}(\pi)$ under specialization and considering the classification of rational elliptic surfaces [9]. In Section 4, we discuss the possible singularities for quartic and sextic plane curves. This yields a classification of possible singularities on Y. In Section 5, we calculate the local cohomology $H^4_p(Y)$ for each possible type of singularity on Y. In Section 6, we give some details on how to calculate $\operatorname{rank} \mathrm{MW}(\pi)$. In the following three sections, we give a classification for the cases $j(\pi) = 1728$, $j(\pi) = 0$ and $R = 0$ is non-reduced and $j(\pi) = 0$ and $R = 0$ is a cuspidal sextic. In Section 10, we prove Theorem 1.1.

NOTATION 1.3. Let x, y, z_0, z_1, z_2 be coordinates on $\mathbf{P}(2,3,1,1,1)$. Throughout this paper, $Y \subset \mathbf{P}(2,3,1,1,1)$ is a reduced and irreducible degree 6 hypersurface, containing the point $(1 : 1 : 0 : 0 : 0)$, and such that Y corresponds to an elliptic fibration with constant j-invariant, that is, Y has a defining equation of the form

$$y^2 = x^3 + AP^2x + BP^3, \qquad y^2 = x^3 + Qx, \quad \text{or} \quad y^2 = x^3 + R.$$

Here P, Q, R are homogeneous polynomials in z_0, z_1, z_2 of degree $2, 4$ and 6 respectively and $A, B \in \mathbf{C} \setminus \{0\}$. The curve C is the curve defined by $P = 0$, $Q = 0$ or $R = 0$ (depending on the case). The set $\mathcal{Q}$ consists of the singular points of C_{red}.

2. Preliminaries

DEFINITION 2.1. An *elliptic n-fold* is a quadruple (X, B, π, σ_0), with X a smooth projective n-fold, B a smooth projective $n-1$-fold, $\pi : X \to B$ a flat morphism, such that the generic fiber is a genus 1 curve and σ_0 is a section of π.

The *Mordell–Weil group* of π, denoted by $\mathrm{MW}(\pi)$, is the group of rational sections $\sigma : B \dashrightarrow X$ with identity element σ_0.

We will focus on the cases $n = 2, 3$. Note that in the case $n = 2$ any rational section can be extended to a regular section.

Clearly $\mathrm{MW}(\pi)$ is a birational invariant, in the sense that if $\pi_i : X_i \to B_i$, $i = 1, 2$ are elliptic n-folds such that there exists a birational isomorphism $\varphi : X_1 \overset{\sim}{\dashrightarrow} X_2$ mapping the general fiber of π_1 to the general fiber of π_2, then $\varphi^* : \mathrm{MW}(\pi_2) \to \mathrm{MW}(\pi_1)$ is well-defined and is an isomorphism.

We shall now describe in some detail how to associate to an elliptic n-fold $\pi : X \to B$ a degree $6k$ hypersurface Y in the weighted projective space $\mathbf{P} := \mathbf{P}(2k, 3k, 1^{n-1})$ which is birational to X. Here, we restrict ourselves to the case where B is a rational $n-1$-fold. In this case, the morphism π establishes $\mathbf{C}(X)$ as a field extension of $\mathbf{C}(B) = \mathbf{C}(z_1, \ldots, z_{n-1})$. The field $\mathbf{C}(X)$ is the function field of an elliptic curve E over $\mathbf{C}(z_1, \ldots, z_{n-1})$, that is, $\mathbf{C}(X) = \mathbf{C}(x, y, z_1, \ldots, z_{n-1})$ where

$$y^2 = x^3 + f_1(z_1, \ldots, z_{n-1})x + f_2(z_1, \ldots, z_{n-1}) \tag{1}$$

with $f_1, f_2 \in \mathbf{C}(z_1, \ldots, z_{n-1})$. One has a natural isomorphism

$$\mathrm{MW}(\pi) \cong E\big(\mathbf{C}(B)\big),$$

where $E(\mathbf{C}(B))$ is the group of $\mathbf{C}(B)$-rational points of E.

Without loss of generality, we may assume that (1) is a global minimal Weierstrass equation, that is, f_1, f_2 are polynomials and there is no polynomial $g \in \mathbf{C}[z_1, \ldots, z_{n-1}] \setminus \mathbf{C}$ such that g^4 divides f_1 and g^6 divides f_2.

To obtain a hypersurface in $\mathbf{P}$, we need to find a weighted homogeneous polynomial. Let $k = \lceil \max\{\deg(f_1)/4, \deg(f_2)/6\} \rceil$ and define P and Q as the polynomials

$$P = z_0^{4k} f_1(z_1/z_0, \ldots, z_{n-1}/z_0), \qquad Q = z_0^{6k} f_2(z_1/z_0, \ldots, z_{n-1}/z_0).$$

Then

$$y^2 = x^3 + P(z_0, z_1, \ldots, z_{n-1})x + Q(z_0, z_1, \ldots, z_{n-1})$$

defines a hypersurface Y of degree $6k$ in $\mathbf{P}$. Let Σ be the locus where all the partial derivatives of the defining equation vanish. Consider the projection $\tilde{\psi} : \mathbf{P} \dashrightarrow \mathbf{P}^{n-1}$ with center $L = \{z_0 = \cdots = z_{n-1} = 0\}$ and its restriction $\psi = \tilde{\psi}|_Y$ to Y. Then there exists a diagram

$$\begin{array}{ccc} X & \dashrightarrow & Y \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \psi} \\ S & \dashrightarrow & \mathbf{P}^2. \end{array}$$

Note that $Y \cap L = \{(1:1:0:\cdots:0)\}$. If $k = 1$ then $\mathbf{P}_{\mathrm{sing}}$ consists of two points, none of which lie on Y. If $k > 1$, then an easy calculation in local coordinates shows that $\mathbf{P}_{\mathrm{sing}}$ is precisely L, that Σ and L are disjoint and that Y has an isolated singularity at $(1:1:0:\cdots:0)$. For any k, we have that ψ is not defined at $(1:1:0:\cdots:0)$. Let $\tilde{\mathbf{P}}$ be the blow-up of $\mathbf{P}$ along L. Let X_0 be the strict transform of Y in $\tilde{\mathbf{P}}$. An easy calculation in local coordinates shows that $X_0 \to Y$ resolves the singularity of Y at $(1:1:0:\cdots:0)$ and that the induced map $\pi_0 : X_0 \to \mathbf{P}^2$ is a morphism. Moreover, all fibers of π_0 are irreducible curves.

Definition 2.2. The degree of an elliptic n-fold $\pi : X \to B$, with rational base, is the smallest k such that there is a degree $6k$ hypersurface Y in $\mathbf{P}(2k, 3k, 1^{n-1})$ birational to π.

As remarked above, we can consider the generic fiber of π as an elliptic curve E over $\mathbf{C}(z_1, \ldots, z_{n-1})$. In the sequel, we consider only elliptic curves such that $j(\pi) = j(E)$ is constant, that is, $j(E) \in \mathbf{C}$. Most of the sequel will be concentrated on $j(\pi) \in \{0, 1728\}$. If this is the case, then E has complex multiplication.

Lemma 2.3. *Let K be a field, E/K an elliptic curve, such that E has complex multiplication over K. Suppose* $\operatorname{rank} E(K)$ *is finite. Then* $\operatorname{rank}_{\mathbf{Z}} E(K)$ *is even.*

Proof. Since $E(K) \otimes \mathbf{Q}$ is a vector space over $\operatorname{End}(E) \otimes \mathbf{Q}$. Since $\dim \operatorname{End}(E) \otimes \mathbf{Q}$ if a quadratic extension of $\mathbf{Q}$ it follows that $E(K)$ has even $\mathbf{Z}$-rank. □

The following minor results will be used several times.

Lemma 2.4. *Let $V/\mathbf{C}$ be a variety. Let $E/\mathbf{C}(V)$ be an elliptic curve such that $j(E) \in \mathbf{C}$. Suppose $j(E) \neq 0, 1728$, then $(\mathbf{Z}/2\mathbf{Z})^2$ is a subgroup of $E(\mathbf{C}(V))$.*

Proof. Let $E'/\mathbf{C}$ be an elliptic curve with $j(E')=j(E)$. Then we can find a Weierstrass equation $y^2=x^3+ax+b$ for E', with $a,b\in\mathbf{C}$. Let $\alpha_1,\alpha_2,\alpha_3$ be roots of x^3+ax+b.

Since $j(E)\neq 0,1728$, we have that E is given by

$$y^2=x^3+aP^2x+bP^3$$

for some $P\in\mathbf{C}(V)^*$. For $i=1,2,3$, we have that $x=\alpha_iP$ is a root of $x^3+aP^2x+bP^3$, hence $x=\alpha_iP, y=0$ is a point of order 2 on $E(\mathbf{C}(V))$. From this it follows that $(\mathbf{Z}/2\mathbf{Z})^2\subset E(\mathbf{C}(V))$. □

LEMMA 2.5. *Let K be any field not of characteristic $2,3$. Let E/K be an elliptic curve with $j(E)=1728$ then $E(K)$ contains a point of order* 2.

Proof. Since K is not of characteristic $2,3$, we have that E has a Weierstrass equation $y^2=x^3+ax$ with $a\in K$. The point $(0,0)$ is a point of order 2. □

3. Possible Mordell–Weil groups & specialization

We describe now all possible Mordell–Weil groups for elliptic surfaces of degree 1 with constant j-invariant. Using a specialization result this limits the possibilities for Mordell–Weil groups for elliptic threefolds of degree 1. Note that an elliptic surface is rational if and only if its degree is 1. We start by recalling a consequence of the classification of rational elliptic surfaces by Oguiso and Shioda [9].

PROPOSITION 3.1. *Suppose $\pi: S\to\mathbf{P}^1$ is a rational elliptic surface such that $j(\pi)$ is constant, then* $\mathrm{MW}(\pi)$ *is a subgroup of*

- $(\mathbf{Z}/2\mathbf{Z})^2$ *if* $j(\pi)\neq 0,1728$.
- $\mathbf{Z}^8$, $\mathbf{Z}^2\times\mathbf{Z}/3\mathbf{Z}$, *or* $(\mathbf{Z}/2\mathbf{Z})^2$ *if* $j(\pi)=0$.
- $\mathbf{Z}^4\times\mathbf{Z}/2\mathbf{Z}$ *or* $(\mathbf{Z}/2\mathbf{Z})^2$ *if* $j(\pi)=1728$.

The elements in the Mordell–Weil group of an elliptic threefold $\pi: X\to\mathbf{P}^2$ correspond to the $\mathbf{C}(s,t)$-rational points of the generic fiber of π. We can also consider the generic fiber of π as a rational elliptic surface defined over the t-line, defined over the field $\mathbf{C}(s)$, provided that the discriminant curve is not an union of lines through a point. In particular, this shows that $\mathrm{MW}(\pi)$ is a subgroup of the groups mentioned in Proposition 3.1. Using the results of the previous section, we can exclude a few other possibilities.

COROLLARY 3.2. *Suppose $\pi: X\to B$ is an elliptic threefold of degree* 1, *$j(\pi)$ is constant and $j(\pi)\neq 0,1728$. Then* $\mathrm{MW}(\pi)=(\mathbf{Z}/2\mathbf{Z})^2$.

Proof. From Lemma 2.4, it follows that $(\mathbf{Z}/2\mathbf{Z})^2\subset\mathrm{MW}(\pi)$. From Propositions 3.1, it follows that $\mathrm{MW}(\pi)\subset(\mathbf{Z}/2\mathbf{Z})^2$, which yields the corollary. □

COROLLARY 3.3. *Suppose* $\pi : X \to B$ *is an elliptic threefold of degree* 1, $j(\pi)$ *is constant and equals* 1728. *Then* $\mathrm{MW}(\pi)$ *is one of the following*:

$$(\mathbf{Z}/2\mathbf{Z}) \times \mathbf{Z}^r, (\mathbf{Z}/2\mathbf{Z})^2$$

with $r \in \{0,2,4\}$.

Proof. From Lemma 2.5, it follows that $(\mathbf{Z}/2\mathbf{Z}) \subset \mathrm{MW}(\pi)$. From Lemma 2.3, it follows that the rank is even. From Proposition 3.1, it follows that $\mathrm{MW}(\pi)$ is a subgroup of either $(\mathbf{Z}/2\mathbf{Z}) \times \mathbf{Z}^4$ or $(\mathbf{Z}/2\mathbf{Z})^2$, which yields the corollary. □

COROLLARY 3.4. *Suppose* $\pi : X \to B$ *is an elliptic threefold of degree* 1 *and* $j(\pi)$ *is constant and equals* 0. *Then* $\mathrm{MW}(\pi)$ *is one of the following*:

$$\mathbf{Z}^{r_1}, (\mathbf{Z}/3\mathbf{Z}) \times \mathbf{Z}^{r_2}, (\mathbf{Z}/2\mathbf{Z})^2$$

with $r_1 \in \{0,2,4,8\}$, $r_2 \in \{0,2\}$.

Proof. From Proposition 3.1, it follows that $\mathrm{MW}(\pi)$ is a subgroup of either $\mathbf{Z}^8$, $(\mathbf{Z}/2\mathbf{Z})^2$ or $\mathbf{Z}/3\mathbf{Z} \times \mathbf{Z}^2$. From Lemma 2.3, it follows that the rank is even.

To prove the corollary, we have to exclude the group $\mathbf{Z}/2\mathbf{Z}$. Suppose π has a section of order two, that is, Y is given by an equation of the from $y^2 = x^3 - T^3$ and $[z_0, z_1, z_2] \mapsto [T, 0, z_0, z_1, z_2]$ is a section of order 2. Then for $i = 1, 2$ the morphisms $[z_0, z_1, z_2] \mapsto [\omega^i T, 0, z_0, z_1, z_2]$ define also sections of order 2, where $\omega^2 = -\omega - 1$, hence we have complete two-torsion. In particular, $\mathbf{Z}/2\mathbf{Z}$ does not occur as possible Mordell–Weil group. □

Let $Y \subset \mathbf{P}(2,3,1,1,1)$ be an elliptic threefold. Let $\ell = \{a_0z_0 + a_1z_1 + a_2z_2 = 0\} \subset \mathbf{P}^2$ be a line. Let $H_\ell = \{a_0z_0 + a_1z_1 + a_2z_2 = 0\} \subset \mathbf{P}$ be the corresponding hyperplane. Then $Y_\ell = Y \cap H_\ell \subset \mathbf{P}(2,3,1,1)$ is a rational elliptic surface, provided ℓ was not a component of the discriminant curve of π.

The restriction of rational sections to ℓ defines a group homomorphism $\mathrm{MW}(\pi) \to \mathrm{MW}(\pi_\ell)$: we can consider $\mathrm{MW}(\pi_\ell)$ as the $\mathbf{C}(\pi_\ell)$-valued points of the general fiber of π_ℓ and $\mathrm{MW}(\pi)$ as the $\mathbf{C}(\pi_\ell)$-rational sections of an elliptic surface over $\mathbf{C}(\pi_\ell)$. Then the map $\mathrm{MW}(\pi) \to \mathrm{MW}(\pi_\ell)$ is just the specialization map as defined in for example, [11, Section III.11], and this is a group homomorphism if ℓ is not a component of the disciminant curve.

Later on, we need that for a special choice of ℓ the map $\mathrm{MW}(\pi) \to \mathrm{MW}(\pi_\ell)$ is injective. This result is probably known to the experts, but we did not find a reference for this in the literature.

PROPOSITION 3.5. *Let* $\Delta_{\mathrm{red}} \subset \mathbf{P}^2$ *be the reduced curve defined by the vanishing of* $4A^3 + 27B^2$. *Let* $\ell \subset \mathbf{P}^2$ *be a very general line. Then the map* $\mathrm{MW}(\pi) \to \mathrm{MW}(\pi_\ell)$ *is injective.*

Moreover, suppose that Δ_{red} *is neither a union of lines nor an irreducible conic. Then there exists a line* ℓ *such that*

(1) ℓ *is tangent to* Δ_{red} *at some point.*

(2) ℓ *intersects* Δ_{red} *in at least two distinct points.*

(3) *The natural map*

$$\mathrm{MW}(\pi) \to \mathrm{MW}(\pi_\ell)$$

is injective.

Proof. It suffices to prove that there are at most countable many lines such that

$$\mathrm{MW}(\pi) \to \mathrm{MW}(\pi_\ell)$$

is not injective, since if Δ_{red} is not the union of lines nor a conic then there are uncountable many lines that satisfy the first and second property the results follows.

Let $r = \operatorname{rank} \mathrm{MW}(\pi)$. Write $\mathrm{MW}(\pi)_{\mathrm{tor}} = \{\tau_1, \dots, \tau_k\}$. Fix $\sigma_1, \dots, \sigma_r \in \mathrm{MW}(\pi)$ such that the σ_i and τ_j generate $\mathrm{MW}(\pi)$.

Consider a section $\sigma = \tau_j + \sum_{i=1}^r n_i \sigma_i \in \mathrm{MW}(\pi)$. Write σ as

$$[z_0, z_1, z_2] \mapsto [f, g, z_0 h, z_1 h, z_2 h],$$

where f, g, h are homogeneous polynomials in z_0,z_1 and z_2 such that $\deg(f) = 2\deg(h) + 2$ and $\deg(g) = 3\deg(h) + 3$.

If σ lies in the kernel of $\mathrm{MW}(\pi) \to \mathrm{MW}(\pi_\ell)$, then h needs to vanish along ℓ. In particular, there are at most finitely many lines ℓ such that σ is mapped to zero in $\mathrm{MW}(\pi_\ell)$. Since $\mathrm{MW}(\pi)$ is countable there are at most countably many lines ℓ for which some σ is mapped to zero, that is, for which the map $\mathrm{MW}(\pi) \to \mathrm{MW}(\pi_\ell)$ is not injective. □

Later on, we will show that the cases $r = 4$, $r_1 = 8$ and $r_2 = 2$ can only occur in the cone construction case. At this point we will show that in these cases the curve C is a union of lines, but not necessarily through one point.

Assume that the j-invariant is constant. Then it is well known that the rank of $\mathrm{MW}(\pi_\ell)$ equals $2a - 4$, where a is the number of singular fibers of π_ℓ, that is, $a = \# C \cap \ell$ (counted without multiplicities).

LEMMA 3.6. *Suppose C is not the union of lines. Then $y^2 = x^3 + Px$ has Mordell–Weil rank at most* 2.

Proof. If P is of the form P_0^2 for some irreducible polynomial P_0 of degree 2, then for a very general line ℓ the elliptic surface π_ℓ is an elliptic surface with $2I_0^*$ fibers, and therefore has rank 0. Since for a very general line the map $\mathrm{MW}(\pi) \to \mathrm{MW}(\pi_\ell)$ is injective (Proposition 3.5), we have that $\operatorname{rank} \mathrm{MW}(\pi) = 0$.

Otherwise we can apply the second part of Proposition 3.5. Let ℓ be a line satisfying the properties mentioned in this proposition. Then π_ℓ has $j = 1728$ and one fiber either of type I_0^* or III^* and hence by the classification of Oguiso and Shioda [9] has rank at most 2. Since $\mathrm{MW}(\pi) \to \mathrm{MW}(\pi_\ell)$ is injective, it follows that $\mathrm{MW}(\pi)$ has rank at most 2. □

Lemma 3.7. *Suppose C is not the union of lines. Then $y^2 = x^3 + Q$ has Mordell–Weil rank at most* 6.

Proof. The proof is similar to the previous lemma. If Q is a quadratic polynomial to the power three, then for a general line π_ℓ is an elliptic surface with 2 I_0^* fibers has therefore rank 0 and hence rank $\mathrm{MW}(\pi) = 0$.

Otherwise we can apply Proposition 3.5. Using this we find a line ℓ such that π_ℓ has a fiber of type IV, I_0^*, IV^* or II^*, and therefore rank $\mathrm{MW}(\pi)_\ell \leq 6$, and such that $\mathrm{MW}(\pi) \to \mathrm{MW}(\pi_\ell)$ is injective. □

Lemma 3.8. *Suppose $j = 0$ then $3 \mid \#\mathrm{MW}(\pi)_{\mathrm{tor}}$ if and only if Y is given by an equation of the form $y^2 = x^3 + f^2$, where f is a cubic polynomial.*

If $(\mathbf{Z}/3\mathbf{Z}) \times \mathbf{Z}^2$ is a subgroup of $\mathrm{MW}(\pi)$ then $f = 0$ is a union of lines.

Proof. Suppose $3 \mid \#\mathrm{MW}(\pi)_{\mathrm{tor}}$. Since for a very general line ℓ the map $\mathrm{MW}(\pi) \to \mathrm{MW}(\pi_\ell)$ is injective (Proposition 3.5) it follows that for such line ℓ the $\pi_\ell : X_\ell \to \mathbf{P}^1$ is a rational elliptic surface with 3 torsion. It follows from the classification of Oguiso and Shioda [9] that then the intersection $C \cap \ell$ consists of three points with multiplicity 2 or one point with multiplicity 2 and one point with multiplicity 4. Hence, C is a double cubic.

Conversely, if Y is given by $y^2 = x^3 + f^2$ then $x = 0, y = f$ defines a section of order 3.

Suppose $f = 0$ is not the union of lines. Then C is a reduced cubic. Then by Proposition 3.5 there exists a line ℓ, such that $\mathrm{MW}(\pi) \to \mathrm{MW}(\pi_\ell)$ is injective and such that $\#\ell \cap C = 2$, hence $\mathrm{MW}(\pi_\ell)$ is finite. □

4. Singularities of quartic and sextic plane curves

4.1. Quartic curves. The classification of singular quartic curves is well-known. We give a sketch. First, assume that C is reduced. Then either C is the union of four lines through one point, or C has at most ADE singularities. The first case corresponds to the cone construction case, so suppose we are in the latter case. Let $p_1, \dots, p_k$ be the singularities of C, let $M_1, \dots, M_k$ be the corresponding Milnor lattices. Then $\bigoplus M_i$ can be embedded in the lattice corresponding to the affine Dynkin diagram $\tilde{E}_7$. This limits the possibilities to $A_1, \dots, A_7, D_4 \dots, D_7$ and E_6, E_7.

Assume that C is non-reduced and that C_{red} is not the union of lines through one point. Then C is either a double line ℓ together with a (possible reducible) conic T, or a double conic. If C is a double conic, then it has to be irreducible, hence C_{red} is smooth and $\mathcal{P} = \emptyset$.

Let q be a point of the singular locus of C_{red}. The above discussion shows that (C, q) is one of the following singularities

- $t^2 s$, i.e., ℓ and T intersect transversely;
- $t^2(t - s^2)$, i.e., ℓ is tangent to T;
- ts, (A_1 singularity), i.e., T is the union of two lines.

Note that in the second case we have that $\mathcal{P}$ consists of one point.

4.2. Sextic curves.

Sextic curves have more possible singularities.

THEOREM 4.1. *A reduced sextic can have the following singularities* [6]:

- $A_k : x^2 + y^{k+1}$, $k \leq 19$.
- $D_k : y(x^2 + y^{k-1})$, $k \leq 19$.
- $E_6 : x^3 + y^4$.
- $E_7 : x^3 + xy^3$.
- $E_8 : x^3 + y^5$.
- $B_{k,l} : x^k + y^l$, $3 \leq k \leq l$. *If* $k = 3$ *then* $6 \leq l \leq 12$, *if* $k > 3$ *then* $l \leq 6$.
- $xB_{k,l} : x(x^k + y^l)$, $(k, l) \in \{(2, 5), (2, 7), (3, 4), (3, 5), (4, 5)\}$.
- $yB_{k,l} : y(x^k + y^l)$, $(k, l) \in \{(3, 4), (3, 5), (3, 6), (4, 5)\}$.
- $xyB_{k,l} : xy(x^k + y^l)$, $(k, l) \in \{(2, 3), (3, 4)\}$.
- $C_{k,l} : x^k + y^l + x^2y^2$, $k \leq l$, $2/k + 2/l \leq 1$ *and* $k \leq n(l)$, *with* $n(l) = 15, 14, 14, 12, 11, 11, 9$ *for* $l = 3, 4, \ldots, 9$.
- $yC_{k,l} : y(x^k + y^l + x^2y^2)$. $k \leq l$, $2/k + 2/l \leq 1$, $k \in \{3, 5\}$. *If* $k = 3$ *then* $7 \leq l \leq 12$. *If* $k = 5$ *then* $l \in \{5, 6\}$.
- $D_{k,l} : x^k + y^l + x^2y^3$. $2/p + 3/q \leq 1$. *If* $k = 3$ *then* $9 \leq 10 \leq 13$. *Otherwise* $(k, l) \in \{(4, 7), (5, 6), (5, 7), (6, 5), (6, 6), (6, 7)\}$.
- $F_{k,l} : x^k + y^l + x^2y^3 + x^3y^2$. $6 \leq k \leq l \leq 7$.
- $S_{2k-1} : (x^2 + y^3)^2 + (a_0 + a_1y)xy^{4+k}$. $a_0 \neq 0, a_1 \in \mathbf{C}, k = 1, 2, 3$.
- $S_{2k} : (x^2 + y^3)^2 + (a_0 + a_1y)x^2y^{3+k}$. $a_0 \neq 0, a_1 \in \mathbf{C}, k = 1, 2, 3$.

All these singularities are also in Arnol'd's list, so one might also use the names given by Arnol'd. A translation between our name-giving and that of Arnol'd can be found in [6, Remark 1].

Several of these singularities have distinct Milnor and Tjurina number, and are therefore not semi-weighted homogeneous.

We do not present a classification of non-reduced sextics here. Essentially, one has either

- a double line with a quartic,
- a double conic with another conic,
- a double cubic,
- a triple line with a cubic,
- a triple line with a double line and a line,
- a triple conic or
- a quadruple line with conic.

The possibilities for the singularities are a combination of the possibilities of singularities for conics, cubics and quartic, and the possible intersection numbers between the components.

5. Calculating $H^4_p(Y,\mathbf{C})$

In this section, we discuss three approaches to calculate $H^4_p(Y,\mathbf{Q})$. For each singularity that we encounter, one of these methods applies, except for six types of singularities. We list $h^4_p(Y)$ for each of the singularities.

5.1. Dimca's method. Let (Y,p) be a semi-weighted isolated hypersurface singularity. We have a local equation of the form $f_p+g_p=0$, such that (Y,p) is a μ-constant deformation of $f_p=0$ and $f_p=0$ defines a weighted homogeneous isolated hypersurface singularity.

Let $w_1,\dots,w_4$ are the weights of the variables, $w_p=w_1+w_2+w_3+w_4$ and let d_p be the (weighted) degree of f. Then Dimca [3] shows

$$H^4_p(Y)=\bigoplus_{i=1}^{3} R(f)_{id_p-w_p}.$$

Moreover, this direct sum decomposition is just $\bigoplus \mathrm{Gr}^F_{4-i}H^4_p(Y)$. Finally, Dimca shows that $H^4_p(Y)$ has a pure weight 4 Hodge structure.

It turns out that all singularities under consideration satisfy $R(f)_{d_p-w_p}=0$ and $R(f)_{3d_p-w_p}=0$. This follows from the fact that in all cases $d_p<w_p$ and the existence of a non-degenerated pairing $R(f)_{d_p-w_p}\times R(f)_{3d_p-w_p}\to R(f)_{4d_p-2w_p}\cong\mathbf{C}$. This implies that $H^4_p(Y,\mathbf{C})=\mathbf{C}(-2)^k$ with $k=h^4_p(Y)$.

If $j(\pi)=1728$, then all singularities of Y are non-isolated, so for the rest of this subsection assume that $j(\pi)=0$.

We have a semi-weighted hypersurface singularity if and only if the sextic C is reduced at $q=\psi(p)$ and has a weighted homogeneous singularity.[1] This limits us to cases that C has either an ADE singularity, or a $B_{k,l},xB_{k,l},yB_{k,l},xyB_{k,l}$ singularity. We list now the singularities with nontrivial $H^4_p(Y)$.

Proposition 5.1. *Suppose (C,q) is a weighted homogeneous singularity of a sextic curve, not a point of order six, and such that $h^4_p(Y)\neq 0$. Then (C,q) is one of*

- $A_2,A_5,A_8,A_{11},A_{14},A_{17}$,
- E_6,
- $B_{3,6},B_{3,8},B_{3,10},B_{3,12},B_{4,6}$.

The following lemmata yield a proof of this proposition and the proofs provide basis for $H^4_p(Y,\mathbf{C})$ for each nontrivial case.

Lemma 5.2. *Suppose C has an A_k singularity at q. If $k\equiv 2 \bmod 3$, then $H^4_p(Y,\mathbf{C})$ is isomorphic to $\mathbf{C}(-2)^2$. Otherwise, $H^4_p(Y,\mathbf{C})$ vanishes.*

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[1] From the discussion of singular sextics it follows that only the S_k singularities have moduli. Since the S_k singularities are not semi weighted homogeneous it turns out that all semi weighted homogeneous singularities are rigid and therefore weighted homogeneous.

Proof. We have a local equation for Y of the form

$$y^2 = x^3 + t^2 + s^{k+1}.$$

Setting weights $6, 3k+3, 2k+2, 3k+3$ for s, t, x, y, we obtain $d_p = 6k+6$, $w_p = 8k+14$. Hence, $2d_p - w_p = 4k-2$. The Jacobian ideal is generated by y, t, x^2, s^k. Hence, $R(f)_{4k-2}$ is spanned by

$$xs^{(k-2)/3}, s^{(2k-1)/3}.$$

This means that $H^4_p(Y) = 0$ if $k \not\equiv 2 \bmod 3$ and $H^4_p(Y) = \mathbf{C}(-2)$ if $k \equiv 2 \bmod 3$. □

LEMMA 5.3. *Suppose C has an D_k singularity at q then $H^4_p(Y, \mathbf{C}) = 0$.*

Proof. We have a local equation for Y of the form

$$y^2 = x^3 + st^2 + s^{k-1}.$$

Setting weights $6, 3k-6, 2k-2, 3k-3$, we obtain $d_p = 6k-6$, $w_p = 8k-5$. Hence, $2d_p - w_p = 4k-7$. Every monomial in x, s, t has even degree and since y is in the Jacobian ideal it follows that $R(f)_{4k-7} = 0$. □

LEMMA 5.4. *Suppose C has an E_k singularity at q, $k \in \{6, 7, 8\}$ then $H^4_p(Y, \mathbf{C}) = \mathbf{C}(-2)^2$ if $k = 6$ and $H^4_p(Y, \mathbf{C}) = 0$ otherwise.*

Proof. E_6: We have a local equation for Y of the form

$$y^2 = x^3 + t^3 + s^4.$$

Setting weights $3, 4, 4, 6$, we obtain $d_p = 12, w_p = 17, 2d_p - w_p = 7$. The only monomials of weights 7 are xs, ts and their classes provide a basis for $R(f_p)_7$.

E_7: We have a local equation for Y of the form

$$y^2 = x^3 + t^3 + s^3 t.$$

Setting weights $4, 6, 6, 9$, we obtain $2d_p - w_p = 11$. Since there are no monomials of degree 11, we obtain $R(f_p)_{11} = 0$.

E_8: We have a local equation for Y of the form

$$y^2 = x^3 + t^3 + s^5.$$

Setting weights $6, 10, 10, 15$, we obtain $2d_p - w_p = 19$. Since there are no monomials of degree 19, we obtain $R(f_p)_{10} = 0$. □

REMARK 5.5. Suppose (Y, p) is a weighted homogeneous hypersurface singularity. Let $(\{f_p = 0\}, 0)$ be a local equation of (Y, p), where f_p is weighted homogeneous. Then $S_p = \{f_p = 0\}$ defines a surface in $\mathbf{P}(w_0, w_1, w_2, w_3)$. Dimca's method (as well as the method of Hulek–Kloosterman) relies on the isomorphism $H^4_p(Y, \mathbf{C}) \cong H^2(S_p, \mathbf{C})_{\text{prim}}(1)$.

Often one can simplify this calculation. If exactly three of the four weights have a nontrivial common divisor one can apply the following procedure: Suppose $S_p \subset \mathbf{P}(w_0, gw_1, gw_2, gw_3)$ and $g \nmid w_0$. Then there is an isomorphism

$\varphi : \mathbf{P}(w_0, gw_1, gw_2, gw_3) \to \mathbf{P}(w_0, w_1, w_2, w_3)$ by sending $(x_0 : x_1 : x_2 : x_3) \to (x_0^g : x_1 : x_2 : x_3)$.

Let g_p be the equation of $\varphi(S_p)$. Suppose that g_p has an isolated singularity, that is, $\varphi(S_p)$ is quasi-smooth. Since φ is an isomorphism, we have then that $H^{1,1}(S_p)_{\text{prim}} \cong R(g_p)_{d-w_0-w_1-w_2-w_3}$. We often refer to this as the *three weights trick*.

The reason to apply the three weights trick is the following: in several cases it turns out that 1 lies in the Jacobian ideal of g_p. This in turn implies that $R(g_p) = 0$, and $h^{1,1}(S_p)_{\text{prim}} = 0$.

LEMMA 5.6. *Suppose C has a $B_{k,l}$ singularity and $6 \nmid kl$ then $H^4_p(Y, \mathbf{C}) = 0$.*

Proof. We have a local equation for (C, q) of the form

$$y^2 = x^3 + t^k + s^l$$

hence we can set the weights for s, t, x, y to be $6k, 6l, 2kl, 3kl$. If $2 \nmid kl$, we may apply the three weight trick (Remark 5.5). Therefore, it suffices to study $y = x^3 + t^k + s^l$. If $3 \nmid kl$, we can also use the three weights trick, in this case we obtain the singularity $y^2 = x + t^k + s^l$. In both cases, 1 is in the Jacobian ideal, hence the Jacobian ring (and therefore the local cohomology) vanishes. □

Recall that in Theorem 4.1 we give a list of possible values (k, l) such that $B_{k,l}$ occurs as a singularity on a sextic.

LEMMA 5.7. *Suppose C has a $B_{k,l}$ singularity and $6 \mid kl$ then*

- $H^4_p(Y, \mathbf{C}) = \mathbf{C}(-2)^2$ *if* $k = 3$ *and* $l \equiv 2, 4 \bmod 6$.
- $H^4_p(Y, \mathbf{C}) = \mathbf{C}(-2)^4$ *if* $k = 3$ *and* $l \equiv 0 \bmod 6$.
- $H^4_p(Y, \mathbf{C}) = \mathbf{C}(-2)^2$ *if* $(k, l) = (4, 6)$.
- $H^4_p(Y, \mathbf{C}) = 0$ *if* $(k, l) = (5, 6)$.

Proof. An easy computation shows that if $(k, l) = (5, 6)$ then $R_{2d_p - w_p} = 0$ and there is no local cohomology.

If $k = 3$, then $R_{2d_p - w_p}$ is generated by

$$xts^{(l-6)/6}, xs^{(l-2)/2}, ts^{(l-2)/2}, s^{(5l-1)/6}.$$

If $(k, l) = (4, 6)$, then $R_{2d_p - w_p}$ is generated by xts, t^2s. □

LEMMA 5.8. *Suppose C has an $xB_{k,l}$, an $yB_{k,l}$ or an $xyB_{k,l}$ singularity then $H^4_p(Y, \mathbf{C})$ vanishes.*

Proof. We used the computer algebra package Singular to check for every admissible value of (k, l) (see Theorem 4.1) that $R_{2d_p - w_p} = 0$. □

5.2. Method of Brieskorn. A second of class of singularities are non-weighted homogeneous isolated hypersurface singularities.

Let f_p be a local equation for (Y,p). Then $f_p = y^2 + x^3 + g_p(s,t)$. We explain now the method to calculate $H^4_p(Y)$. First, observe that this group equals $H^4(F)^0$ the part of the cohomology of the Milnor fiber that is invariant under the monodromy.

Now $H^4(F)$ is naturally isomorphic to the Milnor algebra of $(f_p, 0)$. The Milnor algebra can be easily calculated. Brieskorn [1] developed a method to calculate the action of the monodromy on $H^4(F)$. We will not explain this method, but use the computer algebra package Singular, which contains an implementation of this method.

For computational reasons, it is better to let Singular calculate the monodromy action on $H^2(F_1)$, where F_1 is the Milnor fiber of $g_p(s,t) = 0$. From this, one can deduce the monodromy on $H^4(F)$ as follows:

For an arbitrary singularity $f(x_1, \ldots, x_n) = 0$, one can identify $H^n(F)$ with the Milnor algebra $M(f) := \mathbf{C}[[x_1, \ldots, x_n]]/(f, f_{x_0}, \ldots, f_{x_0})$.

Now, the Milnor algebra of $x^d_{n+1} + f$ is the direct sum $\bigoplus_{i=0}^{d-2} x^i M(f)$. One easily shows that for $h \in M(f)$ we have $T_{x^d_{n+1}+f}(x^j h) = \exp(2j\pi i/d) x^j T_f(h)$, where T_g is the monodromy operator for the singularity $(g, 0)$. More specific, to find all the eigenvalues of $T_{x^d_{n+1}+f}$ one needs to multiply all the eigenvalues of T_f by all the dth roots of unity except for the root of unity 1. In the case $y^2 + x^3 + g_p(s,t)$, we apply this procedure twice. Hence, the eigenvalues of the monodromy of f get multiplied by $\exp(5/3\pi i)$ and $\exp(1/3\pi i)$, that is, the two primitive sixth roots of unity. So, in order to determine $H^4_p(Y)$ we need to find the eigenspaces for the eigenvalues $\exp(5/3\pi i)$ and $\exp(1/3\pi i)$ on $H^2(F_1)$.

The computer algebra package Singular produced the following results.

PROPOSITION 5.9. *Suppose (C,q) is a $C_{k,l}, yC_{k,l}, D_{k,l}, F_{k,l}$ or S_k singularity on a sextic curve. Then*

- $h^4_p(Y, \mathbf{C}) = 4$ *if (C,q) is $C_{3k,3l}$ singularity.*
- $h^4_p(Y, \mathbf{C}) = 2$ *if (C,q) is $C_{k,l}$ singularity, where exactly one of k, l is divisible by* 3, *or (C,q) is either a S_3 or a S_6 singularity.*
- $h^4_p(Y, \mathbf{C}) = 0$ *otherwise.*

5.3. Method of Hulek–Kloosterman. The final method we use works for non-isolated singularities. Let (Y,p) be such a singularity. Since we have a minimal elliptic threefold, such a singularity is one-dimensional, and the transversal types are ADE surface singularities.

There are three classes to distinguish:

- $j(\pi) = 1728$ and C has an isolated singularity at q.
- $j(\pi) = 1728$ and C has a non-isolated singularity at q.
- $j(\pi) = 0$ and C has a non-isolated singularity at q.

If C is a quartic, then (C,q) is weighted homogeneous. If C is a sextic, then except for six types of singularities, (C,q) is weighted homogeneous. For the rest of this subsection, assume that (C,q) is weighted homogeneous. Then (Y,q) is weighted homogeneous. This implies that we may apply [5, Proposition 7.7], which is a generalization of Dimca's method. We start by giving a short outline of this method:

Let f_p be a local equation for (Y,q), let w_p and d_p be as in Dimca's method. Let $R(f_p)$ be the Jacobian ring of f_p. Hulek and the author proved that $H^4_p(Y)$ has pure Hodge structure of weight 4 with $h^{4,0}=h^{0,4}=0$, $h^{3,1}=h^{1,3}=\dim R(f_p)_{d_p-w_p}$. To determine $h^{2,2}$ we need to introduce some notation. The equation $f_p=0$ defines a surface S_p in weighted projective 3-space $\mathbf{P}'$. Now, Hulek and the author show that $h^{2,2}(H^4_p(Y))=h^{1,1}(S_p)_{\text{prim}}$.

The Hodge number $h^{1,1}(S_p)_{\text{prim}}$ can be determined as follows: Let $q_1, \dots, q_s$ be the points where all the partials of f_p vanish. Then (S_p,q_i) is an ADE singularity. If $q\notin\mathbf{P}'_{\text{sing}}$ then let M_q be the Milnor algebra of (S_p,q).

If $q\in\mathbf{P}'_{\text{sing}}$ we do the following: we have a degree 6 quotient map $\varphi:\mathbf{P}^4\to\mathbf{P}'$ let G be the Galois group of this cover. Let $\tilde{q}\in\varphi^{-1}(q)$. Let G_q be the stabilizer of $\tilde{q}$. Let g be a local equation of (S_p,q) in $\mathbf{P}'$. Then G_q acts on the Milnor algebra of g. Let M_q be the invariant part of M under G_q. One can show that this definition is independent of the choices made. Let

$$\tilde{R}(f_p)_{2d_p-w_p}:=\ker R(f_p)_{2d_p-w_p}\to\bigoplus M_{q_i}.$$

Then it follows from the work of Steenbrink [12] that

$$h^{1,1}(S_p)=\dim\tilde{R}(f_p)_{2d_p-w_p}.$$

This suffices to calculate all the Hodge numbers.

Remark 5.10. In addition to the *three weights trick* (Remark 5.5) there is another trick we can apply. Namely, let $\Sigma(f_p)$ be the locus, where all the partials of f_p vanish. Assume that $\Sigma(f_p)\cap\mathbf{P}'_{\text{sing}}=\emptyset$. Then

$$h^{1,1}(S_p)=h^{1,1}(S)-\sum_{q\in\Sigma(f_p)}\mu_q,$$

where μ_q is the Milnor number of the singularity at q. (This formula holds, since S_p has only ADE surface singularities. For a proof of this, see, for example, [5, Lemma 6.1].)

As written above, we distinguish between three classes. First, assume $j(\pi)=1728$.

Proposition 5.11. *Suppose (C,q) is a singularity of a quartic curve, not a point of order four, such that $h^4_p(Y)\neq 0$. Then (C,q) is isolated and one of A_3, A_7, D_7.*

Assume first that (C,q) is isolated. From the classification of singular quartics, it follows that (C,q) is an ADE singularity.

In all cases, it turns out that $d_p - w_p < 0$, hence $H^4_p(Y)$ is of pure $(2,2)$-type. Since w_p and d_p are listed in every proof, we do not mention that $d_p < w_p$. We prove now:

LEMMA 5.12. *Suppose C has an A_k singularity at q then $H^4_p(Y,\mathbf{C}) = \mathbf{C}(-2)^2$ if $k \equiv 3 \bmod 4$ and $H^4_p(Y,\mathbf{C}) = 0$ otherwise.*

Proof. If C has an A_k singularity at q, then Y is locally of the from $f_p = 0$ with

$$f_p = y^2 + x^3 + \left(t^{k+1} + s^2\right)x.$$

Set the weights of s,t,x,y to be $2k+2, 4, 2k+2, 3k+3$. The sum w_p of the weights equals $7k+11$. The degree d_p equals $6k+6$.

To determine $h^{2,2}$ we start by determining $R_{2d_p - w_p} = R_{5k+1}$. Since $y, x^2 + t^{k+1} + s^2, t^k x$ and sx generate the Jacobian ideal, it follows that

$$R_{5k+1} = \operatorname{span}\left\{t^{(5k+1)/4}, t^{(3k-1)/4}s, t^{(k-3)/4}s^2, xt^{(3k-1)/4}\right\}.$$

Hence, $R_{5k+1} = 0$ if $k \not\equiv 3 \bmod 4$. From this, it follows that $h^4_p(Y) = 0$ if $k \not\equiv 3 \bmod 4$.

Suppose that $k \equiv 3 \bmod 4$, i.e., $k = 3 + 4(m-1)$. Our defining equation is of the form

$$y^2 + x^3 + \left(t^{4m} + s^2\right)x.$$

We set the weights of s,t,x,y to be $2m, 1, 2m, 3m$. Now, $d_p = 6m$, $w_p = 7m+1$. From this, it follows that $R(f_p)_{2d_p - w_p}$ is generated by

$$t^{5m-1}, t^{3m-1}s, t^{m-1}s^2, t^{3m-1}x.$$

Since S has A_1 singularities at $(1:1:0:0)$ and $(1:\ -1:0:0)$. The Milnor algebra is generated by 1, i.e., $\tilde{R}_{2d_p - w_p}$ is spanned by elements of $R_{2d_p - w_p}$ that vanish at $(1:1:0:0)$ and $(1:\ -1:0:0)$, hence it is spanned by

$$xt^{3m-1} \text{ and } \left(t^{4m} - s^2\right)t^{m-1}$$

and $h^4_p(Y) = 2$. □

LEMMA 5.13. *Suppose C has a D_k singularity at q. Then $H^4_p(Y,\mathbf{C}) = \mathbf{C}(-2)^2$ if $k \equiv 3 \bmod 4$ and $H^4_p(Y,\mathbf{C}) = 0$ otherwise.*

Proof. In this case, we have a local equation of the form

$$y^2 + x^3 + t\left(t^{k-2} + s^2\right)x.$$

The weights here are $2k-4, 4, 2k-2, 3k-3$. Hence $d_p = 6k-6$, $w_p = 7k-5$, $2d_p - w_p = 5k-7$. Consider

$$R_{2d_p - w_p} = \left(\mathbf{C}[x,y,t,s]/\left(y, x^2 + t^{k-1} + s^2 t, stx, \left((k-1)t^{k-2} + s^2\right)x\right)\right)_{5k-7}.$$

It is easy to see that $t^{5/4k-7/4}, t^{3/4k-3/4}s, t^{1/4k+1/4}s^2, xt^{3/4k-5/4}$ span this vector space. Hence, a necessary condition to have local cohomology is $k \equiv \pm 1 \bmod 4$.

Consider first the case $k \equiv 3 \bmod 4$, i.e, $k = 4m+3$, then we have a local equation of the form

$$y^2 + x^3 + t\big(t^{4m+1} + s^2\big)x.$$

We can normalize the weights such that they become $4m+1, 2, 4m+2, 6m+3$. The degree is $12m+6$, the sum of the weights equals $14m+8$. The vector space R_{10m+4} is spanned by

$$t^{5m+2}, s^2t^{m+1}, xt^{3m+1}.$$

The partials of f_p vanish if $t = x = y = 0$ or if $t^{4m+1} + s^2 = x = y = 0$. These equations yield points q_1, q_2 where S_p has an A_1 singularity. At such a point the Milnor algebra is generated by 1, hence the kernel $R(f_p) \to M_{q_1} \oplus M_{q_2}$ consists of functions vanishing at q_1 and q_2. So $\tilde{R}$ is generated by

$$\big(t^{4m+1} + s^2\big)t^{m+1}, xt^{3m+1}.$$

Thus $H^4_p(Y, \mathbf{C}) = \mathbf{C}(-2)^2$.

The case $k \equiv 1 \bmod 4$ is different. Set $k = 4m+1$. Then we have a local equation of the form

$$y^2 + x^3 + t\big(t^{4m-1} + s^2\big)x.$$

The weights are $4m-1, 2, 4m, 6m$. This surface is isomorphic to the surface S given by

$$y^2 + x^3 + t\big(t^{4m-1} + s\big)x$$

in $\mathbf{P}(4m-1, 1, 2m, 3m)$. The surface S is of degree $6m$ and the sum of the weights is $9m$. The only monomials of degree $3m$ are y, xt^m, t^{3m}. Since y and xt are in the Jacobian ideal it turns out that $R(f_p)_{2d_p - w_p}$ is generated by t^{3m}.

The surface S has an A_1 singularity at $q = (1 : -1 : 0 : 0)$. At this point we have a trivial stabilizer. The Milnor algebra M_q is generated by 1 in local coordinates. Hence, all elements of $\tilde{R}(f_p)_{2d_p - w_p}$ have to vanish at q. So $t^{3m} \notin \tilde{R}(f_p)_{2d_p - w_p}$, hence $h^4_p(Y) = 0$. □

LEMMA 5.14. *Suppose C has an E_k singularity at q then $H^4_p(Y, \mathbf{C})$ vanishes.*

Proof. Case E_6:

$$y^2 + x^3 + \big(s^3 + t^4\big)x,$$

the weights are $4, 3, 6, 9$. This surface is isomorphic to

$$y^2 + x^3 + \big(s + t^4\big)x$$

in $\mathbf{P}(4, 1, 2, 3)$. The degree is 6, the sum of the weights equals 10, whence $2d_p - w_p = 2$. The only monomials of degree 2 are x and t^2. Since x is in the Jacobian ideal it follows that $R_{2d_p - w_p}$ is generated by t^2. As S has an A_1 singularity at $(1:1:0:0)$, all elements of $\tilde{R}_{2d_p-w_p}$ have to vanish at $(1:1:0:0)$. Since t^2 does not vanish, we obtain that $h^4_p(Y)=0$.

Case E_7:

$$y^2+x^3+(s^3+st^3)x,$$

the weights are $12,8,18,27$. This surface is isomorphic to

$$y+x^3+(s^3+st^3)x$$

in $\mathbf{P}(6,4,9,27)$. Since 1 is in the Jacobian ideal, we obtain R is the zero ring, hence $H^4_p(Y,\mathbf{C})=0$. □

We come now to the case where (C,q) is a not an isolated singularity. From Section 4, it follows that we only have to consider the following two singularities:

LEMMA 5.15. *Suppose* (C,q) *has local equation* $s^2t=0$ *or* $s^2(s-t^2)=0$. *Then* $H^4_p(Y)=0$.

Proof. In the first case, we have a local equation $y^2=x^3+s^2tx$ for (Y,p). This defines a degree 6 surface S in $\mathbf{P}(1,2,2,3)$. Hence $2d_p-w_p=4$. The monomials xt,s^4,t^2,s^2t span $R(f_p)_4$. The surface S has two singularities, namely at $q_1:=(1:0:0:0)$ and $q_2:=(0:1:0:0)$.

The Milnor algebra M_{q_1} is generated by 1 (which translates to s^4 in global coordinates). For q_2, note that the Milnor algebra is generated by $1,x,s,s^2$. The group G_{q_2} is generated by $s\mapsto -s$, hence $M_{q_2}^{G_{q_2}}$ is spanned by t^2,xt,s^2t and $\tilde{R}_{2d_p-w_p}=0$.

Consider now $y^2=x^3+s^2(s-t^2)x$. This defines a surface in $\mathbf{P}(4,2,6,9)$ and is isomorphic to $y=x^3+s^2(s-t^2)x$ in $\mathbf{P}(2,1,3,9)$. This surface has $h^2_{\mathrm{prim}}=0$, hence $H^4_p(Y,\mathbf{C})=0$. □

We turn now to the final case, namely $j(\pi)=0$ and C is non-reduced.

LEMMA 5.16. *Suppose* (Y,p) *is one of the following singularities*

(1) $y^2=x^3+t^2s$,
(2) $y^2=x^3+t^2(t-s^3)$,
(3) $y^2=x^3+t^3s$,
(4) $y^2=x^3+t^3(t-s^2)$,
(5) $y^2=x^3+t^4s$,
(6) $y^2=x^3+t^2s^2$.

Then $H^4_p(Y)=0$.

Proof. For each case, we list a choice for the weights. We then either state that we may apply the the three weights trick (Remark 5.5) or we give an outline on how to compute $\tilde{R}_{2d_p-w_p}$:

(1) $2,2,2,3$: (three weights).
(2) $2,6,6,9$: (three weights).

(3) $3,1,2,3$: In this case we have $2d_p - w_p = 3$. A basis for $R(f_p)_3$ is s, xt. At $(1:0:0:0)$ we have the following stabilizer: $x \mapsto \omega^2 x, t \mapsto \omega t$. The Milnor algebra has basis $1, x, t, xt$. After taking invariants under the stabilizer we find that $1, xt$ span $M_p^{G_p}$. Hence, $\tilde{R}_{2d_p - w_p} = 0$.
(4) $3,6,8,12$: (three weights).
(5) $2,1,2,3$: In this case, we have $2d - w = 4$. A basis for $R(f_p)_4$ is xs, xt^2, $t^2 s$, s^2. At $(1:0:0:0)$ we have $t \to -t$ as stabilizer. The Milnor algebra is spanned by $1, x, t, xt, t^2, xt^2$, hence the invariants under the stabilizer are (in global coordinates) $s^2, xs, t^2 s, xt^2$. Hence, $\tilde{R}(f_p)_{2d_p - w_p} = 0$.
(6) $2,1,2,3$: A basis for $R(f_p)_4$ is xt^2, xs, t^4, s^2. At $(1:0:0:0)$ the stabilizer is generated by $t \to -t$, the Milnor algebra is spanned by $1, x$, hence is invariant under the stabilizer, so we can exclude s^2, xs. At $(0:1:0:0)$ we have no stabilizer, the Milnor algebra is spanned by $1, x$ in local coordinates, hence t^4, xt^2 can be excluded. From this it follows that $\tilde{R}_{2d_p - w_p}(f_p) = 0$. □

LEMMA 5.17. *Suppose T is a reduced quartic with a double and a triple point. Then either*

- *T has exactly two singularities, the triple point is a D_6 singularity and the double point is an A_1 singularity,*
- *T has exactly two singularities, the triple point is a D_5 singularity and the double point is an A_1 singularity or*
- *T has a D_4 singularity an up to 3 A_1 singularities.*

Proof. Suppose q_1 is a double point and q_2 a the triple point. Let ℓ be the line through q_1 and q_2. Since $(T \cdot \ell)_{q_1} \geq 2$ and $(T \cdot \ell)_{q_2} \geq 3$ it follows that ℓ is a component of T. Let K be the residual cubic. Then q_1 is a smooth point of K and q_2 is double point of K. Since T is reduced, we have that ℓ is not a component of K. From this, it follows that $(K \cdot \ell)_{q_i} = i$ for $i = 1, 2$. In particular, at q_1 we have an A_1 singularity. Hence, all double points of T are A_1 singularities.

Note that if K has an A_k singularity at q_1 then T has D_{3+k} singularity. Since K is a cubic we have that $k \leq 3$.

If K has an A_3 singularity at q_1 then K is a conic Q together with a line tangent to Q at q_1. Hence, T has a D_6 and an A_1 singularity and no other singularities.

If K has an A_2 singularity at q_1 then K is an irreducible cubic and smooth outside q_1. Hence T has a D_5 and an A_1 singularities.

If K has an A_1 singularity at q_1 then K has at most 2 other A_1 singularities, hence T has a D_4 singularity and at most three A_1 singularities. □

LEMMA 5.18. *Suppose C is a double line ℓ together with a reduced quartic T. Suppose that Y has at least two singularities such that $H^4_p(Y) \neq 0$ then one of the following occurs*

- *C has at least two cusps, none of them along ℓ.*
- *ℓ is a bitangent of T, and C might be smooth or has double points along $C \cap \ell$.*
- *C has an E_6 singularity, but not along $C \cap \ell$, and there is a point $q \in C \cap \ell$ such that $(C \cdot \ell)_q \in \{2, 4\}$.*
- *C has an A_2 or A_5 singularity, C is smooth along $C \cap \ell$ and there is a point $q \in C \cap \ell$ such that $(C \cdot \ell)_q \in \{2, 4\}$.*
- *C has an A_2 or A_5 singularity not along $C \cap \ell$ and C has a double point along $C \cap \ell$.*

Proof. Suppose first that T is smooth outside $T \cap \ell$. Since we have at least two singularities that are not rationally smooth, and the singularity $y^2 = x^3 + t^2 s$ is rationally smooth (Lemma 5.16), it follows that $(T \cdot \ell) \geq 2$ for at least 2 points in the intersection. Hence, ℓ is a bitangent.

Suppose that there is a singularity $(T, q'), q' \notin \ell$ such that $H^4_p(Y) \neq 0$. Then (T, q') is a A_2, A_5 or E_6 singularity. Let $q \in T \cap \ell$. If T is smooth at q it follows from Lemma 5.16 that $(T \cdot \ell)_q \in \{2, 4\}$.

If (T, q') is an E_6 singularity then it follows from Lemma 5.17 that T has no double points. Since a reduced quartic has at most one triple point, this implies that T is smooth outside q'. Hence the second singularity such that $H^4_p(Y) \neq 0$, comes from a point in $q \in T \cap \ell$. From Lemma 5.16 it follows that $(T \cdot \ell)_q \in \{2, 4\}$.

If (T, q') is an A_2 or A_5 singularity, then (T, q) might be smooth and the intersection number $(T \cdot \ell)_q$ is 2 or 4, or (T, q) is an A_k singularity.

Suppose none of the intersections points of T and ℓ yields a non-trivial $H^4_p(Y)$. Then T has at least two singularities with types A_2, A_5, E_6. Since the combinations $2E_6$, $2A_5$, $E_6 + A_2$ and $A_5 + A_2$ are not possible, it follows that T has at least two A_2 singularities. □

Lemma 5.19. *Suppose Q is a quartic with an A_k singularity at q and ℓ is a line through p, not contained in Q. Then $(k, (Q \cdot \ell))$ is one of the following*

- $(k, 2)$, $1 \leq k \leq 7$.
- $(k, 3)$, $1 \leq k \leq 2$.
- $(k, 4)$, $1 \leq k \leq 7$, $k \neq 2$.

Proof. Since Q is a quartic, we have $1 \leq k \leq 7$. For a general line ℓ we have $(Q \cdot \ell) = 2$. This yields the case $(k, 2)$.

Suppose now $k = 2$ and ℓ is given by $t = 0$. The quartics locally given by $st + s^3$ or $st + s^4$ yield the cases $(1, 3)$ and $(1, 4)$.

Suppose now that we have $k > 1$ then we have a local equation of the form

$$t^2 + a_{30}s^3 + a_{21}s^2t + a_{12}st^2 + a_{03}t^3 + a_{40}s^4 + a_{31}s^3t + a_{22}s^2t^2 + a_{03}st^3 + a_{04}t^4.$$

Since ℓ is not a component of Q, we have that either a_{30} or a_{40} is nonzero.

If $a_{30} \neq 0$, then we have an A_2 singularity and $C \cdot \ell = 3$.

If $a_{30} = 0$, then $k \geq 3$ and $C \cdot \ell = 4$. □

Remark 5.20. A straightforward calculation shows that the contact equivalence class of $t^2 f(t,s)$, where f is a singularity on a quartic curve, is determined by the type of singularity of f and the intersection number of $t=0$ with $f(t,s)$.

Lemma 5.21. *Suppose C is a triple line ℓ together with a reduced cubic K. Suppose that Y has at least two singularities such that $H^4_p(Y) \neq 0$ then K is a cuspidal cubic, and ℓ is a flex line at a smooth point of K.*

Proof. Suppose K has two points q_1, q_2 not on ℓ yielding nonzero $H^4_p(Y)$, then K has an A_2 singularity at q_1, q_2. Since a cubic has at most one cusp, this is not possible.

Suppose there is a point $q \in K \cap \ell$ yielding a nontrivial $H^4_p(Y)$. Then from Lemma 5.16 it follows that K is singular at q, or q is a flex point and ℓ is a flex line. This implies that there is at least one point q' not on ℓ yielding non trivial local cohomology.

Since a cubic has only A_1, A_2 or D_4 singularities, and A_1, D_4 singularities yield rationally smooth points on Y, it follows that (K,q) is an A_2 singularity.

Since cuspidal cubics have exactly one singularity, it follows that (K,q) is smooth, hence ℓ is the flex line of K at q. □

Lemma 5.22. *Suppose C is a quadruple line ℓ together with a reduced conic T. Then Y has at most one singular point p with $H^4_p(Y) \neq 0$.*

Proof. Let q_1 and q_2 be points yielding nontrivial local cohomology. Since T is a conic it is either smooth or has an A_1 singularity. Since an isolated A_1 singularity yields a rational smooth singularity on Y, we have that $q_1, q_2 \in \ell$. In particular ℓ is not a tangent of T. From Lemma 5.16, it follows that $H^4_p(Y) = 0$ in this case. □

Lemma 5.23. *Suppose C consists of two double lines ℓ_1, ℓ_2 together with a reduced conic T. Suppose that Y has at least two singularities such that $H^4_p(Y) \neq 0$. Then ℓ_1 and ℓ_2 are tangent to T.*

Proof. A point on T but not in $T \cap (\ell_1 \cup \ell_2)$ is either an isolated A_1 singularity of C or smooth, hence has no no-trivial local cohomology.

From Lemma 5.16, it follows that transversal intersections of ℓ_1 with T has trivial local cohomology. Hence ℓ_1 and ℓ_2 are tangent to C. □

Lemma 5.24. *Suppose C consists of a smooth double conic K together with a reduced conic T. Suppose that Y has at least two singularities such that $H^4_p(Y) \neq 0$ then C and K have common tangents at two intersections points.*

Proof. A point on T but not in $T \cap K$ is either an isolated A_1 singularity of C or smooth, hence has trivial local cohomology.

Transversal intersections of K with T have trivial local cohomology. Hence we need at least two points such that $(K\cdot T)_q\geq 2$. Since K and T are conics, this implies that K and T have two intersections points with intersection multiplicity 2. □

LEMMA 5.25. *Suppose C consists of three double lines, not passing through one point or C consists of the union of a triple line with a double and single line, not all three passing through one point. Then there is no point with non-trivial local cohomology.*

Proof. Note that all intersections are transversal. Hence, the result follows directly from Lemma 5.16. □

We still need to determine $H^4_p(Y)$ for singularities of type (A_k,m). Note that for $(A_k,4),k\geq 4$ we have local equations

$$\left(t+s^2\right)^2\left(t^2+s^{k+1}\right)$$

which are not weighted homogeneous. For singularities of type (A_2,k), for $k\in\{3,4\}$ we have local equations

$$t^2\left(ts+(t-s)^k\right)$$

which are not weighted homogeneous. In total we have six types of singularities for which we do not have a method to calculate $H^4_p(Y)$.

It remains to consider the cases $(A_k,2)$, for $1\leq k\leq 7$, $(A_2,3)$, $(A_3,4)$ and the case that Q is smooth at the intersection points with ℓ, and ℓ is a bitangent or a quadruple tangent to Q.

LEMMA 5.26. *Suppose we have a singularity of the form*

$$y^2=x^3+t^2\left(t+s^{2k}\right)$$

with $k\in\{1,2\}$. Then $H^4_p(Y)$ is two-dimensional.

Proof. Setting weights $1,2k,2k,3k$, yields $2d-w=5k-1$. Clearly, the degree $2d_p-w_p$ part of $R(f_p)_{2d_p-w_p}$ is spanned by $s^{5k-1},xs^{3k-1},ts^{3k-1},xts^{k-1}$. At $(1:0:0:0)$ we have an A_2 singularity. The images of s^{5k-1} and xs^{3k-1} generate the local Milnor algebra, hence $H^4_p(Y)$ is 2-dimensional. □

LEMMA 5.27. *Suppose we have an $(A_k,2)$ singularity then $H^4_p(Y)$ is non-zero if and only if $k\in\{3,6\}$. If $k\in\{3,6\}$, then $H^4_p(Y)=\mathbf{C}(-2)^2$.*

Proof. A local equation is of the form

$$y^2=x^3+s^2\left(t^2+s^{k+1}\right).$$

Setting weights $6,3k+3,2k+6,3k+9$ shows that we can apply the three weight trick if $3\nmid 2k+6$. Hence, if $k\not\equiv 0 \bmod 3$ then $H^4_p(Y)=0$.

If $k=3$, we have that $R_{2d_p-w_p}$ is spanned by the images of xs^2, xt, s^4, t^2. The local Milnor algebra at $y=x=s=0$ is generated by $1, x$, hence $\tilde{R}_{2d_p-w_p}$ is spanned by xs^2, s^4 and $h_p^4(Y)=2$.

If $k=6$, we have that $R_{2d_p-w_p}$ is spanned by the images of xs^4, s^6. The local Milnor algebra at $y=x=s=0$ is generated by $1, x$, hence $\tilde{R}_{2d_p-w_p} = R_{2d_p-w_p}$ is spanned by xs^3, s^6 and $h_p^4(Y)=2$. □

LEMMA 5.28. *Suppose we have an* $(A_2, 3)$ *or an* $(A_3, 4)$ *singularity then* $H_p^4(Y)=0$.

Proof. In the first case, we have local equation $y^2=x^3+t^2(t^2+s^3)$. Setting weights $3,2,4,6$, shows that we can apply the three weights trick to reduce to the singularity $y^2=x^3+s(s+t^3)$ with weights $3,1,2,3$. From Lemma 5.16, it follows that this singularity has no local cohomology.

In the second case, we have local equation $y^2=x^3+t^2(t^2+s^4)$. Setting weights $6,3,8,12$ shows that we can apply the three weights trick. Hence, there is no local cohomology. □

Finally, in the case of a triple line and a cubic curve we have the following singularity.

LEMMA 5.29. *Suppose* (Y,p) *is a singularity of type*

$$y^2=x^3+t^3(t+s^3).$$

Then $H_p^4(Y)$ *is two-dimensional.*

Proof. If we set weights of s,t,x,y to be $1,3,4,6$, we obtain $2d-w=10$. The vector space $R(f_p)_{10}$ is spanned by $xs^6, xts^3, xt^2, t^2s^4ts^7s^{10}$. We have a singularity at $(1:0:0:0)$. The stabilizer at this point is trivial and the Milnor algebra is generated (in global coordinates) by $s^{10}, xs^6, ts^7, xts^3$, hence $\tilde{R}$ is generated by t^2s^4, xt^2. □

6. Determining the Mordell–Weil rank

To determine the Mordell–Weil rank of an elliptic threefold, we use the main result of [5]: Let $Y \subset \mathbf{P}(2,3,1,1,1)$ be a hypersurface given by

$$y^2=x^3+Px+Q,$$

where P and Q are polynomials in z_0, z_1, z_2 of degree 4 and 6 respectively.

Let $\psi : Y \dashrightarrow \mathbf{P}^2$ be the projection from $(1:1:0:0:0)$ onto the plane $\{x=y=0\}$. The map ψ is not defined at $(1:1:0:0:0)$. Let X_0 be the blow-up of Y at $(1:1:0:0:0)$. This blow-up resolves the singularity of ψ and endows X_0 with the structure of a Weierstrass fibration in the sense of Miranda. Miranda gave a description of which birational transformations one needs to apply in order to obtain an elliptic threefold $\pi : X \to S$.

The torsion part of $\mathrm{MW}(\pi)$ can be determined by specialization and we will come back to this later. In [5] we gave together with Klaus Hulek a procedure that for general Y calculates $\operatorname{rank}\mathrm{MW}(\pi)$. To determine the rank of $\mathrm{MW}(\pi)$ one can use that if $H^4(Y,\mathbf{C})$ has a pure weight 4 Hodge structure then

$$\operatorname{rank}\mathrm{MW}(\pi) = \operatorname{rank} H^{2,2}\big(H^4(Y,\mathbf{C})\big) \cap H^4(Y,\mathbf{Z}) - 1.$$

In general, intersections of the type $H^{2,2}(H^4(Y,\mathbf{C})) \cap H^4(Y,\mathbf{Z})$ are hard to calculate. An exception is the case where $H^4(Y,\mathbf{C}) = H^{2,2}(Y,\mathbf{C})$. This is actually always the case in all our examples.

LEMMA 6.1. *Suppose every non-isolated singularity of Y is weighted homogeneous. Then $H^4(Y,\mathbf{C})$ is pure of type* $(2,2)$.

Proof. Consider the exact sequence

$$\cdots \to \bigoplus_{p\in\Sigma} H^4_p(Y) \to H^4(Y) \to H^4\big(Y^*\big).$$

We start by proving that the mixed Hodge structure on $H^4(Y)$ is pure of weight 4. Since Y^* is smooth, it follows that $H^4(Y^*)$ has only Hodge weights ≥ 4, whereas $H^4(Y)$ has only Hodge weights ≤ 4, since Y is proper (both statements can be found in [10, Section 5.3]). Hence to prove the above claim, it suffices to prove that the Hodge structure on $H^4_p(Y)$ is of pure weight 4.

Suppose $p \in Y_{\mathrm{sing}}$ and suppose we have a weighted homogeneous singularity at p. Then by the results of Dimca [2] and of [5], it follows that $H^4_p(Y)$ has pure weight 4. If (Y,p) is not weighted homogeneous then this singularity is isolated. The procedures in the Singular library `gmssing.lib` allow us to calculate the weight filtration on $H^4_p(Y)$. It turns out that for all singularities mentioned in Theorem 4.1 the Hodge structure on $H^4_p(Y)$ is pure of weight 4. From this it follows that $H^4(Y)$ is pure of weight 4.

In order to prove that $H^4(Y)$ is pure of type $(2,2)$, consider $f : \tilde{Y} \to Y$, a resolution of singularities of Y. Let $\ell \subset \mathbf{P}^2$ be a general line. Then $Y_\ell := f^{-1}(\psi^{-1}(\ell))$ is irreducible and is a rational elliptic surface. Moreover, since ℓ is ample, we have by Lefschetz' hyperplane theorem that $H^2(\tilde{Y}) \to H^2(Y_\ell)$ is injective. From the rationality of Y_ℓ it follows that $h^{2,0}(Y_\ell) = 0$ and therefore $h^{2,0}(\tilde{Y}) = 0$. Using Poincaré duality one obtains $h^{3,1}(\tilde{Y}) = h^{1,3}(\tilde{Y}) = 0$. In particular $H^{2,2}(\tilde{Y}) = H^4(\tilde{Y})$.

We have an exact sequence $H^3(E) \to H^4(Y) \to H^4(\tilde{Y})$. Since E is proper it turns out that there the graded piece of weight 4 in $H^3(E)$ is trivial. Since $H^4(Y)$ is pure of weight 4 this exact sequence implies that $H^4(Y)$ injects in $H^4(\tilde{Y})$. The latter Hodge structure is pure of type $(2,2)$, so the same holds for $H^4(Y)$. □

PROPOSITION 6.2. *Suppose (Y,p) is a semi-weighted homogeneous hypersurface singularity. Then $H^4(\mathbf{P}\setminus Y,\mathbf{C}) \to H^4_p(Y)$ is surjective.*

Proof. Suppose first that $j=0$. Then there exist positive integers d (divisible by 6), v_1, v_2, α, β, γ, δ such that $v_1\alpha+v_2\beta=v_1\gamma+v_2\delta=d$, and the gcd of $d/6$, v_1 and v_2 equals 1 and (Y,p) is locally given by $y^2+x^3+s^\alpha t^\beta+s^\gamma t^\delta$ plus terms of the same or higher (weighted) degree. Moreover, since C is a sextic we may assume that $\alpha+\beta$ and $\gamma+\delta$ are at most 6.

If both v_1 and v_2 are divisible by 2 then three of the weights are divisible by 2 and we can apply the 3 weights trick and obtain that $H^4_p(Y)=0$. The same conclusion holds if both v_1 and v_2 are divisible by 3.

For all other choices of (v_1,v_2) we used the computer program Singular to calculate $2d-w$ and $d-w$. If $\mathbf{C}[x,y,s,t]_{2d-w}$ is spanned by elements of (usual) degree at most 4, then the map $H^4(U)\to H^4_p(Y)$ is surjective. The only triples (d,v_1,v_2) not satisfying this criterion are $d=12, v_1=1, v_2=3$ and $d=12, v_1=1, v_2=4$. Since the singularity lies on a sextic it turns out that this corresponds to the singularities

$$y^2=x^3+t^3\bigl(t+s^3\bigr) \quad \text{resp.} \quad y^2=x^3+t^2\bigl(t+s^4\bigr).$$

For both singularities we know $H^4_p(Y)=0$.

The case $j=1728$ can be treated similarly, but turns out to be easier. This finishes the proof. □

Summarizing, we have that $\operatorname{rank}\mathrm{MW}(\pi)=h^4(Y)-1$, that $h^4(Y)-1$ equals the dimension of the cokernel of

$$H^4(\mathbf{P}\setminus Y,\mathbf{C})\to\bigoplus_{p\in\Sigma}H^4_p(Y)$$

and that if Σ consists of one point at which Y has a weighted homogeneous singularity then this cokernel is trivial.

To calculate in practice the cokernel, we might use that this cokernel is of pure $(2,2)$-type, hence it suffices to calculate

$$\operatorname{coker}\bigl(\mathrm{Gr}^2_F H^4(U,\mathbf{C})\to\mathrm{Gr}^2_F H^4_p(Y)\bigr).$$

In the sequel, we will only calculate the rank in the case that (Y,p) is weighted homogeneous, hence for the rest of this section assume that Y has only weighted homogenous singularities. In the previous section, we showed for each weighted homogeneous singularity that $H^4_p(Y)$ is pure of type $(2,2)$. Hence, it suffices to calculate

$$\operatorname{coker}\bigl(\mathrm{Gr}^2_F H^4(U,\mathbf{C})\to H^4_p(Y)\bigr).$$

There is a natural map $\mathbf{C}[x,y,z_0,z_1,z_2]_4\to\mathrm{Gr}^2_F H^4(U,\mathbf{C})$ given by

$$g\mapsto\frac{g}{f^2}\Omega.$$

Here f is a defining equation for Y and Ω is the "standard" 4-form on $\mathbf{P}$ (cf. [5, Section 5]). The Jacobian ideal lies in the kernel of this map (see e.g.,

[2]). Since y is in the Jacobian ideal, we get a surjection $\mathbf{C}[x, z_0, z_1, z_2]_4 \to H^4(U, \mathbf{C})$.

The map $H^4(U, \mathbf{C}) \to H^4_p(Y, \mathbf{C})$ can be calculated as follows. In the previous section we provided generators $g_1, \ldots, g_k$ for $H^4_p(Y, \mathbf{C})$. Now the map $\mathbf{C}[z_0, z_1, z_2]_2 x \oplus \mathbf{C}[z_0, z_1, z_2]_4 \to H^4_p(Y, \mathbf{C})$ is given by

$$G \to \left(\frac{\partial G}{\partial g_1}(p), \ldots, \frac{\partial G}{\partial g_k}(p) \right).$$

We can simplify the calculation of the Mordell–Weil rank further: the only interesting cases are $j(\pi) = 0, 1728$. In that case, the fibration with section has an extra automorphism, namely

$$\omega : (x, y, z_0, z_1, z_2) \to (\omega x, y, z_0, z_1, z_2)$$

with $\omega^2 = -\omega - 1$ (if $j(\pi) = 0$) or

$$i : (x, y, z_0, z_1, z_2) \to (-x, iy, z_0, z_1, z_2) \quad \text{if } j(\pi) = 1728.$$

Let σ either be ω or i. The action of σ gives $\mathrm{MW}(\pi)$ the structure of a $\mathbf{Z}[\sigma]$-module. In particular the $\mathbf{Z}$-rank of $\mathrm{MW}(\pi)$ is twice the $\mathbf{Z}[\sigma]$-rank of $\mathrm{MW}(\pi)$. If we fix a basis $P_1, \ldots, P_r$ for $\mathrm{MW}(\pi)/\mathrm{MW}(\pi)_{\mathrm{tor}}$ as $\mathbf{Z}[\sigma]$-module, then $P_1, \sigma P_1, \ldots, P_r, \sigma P_r$ is a basis for $\mathrm{MW}(\pi)/\mathrm{MW}(\pi)_{\mathrm{tor}}$ as $\mathbf{Z}$-module.

Then σ acts on $P_i, \sigma P_i$ as

$$\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \quad \text{resp.} \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

This implies that on $\mathrm{MW}(\pi) \otimes_{\mathbf{Z}} \overline{\mathbf{Q}}$ the only eigenvalues of σ are ω, ω^2 resp. $i, -i$, and the corresponding two eigenspaces have the same dimension.

The automorphism σ induces actions on $H^4(Y, \mathbf{C})_{\mathrm{prim}}$, $H^4_p(Y, \mathbf{C})$ and the graded piece $\mathrm{Gr}^k_F H^4(U, \mathbf{C})$. Recall that we are interested in the calculation of the cokernel of

$$F^3 H^4(U, \mathbf{C}) \to \bigoplus_{p \in \mathcal{P}} H^4_p(Y).$$

The cokernel is a direct sum of the two eigenspaces and both eigenspaces have the same dimension. Hence, it suffices the calculate the dimension of the ω^2 (resp. i) eigenspace of the cokernel.

Since $\sigma(\Omega) = \omega\Omega$ if $j(\pi) = 0$ (resp. $-i\Omega$ if $j(\pi) = 1728$) and $F^3 H^4(U, \mathbf{C})$ is a quotient of

$$\left(x\mathbf{C}[z_0, z_1, z_2]_2 \oplus \mathbf{C}[z_0, z_1, z_2]_4 \right) \cdot \frac{1}{f^2}\Omega,$$

it follows that the ω^2-eigenspace, respectively, the i eigenspace is the co-kernel of

$$x\mathbf{C}[z_0, z_1, z_2]_2 \cdot \frac{1}{f^2}\Omega \to \bigoplus H^4_p(Y, \mathbf{C})^{\sigma - \omega^2, \sigma + i}.$$

At the level of the local cohomology the same phenomena happens i.e., σ acts on monomials of the form $xh(t,s)$ as multiplication by ω^2 resp. i, and on monomials of the form $h(t,s)$ it acts as ω, resp. $-i$.

Remark 6.3. It should be remarked that on $H^4_p(Y)$ the two eigenspaces have the same dimension. However, on $F^3H^4(U,\mathbf{C})$ the two eigenspaces have different dimensions, namely 6 and 15. For computational reasons we choose to work with the 6-dimensional space.

7. Classification I: $j(\pi)=1728$

This case is rather easy.

Lemma 7.1. *Suppose* $j(\pi)=1728$. *Then* $\mathrm{MW}(\pi)_{\mathrm{tor}} \not\cong \mathbf{Z}/2\mathbf{Z}$ *if and only if* C *is a double conic. If* C *is a double conic then* $\mathrm{MW}(\pi)_{\mathrm{tor}} = (\mathbf{Z}/2\mathbf{Z})^2$.

Proof. From Lemma 2.5 it follows that $\mathbf{Z}/2\mathbf{Z}$ is a subgroup of $\mathrm{MW}(\pi)$. Suppose that $\#\mathrm{MW}(\pi)_{\mathrm{tor}} > 2$, then for a general line ℓ the specialization map $\mathrm{MW}(\pi) \to \mathrm{MW}(\pi_\ell)$ is injective hence π_ℓ is a rational elliptic surface $j(\pi_\ell) = 1728$ and the torsion subgroup of $\mathrm{MW}(\pi_\ell)$ consists of at least 3 elements. From the classification of rational elliptic surfaces it follows that $C \cap \ell$ consists of two points with multiplicity 2. Hence C is a double conic. Conversely, if C is a double conic, then Y is given by $y^2 = x^3 + f^2x$. This threefold has $x = \pm f, y = 0$ and $x = 0, y = 0$ as sections of order 2. Hence $\mathrm{MW}(\pi)_{\mathrm{tor}} \supset (\mathbf{Z}/2\mathbf{Z})^2$. From Corollary 3.3 it follows that then $\mathrm{MW}(\pi)_{\mathrm{tor}} = (\mathbf{Z}/2\mathbf{Z})^2$. □

Theorem 7.2. *Suppose* $j(\pi)=1728$ *and that* C_{red} *is not the union of lines through one point. Then* $\mathrm{MW}(\pi)$ *is infinite if and only if* C *is a quartic with two* A_3 *singularities.*

Moreover, we have

- $\mathrm{MW}(\pi) \cong \mathbf{Z}/4\mathbf{Z}$ *if and only if* C *is a double conic,*
- $\mathrm{MW}(\pi) \cong \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}^2$ *if and only if* C *is a quartic with two* A_3 *singularities.*
- $\mathrm{MW}(\pi) \cong \mathbf{Z}/2\mathbf{Z}$ *otherwise.*

Proof. Suppose C is a quartic with two A_3 singularities. A smooth degree 4 curve has Euler characteristic -4. Since the Milnor number of an A_3 singularity is 3, we obtain that C has Euler characteristic $-4+6=2$, hence $h^1(C)=0$, because $h^0(C)=h^2(C)=1$. This implies that C is a rational curve. Hence without loss of generality we may assume that C is given by $z_0^4 - z_1^2z_2^2$. It remains to show that

$$y^2 = x^3 - \left(z_0^4 - z_1^2z_2^2\right)x$$

has Mordell–Weil rank 2. Since $h^4_p(Y)=2$ and Σ consists of two points, we have $\operatorname{rank}\mathrm{MW}(\pi) \le 4$. From the surjectivity of $H^4(U) \to H^4_p(Y)$ (Proposition 6.2), it follows that the cokernel $H^4(U) \to H^4_\Sigma(Y)$ has dimension at most

3, and, since this dimension is even, it follows that $\operatorname{rank} \mathrm{MW}(\pi) \in \{0,2\}$. Note that $x = z_0^2, y = z_0 z_1 z_2$ is a point of infinite order. Hence, $\operatorname{rank} \mathrm{MW}(\pi) = 2$.

Conversely, we have that $H^4_p(Y, \mathbf{C})$ is non-zero if and only if (C, q) is an isolated singularity if type A_3, A_7 or D_7. Since $H^4(U, \mathbf{C}) \to H^4_p(Y, \mathbf{C})$ is surjective for each such singularity, we need to have at least two singularities for positive rank. This means that C is a quartic with 2 A_3 singularities.

To finish the proof, note that from Corollary 3.3 implies that if $\mathrm{MW}(\pi)$ is finite then it is isomorphic to $\mathbf{Z}/2\mathbf{Z}$ or $(\mathbf{Z}/2\mathbf{Z})^2$. From the previous lemma, it follows that the latter only occurs if C is a double conic. □

8. Partial classification: Case $j(\pi) = 0$ and C is non-reduced

Suppose C is a non-reduced sextic. Consider first the case that C is a reduced quartic with a double line. In this case, we cannot calculate $H^4_p(Y)$ for six types of the singularities that occur in this case. For this reason, we give we give a few examples with positive rank.

EXAMPLE 8.1. Suppose C is the union of a double line ℓ and a quartic Q. Then $\mathrm{MW}(\pi)$ has rank 2 if one of the following occurs

- C has an E_6 singularity, and ℓ intersects Q with multiplicity 4 in a smooth point or
- C has two A_3 singularities along ℓ.

Proof. In the first case we may assume that, after a change of coordinates if necessary, Y is given by $y^2 = x^3 + z_0^2(z_1^4 + z_0 z_2^3)$. Since $H^4_\Sigma(Y)$ is four-dimensional, $H^4(U) \to H^4_\Sigma(Y)$ is not the zero map, and the cokernel has even dimension, we have that $\operatorname{rank} \mathrm{MW}(\pi) \in \{0,2\}$. Now $x = z_0 z_2$ and $y = z_0 z_1^2$ is a point of infinite order, showing that $\operatorname{rank} \mathrm{MW}(\pi) = 2$.

In the case, we may assume that, after a change of coordinates if necessary, Y is given by $y^2 = x^3 + z_0^2(z_0^4 + z_1^2 z_2^2)$. Since C has two A_3 singularities it follows that $H^4_\Sigma(Y)$ is four-dimensional. By the same reasoning as above, we have that $\operatorname{rank} \mathrm{MW}(\pi) \in \{0,2\}$. The point $x = z_0 z_1 z_2, y = z_0 z_1^2$ has clearly infinite order, hence the rank equals 2. □

From the results in Section 5, it follows that there are non-reduced sextics, not being a double line with a quartic, that might yield elliptic threefolds with positive rank. In all these cases, it turns out that the rank equals 2.

THEOREM 8.2. *Suppose C is one of the following*

- *C is a triple line ℓ together with cuspidal cubic K, and ℓ is a flex line at a smooth point of K,*
- *C is a conic together with two double lines ℓ_1, ℓ_2, such that the $\ell_{i,\mathrm{red}}$ are tangent to C or*
- *C is a conic together C_1 with a double conic C_2, and C_1 and $C_{2,\mathrm{red}}$ intersect in precisely two points with multiplicity 2.*

Then $\mathrm{MW}(\pi)=\mathbf{Z}^2$.

Proof. Using a specialization argument, it follows that in all these cases the torsion part is trivial. In all cases, Σ consists of two points, and both points have $h^4_p(Y)=2$. The map $H^4(\mathbf{P}\setminus Y)\to H^4_{p_1}(Y)\oplus H^4_{p_2}(Y)$ is not the zero map by Proposition 6.2, hence the cokernel has dimension at most 3 and therefore $\operatorname{rank}\mathrm{MW}(\pi)\leq 3$. Since the rank is even, one has $\operatorname{rank}\mathrm{MW}(\pi)\in\{0,2\}$. In order to prove the results, it suffices to give a non-trivial section.

In the first case, without loss of generality we may assume that Y is given by

$$y^2=x^3+z_0^3\left(z_0^2z_1-z_2^3\right).$$

Then the section $x=z_0z_2$, $y=z_0^2z_1$ is non-torsion.

In the second case, without loss of generality we may assume that Y is given by

$$y^2=x^3+\left(z_0^2+z_1z_2\right)z_1^2z_2^2.$$

Then the section $x=-z_1z_2$, $y=z_0z_1z_2$ is non-torsion.

In the third case, without loss of generality we may assume that without loss of generality Y is given by

$$y^2=x^3+\left(z_0^2+z_1z_2\right)\left(\alpha z_0^2+z_1z_2\right)^2,$$

with $\alpha\in\mathbf{C}$. The section $x=(\alpha z_0^2+z_1z_2)$, $y=(\sqrt{1-\alpha})z_0^2(\alpha z_0^2+z_1z_2)$ is non-torsion. □

9. Case $j(\pi)=0$ and C is a cuspidal curve

Suppose C is a sextic with only cusps. It is well known that C can have at most 9 cusps. Moreover, at most 3 of such cusps can lie on a line and at most 6 of them on a conic.

We need the following lemma.

LEMMA 9.1. *Let* $\{p_1,\dots,p_m\}$, $m\leq 9$ *be a set of distinct points in* $\mathbf{P}^2$, *with no four points collinear and no seven points lying on the same conic. Let* K *be the cokernel of the evaluation map at* $p_1,\dots,p_m$:

$$\varphi:\mathbf{C}[z_0,z_1,z_2]_2\to\mathbf{C}^m.$$

Then $\dim K=m-6$ *for* $m\geq 7$, *and* $\dim K=0$ *for* $m\leq 5$. *For* $m=6$ *we have* $\dim K=1$ *if all the points lie on a conic*, $\dim K=0$ *otherwise.*

Proof. If $m\geq 7$, then the m points do not lie on a conic, hence the kernel of φ is trivial and the cokernel has dimension $6-m$.

If $m=6$ and the points do not lie on a conic, then the kernel of φ is again trivial and $\dim K=0$.

If $m=6$ and the points do lie on a conic, then the kernel of φ is one-dimensional and so is the cokernel.

If $m<6$, then K is nontrivial only if the elements in the kernel have a common component. Such a component is necessarily a line and $m\geq 3$. A straightforward calculation shows that if $3\leq m\leq 5$ and precisely three of the m points are collinear then the kernel of φ has dimension $6-m$, so $\dim K=0$. $\square$

Let Y be an elliptic threefold of the form $y^2=x^3+f(z_0,z_1,z_2)$ where $f=0$ is a reduced sextic with only cusps as singularities. For each cusp p_i of $f=0$, fix a direction ℓ_i such that C intersects ℓ_i with multiplicity 3 at p_i.

In Lemma 5.2, we studied the singularity $y^2=x^3+t^3+s^2$. It turns out that $H^4_p(Y)$ is generated by the class of x and t.

This implies that we can determine the cokernel of the map $H^4(U,\mathbf{C})\to H^4_\Sigma(Y)$ as follows:

$$x\mathbf{C}[z_0,z_1,z_2]_2\oplus\mathbf{C}[z_0,z_1,z_2]_4\to\left(\mathbf{C}^2\right)^m,$$

where (xf_2+f_4) is mapped to $(f_2(p_i),\frac{\partial}{\partial\ell_i}f_4)$. To simplify matters, we can decompose the cokernel into eigenspaces for the complex multiplication. One eigenspace is the cokernel of

$$\mathbf{C}[z_0,z_1,z_2]_4\to\mathbf{C}^m,\qquad f_4\to\frac{\partial}{\partial\ell_i}f_4,$$

where the other is the cokernel of

$$x\mathbf{C}[z_0,z_1,z_2]_2\to\mathbf{C}^m,\qquad xf_2\mapsto f_2(p_i).$$

By the above lemma, this map has one dimensional cokernel if $m=6$ and the cusps lie on a conic or $m=7$, a two-dimensional cokernel if $m=8$ and a three-dimensional cokernel of $m=9$. The latter case is well known, it means that the curve C is the dual of a smooth cubic.

Since both eigenspaces have equal dimension we obtain the following result.

THEOREM 9.2. *Let $f=0$ be a reduced sextic, with only cusps as singularities. Suppose the cusps are at $p_1,\ldots,p_m$. Then the elliptic threefold*

$$y^2=x^3+f$$

has the following Mordell–Weil group:

- *If $m\leq 5$ or $m=6$ and the p_i do not lie on a conic then* $\mathrm{MW}(\pi)=0$.
- *If $m=6$ and the p_i lie on a conic then* $\mathrm{MW}(\pi)=\mathbf{Z}^2$.
- *If $m\geq 7$ then* $\mathrm{MW}(\pi)=\mathbf{Z}^{2(m-6)}$.

In particular, this shows the existence of the Mordell–Weil groups $\mathbf{Z}^{2r}$ for $r=0,1,2,3$.

REMARK 9.3. Suppose C is a sextic with 9 cusps. Then C is the dual curve of a smooth cubic. Hence there is a one-dimensional family of sextics with 9 cusps, and hence a one-dimensional family of elliptic threefolds with Mordell–Weil rank 6. Since the Mordell–Weil rank is six and $H^4(Y)$ is pure of type 2, 2 it follows that $H^4(Y,\mathbf{Q}) = \mathbf{Q}(-2)^7$. All other cohomology groups, except for H^3, can be calculated using the Lefschetz hyperplane theorem, i.e., $H^{2i}(Y,\mathbf{Q}) = \mathbf{Q}(-i)$ for $i = 0,1,3$ and $H^i(Y,\mathbf{Q}) = 0$ for $i \notin \{0,2,3,4,6\}$.

As explained in [7, Section 3], it follows that $H^3(Y,\mathbf{Q}) = \mathbf{Q}(-1)^{12}$. All cohomology groups have Hodge structures of Tate type, and there is no variation of Hodge structures possible. In particular, a Torelli type result as obtained by Grooten–Steenbrink [4, Section 6] in a similar setting is not possible in our case.

10. Possible Mordell–Weil groups

In the previous section, we have seen the existence of the groups $\mathbf{Z}^{2r}$ for $r = 0,1,2,3$. In order to prove Theorem 1.1 we have to show the existence of the groups $\mathbf{Z}/3\mathbf{Z}, (\mathbf{Z}/2\mathbf{Z})^2$.

REMARK 10.1. We have that $\mathbf{Z}/3\mathbf{Z} \subset \mathrm{MW}(\pi)$ if and only if Y has an equation of the form
$$y^2 = x^3 + f^2,$$
where $f = 0$ is a cubic. We showed Lemma 3.8 that then $\mathrm{MW}(\pi) = \mathbf{Z}/3\mathbf{Z}$ unless $f = 0$ is the union of three lines, and since we have excluded the cone construction case, $f = 0$ is the union of three lines ℓ_1, ℓ_2, ℓ_3 without a common intersection point. That means that Σ consists of three points $\{p_1, p_2, p_3\}$ and at each point we have a local equation $y^2 = x^3 + (ts)^2$. As explained in Lemma 5.16, we have that $H^4_{p_i}(Y) = 0$, whence $\mathrm{MW}(\pi) = \mathbf{Z}/3\mathbf{Z}$ in this case, and $\mathbf{Z}/3\mathbf{Z} \times \mathbf{Z}^2$ is not possible.

REMARK 10.2. Suppose we have that $\mathrm{MW}(\pi) = \mathbf{Z}^8$. We showed before that than C is a reduced sextic, and is a union of six lines, not through one point. That means that for each $p \in \Sigma$ we have a local equation of the form $y^2 = x^3 + t^m + s^m$ with $2 \leq m \leq 5$. For each such singularity, we have $H^4_p(Y) = 0$, so if C is the union of lines then $\mathrm{MW}(\pi)$ is finite. This shows that $\mathbf{Z}^8$ is not possible.

Summarizing we get the following theorem.

THEOREM 10.3. *Let $y^2 = x^3 + f$ be an elliptic threefold, $f = 0$ is not the union of lines through one point.*

- $\mathrm{MW}(\pi) \cong (\mathbf{Z}/2\mathbf{Z})^2$ *if and only if $f = 0$ is triple conic.*
- $\mathrm{MW}(\pi) \cong (\mathbf{Z}/3\mathbf{Z})$ *if and only if $f = 0$ is double cubic.*
- *Otherwise* $\mathrm{MW}(\pi)$ *is one of* $0, \mathbf{Z}^2, \mathbf{Z}^4, \mathbf{Z}^6$, *and all these cases occur.*

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