Illinois Journal of Mathematics
Volume 55, Number 1, Spring 2011, Pages 261–285
S 0019-2082

EQUIVARIANT PRINCIPAL BUNDLES OVER THE COMPLEX PROJECTIVE LINE

INDRANIL BISWAS

ABSTRACT. Let G be a connected complex reductive linear algebraic group, and let $K \subset G$ be a maximal compact subgroup of it. Let E_G be a holomorphic principal G-bundles over the complex projective line $\mathbb{CP}^1$ and $E_K \subset E_G$ a C^∞ reduction of structure group of E_G to K. We consider all pairs (E_G, E_K) of this type such that the total space of E_K is equipped with a C^∞ lift of the standard action of $\mathrm{SU}(2)$ on $\mathbb{CP}^1$ which satisfies the following two conditions: the actions of K and $\mathrm{SU}(2)$ on E_K commute, and for each element $g \in \mathrm{SU}(2)$, the induced action of g on E_G is holomorphic. We give a classification of the isomorphism classes of all such objects.

1. Introduction

The projection $\mathbb{C}^2 \setminus \{0\} \longrightarrow \mathbb{CP}^1$ that sends any v to the line in $\mathbb{C}^2$ generated by v defines a holomorphic principal $\mathbb{C}^*$-bundle on $\mathbb{CP}^1$. This holomorphic principal $\mathbb{C}^*$-bundle will be denoted by $E^0_{\mathbb{C}^*}$.

Let G be a connected reductive linear algebraic group defined over the field of complex numbers. A theorem due to Grothendieck shows that all holomorphic principal G-bundles over $\mathbb{CP}^1$ are constructed from the above tautological principal $\mathbb{C}^*$-bundle $E^0_{\mathbb{C}^*}$. More precisely, given a holomorphic principal G-bundle E_G over $\mathbb{CP}^1$, there is a homomorphism

$$\chi : \mathbb{C}^* \longrightarrow G$$

such that E_G is holomorphically isomorphic to the principal G-bundle obtained by extending the structure group of $E^0_{\mathbb{C}^*}$ using χ.

Received July 27, 2009; received in final form November 10, 2010.
2010 *Mathematics Subject Classification.* 53B35, 14F05.

Two homomorphisms from $\mathbb{C}^*$ to G that differ by an inner automorphism of G produce isomorphic principal G-bundles. Therefore, the above homomorphism χ can be assumed to have the property that it factors through a fixed maximal torus T of G. Consequently, the isomorphism classes of all holomorphic principal G-bundles over $\mathbb{CP}^1$ are parametrized by $\mathrm{Hom}(\mathbb{C}^*, T)/W$, where W is the Weyl group for the maximal torus T (it is the quotient by T of the normalizer of T in G); see [5, p. 122, Théorème 1.1].

Our aim here is to understand the holomorphic Hermitian principal G-bundles over $\mathbb{CP}^1$ that are SU(2)-equivariant.

Fix a maximal compact subgroup K of E. A holomorphic Hermitian principal G-bundle over $\mathbb{CP}^1$ is a holomorphic principal G-bundle E_G together with a C^∞ reduction of structure group $E_K \subset E_G$ of E_G to K.

A SU(2)-equivariant holomorphic Hermitian principal G-bundle over $\mathbb{CP}^1$ is a triple $(E_G, E_K; \rho)$, where (E_G, E_K) is a holomorphic Hermitian principal G-bundle as above, and ρ is a smooth action of SU(2) on E_K satisfying the following conditions: it lifts the standard action on $\mathbb{CP}^1$, preserves the principal K-bundle structure, and the induced action on E_G is by holomorphic automorphisms.

The unit sphere $S^3 \subset \mathbb{C}^2 \setminus \{0\}$ for the standard inner product on $\mathbb{C}^2$ is a smooth reduction of structure group of the tautological principal $\mathbb{C}^*$-bundle $E^0_{\mathbb{C}^*}$ to the subgroup $S^1 = \mathrm{U}(1) \subset \mathbb{C}^*$. This pair $(E^0_{\mathbb{C}^*}, S^3)$ equipped with the standard action of SU(2) define a SU(2)-equivariant holomorphic Hermitian principal $\mathbb{C}^*$-bundle. We will refer to it as the tautological SU(2)-equivariant holomorphic Hermitian principal $\mathbb{C}^*$-bundle.

Take any homomorphism

$$\gamma : \mathrm{U}(1) \longrightarrow K.$$

It extends uniquely to a holomorphic homomorphism $\widetilde{\gamma} : \mathbb{C}^* \longrightarrow G$. Let $(E_G, E_K; \rho)$ be the SU(2)-equivariant holomorphic Hermitian principal G-bundle over $\mathbb{CP}^1$ obtained by extending the structure group of the tautological SU(2)-equivariant holomorphic Hermitian principal $\mathbb{C}^*$-bundle using $\widetilde{\gamma}$. The Lie algebra $\mathfrak{g}$ of G will be considered as a U(1)-module using γ and the adjoint action of G on $\mathfrak{g}$. Let $\mathrm{ad}(E_G)$ denote the adjoint vector bundle for E_G.

Fix a point

$$x \in \mathbb{CP}^1.$$

The isotropy subgroup H_x of x for the standard action of SU(2) on $\mathbb{CP}^1$ is identified with U(1) (see Equation (3.2)). The actions of H_x on $(T^{0,1}_x)^*$ and $\mathrm{ad}(E_G)_x$ together induce an action of H_x on $(T^{0,1}_x)^* \otimes \mathrm{ad}(E_G)_x$. Let

$$\mathcal{V}_\gamma := \left(\left(T^{0,1}_x\right)^* \otimes \mathrm{ad}(E_G)_x\right)^{H_x} \subset \left(T^{0,1}_x\right)^* \otimes \mathrm{ad}(E_G)_x$$

be the space of invariants for this induced action of H_x.

We prove the following theorem (see Theorem 5.1).

THEOREM 1.1. *Consider all pairs of the form* $\{\gamma, v\}$, *where*

$$\gamma : \mathrm{U}(1) \longrightarrow K$$

is a homomorphism, and

$$v \in \mathcal{V}_\gamma.$$

There is a natural map from such pairs to the SU(2)*-equivariant holomorphic Hermitian principal* G*-bundles on* $\mathbb{CP}^1$.

Given any SU(2)*-equivariant holomorphic Hermitian principal* G*-bundle* $(E_G, E_K; \rho)$ *on* $\mathbb{CP}^1$, *there is a pair* $\{\gamma, v\}$ *of the above type such that the corresponding* SU(2)*-equivariant holomorphic Hermitian principal* G*-bundle is isomorphic to* $(E_G, E_K; \rho)$.

Let $\{\gamma, v\}$ (*respectively,* $\{\gamma', v'\}$) *be a pair of the above type, and let*

$$(E_G, E_K; \rho) \quad \big(\textit{respectively, } \big(E'_G, E'_K; \rho'\big)\big)$$

be the corresponding SU(2)*-equivariant holomorphic Hermitian principal* G*-bundle. Then* $(E_G, E_K; \rho)$ *is isomorphic to* $(E'_G, E'_K; \rho')$ *if and only if there is an element* $g_0 \in K$ *that satisfies the following two conditions*:

- $\gamma'(g) = g_0^{-1}\gamma(g)g_0$ *for all* $g \in \mathrm{SU}(1)$, *and*
- $v' = (\mathrm{Id}_{(T_x^{0,1})^*} \otimes \delta_{g_0})(v)$, *where* $\delta_{g_0} : \mathrm{ad}(E_G) \longrightarrow \mathrm{ad}(E'_G)$ *is the the natural isomorphism.*

The above mentioned isomorphism δ_{g_0} is constructed in Equation (5.3).

2. Projective line and principal bundles

2.1. Action on the projective line. Let $\mathbb{CP}^1$ denote the complex projective line. So $\mathbb{CP}^1$ parametrizes all one-dimensional linear subspaces of $\mathbb{C}^2$. The group of all holomorphic automorphisms of $\mathbb{CP}^1$ will be denoted by $\mathrm{Aut}(\mathbb{CP}^1)$.

The special unitary group SU(2) has the standard action on $\mathbb{C}^2$. This action clearly induces an action of SU(2) on $\mathbb{CP}^1$. Let

$$f : \mathrm{SU}(2) \longrightarrow \mathrm{Aut}\big(\mathbb{CP}^1\big) \tag{2.1}$$

be the homomorphism giving this action of SU(2) on $\mathbb{CP}^1$. The kernel of f is $\pm I$, which is also the center of SU(2).

Let

$$\psi : \mathbb{C}^2 \setminus \{0\} \longrightarrow \mathbb{CP}^1 \tag{2.2}$$

be the natural projection that sends any nonzero vector to the line generated by it. Let

$$\omega_0 := \frac{\sqrt{-1}}{2} \cdot \frac{\mathrm{d}x \wedge \mathrm{d}\overline{x} + \mathrm{d}y \wedge \mathrm{d}\overline{y}}{|x|^2 + |y|^2}$$

be the positive (1,1)-form on $\mathbb{C}^2 \setminus \{0\}$. Consider the restriction of ω_0 to the direction orthogonal to the radial vector field on $\mathbb{C}^2 \setminus \{0\}$. This restriction descends, using the projection ψ in Equation (2.2), to a Hermitian $(1,1)$-form on $\mathbb{CP}^1$. Let

$$\omega \in C^\infty\big(\mathbb{CP}^1; \Omega^{1,1}_{\mathbb{CP}^1}\big) \tag{2.3}$$

be this descended form. Since ω_0 is positive, it follows that ω is also positive. Since $\dim_{\mathbb{C}} \mathbb{CP}^1 = 1$, the form ω defines a Kähler structure on $\mathbb{CP}^1$. It is easy to check that ω is the unique Kähler form on $\mathbb{CP}^1$ of total volume $2/3$ which is left invariant by the action of $\mathrm{SU}(2)$ on $\mathbb{CP}^1$.

2.2. Principal bundles. Let G be a connected reductive linear algebraic group defined over $\mathbb{C}$. Fix a maximal compact subgroup

$$K \subset G. \tag{2.4}$$

It is known that any two maximal compact subgroups of G are conjugate [6, p. 256, Theorem 2.1].

Let E_G be a C^∞ principal G-bundle on $\mathbb{CP}^1$. A *Hermitian structure on* E_G is a C^∞ reduction of structure group of E_G

$$E_K \subset E_G$$

to the subgroup K in Equation (2.4). By a *holomorphic Hermitian principal G-bundle* on $\mathbb{CP}^1$, we will mean a holomorphic principal G-bundle E_G on $\mathbb{CP}^1$ together with a Hermitian structure E_K on E_G.

Let (E_G, E_K) and (E'_G, E'_K) be two holomorphic Hermitian principal G-bundles on $\mathbb{CP}^1$. Any C^∞ isomorphism of principal K-bundles

$$\beta : E_K \longrightarrow E'_K$$

extends uniquely to a C^∞ isomorphism

$$\widetilde{\beta} : E_G \longrightarrow E'_G$$

of principal G-bundles. Indeed, the diffeomorphism

$$\beta \times \mathrm{Id}_G : E_K \times G \longrightarrow E'_K \times G$$

descends to the isomorphism $\widetilde{\beta}$ of $E_G := E_K \times^K G$ with $E'_G := E'_K \times^K G$; we recall that $E_K \times^K G$ is the quotient of $E_K \times G$ obtained by identifying $(z, g) \in E_K \times G$ with $(zk, k^{-1}g)$, where $k \in K$.

The isomorphism β is called a *holomorphic isometry* if $\widetilde{\beta}$ is holomorphic. If β is a holomorphic isometry, then $\widetilde{\beta}$ is also called a *holomorphic isometry*. Note that β, being the restriction of $\widetilde{\beta}$, is uniquely determined by $\widetilde{\beta}$. Therefore, there is no abuse of terminology.

Two holomorphic Hermitian principal G-bundles are called *holomorphically isometric* if there exists a holomorphic isometry between them.

Let

$$\tau : \mathbb{CP}^1 \longrightarrow \mathbb{CP}^1$$

be a holomorphic map. Given any holomorphic Hermitian principal G-bundle (E_G, E_K) on $\mathbb{CP}^1$, its *pullback* by τ is defined to be the holomorphic Hermitian principal G-bundle $(\tau^* E_G, \tau^* E_K)$.

DEFINITION 2.1. A holomorphic Hermitian principal G-bundle (E_G, E_K) on $\mathbb{CP}^1$ is called SU(2)-*homogeneous* if for each $U \in \mathrm{SU}(2)$, the pulled back holomorphic Hermitian principal G-bundle $(f(U)^* E_G, f(U)^* E_K)$ is holomorphically isometric to (E, h), where f is the homomorphism in Equation (2.1).

DEFINITION 2.2. A SU(2)-*equivariant holomorphic Hermitian principal G-bundle* on $\mathbb{CP}^1$ is a triple $(E_G, E_K; \rho)$, where

- (E_G, E_K) is a holomorphic Hermitian principal G-bundle (E_G, E_K) on $\mathbb{CP}^1$, and
- ρ is a smooth action of SU(2) on the total space of E_G

$$\rho:\ \mathrm{SU}(2) \times E_G \longrightarrow E_G \tag{2.5}$$

  such that the following four conditions hold:
  1. $p \circ \rho(U, z) = f(U)(p(z))$ for all $(U, z) \in \mathrm{SU}(2) \times E_G$, where p is the projection of E_G to $\mathbb{CP}^1$ and f is the homomorphism in Equation (2.1),
  2. the actions of G and SU(2) on E_G commute,
  3. $\rho(\mathrm{SU}(2) \times E_K) = E_K$, and
  4. for each $U \in \mathrm{SU}(2)$, the map $E_G \longrightarrow E_G$ defined by $z \longmapsto \rho(U, z)$ is holomorphic.

Two SU(2)-equivariant holomorphic Hermitian principal G-bundles

$$(E_G, E_K; \rho) \quad \text{and} \quad \left(E'_G, E'_K; \rho'\right)$$

are called *isomorphic* if there is a holomorphic isometry

$$\widetilde{\beta} : E_G \longrightarrow E'_G$$

such that $\widetilde{\beta} \circ \rho = \rho' \circ (\mathrm{Id}_{\mathrm{SU}(2)} \times \widetilde{\beta})$.

We note that for any SU(2)-equivariant holomorphic Hermitian principal G-bundle $(E_G, E_K; \rho)$, the action on E_G of each element $U \in \mathrm{SU}(2)$ is a holomorphic isometry of the pulled back holomorphic principal G-bundle $(f(U^{-1})^* E_G, f(U^{-1})^* E_K)$ with (E_G, E_K).

2.3. SU(2)-homogeneous bundles are SU(2)-equivariant. Comparing Definition 2.2 with Definition 2.1 it follows immediately that every SU(2)-equivariant holomorphic Hermitian principal G-bundle on $\mathbb{CP}^1$ is SU(2)-homogeneous. A weak converse also holds as shown by the following lemma.

LEMMA 2.3. *Let (E_G, E_K) be a* SU(2)*-homogeneous holomorphic Hermitian principal G-bundle over $\mathbb{CP}^1$. Then the principal G-bundle E_G admits a smooth action ρ of* SU(2) *such that the triple $(E_G, E_K; \rho)$ is a* SU(2)*-equivariant holomorphic Hermitian principal G-bundle.*

Proof. Consider the homomorphism f in Equation (2.1). For any $U \in \mathrm{SU}(2)$, let $T(U)$ denote the space of all holomorphic isometries of the holomorphic Hermitian principal G-bundle $(f(U^{-1})^*E_G, f(U^{-1})^*E_K)$ with (E_G, E_K). Since (E_G, E_K) is SU(2)-homogeneous, we know that this space of holomorphic isometries is nonempty. The union

$$U_{E_G} := \bigcup_{U \in \mathrm{SU}(2)} T(U) \tag{2.6}$$

has a natural structure of a finite dimensional Lie group. The group operation is defined as follows: for $A_1 \in T(U_1)$ and $A_2 \in T(U_2)$,

$$A_1A_2 = \left(f\left(U_1^{-1}\right)^* A_2\right) \circ A_1 \in T(U_1U_2)$$

is simply the composition of the holomorphic isometry

$$A_1 : E_G \longrightarrow f\left(U_1^{-1}\right)^* E_G$$

with the holomorphic isometry

$$f\left(U_1^{-1}\right)^* A_2 : f\left(U_1^{-1}\right)^* E_G \longrightarrow f\left(U_1^{-1}\right)^* f\left(U_2^{-1}\right)^* E_G = f\left((U_1U_2)^{-1}\right)^* E_G.$$

We have a forgetful homomorphism of Lie groups from U_E in Equation (2.6)

$$H : U_{E_G} \longrightarrow \mathrm{SU}(2) \tag{2.7}$$

that sends any $A \in T(U)$ to U. It was noted above that H is surjective since (E_G, E_K) is SU(2)-homogeneous. Consequently, we have a short exact sequence of groups

$$e \longrightarrow \mathrm{Aut}(E_G, E_K) \longrightarrow U_{E_G} \xrightarrow{H} \mathrm{SU}(2) \longrightarrow e, \tag{2.8}$$

where $\mathrm{Aut}(E_G, E_K)$ is the group of all holomorphic isometries of the holomorphic Hermitian principal G-bundle (E_G, E_K), and H is constructed in Equation (2.7).

The Lie algebra of the Lie group U_{E_G} (respectively, $\mathrm{Aut}(E_G, E_K)$) will be denoted by $\widetilde{\mathfrak{g}}$ (respectively, $\mathfrak{g}_0$). Let

$$0 \longrightarrow \mathfrak{g}_0 \longrightarrow \widetilde{\mathfrak{g}} \xrightarrow{h} \mathrm{su}(2) \longrightarrow e \tag{2.9}$$

be the short exact sequence of Lie algebras associated to the short exact sequence of Lie groups in Equation (2.8). The Lie algebra su(2) of SU(2) is simple. Hence the homomorphism h in Equation (2.9) splits (see [4, p. 91, Corollaire 3]). In other words, there is a Lie algebra homomorphism

$$h' : \mathrm{su}(2) \longrightarrow \widetilde{\mathfrak{g}} \tag{2.10}$$

such that $h \circ h' = \mathrm{Id}_{\mathrm{su}(2)}$. Fix a splitting h' as in Equation (2.10). The Lie group SU(2) is simply connected. Hence, the homomorphism h' integrates into a homomorphism of Lie groups. In other words, there is a unique homomorphism of Lie groups

$$\rho' : \mathrm{SU}(2) \longrightarrow U_{E_G} \tag{2.11}$$

whose differential, at the identity element, is the homomorphism h' in Equation (2.10). Since the differential $h \circ h'$ of the homomorphism $H \circ \rho'$ is the identity automorphism of su(2), it follows that $H \circ \rho' = \mathrm{Id}_{\mathrm{SU}(2)}$.

Define

$$\rho:\ \mathrm{SU}(2) \times E_G \longrightarrow E_G \tag{2.12}$$

as follows:

$$\rho(A, z) = \rho'\big(A^{-1}\big)(z)$$

for all $(A, z) \in \mathrm{SU}(2) \times E_G$, where ρ' is the homomorphism in Equation (2.11). It is now straight-forward to check that ρ in Equation (2.12) is a smooth action of SU(2) on the total space of E_G that satisfies all the four conditions in Definition 2.2. In other words, $(E_G, E_K; \rho)$ is a SU(2)-equivariant holomorphic Hermitian principal G-bundle. This completes the proof of the lemma. □

Remark 2.4. A given SU(2)-homogeneous holomorphic Hermitian principal G-bundle can have many non-isomorphic SU(2)-equivariant structures. To explain this, take any homomorphism

$$\beta:\ \mathrm{SU}(2) \longrightarrow K.$$

Let E_G be the trivial holomorphic principal G-bundle $\mathbb{CP}^1 \times G$, and let

$$E_K := \mathbb{CP}^1 \times K \subset \mathbb{CP}^1 \times G = E_G$$

be the natural reduction of structure group to K. The group SU(2) acts on K as left translations using the homomorphism β. Consider the diagonal action of SU(2) on $E_K = \mathbb{CP}^1 \times K$ with SU(2) acting on $\mathbb{CP}^1$ using f in Equation (2.1). This diagonal action will be denoted by ρ_β. The triple $(E_G, E_K; \rho_\beta)$ is a SU(2)-equivariant holomorphic Hermitian principal G-bundle.

It is easy to see that for another homomorphism $\beta' :\ \mathrm{SU}(2) \longrightarrow K$, the corresponding SU(2)-equivariant holomorphic Hermitian principal G-bundle $(E_G, E_K; \rho_{\beta'})$ is isomorphic to $(E_G, E_K; \rho_\beta)$ if and only if there is a fixed element $g_0 \in K$ such that

$$\beta'(g) = g_0^{-1}\beta(g)g_0$$

for all $g \in \mathrm{SU}(2)$. In particular, if the homomorphism β is nontrivial, then $(E_G, E_K; \rho_\beta)$ is not isomorphic to the SU(2)-equivariant holomorphic Hermitian principal G-bundle corresponding to the trivial homomorphism of SU(2) to K.

3. Action of the isotropy subgroups

Consider the action of SU(2) on $\mathbb{CP}^1$ in Equation (2.1). For any point $x \in \mathbb{CP}^1$, let

$$H_x \subset \mathrm{SU}(2) \tag{3.1}$$

be the isotropy subgroup of x for this action. Consider the line L^x in $\mathbb{C}^2$ represented by x. Since H_x fixes x, it acts on this line L^x. This action defines a homomorphism of Lie groups

$$\chi^x : H_x \longrightarrow \mathrm{U}(1) = \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}. \tag{3.2}$$

It is easy to see that this homomorphism χ^x is an isomorphism. In other words, χ^x in Equation (3.2) identifies the isotropy subgroup H_x with U(1).

For a principal G-bundle E_G over $\mathbb{CP}^1$, its adjoint bundle will be denoted by $\mathrm{Ad}(E_G)$. We recall that

$$\mathrm{Ad}(E_G) := E_G \times^G G$$

is the fiber bundle over $\mathbb{CP}^1$ associated to E_G for the adjoint action of G on itself.

REMARK 3.1. Since the adjoint action of G on itself preserves the group structure of G, the fibers of $\mathrm{Ad}(E_G)$ are groups isomorphic to G. More precisely, for any $x \in \mathbb{CP}^1$, there is an isomorphism of G with the fiber $\mathrm{Ad}(E_G)_x$ over x which is unique up to an inner automorphism of G. Indeed, fixing a point $z \in (E_G)_x$ we get an isomorphism

$$f_z : G \longrightarrow \mathrm{Ad}(E_G)_x \tag{3.3}$$

that sends any $g \in G$ to the image of (z, g) in $\mathrm{Ad}(E_G)_x$ (recall that $\mathrm{Ad}(E_G)_x$ is a quotient of $(E_G)_x \times G$). If we replace z by zg_0, where $g_0 \in G$, then the above isomorphism $G \longrightarrow \mathrm{Ad}(E_G)_x$ gets pre-composed with the inner automorphism of G that sends any $g \in G$ to $g_0 g g_0^{-1}$.

Let $(E_G, E_K; \rho)$ be a SU(2)-equivariant holomorphic Hermitian principal G-bundle over $\mathbb{CP}^1$. The action of SU(2) on E_G defined by ρ induces an action of SU(2) on the total space of $\mathrm{Ad}(E_G)$ that lifts the action of SU(2) on $\mathbb{CP}^1$.

Take any point $x \in \mathbb{CP}^1$. Let $(E_G)_x$ denote the fiber of E_G over x. The isotropy subgroup H_x in Equation (3.1) acts on $(E_G)_x$ using ρ. The automorphisms of the principal G-bundle $(E_G)_x$ over x given by this action of H_x define a homomorphism of groups

$$\gamma_x : H_x \longrightarrow \mathrm{Ad}(E_G)_x. \tag{3.4}$$

Indeed, $\mathrm{Ad}(E_G)_x$ is the group of all diffeomorphisms of $(E_G)_x$ that commute with the action of G on $(E_G)_x$. Hence, the action of H_x on $(E_G)_x$ gives a homomorphism γ_x as in Equation (3.4).

REMARK 3.2. Take two principal G-bundles E_G^1 and E_G^2 on $\mathbb{CP}^1$. The corresponding adjoint bundles will be denoted by $\mathrm{Ad}(E_G^1)$ and $\mathrm{Ad}(E_G^2)$, respectively. Fix two points x_1 and x_2 in $\mathbb{CP}^1$. Both the fibers $\mathrm{Ad}(E_G^1)_{x_1}$ and $\mathrm{Ad}(E_G^2)_{x_2}$ are identified with the group G up to inner automorphisms of G (this was explained in Remark 3.1). Hence, the class of inner isomorphisms between $\mathrm{Ad}(E_G^1)_{x_1}$ and $\mathrm{Ad}(E_G^2)_{x_2}$ has a precise meaning. It is an isomorphism of groups

$$\beta : \mathrm{Ad}\big(E_G^1\big)_{x_1} \longrightarrow \mathrm{Ad}\big(E_G^2\big)_{x_2}$$

such that after fixing isomorphisms of $\mathrm{Ad}(E_G^1)_{x_1}$ and $\mathrm{Ad}(E_G^2)_{x_2}$ with G in the class of natural isomorphisms (which differ by inner automorphisms of G) the isomorphism β is transported to an inner automorphism of G.

The group H_x is identified with $\mathrm{U}(1)$ (see Equation (3.2)). Take any $x' \in \mathbb{CP}^1$. Since the action of $\mathrm{SU}(2)$ on $\mathbb{CP}^1$ is transitive, the homomorphism γ_x (see Equation (3.4)) is equivalent, in the following sense, to the homomorphism

$$\gamma_{x'} : H_{x'} = \mathrm{U}(1) \longrightarrow \mathrm{Ad}(E_G)_{x'}$$

constructed as in Equation (3.4) for x'. There is an inner isomorphism of the group $\mathrm{Ad}(E_G)_x$ with $\mathrm{Ad}(E_G)_{x'}$ that transports the homomorphism γ_x to $\gamma_{x'}$; see Remark 3.2 for inner isomorphism. To construct such an isomorphism of $\mathrm{Ad}(E_G)_x$ with $\mathrm{Ad}(E_G)_{x'}$, fix an element $A \in \mathrm{SU}(2)$ such that $f(A)(x) = x'$, where f is the homomorphism in Equation (2.1). The action of A on $\mathrm{Ad}(E_G)$ takes the fiber $\mathrm{Ad}(E_G)_x$ to $\mathrm{Ad}(E_G)_{x'}$. This isomorphism of $\mathrm{Ad}(E_G)_x$ with $\mathrm{Ad}(E_G)_{x'}$ given by the action of A intertwines the homomorphisms γ_x and $\gamma_{x'}$ from $\mathrm{U}(1)$ to $\mathrm{Ad}(E_G)_x$ and $\mathrm{Ad}(E_G)_{x'}$, respectively.

Remark 3.3. Since $\gamma_{x'}$ is equivalent to γ_x, if the image of γ_x lies in the center of the group $\mathrm{Ad}(E_G)_x$, then the image of $\gamma_{x'}$ also lies in the center of $\mathrm{Ad}(E_G)_{x'}$.

Corollary 3.4. *Let $(E_G, E_K; \rho)$ be a $\mathrm{SU}(2)$-equivariant holomorphic Hermitian principal G-bundle over $\mathbb{CP}^1$ such that the image of the homomorphism γ_x (see Equation (3.4)) lies in the center of the group $\mathrm{Ad}(E_G)_x$ for some $x \in X$ (hence for all $x \in X$ by Remark 3.3). Let $(E'_G, E'_K; \rho')$ be another $\mathrm{SU}(2)$-equivariant holomorphic Hermitian principal G-bundle over $\mathbb{CP}^1$ such that there is an inner isomorphism*

$$\mathrm{Ad}(E_G)_x \longrightarrow \mathrm{Ad}\big(E'_G\big)_x$$

(see Remark 3.2) that takes the homomorphism γ_x to the homomorphism

$$\gamma'_x : H_x \longrightarrow \mathrm{Ad}\big(E'_G\big)_x$$

constructed as in Equation (3.4) for $(E'_G, E'_K; \rho')$. Then the two $\mathrm{SU}(2)$-equivariant holomorphic Hermitian principal G-bundles $(E_G, E_K; \rho)$ and $(E'_G, E'_K; \rho')$ are isomorphic.

The C^∞ principal G-bundle E_G equipped with the action ρ of $\mathrm{SU}(2)$ does not admit a different holomorphic structure $\widehat{E}_G$ satisfying the condition that $(\widehat{E}_G, E_K; \rho)$ is also a $\mathrm{SU}(2)$-equivariant holomorphic Hermitian principal G-bundle.

Proof. Let

$$Z(G) \subset G \tag{3.5}$$

be the center, which is a complex Abelian reductive group. Since the right-translation action of G on itself commutes with the right-translation action of $Z(G)$ on G, for any $g \in Z(G)$, the map

$$E_G \longrightarrow E_G$$

defined by

$$z \longmapsto zg \tag{3.6}$$

is a holomorphic automorphism of the principal G-bundle E_G. Therefore, we have a homomorphism

$$\zeta : Z(G) \longrightarrow H^0\big(\mathbb{CP}^1, \mathrm{Ad}(E_G)\big), \tag{3.7}$$

where $H^0(\mathbb{CP}^1, \mathrm{Ad}(E_G))$ is the group of all holomorphic sections of $\mathrm{Ad}(E_G)$ (which is same as the group of all holomorphic global automorphisms of the principal G-bundle E_G). The homomorphism ζ takes any $g \in Z(G)$ to the automorphism of E_G defined in Equation (3.6). For any point $y \in \mathbb{CP}^1$, the image

$$\zeta(y)\big(Z(G)\big) \subset \mathrm{Ad}(E_G)_y$$

is the center of the group $\mathrm{Ad}(E_G)_y$.

Fix an element

$$\kappa \in (E_K)_x \tag{3.8}$$

in the fiber of E_K over x, and also fix an element

$$\kappa' \in \big(E'_K\big)_x, \tag{3.9}$$

where x is the point of $\mathbb{CP}^1$ in the statement of the proposition. Let

$$\tau_x : (E_G)_x \longrightarrow \big(E'_G\big)_x \tag{3.10}$$

be the isomorphism defined by

$$\kappa g \longmapsto \kappa' g$$

for all $g \in G$, where κ and κ' are the points in Equation (3.8) and Equation (3.9).

From the two conditions in the proposition that the image of the homomorphism γ_x lies in the center of $\mathrm{Ad}(E_G)_x$, and there is an inner isomorphism

$$\mathrm{Ad}(E_G)_x \longrightarrow \mathrm{Ad}\big(E'_G\big)_x$$

that takes γ_x to γ'_x, it follows immediately that the image of the homomorphism γ'_x also lies in the center of $\mathrm{Ad}(E'_G)_x$. It is now straight-forward to check that the map τ_x in Equation (3.10) intertwines the actions of the group H_x in Equation (3.1) on $(E_G)_x$ and $(E'_G)_x$.

We will show that τ_x extends uniquely to a C^∞ isomorphism of E_G with E'_G that intertwines the actions of SU(2) on E_G and E'_G.

Take any point
$$y \in \mathbb{CP}^1.$$
Fix
$$A_y \in \mathrm{SU}(2) \tag{3.11}$$
such that $f(A_y)(x) = y$, where f is the homomorphism in Equation (2.1). Let
$$\tau_y : (E_G)_y \longrightarrow \big(E'_G\big)_y \tag{3.12}$$
be the isomorphism defined by
$$\rho(A_y, z) \longmapsto \rho'\big(A_y, \tau_x(z)\big) \tag{3.13}$$
for all $z \in (E_G)_x$, where τ_x is defined in Equation (3.10). Since the actions of A_y on E_G defined by ρ sends $(E_G)_x$ isomorphically to $(E_G)_y$, the map τ_y in Equation (3.13) is well defined.

Using the fact that τ_x intertwines the actions of the isotropy subgroup H_x on $(E_G)_x$ and $(E'_G)_x$ it can be shown that the isomorphism τ_y in Equation (3.12) also intertwines the actions of H_y on the fibers $(E_G)_y$ and $(E'_G)_y$. Indeed, for any $z \in (E_G)_x$ and any $g \in H_y$, we have
$$\begin{aligned}\tau_y\big(\rho\big(g, \rho(A_y, z)\big)\big) &= \tau_y\big(\rho\big(A_y A_y^{-1} g, \rho(A_y, z)\big)\big)\\ &= \tau_y\big(\rho\big(A_y, \rho\big(A_y^{-1} g A_y, z\big)\big)\big)\end{aligned} \tag{3.14}$$
(the second equality follows from the fact that ρ is an action of the group SU(2)). Now from the definition of τ_y we have
$$\tau_y\big(\rho\big(A_y, \rho\big(A_y^{-1} g A_y, z\big)\big)\big) = \rho'\big(A_y, \tau_x\big(\rho\big(A_y^{-1} g A_y, z\big)\big)\big). \tag{3.15}$$
Clearly, $A_y^{-1} g A_y \in H_x$, hence τ_x intertwines the actions of $A_y^{-1} g A_y$ on $(E_G)_x$ and $(E'_G)_x$. In other words,
$$\tau_x\big(\rho\big(A_y^{-1} g A_y, z\big)\big) = \rho'\big(A_y^{-1} g A_y, \tau_x(z)\big). \tag{3.16}$$
Since ρ' is an action of the group SU(2), from Equation (3.16) and the definition of τ_y we have
$$\begin{aligned}\rho'\big(A_y, \tau_x\big(\rho\big(A_y^{-1} g A_y, z\big)\big)\big) &= \rho'\big(A_y, \rho'\big(A_y^{-1} g A_y, \tau_x(z)\big)\big)\\ &= \rho'\big(A_y A_y^{-1} g, \rho'\big(A_y, \tau_x(z)\big)\big).\end{aligned} \tag{3.17}$$
Also, $\rho'(A_y A_y^{-1} g, \rho'(A_y, \tau_x(z))) = \rho'(g, \tau_y(\rho(A_y, z)))$. Therefore, combining Equation (3.14), Equation (3.15) and Equation (3.17) we have
$$\tau_y\big(\rho\big(g, \rho(A_y, z)\big)\big) = \rho'\big(g, \tau_y\big(\rho(A_y, z)\big)\big).$$
In other words, the isomorphism τ_y intertwines the actions of H_y on the fibers $(E_G)_y$ and $(E'_G)_y$.

For any $A'_y \in \mathrm{SU}(2)$ such that $f(A'_y)(x) = y$, we have $A'_y = g A_y$, where $g \in H_y$. Since τ_x (respectively, τ_y) intertwines the action of H_x (respectively,

H_y) on $(E_G)_x$ and $(E'_G)_x$ (respectively, $(E_G)_y$ and $(E'_G)_y$), the isomorphism τ_y is actually independent of the choice of the element A_y in Equation (3.11) (but of course it depends on the map τ_x). Also, if $x = y$, then τ_y clearly coincides with τ_x. Hence, we have a diffeomorphism

$$\tau : E_G \longrightarrow E'_G \tag{3.18}$$

defined by

$$\tau(z) = \tau_{p(z)}(z),$$

where $p : E_G \longrightarrow \mathbb{CP}^1$ is the natural projection. This map τ clearly intertwines the actions of G on E_G and E'_G. Hence, τ in Equation (3.18) is a C^∞ isomorphism of principal bundles.

It is straight-forward to check that

$$\tau(E_K) = E'_K$$

as well as that τ intertwines the actions of SU(2) on E_G and E'_G.

Therefore, to prove that τ is an isomorphism between the two SU(2)-equivariant holomorphic Hermitian principal G-bundles

$$(E_G, E_K; \rho) \quad \text{and} \quad \big(E'_G, E'_K; \rho'\big)$$

it suffices to show that τ is holomorphic.

Pull back to E'_G the holomorphic structure on the principal G-bundle E'_G using the isomorphism τ in Equation (3.18). Any two holomorphic structures on the smooth principal G-bundle E_G differ by a smooth $(0,1)$-form with values in the adjoint vector bundle $\mathrm{ad}(E_G)$. We recall that $\mathrm{ad}(E_G) = E_G \times^G \mathfrak{g}$ is the vector bundle over $\mathbb{CP}^1$ associated to the principal G-bundle E_G for the adjoint action of G on its Lie algebra $\mathfrak{g}$. Let

$$\theta \in C^\infty\big(\Omega^{0,1}\big(\mathrm{ad}(E_G)\big)\big) \tag{3.19}$$

be the $(0,1)$-form with values in $\mathrm{ad}(E_G)$ obtained by taking the difference of the pulled back, by τ, of the holomorphic structure and the original holomorphic structure on E_G.

Since τ intertwines the actions of SU(2) on E_G and E'_G, and SU(2) acts on E_G and E'_G as holomorphic automorphisms, it follows immediately that the section θ in Equation (3.19) is left invariant by the action of SU(2) on the C^∞ vector bundle

$$\Omega^{0,1}\big(\mathrm{ad}(E_G)\big) = \big(T^{0,1}\mathbb{CP}^1\big)^* \otimes \mathrm{ad}(E_G).$$

(The action of SU(2) on $(T^{0,1}\mathbb{CP}^1)^* \otimes \mathrm{ad}(E_G)$ is the tensor product of its actions on $(T^{0,1}\mathbb{CP}^1)^*$ and $\mathrm{ad}(E_G)$.) Consider the action of the isotropy subgroup H_x on the fiber

$$\Omega^{0,1}\big(\mathrm{ad}(E_G)\big)_x = \big(T^{0,1}_x\big)^* \otimes \mathrm{ad}(E_G)_x.$$

The group $H_x = \mathrm{U}(1)$ (see Equation (3.2)) acts on the fiber $(T^{0,1}_x)^*$ as follows: any $\lambda \in \mathrm{U}(1)$ acts on $(T^{0,1}_x)^*$ as multiplication by $1/\lambda^2$. On the other hand, since the image of the homomorphism γ_x (see Equation (3.4)) lies in the center of the group $\mathrm{Ad}(E_G)_x$, it follows immediately that H_x acts trivially on the fiber $\mathrm{ad}(E_G)_x$ (the Lie algebra of the group $\mathrm{Ad}(E_G)_x$ is $\mathrm{ad}(E_G)_x$, and the adjoint action on $\mathrm{ad}(E_G)_x$ of the center of $\mathrm{Ad}(E_G)_x$ is trivial). Consequently, no nonzero element of the fiber $\Omega^{0,1}(\mathrm{ad}(E_G))_x$ is preserved by the action of H_x.

Since θ in Equation (3.19) is left invariant by the action of SU(2) on E_G, we now conclude that $\theta = 0$. In other words, the isomorphism τ in Equation (3.18) is holomorphic. This completes the proof of the first part of the proposition.

To prove the second statement of the proposition, assume that the C^∞ principal G-bundle E_G equipped with the action ρ of SU(2) admits another holomorphic structure $\widehat{E}_G$ such that $(\widehat{E}_G, E_K; \rho)$ is also a SU(2)-equivariant holomorphic Hermitian principal G-bundle. Let

$$\theta \in C^\infty\big(\Omega^{0,1}\big(\mathrm{ad}(E_G)\big)\big)$$

be the difference of the two holomorphic structures on the C^∞ principal G-bundle E_G. Clearly, θ is left invariant by the action of SU(2) on $\Omega^{0,1}(\mathrm{ad}(E_G))$. We have already shown above that such a section must vanish identically. Hence, the holomorphic structure $\widehat{E}_G$ actually coincides with the original holomorphic structure on E_G. This completes the proof of the proposition. □

The projection ψ in Equation (2.2) defines a holomorphic principal $\mathbb{C}^*$-bundle on $\mathbb{CP}^1$. Let

$$\psi : E_{\mathbb{C}^*} \longrightarrow \mathbb{CP}^1 \tag{3.20}$$

be the principal $\mathbb{C}^*$-bundle defined by ψ. We will construct a C^∞ reduction of structure group of $E_{\mathbb{C}^*}$ to the subgroup $\mathrm{U}(1) \subset \mathbb{C}^*$.

Take a point

$$x \in \mathbb{CP}^1, \tag{3.21}$$

and also fix a point $\widetilde{x} = (z_1, z_2) \in \mathbb{C}^2 \setminus \{0\}$ such that $|z_1|^2 + |z_2|^2 = 1$, and $\psi(\widetilde{x}) = x$, where ψ is the projection in Equation (2.2). Let

$$\mathcal{O}(\widetilde{x}) := \mathrm{SU}(2)(\widetilde{x}) \subset \mathbb{C}^2 \setminus \{0\}$$

be the orbit of $\widetilde{x}$ for the standard action of SU(2) on $\mathbb{C}^2 \setminus \{0\}$. Note that the action of SU(2) on $\mathbb{C}^2 \setminus \{0\}$ is free, hence $\mathcal{O}(\widetilde{x})$ is identified with SU(2). The restriction of the projection ψ to $\mathcal{O}(\widetilde{x})$

$$\psi_x : \mathcal{O}(\widetilde{x}) \longrightarrow \mathbb{CP}^1 \tag{3.22}$$

is a principal H_x-bundle, where H_x is the isotropy subgroup in Equation (3.1). We noted earlier that $H_x = \mathrm{U}(1)$ (see Equation (3.2)), hence ψ_x in Equation (3.22) defines a principal U(1)-bundle. The inclusion map $\mathcal{O}(\widetilde{x}) \hookrightarrow$

$\mathbb{C}^2 \setminus \{0\}$ is a C^∞ reduction of structure group of $E_{\mathbb{C}^*}$ (see Equation (3.20)) to U(1). The standard action of SU(2) on $\mathbb{C}^2 \setminus \{0\}$ makes the pair

$$\big(E_{\mathbb{C}^*}, \mathcal{O}(\widetilde{x})\big) \tag{3.23}$$

a SU(2)-equivariant holomorphic Hermitian principal $\mathbb{C}^*$-bundle over $\mathbb{CP}^1$.

Take a homomorphism of Lie groups

$$\gamma : \mathrm{U}(1) \longrightarrow K, \tag{3.24}$$

where K is the group in Equation (2.4). Let

$$\widetilde{\gamma} : \mathbb{C}^* \longrightarrow G \tag{3.25}$$

be an extension of γ in Equation (3.24) as a holomorphic homomorphism between complex Lie groups. We note that there is exactly one such extension.

Let (E^γ_G, E^γ_K) be the SU(2)-equivariant holomorphic Hermitian principal G-bundle over $\mathbb{CP}^1$ obtained by extending the structure group of the SU(2)-equivariant holomorphic Hermitian principal $\mathbb{C}^*$-bundle in Equation (3.23) using the homomorphism γ in Equation (3.24). More precisely, the principal G-bundle E^γ_G (respectively, the principal K-bundle E^γ_K) is obtained by extending the structure group of $E^0_{\mathbb{C}^*}$ (respectively, $\mathcal{O}(\widetilde{x})$) using the homomorphism $\widetilde{\gamma}$ (respectively, γ) in Equation (3.25) (respectively, Equation (3.24)). Note that the action of SU(2) on $\mathcal{O}(\widetilde{x})$ induces an action of SU(2) on $E^\gamma_K = \mathcal{O}(\widetilde{x}) \times^{\mathrm{U}(1)} K$.

Since E^γ_K is the extension of structure group of the principal U(1)-bundle $\mathcal{O}(\widetilde{x})$ in Equation (3.23), we have a map

$$\phi : \mathcal{O}(\widetilde{x}) \longrightarrow E^\gamma_K.$$

The fiber $\mathrm{Ad}(E^\gamma_K)_x$ over the point x in Equation (3.21) is identified with K as follows: send any $g \in K$ to the point in $\mathrm{Ad}(E^\gamma_K)_x$ defined by $(\phi(\widetilde{x}), g)$ (recall that $\mathrm{Ad}(E^\gamma_K)$ is a quotient of $E^\gamma_K \times K$). This identification of $\mathrm{Ad}(E^\gamma_K)_x$ with K extends to an identification of $\mathrm{Ad}(E^\gamma_G)_x$ with G.

Construct the homomorphism γ_x as in Equation (3.4) for the SU(2)-equivariant holomorphic Hermitian principal G-bundle (E^γ_G, E^γ_K) constructed above from the holomorphic Hermitian principal $\mathbb{C}^*$-bundle in Equation (3.23). It is straight-forward to check that γ_x coincides with the homomorphism γ in Equation (3.24) after we identify U(1) with H_x using the character χ^x in Equation (3.2).

Therefore, we have the following lemma which also complements Proposition 3.4.

LEMMA 3.5. *Take a homomorphism γ as in Equation (3.24). Associated to γ, there is a natural* SU(2)*-equivariant holomorphic Hermitian principal G-bundle $(E_G, E_K; \rho)$ over $\mathbb{CP}^1$ such that the homomorphism γ_x (see Equation (3.4)) coincides with γ after identifying* U(1) *with H_x using the character χ^x in Equation (3.2).*

4. A construction of SU(2)-equivariant holomorphic Hermitian principal bundles

Let $(E_G, E_K; \rho)$ be a SU(2)-equivariant holomorphic Hermitian principal G-bundle over $\mathbb{CP}^1$. Fix a point $x \in \mathbb{CP}^1$, and consider the homomorphism γ_x constructed in Equation (3.4). The action of SU(2) on E_G is induced by an action on E_K, namely ρ, of SU(2). Hence, the image of γ_x lies inside the subgroup $\mathrm{Ad}(E_K)_x \subset \mathrm{Ad}(E_G)_x$. Fix a point

$$z \in (E_K)_x.$$

Let f_z be the isomorphism constructed as in Equation (3.3). Since $z \in (E_K)_x$, the isomorphism f_z takes the subgroup $K \subset G$ to $\mathrm{Ad}(E_K)_x \subset \mathrm{Ad}(E_G)_x$. Define

$$\alpha_0 := f_z^{-1} \circ \gamma_x \circ \left(\chi^x\right)^{-1} : \mathrm{U}(1) \longrightarrow K \subset G \tag{4.1}$$

to be the homomorphism, where χ^x is constructed in Equation (3.2).

Now, for any point $y \in \mathbb{CP}^1 \setminus \{x\}$, consider the homomorphism

$$\gamma_y : H_y \longrightarrow \mathrm{Ad}(E_K)_y$$

constructed as in Equation (3.4). This homomorphism γ_y is conjugate to the homomorphism α_0 in Equation (4.1) after H_y is identified with U(1) using χ^y is constructed as in Equation (3.2). To see this, fix an element $g \in \mathrm{SU}(2)$ such that $f(g)(x) = y$, where f is the homomorphism in Equation (2.1). It is now straight-forward to check that the isomorphism $f_{\rho(g,z)}^{-1} \circ \gamma_y \circ (\chi^y)^{-1}$ coincides with α_0, where $f_{\rho(g,z)}$ is defined in Equation (3.3).

Using the homomorphism α_0, we will construct a smooth reduction of structure group of E_K.

Let

$$K_0 := C\big(\alpha_0\big(\mathrm{U}(1)\big)\big) \subset K \tag{4.2}$$

be the centralizer of the subgroup $\alpha_0(\mathrm{U}(1))$ of K, where α_0 is constructed in Equation (4.1). This subgroup K_0 of K is compact and connected.

Corollary 4.1. *Let $(E_G, E_K; \rho)$ be a SU(2)-equivariant holomorphic Hermitian principal G-bundle over $\mathbb{CP}^1$. The principal K-bundle E_K has a natural smooth reduction of structure group*

$$E_{K_0} \subset E_K,$$

to the subgroup K_0 in Equation (4.2), which is left invariant by the action ρ of SU(2) on E_K.

Proof. For any point $y \in \mathbb{CP}^1$, and any point $z' \in E_K$, let

$$\delta_{z'} := f_{z'}^{-1} \circ \gamma_y \circ \left(\chi^y\right)^{-1} : \mathrm{U}(1) \longrightarrow K \subset G$$

be the homomorphism, where $f_{z'} : G \longrightarrow \mathrm{Ad}(E_G)_y$ is constructed as in Equation (3.3), the homomorphism γ_y is constructed in Equation (3.4) and χ^y is defined as in Equation (3.2). We note that $\delta_z = \alpha_0$, where α_0 is constructed in Equation (4.1). Define

$$(E_{K_0})_y := \left\{ z' \in (E_K)_y \mid \delta_{z'} = \alpha_0 \right\} \subset (E_K)_y. \tag{4.3}$$

Let

$$E_{K_0} \subset E_K$$

be the sub-fiber bundle whose fiber over any point y is $(E_{K_0})_y$ defined in Equation (4.3).

It is straight-forward to check that the subgroup K_0 in Equation (4.2) acts transitively on the fibers of E_{K_0}. Therefore, E_{K_0} is a smooth reduction of structure group of E_K to K_0. The action of SU(2) on E_K evidently preserves the submanifold E_{K_0}. This completes the proof of the proposition. □

Let

$$G_0 \subset G \tag{4.4}$$

be the Zariski closure of the subgroup K_0 defined in Equation (4.2). The group G_0 is reductive, because K_0 is a compact subgroup of G. Since K_0 is connected it also follows that G_0 is connected.

Let E_{G_0} be the C^∞ principal G_0-bundle over $\mathbb{CP}^1$ obtained by extending the structure group of E_{K_0} constructed in Proposition 4.1 using the inclusion of K_0 in the group G_0 in Equation (4.4). We recall that E_{K_0} is a reduction of structure group of E_K to the subgroup $K_0 \subset K$. On the other hand, E_K is a reduction of structure group of E_G to K. Hence E_{G_0} is also a C^∞ reduction of structure group of E_G to G_0.

Since the reduction E_{K_0} is preserved by the action ρ of SU(2) on E_{K_0}, the principal K_0-bundle E_{K_0} gets an induced action. This induced action of SU(2) on E_{K_0} will be denoted by ρ_0. Now, ρ_0 induces an action of SU(2) on E_{G_0}; this induced action of SU(2) on E_{G_0} will also be denoted by ρ_0. We will show that $(E_{G_0}, E_{K_0}; \rho_0)$ has a natural structure of a SU(2)-equivariant holomorphic Hermitian principal G_0-bundle.

Take a point $x \in \mathbb{CP}^1$. Let

$$\gamma_x^0 : H_x \longrightarrow \mathrm{Ad}(E_{G_0})_x \tag{4.5}$$

be the homomorphism constructed as in Equation (3.4) for the action ρ_0 of SU(2) on E_{G_0}. Since K_0 is centralizer of $\alpha_0(\mathrm{U}(1))$ in K (see Equation (4.2)), the image $\alpha_0(\mathrm{U}(1))$ lies inside the center of K_0. Therefore, $\alpha_0(\mathrm{U}(1))$ lies inside the center of G_0. Now comparing the definitions of α_0 and γ_x (see Equation (4.1)) we conclude that the image of the homomorphism γ_x^0 in Equation (4.5) lies inside the center of $\mathrm{Ad}(E_{G_0})_x$.

Since the image of the homomorphism γ_x^0 lies inside the center of $\mathrm{Ad}(E_{G_0})_x$, from the second part of Proposition 3.4 and Lemma 3.5 we conclude that there is exactly one holomorphic structure on the principal G_0-bundle on E_{G_0}

that makes $(E_{G_0}, E_{K_0}; \rho_0)$ into a SU(2)-equivariant holomorphic Hermitian principal G_0-bundle.

The holomorphic principal G_0-bundle defined by this unique holomorphic structure on the C^∞ principal G_0-bundle E_{G_0} will be denoted by $\widehat{E}_{G_0}$. Let $\widehat{E}_G$ denote the holomorphic principal G-bundle over $\mathbb{CP}^1$ obtained by extending the structure group of $\widehat{E}_{G_0}$ using the inclusion of G_0 in G.

We noted earlier that E_{G_0} is a C^∞ reduction of structure group of E_G. Therefore, the C^∞ principal G-bundle underlying the holomorphic principal G-bundle $\widehat{E}_G$ is identified with that of E_G. Therefore, $\widehat{E}_G$ and E_G are holomorphic structures on the same C^∞ principal G-bundle such that both $(\widehat{E}_G, E_K; \rho)$ and $(E_G, E_K; \rho)$ are SU(2)-equivariant holomorphic Hermitian principal G-bundles.

We note that the SU(2)-equivariant holomorphic Hermitian principal G-bundle $(\widehat{E}_G, E_K; \rho)$ has the following property.

If we set the homomorphism γ in Lemma 3.5 to be α_0 defined in Equation (4.1), then the SU(2)-equivariant holomorphic Hermitian principal G-bundle in Lemma 3.5 is isomorphic to $(\widehat{E}_G, E_K; \rho)$. Indeed, this follows from the above construction of $(\widehat{E}_G, E_K; \rho)$, and the construction in Lemma 3.5.

The above constructions and observations are put down as the following lemma.

LEMMA 4.2. *Given any* SU(2)*-equivariant holomorphic Hermitian principal G-bundle* $(E_G, E_K; \rho)$ *on* $\mathbb{CP}^1$, *there is a natural construction, using* $(E_G, E_K; \rho)$, *of another* SU(2)*-equivariant holomorphic Hermitian principal G-bundle. Only the holomorphic structure of the principal G-bundle of the new* SU(2)*-equivariant holomorphic Hermitian principal G-bundle is different from E_G. More precisely, the underlying C^∞ principal G-bundle, the reduction of structure group to K as well as the action of* SU(2) *on the principal G-bundle remain unchanged.*

The SU(2)*-equivariant holomorphic Hermitian principal G-bundle constructed from* $(E_G, E_K; \rho)$ *is also one those constructed in Lemma* 3.5.

5. Classification of SU(2)-equivariant holomorphic Hermitian bundles

As before, let G be a connected reductive linear algebraic group defined over $\mathbb{C}$ and $K \subset G$ a maximal compact subgroup. The Lie algebra of G will be denoted by $\mathfrak{g}$.

Take any homomorphism

$$\gamma : \mathrm{U}(1) \longrightarrow K$$

as in Equation (3.24). Let $(E_G, E_K; \rho)$ be the SU(2)-equivariant holomorphic Hermitian principal G-bundle over $\mathbb{CP}^1$ obtained by extending the structure group of the SU(2)-equivariant holomorphic Hermitian principal $\mathbb{C}^*$-bundle $(E^0_{\mathbb{C}^*}, S^3)$ in Equation (3.23) using γ (see also Lemma 3.5).

The Lie algebra $\mathfrak{g}$ of G will be considered as a U(1)-module using γ and the adjoint action of G on $\mathfrak{g}$.

Fix a point

$$x \in \mathbb{CP}^1. \tag{5.1}$$

We recall that the action of any $\lambda \in H_x = \mathrm{U}(1)$ on the line $(T^{0,1}_x)^*$ is multiplication by $1/\lambda^2$ (see the proof of Proposition 3.4); as before, H_x is identified with U(1) using χ^x defined in Equation (3.2). Consider the tensor product $(T^{0,1}_x)^* \otimes \mathrm{ad}(E_G)_x$ of U(1)-modules. The U(1)-module $(T^{0,1}_x)^* \otimes \mathrm{ad}(E_G)_x$ is isomorphic to the tensor product $\mathbb{C} \otimes_{\mathbb{C}} \mathfrak{g}$ of U(1)-modules, where the action of any $\lambda \in H_x$ on $\mathbb{C}$ is multiplication by $1/\lambda^2$. Let

$$\mathcal{V}_\gamma \subset \left(T^{0,1}_x\right)^* \otimes \mathrm{ad}(E_G)_x \tag{5.2}$$

be the space of invariants for the action of $H_x = \mathrm{U}(1)$ on $(T^{0,1}_x)^* \otimes \mathrm{ad}(E_G)_x$.

Take any $g_0 \in K$. Let

$$\gamma' : \mathrm{U}(1) \longrightarrow K$$

be the homomorphism defined by $g \longmapsto g_0^{-1}\gamma(g)g_0$. Let E'_G be the holomorphic principal G-bundle over $\mathbb{CP}^1$ obtained by extending the structure group of the holomorphic principal $\mathbb{C}^*$-bundle $E^0_{\mathbb{C}^*}$ in Equation (3.23) using the (unique) homomorphism $\mathbb{C}^* \longrightarrow G$ that extends γ' (see Equation (3.25)). Let

$$\mathrm{Ad}(g_0) : \mathfrak{g} \longrightarrow \mathfrak{g}$$

be the automorphism of the Lie algebra given by the automorphism of G that sends any g to $g_0^{-1}gg_0$. This automorphism $\mathrm{Ad}(g_0)$ of $\mathfrak{g}$ induces a holomorphic isomorphism

$$\delta_{g_0} : \mathrm{ad}(E_G) \longrightarrow \mathrm{ad}\left(E'_G\right) \tag{5.3}$$

of Lie algebra bundles. We note that since the principal G-bundle E_G (respectively, E'_G) is the one obtained by extending the structure group of the principal U(1)-bundle S^3 in Equation (3.23) using γ (respectively, γ'), the adjoint vector bundles $\mathrm{ad}(E_G)$ (respectively, $\mathrm{ad}(E'_G)$) is identified with the one associated to the principal U(1)-bundle S^3 for $\mathfrak{g}$ considered as a U(1)-module using γ (respectively, γ').

Theorem 5.1. *Consider all pairs of the form* $\{\gamma, v\}$, *where*

$$\gamma : \mathrm{U}(1) \longrightarrow K$$

is a homomorphism, and

$$v \in \mathcal{V}_\gamma$$

(*see Equation* (5.2)). *There is a natural map from such pairs to the* SU(2)-*equivariant holomorphic Hermitian principal* G-*bundles on* $\mathbb{CP}^1$.

Given any SU(2)*-equivariant holomorphic Hermitian principal* G*-bundle* $(E_G, E_K; \rho)$ *on* $\mathbb{CP}^1$*, there is a pair* $\{\gamma, v\}$ *of the above type such that the corresponding* SU(2)*-equivariant holomorphic Hermitian principal* G*-bundle is isomorphic to* $(E_G, E_K; \rho)$.

Let $\{\gamma, v\}$ *and* $\{\gamma', v'\}$ *be two pairs of the above type. Let* $(E_G, E_K; \rho)$ *and* $(E'_G, E'_K; \rho')$ *be the corresponding* SU(2)*-equivariant holomorphic Hermitian principal* G*-bundles. Then* $(E_G, E_K; \rho)$ *and* $(E'_G, E'_K; \rho')$ *are isomorphic if and only if there is an element* $g_0 \in K$ *that satisfies the following two conditions*:

- $\gamma'(g) = g_0^{-1}\gamma(g)g_0$ *for all* $g \in \mathrm{SU}(1)$, *and*
- $v' = (\mathrm{Id}_{(T_x^{0,1})^*} \otimes \delta_{g_0})(v)$, *where* δ_{g_0} *is the isomorphism in Equation* (5.3), *and* x *is the point in Equation* (5.1).

Proof. Take any pair

$$\{\gamma, v\} \tag{5.4}$$

as in the statement of the theorem. First, using γ, we get a SU(2)-equivariant holomorphic Hermitian principal G-bundle $(E^\gamma_G, E^\gamma_K, \rho)$ (see Lemma 3.5). Using v, we will construct from $(E^\gamma_G, E^\gamma_K, \rho)$ a new SU(2)-equivariant holomorphic Hermitian principal G-bundle.

Consider the C^∞ principal G-bundle underlying the holomorphic principal G-bundle E^γ_G; we will denote this C^∞ principal G-bundle by E^0_G. The Dolbeault operator defining the holomorphic structure of E^γ_G will be denoted by $\overline{\partial}_{E^\gamma_G}$.

Consider the C^∞ vector bundle $\Omega^{0,1}(\mathrm{ad}(E_G)) = (T^{0,1}\mathbb{CP}^1)^* \otimes \mathrm{ad}(E_G)$ on $\mathbb{CP}^1$. The actions of SU(2) on $\mathbb{CP}^1$ and E_G together induce an action of SU(2) on $\Omega^{0,1}(\mathrm{ad}(E_G))$ (see the proof of Proposition 3.4).

Since the isotropy group H_x of the point x in Equation (5.1) acts trivially on v in Equation (5.4), translating v by the action of SU(2) on $(T^{0,1}\mathbb{CP}^1)^* \otimes \mathrm{ad}(E_G)$ we get a section

$$\widetilde{v} \in C^\infty\big(\mathbb{CP}^1, \big(T^{0,1}\mathbb{CP}^1\big)^* \otimes \mathrm{ad}(E_G)\big). \tag{5.5}$$

Therefore, $\widetilde{v}$ is the unique SU(2)-invariant section of $(T^{0,1}\mathbb{CP}^1)^* \otimes \mathrm{ad}(E_G)$ such that $\widetilde{v}(x) = v$.

Consider the Dolbeault operator

$$\overline{\partial}'_{E^\gamma_G} := \overline{\partial}_{E^\gamma_G} + \widetilde{v}$$

on the C^∞ principal G-bundle E^0_G underlying E_G, where $\widetilde{v}$ is constructed in Equation (5.5); recall that $\overline{\partial}_{E^\gamma_G}$ is the Dolbeault operator on E_G. Let E'_G denote the holomorphic principal G-bundle defined by this Dolbeault operator $\overline{\partial}'_{E^\gamma_G}$. Now $(E'_G, E_K; \rho)$ is clearly a SU(2)-equivariant holomorphic Hermitian principal G-bundle on $\mathbb{CP}^1$. Note that since $\widetilde{v}$ is invariant under the action of SU(2), the Dolbeault operator $\overline{\partial}'_{E^\gamma_G}$ is also fixed by the action of SU(2).

Let F denote the map from the pairs of the type in Equation (5.4) to the SU(2)-equivariant holomorphic Hermitian principal G-bundles on $\mathbb{CP}^1$ that sends any $\{\gamma, v\}$ to the SU(2)-equivariant holomorphic Hermitian principal G-bundle $(E'_G, E_K; \rho)$ constructed above from $\{\gamma, v\}$.

We will show that the map F defined above satisfies all the conditions in the theorem.

Take any SU(2)-equivariant holomorphic Hermitian principal G-bundle $(E_G, E_K; \rho)$ $\mathbb{CP}^1$. To show that there is a pair $\{\gamma, v\}$ such that $F(\{\gamma, v\})$ is isomorphic to $(E_G, E_K; \rho)$, first consider the homomorphism α_0 in Equation (4.1) which is constructed by fixing a point z in the fiber $(E_K)_x$. Set γ to be α_0. The SU(2)-equivariant holomorphic Hermitian principal G-bundle $F(\{\alpha_0, 0\})$ clearly coincides with the SU(2)-equivariant holomorphic Hermitian principal G-bundle

$$(\widehat{E}_G, E_K; \rho)$$

constructed in Lemma 4.2 from $(E_G, E_K; \rho)$.

The Dolbeault operator for the holomorphic principal G-bundle E_G (respectively, $\widehat{E}_G$) will be denoted by $\overline{\partial}_{E_G}$ (respectively, $\overline{\partial}_{\widehat{E}_G}$). Set

$$\theta := \overline{\partial}_{E_G} - \overline{\partial}_{\widehat{E}_G} \in C^\infty\big(\Omega^{0,1}\big(\mathrm{ad}(E_G)\big)\big). \tag{5.6}$$

(Recall that the underlying C^∞ principal G-bundle for $\widehat{E}_G$ is identified with that for E_G, hence θ is a smooth section of $\Omega^{0,1}(\mathrm{ad}(E_G))$.) Since both the operators $\overline{\partial}_{E_G}$ and $\overline{\partial}_{\widehat{E}_G}$ are fixed by the action of SU(2), it follows immediately that θ is also fixed by the action of SU(2).

Let

$$v := \theta(x) \in \Omega^{0,1}\big(\mathrm{ad}(E_G)\big)_x \tag{5.7}$$

be the evaluation at the point x (see Equation (5.1)) of the section θ constructed in Equation (5.6). It is now straight-forward to verify that the SU(2)-equivariant holomorphic Hermitian principal G-bundle $F(\{\gamma, v\})$, where v is defined in Equation (5.7), is isomorphic to $(E_G, E_K; \rho)$.

Take any pair $\{\gamma, v\}$ as in Equation (5.4). Fix an element $g_0 \in K$. Let

$$\gamma' : \mathrm{U}(1) \longrightarrow K$$

be the homomorphism defined by $g \longmapsto g_0^{-1}\gamma(g)g_0$. Set

$$v' := (\mathrm{Id}_{(T^{0,1}_x)^*} \otimes \delta_{g_0})(v) \in \big(T^{0,1}_x\big)^* \otimes \mathrm{ad}\big(E'_G\big)_x,$$

where δ_{g_0} is defined in Equation (5.3). We will show that that the SU(2)-equivariant holomorphic Hermitian principal G-bundle $F(\{\gamma, v\})$ is isomorphic to $F(\{\gamma', v'\})$, where γ' and v' are defined above.

To prove this, first consider the automorphism

$$S^3 \times G \longrightarrow S^3 \times G$$

defined by $(z, g) \longmapsto (z, g_0^{-1} g g_0)$, where S^3 is the principal U(1)-bundle in Equation (3.23). This automorphism descends to an isomorphism of principal G-bundles

$$E_G := S^3 \times^{\gamma} G \longrightarrow S^3 \times^{\gamma'} G =: E'_G.$$

Here $S^3 \times^{\gamma} G$ (respectively, $S^3 \times^{\gamma'} G$) denotes the quotient of $S^3 \times G$ that identifies any $(z, g) \in S^3 \times G$ with $(zh^{-1}, \gamma(h)g)$ (respectively, $(zh^{-1}, \gamma'(h)g)$), where $h \in \mathrm{U}(1)$. We now note that E_G (respectively, E'_G) is the C^∞ principal G-bundle underlying the SU(2)-equivariant holomorphic Hermitian principal G-bundle $F(\{\gamma, v\})$ (respectively, $F(\{\gamma', v'\})$). The above isomorphism $E_G \longrightarrow E'_G$ is holomorphic with respect to the holomorphic structures underlying $F(\{\gamma, v\})$ and $F(\{\gamma', v'\})$, and it in fact gives an isomorphism of the SU(2)-equivariant holomorphic Hermitian principal G-bundle $F(\{\gamma, v\})$ with $F(\{\gamma', v'\})$.

Take two pairs $\{\gamma, v\}$ and $\{\gamma', v'\}$ as in Equation (5.4) such that the SU(2)-equivariant holomorphic Hermitian principal G-bundle

$$F\big(\{\gamma, v\}\big) =: (E_G, E_K; \rho)$$

is isomorphic to $F(\{\gamma', v'\}) =: (E'_G, E'_K; \rho')$. To complete the proof of the theorem we need to show that there is an element $g_0 \in K$ that satisfies the two conditions in the final part of the theorem.

Let

$$\varphi : E_K \longrightarrow E'_K \tag{5.8}$$

be an isomorphism of principal K-bundles that induces an isomorphism of the SU(2)-equivariant holomorphic Hermitian principal G-bundle $(E_G, E_K; \rho)$ with $(E'_G, E'_K; \rho')$. Fix a point

$$z \in (E_K)_x$$

(respectively, $z' \in (E'_K)_x$), in the fiber over the point x in Equation (5.1), such that γ (respectively, γ') coincides with the homomorphism constructed as in Equation (4.1) using z (respectively, z').

Let $g_0 \in K$ be the unique element that satisfies the condition

$$z' = \varphi(z) g_0,$$

where φ is the isomorphism in Equation (5.8). It is now straight-forward to verify that this element g_0 satisfies the two conditions in the final part of the theorem. This completes the proof of the theorem. □

REMARK 5.2. Take a homomorphism $\gamma : \mathrm{U}(1) \longrightarrow K$. Consider the Lie algebra $\mathfrak{g}$ as a U(1)-module using γ and the adjoint action of K on $\mathfrak{g}$. Let

$$\mathfrak{g}_2 \subset \mathfrak{g} \tag{5.9}$$

be the isotypical component of the U(1)-module $\mathfrak{g}$ on which each element $\lambda \in \mathrm{U}(1)$ acts as multiplication by λ^2.

As before, let $(E_G, E_K; \rho)$ be the SU(2)-equivariant holomorphic Hermitian principal G-bundle over $\mathbb{CP}^1$ obtained by extending the structure group of the SU(2)-equivariant holomorphic Hermitian principal $\mathbb{C}^*$-bundle $(E^0_{\mathbb{C}^*}, S^3)$ in Equation (3.23) using γ. The subspace $\mathcal{V}_\gamma$ in Equation (5.2) is isomorphic to $\mathfrak{g}_2$ in Equation (5.9). To construct such an isomorphism, fix a nonzero element

$$u_0 \in \left(T^{0,1}_x\right)^*,$$

and also fix an element z_0 in the fiber, over x, of the principal U(1)-bundle S^3 in Equation (3.23). Now we have an isomorphism

$$\mathfrak{g}_2 \longrightarrow \mathcal{V}_\gamma$$

that sends any v to $u_0 \otimes \widetilde{v}$, where $\widetilde{v} \in \mathrm{ad}(E_G)_x$ is the image, in $\mathrm{ad}(E_G)_x$, of (z_0, v).

Fix a maximal torus $T \subset G$ such that the (unique) maximal compact subgroup of T is contained in K.

Take any homomorphism

$$\rho : \mathbb{C}^* \longrightarrow T \overset{\iota}{\hookrightarrow} G. \tag{5.10}$$

Let E^ρ_G denote the holomorphic principal G-bundle over $\mathbb{CP}^1$ obtained by extending the structure group of the tautological principal $\mathbb{C}^*$-bundle $E^0_{\mathbb{C}^*}$ (see Equation (3.23)) using the homomorphism $\iota \circ \rho$.

Let E_G be a holomorphic principal G-bundle over $\mathbb{CP}^1$. A theorem due to Grothendieck says that there is a homomorphism ρ as in Equation (5.10) such that the holomorphic principal G-bundle E^ρ_G is holomorphically isomorphic to E_G [5, p. 123, Théorème 1.2].

Since $K \cap T$ is the maximal compact subgroup of T, we know that

$$\rho\big(\mathrm{U}(1)\big) \subset K,$$

where ρ is the homomorphism in Equation (5.10). Let γ denote the restriction of ρ to $\mathrm{U}(1) \subset \mathbb{C}^*$. Let $(E^\gamma_G, E^\gamma_K; \rho^\gamma)$ denote the SU(2)-equivariant holomorphic Hermitian principal G-bundle over $\mathbb{CP}^1$ constructed in Lemma 3.5 from $\gamma = \rho|_{\mathrm{U}(1)}$.

The unique extension of γ to a homomorphism $\mathbb{C}^* \longrightarrow G$ (see Equation (3.25)) clearly coincides with ρ. Therefore, the principal G-bundle E_G is holomorphically isomorphic to E^γ_G. Consequently, any holomorphic principal G-bundle over $\mathbb{CP}^1$ admits the structure of a SU(2)-equivariant holomorphic Hermitian principal G-bundle.

6. The case of $G = \mathrm{GL}(r, \mathbb{C})$

To illustrate Theorem 5.1, we consider the special case where

$$G = \mathrm{GL}(r, \mathbb{C}),$$

and $K = \mathrm{U}(r)$. This case is already well understood (see [1], [2], [3], [7], [8]).

Let E be a holomorphic vector bundle of rank r over $\mathbb{CP}^1$. A theorem of Grothendieck says that E is holomorphically isomorphic to a vector bundle of the form $\bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{CP}^1}(d_i)$ [5]. The action of the group SU(2) on $\mathbb{CP}^1$ has a canonical lift to the holomorphic line bundle $\mathcal{O}_{\mathbb{CP}^1}(1)$. Therefore, the action of SU(2) on $\mathbb{CP}^1$ lifts to E. Let h be a Hermitian structure on E, and let ρ be a C^∞ lift of the action of SU(2) to E, such that the following conditions hold:

(1) The action of SU(2) to E preserves h.
(2) For each $U \in \mathrm{SU}(2)$, the diffeomorphism of the complex manifold E defined by $v \longmapsto \rho(U, v)$ is holomorphic.

Such a triple (E, h, ρ) is called a SU(2)-*equivariant* holomorphic Hermitian vector bundle.

We note that a SU(2)-equivariant holomorphic Hermitian vector bundle (E, h, ρ) is a SU(2)-equivariant holomorphic Hermitian principal $\mathrm{GL}(r, \mathbb{C})$, where $r = \mathrm{rank}(E)$.

Take a pair

$$\big(\{\mathcal{H}_n\}_{n\in\mathbb{Z}}, T\big), \tag{6.1}$$

where

(1) each $\mathcal{H}_n$ is a finite dimensional Hilbert space, and $\mathcal{H}_n = 0$ for all but finitely many n, and
(2) T is a linear operator on the direct sum $\bigoplus_{n\in\mathbb{Z}} \mathcal{H}_n$ satisfying the condition
$$T(\mathcal{H}_n) \subset \mathcal{H}_{n+2}$$
for all $n \in \mathbb{Z}$.

We will associate a SU(2)-equivariant holomorphic Hermitian vector bundle to it.

Consider the C^∞ vector bundle

$$E := \bigoplus_{n\in\mathbb{Z}} \big(\mathcal{O}_{\mathbb{CP}^1}(n) \otimes_{\mathbb{C}} \mathcal{H}_n\big). \tag{6.2}$$

The action of SU(2) on the line bundles $\mathcal{O}_{\mathbb{CP}^1}(n)$ and the trivial action of SU(2) on the vector spaces $\mathcal{H}_n$ together define an action of SU(2) on E. The inner product on the Hilbert spaces $\mathcal{H}_n$ and the Hermitian structure on the line bundles $\mathcal{O}_{\mathbb{CP}^1}(n)$ combine together to produce a SU(2)-invariant Hermitian structure on E.

The natural holomorphic structures of the line bundles $\mathcal{O}_{\mathbb{CP}^1}(n)$ together define a SU(2)-invariant holomorphic structure on E. We will construct a new SU(2)-invariant holomorphic structure on E by altering this holomorphic structure using the endomorphism T in Equation (6.1).

Using the canonical trivialization of the line $\bigwedge^2 \mathbb{C}^2$, we get an identification of $\mathcal{O}_{\mathbb{CP}^1}(2)$ with the holomorphic tangent bundle $T\mathbb{CP}^1 \longrightarrow \mathbb{CP}^1$. Contracting the Kähler form ω in Equation (2.3) with $T\mathbb{CP}^1$, the C^∞ line bundle $T\mathbb{CP}^1$

gets identified with $\Omega^{0,1}_{\mathbb{CP}^1}$. Therefore, we have a C^∞ isomorphism of line bundles

$$\mathcal{H}om\big(\mathcal{O}_{\mathbb{CP}^1}(n),\mathcal{O}_{\mathbb{CP}^1}(n+2)\big)=\mathcal{O}_{\mathbb{CP}^1}(2)=\Omega^{0,1}_{\mathbb{CP}^1}.$$

Using this isomorphism, the endomorphism T in Equation (6.1) produces a C^∞ section

$$\widehat{T}\in C^\infty\big(\mathbb{CP}^1,\Omega^{0,1}_{\mathbb{CP}^1}\otimes\mathcal{E}nd(E)\big).$$

If $\overline{\partial}_0$ is the Dolbeault operator on E defining its standard holomorphic structure, then

$$\overline{\partial}_T:=\overline{\partial}_0+\widehat{T}$$

is a new holomorphic structure on E. This new holomorphic structure is SU(2)-invariant because both $\overline{\partial}_0$ and $\widehat{T}$ are SU(2)-invariant. Therefore, we have constructed a SU(2)-equivariant holomorphic Hermitian vector bundle from the pair $(\{\mathcal{H}_n\}_{n\in\mathbb{Z}},T)$.

The above construction is bijective. More precise, this construction produces a bijection between the isomorphism classes of SU(2)-equivariant holomorphic Hermitian vector bundles on $\mathbb{CP}^1$ and the isomorphism classes of pairs of the form $(\{\mathcal{H}_n\}_{n\in\mathbb{Z}},T)$ as in (6.1).

Note that for any pair $(\{\mathcal{H}_n\}_{n\in\mathbb{Z}},T)$ as above, the rank of the corresponding SU(2)-equivariant holomorphic Hermitian vector bundle is $\sum_n \dim\mathcal{H}_n$.

The above bijective correspondence is equivalent to the one in Theorem 5.1 for $G=\mathrm{GL}(r,\mathbb{C})$. To see this, take a homomorphism $\gamma:\mathrm{U}(1)\longrightarrow\mathrm{U}(r)$ and an element $v\in\mathcal{V}_\gamma$ as in Theorem 5.1. The homomorphism γ gives the isotypical decomposition

$$\mathbb{C}^r=\bigoplus_{\chi\in\mathrm{U}(1)^*}W_\chi, \tag{6.3}$$

where $\mathrm{U}(1)^*=\mathbb{Z}$ is the group of characters of U(1) (the character for $n\in\mathbb{Z}$ is $z\longmapsto z^n$). This isotypical decomposition is orthogonal, and each subspace $W_\chi\in\mathbb{C}^r$ is equipped with the induced inner product. For each $n\in\mathbb{Z}$, associate the Hilbert space W_n in Equation (6.3).

Recall the action of SU(2) on $\mathbb{CP}^1$. It was noted in the proof of Proposition 3.4 that the isotropy group $H_x=\mathrm{U}^1$ (see Equation (3.2)) acts on the fiber $(T^{0,1}_x)^*$ as the character $-2\in\mathbb{Z}=\mathrm{U}(1)^*$. Therefore, the space of invariants $\mathcal{V}_\gamma$ in Equation (5.2) is simply

$$\mathcal{V}_\gamma=\bigoplus_{n\in\mathbb{Z}}\mathrm{Hom}_{\mathbb{C}}(W_n,W_{n+2})=\bigoplus_{n\in\mathbb{Z}}W_n^*\otimes W_{n+2}. \tag{6.4}$$

Let $T\in\bigoplus_{n\in\mathbb{Z}}\mathrm{Hom}_{\mathbb{C}}(W_n,W_{n+2})$ be the element corresponding to the element $v\in\mathcal{V}_\gamma$. So, the pair (γ,v) gives the pair $(\{W_n\}_{n\in\mathbb{Z}},T)$ which satisfies in conditions in Equation (6.1); note that

$$\sum_{n\in\mathbb{Z}}\dim W_n=r.$$

Conversely, given any pair $(\{\mathcal{H}_n\}_{n\in\mathbb{Z}}, T)$ as in (6.1), fix a linear isometry

$$\bigoplus_{n\in\mathbb{Z}} \mathcal{H}_n \xrightarrow{\sim} \mathbb{C}^r,$$

where $\sum_{n\in\mathbb{Z}} \dim \mathcal{H}_n = r$, such that the decomposition of $\mathbb{C}^r$ is orthogonal. Let

$$\gamma : \mathrm{U}(1) \longrightarrow \mathrm{U}(r)$$

be the homomorphism such that the action of $z \in \mathrm{U}(1)$ on the subspace $\mathcal{H}_n \subset \mathbb{C}^r$ is multiplication by z^n. Using the isomorphism in Equation (6.4), the element T defines an element of $\mathcal{V}_\gamma$.

References

[1] L. Álvarez-Cónsul and O. García-Prada, *Dimensional reduction and quiver bundles*, J. Reine Angew. Math. **556** (2003), 1–46. MR 1971137

[2] I. Biswas, *Holomorphic Hermitian vector bundles over the Riemann sphere*, Bull. Sci. Math. **132** (2008), 246–356. MR 2406829

[3] A. I. Bondal and M. M. Kapranov, *Homogeneous bundles*, Helices and vector bundles, London Math. Soc. Lecture Note Ser., vol. 148, Cambridge Univ. Press, Cambridge, 1990, pp. 45–55. MR 1074782

[4] N. Bourbaki, *Éléments de mathématique. XXVI. Groupes et algèbres de Lie. Chapitre 1: Algèbres de Lie*, Actualités Sci. Ind. No. 1285, Hermann, Paris, 1960. MR 0132805

[5] A. Grothendieck, *Sur la classification des fibrés holomorphes sur la sphère de Riemann*, Amer. J. Math. **79** (1957), 121–138. MR 0087176

[6] S. Helgason, *Differential geometry, lie groups, and symmetric spaces*, Graduate Studies in Mathematics, vol. 34. American Mathematical Society, Providence, RI, 2001. MR 1834454

[7] L. Hille, *Homogeneous vector bundles and Koszul algebras*, Math. Nachr. **191** (1998), 189–195. MR 1621314

[8] G. Ottaviani and E. Rubei, *Quivers and the cohomology of homogeneous vector bundles*, Duke Math. J. **132** (2006), 459–508. MR 2219264

Indranil Biswas, School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400005, India

E-mail address: indranil@math.tifr.res.in